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Abstract

We consider interregional migration, where regions may be interpreted as clubs, social subgroups, species, or strategies. Using the positive definite adaptive (PDA) dynamics, which include the replicator dynamics, we examine the evolutionary stable state (ESS) and the asymptotic stability of the spatial distribution of economic activities in a multiregional system. We derive an exact condition for the equivalence between ESS and asymptotically stable equilibrium in each PDA dynamic. We show that market outcome yields the efficient allocation of population with an additional condition. We also show that interior equilibria are stable in the presence of strong congestion diseconomies but unstable in the presence of strong agglomeration economies with a further condition.

Keywords: asymptotic stability, ESS, positive definite adaptive dynamics, replicator dynamics, economic geography.

J.E.L. Classification: C62, C73, R23.

1 Introduction

General equilibrium analysis in international economics or in economic geography has usually dealt with two regions only. It is often said that analyzing more than two regions is no easier than analyzing the universal gravitation among more than two particles in physics. General equilibrium analysis of multiregional dynamical systems is very complicated since we must simultaneously consider not only prices and quantities in all regions but also interregional migration of firms and households.

In this paper, assuming that all economic variables constituting utility functions are expressed by the spatial distribution of population as reduced forms, we pay attention to spatial equilibrium and the optimum population distribution. It is important to know if there exists a spatial equilibrium such that no economic agent has an incentive to migrate in a multiregional system. It is also important to know whether or not these equilibria are stable against any perturbations. However, Fujita, Krugman and Venables (1999), Fujita and Thisse (2001), Ottaviano, Tabuchi and Thisse (2001) and Tabuchi, Thisse and Zeng (2001) among others, analyze spatial equilibria in multiregional systems without fully examining the existence and stability of these equilibria.

Concerning the *existence* of equilibrium, Miyao (1978) showed that a spatial equilibrium in a dynamical system exists when migration is probabilistic for any continuous probability

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function, any continuous utility function, and any number of regions. Ginsburgh, Papageorgiou and Thisse (1985) have shown the existence of spatial equilibrium for a dynamical system when migration is determined by interregional utility differentials with any continuous utility function and for any number of regions.

However, very little is known concerning the *stability* of spatial equilibrium except for the fact that the number of regions is two. Miyao (1978) and Ginsburgh *et al.* (1985) analyze stability conditions in a multiregional system using probabilistic models. Although they provide some sufficient conditions for stability, these conditions seem to be far from necessary for stability. Using a deterministic model of a multiregional system, Tabuchi (1986) obtained instability conditions of interior equilibrium, while Zeng (2001) derived stability conditions of interior and corner equilibria. In both cases, however, they use a specific dynamic.

In this paper, we establish stability and instability conditions of spatial equilibrium in positive definite adaptive (PDA) dynamics including the replicator dynamics (Taylor and Jonker, 1978).

We would like to emphasize that the model of our paper is general enough in three respects. First, although we deal with regions, they may be interpreted as various social subgroups – for example, clubs in local public finance and strategies in population games. Second, our results are for any number of regions, whereas almost all results in the previous literature are limited to two regions. Finally, our dynamics is not confined to a simple economic model of utility differentials. It includes not only the gravity models in international and regional economics, but also the replicator dynamics in biology and game theory as special cases.

The remainder of the paper is organized as follows. Spatial equilibrium and its asymptotic stability are defined formally in Sections 2. We use PDA dynamics to describe migration behavior in Section 3. Section 4 defines ESS and asymptotic stability in PDA dynamics and clarifies their mathematical relationship. Based on this, we establish a theorem on the equivalence of asymptotic stability and ESS conditions under somewhat general conditions on the utility functions in Section 5. We find that the market outcome and the social optimum of population distribution coincide by imposing the symmetric condition on the utility functions in Section 6. We then identify conditions that generate market distortion. In Section 7, we derive the simple stability and instability conditions of interior equilibrium with a further assumption on the utility functions. It is clarified that the system tends to be stable (unstable) in the presence of negative (positive) externalities. Section 8 concludes.

2 Spatial equilibrium

The space-economy is made of $n \geq 2$ regions. The total population is fixed and normalized to 1. Let $x_i \in [0,1]$ denote the population share in region $i = 1, \ldots, n$ and let

$$X \equiv \left\{ \boldsymbol{x} = (x_1, \ldots, x_n)'; \quad \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0 \right\}.$$

Markets are monopolistically competitive as in Fujita *et al.* (1999) or markets are competitive with Marshallian externalities as in Henderson (1974). Each firm produces a differentiated product in a region. Each firm selects a price so as to maximize its profit in monopolistically competitive markets with free entry and free migration of firms. Each homogeneous household consumes a variety of differentiated products so as to maximize its utility under an income constraint. In addition, each household chooses a region with the highest utility under free interregional migration. Suppose that all prices and quantities are uniquely determined by solving a set of the first-order conditions for maximum, together with the zero profit condition and the free migration condition. Then, we would be able to express the indirect utility in each region as a function of population distribution in a general equilibrium context.[1]

Let $u_i(\boldsymbol{x})$ be the (indirect) utility level residing in region $i = 1, \ldots, n$, where $u_i(\boldsymbol{x})$ satisfies the so-called Lipschitz condition and is hence continuous in $\boldsymbol{x}$. Assuming a zero cost of migration, each household freely migrates between regions so as to maximize its utility, resulting in an equal utility level u^*. The distribution $\boldsymbol{x}^* \in X$ is a *spatial equilibrium* when no individual can receive a higher utility level by migrating to another region. Formally, a distribution $\boldsymbol{x}^*$ is an equilibrium if u^* exists such that

$$\begin{aligned} u_i(\boldsymbol{x}^*) &= u^* \quad \text{if} \quad x_i^* > 0, \\ u_i(\boldsymbol{x}^*) &\leq u^* \quad \text{if} \quad x_i^* = 0. \end{aligned}$$

The equality means that the utility level is constant across regions with positive population. On the other hand, the inequality implies that some regions may have zero population in equilibrium with lower utility levels than other regions. Because $u_i(\boldsymbol{x})$ is continuous, we know that a spatial equilibrium always exists from Proposition 1 of Ginsburgh *et al.* (1985).

3 Dynamics for migration

Consider an interior equilibrium $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)'$ with $x_i^* > 0$ for each region i so that we can limit our concern to a neighborhood of $\boldsymbol{x}^*$. The population (share) migrating from the origin region j to the destination region $i (\neq j)$ during the unit time period is

$$\begin{aligned} \dot{x}_{ji} &= M_{ji}(\boldsymbol{x}, u_i(\boldsymbol{x}) - u_j(\boldsymbol{x})) \\ &= f_{ij}(\boldsymbol{x})[u_i(\boldsymbol{x}) - u_j(\boldsymbol{x})] \qquad \text{for } i, j \in \{1, \ldots, n\}, \end{aligned} \tag{1}$$

where the adjustment speed of migration is

$$f_{ij}(\boldsymbol{x}) \equiv \frac{\partial M_{ji}(\boldsymbol{x}, 0)}{\partial(u_i - u_j)}$$

by Taylor's expansion with respect to $u_i - u_j$.[2] By definition of x_{ji}, the symmetry $f_{ij} = f_{ji}$ holds for all $i, j = 1, \ldots, n$. For convenience, we set $f_{ii} = 0$ for all i.

Summing (1) over j yields a dynamical system of multi-regions as:

$$\begin{aligned} \dot{x}_i &= \sum_{j \neq i} \dot{x}_{ji} \\ &= \sum_{j \neq i} f_{ij} \sum_k (u_{ik} - u_{jk}) x_k \end{aligned} \tag{2}$$

[1]See Krugman (1993) and Fujita, Krugman and Mori (1999) *inter alia* for more detailed economic descriptions of firm and household behaviors.

[2]Note that f_{ij} is related to the attributes of the distribution $\boldsymbol{x}$ and origin-destination relationships, such as the distance. For example, the gravity model sets

$$f_{ij} = \kappa \frac{x_i x_j}{d_{ij}^2},$$

where d_{ij} is the distance between regions i and j and κ is a positive constant. The term x_i is interpreted as the supply of migrants in the origin region, and the term x_j is the demand for migrants, such as job opportunities in the destination region. This gravity model is widely used in empirical analyses of migration and international trade—for example, Greenwood (1975) and Bergstrand (1985, 1989)—and there are some microeconomic foundations in Wilson (1967), Anderson (1979), Bergstrand (1985), and Sen and Smith (1995).

where

$$u_{ik} \equiv \frac{\partial u_i(\boldsymbol{x})}{\partial x_k}$$

By Taylor's expansion again, the system (2) is linearized in the neighborhood of $\boldsymbol{x} = \boldsymbol{x}^*$ as

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x})U(\boldsymbol{x})\boldsymbol{x}, \tag{3}$$

where

$$F \equiv \begin{pmatrix} f_1 & -f_{12} & \cdots & -f_{1n} \\ -f_{12} & f_2 & \cdots & -f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -f_{1n} & -f_{2n} & \cdots & f_n \end{pmatrix} \qquad U \equiv \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix}$$

and $f_i \equiv \sum_{j \neq i}^{n} f_{ij}$ for $i = 1, \ldots, n$. In evolutionary game theory, $u_i(\boldsymbol{x})$ is interpreted as absolute fitness, and $(F(\boldsymbol{x})U(\boldsymbol{x})\boldsymbol{x})_i$ is relative fitness.

Following Hopkins and Seymour (2000), we call (3) a *positive definite adaptive (PDA) dynamic*[3] if the following conditions additionally hold.

(i) Every element of F is continuously differentiable in $\boldsymbol{x}$. (4)

(ii) $\boldsymbol{y}'F\boldsymbol{y} > 0$ for all $\boldsymbol{y} \in \mathbb{R}^n$ which is not a multiple of $\mathbf{1} = (1, \ldots, 1)'$. (5)

Together with the Lipschitz condition on $u_i(\boldsymbol{x})$, condition (4) ensures a unique solution for each PDA dynamic.

The class of PDA dynamics is large enough to allow $f_{ij} \leq 0$ for several i and j. If $f_{ij} > 0$ for all $i \neq j$, then it is consistent with a well-established tradition in migration theory (Greenwood, 1975): people migrate from low- to high-utility regions from (1). If we specify $f_{ij} = \kappa x_i x_j$ for all i, j, then the PDA dynamics (3) turn out to be the replicator dynamics:

$$\dot{x}_i = \kappa x_i \left[u_i(\boldsymbol{x}) - \sum_j x_j u_j(\boldsymbol{x})\right] \qquad \text{for } i = 1, \ldots, n.$$

If we set $f_{ij} = \kappa / n$, then (3) is reduced to the simple dynamics (Tabuchi, 1986; Friedman, 1991; Zeng, 2001):

$$\dot{x}_i = \kappa \left[u_i(\boldsymbol{x}) - \frac{1}{n}\sum_j u_j(\boldsymbol{x})\right] \qquad \text{for } \; i = 1, \ldots, n.$$

4 Asymptotic stability and evolutionarily stable state

A spatial equilibrium $\boldsymbol{x}^*$ is *asymptotically stable* if, for any positive ϵ, there exists a neighborhood $N(\boldsymbol{x}^*)$ of $\boldsymbol{x}^*$ such that for any $\boldsymbol{x}^0 \in N(\boldsymbol{x}^*)$, the solution $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))'$ of a given dynamical system with an initial value $\boldsymbol{x}^0(0) = \boldsymbol{x}^0$ satisfies $||\boldsymbol{x}(t) - \boldsymbol{x}^*|| (\equiv \max_{i=1,\ldots,n} |x_i(t) - x_i^*|) < \epsilon$ for any time $t \geq 0$ and $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}^*$. It is known that equilibrium $\boldsymbol{x}^*$ of (3) is asymptotically stable if all the real parts of eigenvalues of $F(\boldsymbol{x}^*)U(\boldsymbol{x}^*)$ are negative.

[3]Hofbauer and Sigmund (1990, 1998) call such a dynamic simply *adaptive dynamic*. This dynamic is *weak compatible* with a fitness function $u_i(\boldsymbol{x})$ (Friedman, 1991).

On the other hand, a spatial equilibrium $\boldsymbol{x}^*$ is an evolutionarily stable state (ESS) if for every perturbation (strategy) $\boldsymbol{x} \neq \boldsymbol{x}^*$,

$$\sum_{i=1}^{n} x_i u_i((1-\epsilon)\boldsymbol{x}^* + \epsilon \boldsymbol{x}) < \sum_{i=1}^{n} x_i^* u_i((1-\epsilon)\boldsymbol{x}^* + \epsilon \boldsymbol{x}) \tag{6}$$

holds for any sufficiently small $\epsilon > 0$.[4] Linearizing $u_i(\cdot)$ in the neighborhood of ϵ, (6) is approximated as

$$(\boldsymbol{x} - \boldsymbol{x}^*)' U (\boldsymbol{x} - \boldsymbol{x}^*) < 0. \tag{7}$$

Hence, $\boldsymbol{x}^*$ is ESS if and only if the following optimization problem has a unique maximum at $\boldsymbol{x} = \boldsymbol{x}^*$.

$$\begin{aligned} &\max_{\boldsymbol{x}} \ (\boldsymbol{x} - \boldsymbol{x}^*)' U (\boldsymbol{x} - \boldsymbol{x}^*) \\ &\text{subject to } \sum_i x_i = 1 \end{aligned} \tag{8}$$

With the Lagrange multiplier λ_i, the first-order conditions for maximum are

$$2(x_i - x_i^*)u_{ii} + \sum_{j \neq i} (x_j - x_j^*)(u_{ij} + u_{ji}) - \lambda_i = 0 \qquad \text{for } i = 1, \dots, n,$$

Equilibrium $\boldsymbol{x}^*$ is ESS only if the following second-order conditions of semi-negative definiteness are satisfied:

$$(-1)^m U_E[1 \cdots m] \equiv (-1)^m \begin{vmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & \dfrac{U(\{1 \cdots m\}) + U(\{1 \cdots m\})'}{2} \end{vmatrix} \geq 0 \qquad \forall m \geq 2, \tag{9}$$

where

$$U(\{i_1 i_2 \cdots i_m\}) \equiv \begin{pmatrix} u_{i_1 i_1} & u_{i_1 i_2} & \cdots & u_{i_1 i_m} \\ u_{i_2 i_1} & u_{i_2 i_2} & \cdots & u_{i_2 i_m} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i_m i_1} & u_{i_m i_2} & \cdots & u_{i_m i_m} \end{pmatrix}.$$

Note that $-U_E[1 \cdots m]$ is the sum of all the cofactors of $[U(\{1 \cdots m\}) + U(\{1 \cdots m\})']/2$. If the inequalities in (9) hold strictly, then $\boldsymbol{x}^*$ becomes an ESS. For later purposes, we define

$$U_A[1 \cdots m] \equiv \begin{vmatrix} 0 & \mathbf{1}' \\ \mathbf{1} & U(\{1 \cdots m\}) \end{vmatrix},$$

which is the sum of all the cofactors of $U(\{1 \cdots m\})$.

The following lemma illuminates the important relationship between the ESS property and the asymptotic stability. Although this result is partly obtained by Samuelson (1947), Hines (1980) first derives the following lemma with respect to $\mathbb{R}^n$, and Hopkins (1999) with respect to $\mathbb{R}_0^n = \{(z_1, \dots, z_n)' | \sum_{i=1}^n z_i = 0\}$.

[4] An ESS is called *regular* if all strategies that are a best reply to $\boldsymbol{x}^*$ are in its support. Following most of the literature, we only consider a regular ESS in this paper.

Lemma 1 *The following two statements are equivalent;*

(i) Matrix $U + U'$ is negative definite when constrained to $\mathbb{R}^n_0$;

(ii) For any matrix F which is positive definite when constrained to $\mathbb{R}^n_0$, the eigenvalues of FU for all eigenvectors in $\mathbb{R}^n_0$ have negative real parts.

Proposition 1 *Suppose $U_E[i_1 \cdots i_m] \neq 0$ for any $m \geq 2$, then an equilibrium is ESS if and only if it is asymptotically stable for any PDA dynamic.*

Hopkins (1999) gives a proof for the *only if* part of Proposition 1.[5] The *if* part holds evidently.

Proposition 1 shows that the ESS is equivalent to the asymptotic stability of the whole class of PDA dynamics. When $\boldsymbol{x}^*$ is asymptotically stable for several PDA dynamics, it is not necessarily ESS. However, we will show that this is true under a certain condition in the next section.

5 Equivalence between asymptotic stability and ESS

We show in this section that the converse of Proposition 1 becomes true when one of the conditions in Lemma 2 holds. That is, the asymptotic stability conditions for any PDA dynamic coincide with the ESS conditions: if $\boldsymbol{x}^*$ is asymptotically stable in a PDA dynamic, then it is ESS, and vice versa. For this purpose, we begin with lemmas.

Lemma 2 *For $n \geq 3$, the following four statements are equivalent:*
(i) For any $m \in \{3, \ldots, n\}$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$, $U_A[i_1 \cdots i_m] = U_E[i_1 \cdots i_m]$ holds.
(ii) For any distinct $i, j, k \in \{1, \ldots, n\}$, $U_A[ijk] = U_E[ijk]$ holds.
(iii) For any distinct $i, j, k \in \{1, \ldots, n\}$, it holds that

$$u_{ij} + u_{jk} + u_{ki} = u_{ik} + u_{kj} + u_{ji}. \tag{10}$$

(iv) There exists $\boldsymbol{a} = (a_1, \ldots, a_n)' \in \mathbb{R}^n$ such that $u_{ij} - u_{ji} = a_j - a_i$ for all $i, j = 1, \ldots, n$.

A proof is in the Appendix. Lemma 2 gives exact conditions for the equivalence between the asymptotic stability and the ESS property shown in Theorem 1 below.

Lemma 3 *Suppose $U_E[i_1 \cdots i_m] \neq 0$ for any $m \geq 2$, then for any symmetric U at spatial equilibrium, the asymptotically stable equilibria are precisely the ESS in every PDA dynamic.*

Proof. For any specific PDA dynamic, each ESS is asymptotically stable by Proposition 1. In the following, we prove the converse. Due to the symmetry of U, we have

$$\begin{aligned}
\frac{d(\boldsymbol{x}'U\boldsymbol{x})}{dt} &= \dot{\boldsymbol{x}}'U\boldsymbol{x} + \boldsymbol{x}'U\dot{\boldsymbol{x}} \\
&= 2\boldsymbol{x}'U\dot{\boldsymbol{x}} \\
&= 2\sum_{i=1}^{n} u_i(\boldsymbol{x}) \sum_{j=1}^{n} [u_i(\boldsymbol{x}) - u_j(\boldsymbol{x})] f_{ij}
\end{aligned}$$

[5] The strict inequality condition of (9) is not clearly stated in Hopkins (1999). However, it is necessary because (7) is just an approximation of (6).

$$
\begin{aligned}
&= 2\left[\sum_{i=1}^{n}(u_i(\boldsymbol{x}))^2\sum_{j=1}^{n} f_{ij} - \sum_{i,j=1}^{n} u_i(\boldsymbol{x})u_j(\boldsymbol{x})f_{ij}\right] \\
&= 2(U\boldsymbol{x})'F(U\boldsymbol{x}) \\
&\geq 0.
\end{aligned}
$$

The third equality is because $u_i(\boldsymbol{x}) = (U\boldsymbol{x})_i$. The inequality is due to the definition of PDA dynamics (5), with equality if and only if $\boldsymbol{x}$ is an interior equilibrium, so that $u_i(\boldsymbol{x}) = u_j(\boldsymbol{x})$ for all $i, j = 1, \dots, n$.

Let $\boldsymbol{x}^*$ be any asymptotically stable equilibrium. Then, $\boldsymbol{x}'U\boldsymbol{x} < \boldsymbol{x}^{*\prime}U\boldsymbol{x}^*$ holds for all $\boldsymbol{x} \neq \boldsymbol{x}^*$ in the neighborhood of $\boldsymbol{x}^*$. Replacing $\boldsymbol{x}$ by $2\boldsymbol{x} - \boldsymbol{x}^*$ (which is also close to $\boldsymbol{x}^*$), we obtain $\boldsymbol{x}'U\boldsymbol{x} < \boldsymbol{x}^{*\prime}U\boldsymbol{x}$, implying that $\boldsymbol{x}^*$ is an ESS. □

Based on the lemmas, we then establish our main result.

Theorem 1 *For any U satisfying condition (10) at spatial equilibrium, the asymptotically stable equilibria are precisely the ESS in any PDA dynamic.*

Proof. Let

$$v_i(\boldsymbol{x}) = u_i(\boldsymbol{x}) - \sum_k a_k x_k.$$

First, we show that the asymptotic stability properties are the same between $v_i(\boldsymbol{x})$ and $u_i(\boldsymbol{x})$. For any distinct i, j, we readily have

$$u_i(\boldsymbol{x}) - u_j(\boldsymbol{x}) = v_i(\boldsymbol{x}) - v_j(\boldsymbol{x}), \tag{11}$$

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} - a_j = \frac{\partial u_j}{\partial x_i} - a_i = \frac{\partial v_j}{\partial x_i}, \tag{12}$$

where the second equality in (12) is due to condition (iv) in Lemma 2. From (11), the solutions $\boldsymbol{x}^v(t)$ of the dynamics (13) and $\boldsymbol{x}^u(t)$ of (14) should be the same because of the uniqueness of solution

$$\begin{cases} \dot{x}_i^v = \sum_j f_{ij}(\boldsymbol{x})[v_i(\boldsymbol{x}) - v_j(\boldsymbol{x})] \\ \boldsymbol{x}^v(0) = \boldsymbol{x}^0 \end{cases} \tag{13}$$

$$\begin{cases} \dot{x}_i^u = \sum_j f_{ij}(\boldsymbol{x})[u_i(\boldsymbol{x}) - u_j(\boldsymbol{x})] \\ \boldsymbol{x}^u(0) = \boldsymbol{x}^0 \end{cases} \tag{14}$$

implying that the asymptotic stability properties are the same between them. Second, we know from Lemma 3 that the asymptotically stable equilibria are precisely the ESS in (13) for any relative fitness function $v(\boldsymbol{x})$ satisfying the symmetry condition (12).

Third, since

$$
\begin{aligned}
\sum_{i,j}(x_i - x_i^*)v_{ij}(x_j - x_j^*) &= \sum_{i,j}(x_i - x_i^*)u_{ij}(x_j - x_j^*) - \sum_i (x_i - x_i^*)\sum_j a_j(x_j - x_j^*) \\
&= \sum_{i,j}(x_i - x_i^*)u_{ij}(x_j - x_j^*),
\end{aligned}
$$

the ESS property for $v(\boldsymbol{x})$ is identical to that for $u(\boldsymbol{x})$.

Finally, by these three equivalence, the ESS property for $u(\boldsymbol{x})$ is shown to be the same as the asymptotic stability for any $u(\boldsymbol{x})$ satisfying condition (iv) in Lemma 2. □

It is known that asymptotically stable equilibria in replicator dynamics are precisely ESS in doubly symmetric games (Losert and Akin, 1983). Theorem 1 is more general. The equivalence property between asymptotic stability and ESS holds for each PDA dynamic and for each symmetric game with a payoff matrix U satisfying (10). Note that the class of PDA dynamics includes the replicator dynamics and the class of symmetric games with payoff matrices satisfying (10) includes doubly symmetric games.

It is also known that if U is negative definite, the equilibrium is asymptotically stable from Proposition 1; and if U is positive definite, the equilibrium is unstable from Proposition 3 in Hopkins and Seymour (2000). If U is neither negative nor positive definite, then the stability depends on the dynamics F. However, the equilibrium turns out to be always unstable under condition (10) when U is not negative definite by Theorem 1, which provides nearly unifying conditions that are both sufficient and necessary.[6] Hence, Taylor-Jonker's (1978, p.151) example, that the equilibrium is not ESS but asymptotically stable, never appears under condition (10).

In the context of economic geography, Theorem 1 is interpreted as follows. ESS is based on "corporate rationality" in that *a mutant entrepreneur maximizes the average utility level of employees* by employing a small number of them from various regions and relocating them to several regions. That is, the entrepreneur optimizes the distribution of employees among multilocational branch firms. On the other hand, asymptotic stability in PDA dynamics is based on "individual rationality" in that *each individual chooses a region so as to maximize her utility level in the long run.* Note that individuals do not always migrate from lower- to higher-utility regions (i.e., $f_{ij} > 0$) in PDA dynamics. They may temporarily migrate from higher- to lower-utility regions (i.e., $f_{ij} < 0$), but they migrate so that their utilities become the highest in the long run. Under (10), the corporate behavior depicted by ESS is shown to coincide with the individual behavior described by asymptotic stability in PDA dynamics.

Not surprisingly, Theorem 1 shows that the stability conditions are determined only by the derivatives of the utilities u_{ij}'s but independent of the interaction terms f_{ij}'s for any PDA dynamic. Such independence of f_{ij} is also found in computing the spatial equilibrium, which is $u_i(\boldsymbol{x})$ =constant.

The assumption (10) is not so restrictive. From Lemma 2, (10) is the same as

$$u_{ij} = b_i + b_{ij} \qquad \text{with} \qquad b_{ij} = b_{ji} \qquad \forall i, j = 1, \ldots, n.$$

Thus, Theorem 1 applies if the utility function is linearized as

$$\begin{aligned} u_i(\boldsymbol{x}) &\simeq \bar{u}_i + \sum_j u_{ij} x_j \\ &= (\bar{u}_i + b_i) + b_{ii} x_i + \sum_{j \neq i} b_{ij} x_j \end{aligned} \tag{15}$$

in the neighborhood of equilibrium $\boldsymbol{x}^*$ as in Tabuchi (1982). The first term $(\bar{u}_i + a_i)$ is the exogenous net amenities in region i, while the second and the third terms are endogenously determined net benefits of intraregional and interregional market interactions. One may consider the population fictitious play in which one individual chooses region i and another individual selects region j with probability x_j. The second term is the case of $i = j$, where her payoff b_{ii}

[6] The reason for "nearly" is that there remain the cases of $U_E[i_1 \cdots i_m] = 0$ ($m \geq 2$) in (9). However, such critical cases are indeterminate without information on the higher-order partial derivatives.

represents the intraregional net benefits or spatial externalities of agglomeration net benefits. The third term is the case of $i \neq j$, where her payoff b_{ij} expresses the interregional net benefits, which is assumed symmetric $b_{ij} = b_{ji}$. In reality, this assumption may be justified since interregional transportation and communication benefits and costs, such as telephone charges and airfares, are symmetric. In this situation, the asymptotic stability is equivalent to the negative definiteness of U only.

6 Social optimum versus market equilibrium

Social optimum configuration $\boldsymbol{x}^o$ is defined by the most efficient allocation of population, which is the solution of

$$\max_{\boldsymbol{x}} \boldsymbol{x}'\boldsymbol{u}(\boldsymbol{x}) \tag{16}$$
$$\text{subject to } \sum_i x_i = 1.$$

As before, linearizing $\boldsymbol{u}(\boldsymbol{x})$ around $\boldsymbol{x}^o$ yields the first-order condition for optimum:

$$\sum_j (u_{ij} + u_{ji}) x_j = \text{constant}. \tag{17}$$

On the other hand, the spatial equilibrium condition is $u_i(\boldsymbol{x}) = u^*$ or

$$\sum_j u_{ij} x_j = \text{constant}. \tag{18}$$

If the symmetry condition $u_{ij} = u_{ji}$ is met for all i and j, the necessary conditions for optimum (17) and equilibrium (18) are equalized. Similar to the proof of Lemma 3, we have the following.

Theorem 2 *For any symmetric U at spatial equilibrium, the socially optimum allocation of population is automatically attained by the market mechanism for any PDA dynamic if the equilibrium is unique.*

It is often the case that true migration behavior is impossible to observe, and hence exact dynamics cannot be depicted. However, insofar as the dynamics are in the class of PDA dynamics, the market equilibrium is shown to be equal to the socially optimum if the symmetry holds near the equilibrium and the equilibrium is unique. In this situation, no market intervention is necessary. In other words, the reasons to prevent equilibrium paths approaching social optimum are the asymmetric marginal utilities and multiple equilibria.

It is worth noting that *the intraregional externalities u_{ii} do not cause any market distortion if the conditions in Theorem 2 are met.* This seems inconsistent with the well-known result that the market outcome with positive (negative) externalities $u_{ii} > (<) 0$ involves too little (much) agglomeration if we can change the number of regions n (Henderson, 1974; Kanemoto, 1980). However, since n is fixed here, such distortions do not occur. In fact, changing n is prohibitively difficult in practice since emergence and disappearance of cities are not realized by infinitesimal lows of migration.

Another important issue is the direction of market distortion. For $n = 2$ with linear utility functions (15), we have

$$x_1^* - x_1^o = \frac{u_{12} - u_{21}}{2U_A[12]}$$

if they are interior solutions. Since $U_A[12] > 0$ holds for asymptotic stability and ESS, we can say that *region 1 is overpopulated (underpopulated) if* $u_{12} > u_{21}$ *(*$u_{12} < u_{21}$*), and is the socially optimum size if* $u_{12} = u_{21}$. Unfortunately, such a result cannot be generalized for $n \geq 3$ except for the symmetric case (Theorem 2).

Finally, the social welfare function (16) resembles the objective function (8) in deriving the ESS condition. However, the former does not have x_i^* and x_j^*, while the latter does. The social planner maximizes the sum of all utilities (16) in optimum. In equilibrium, on the other hand, each individual simply chooses a higher-utility region, or each multilocational firm allocates employees in order to maximize their average utility level. Since each firm is small enough, it is unable to maximize the sum of all utilities without the symmetry and uniqueness conditions.

7 Market interactions within regions

Due to the spatial proximity, the intraregional market interactions are usually much stronger than the interregional market interactions. In order to crystallize the discussion, we consider the special, but important, case of

$$u_i(\boldsymbol{x}) = u_i(x_i). \tag{19}$$

Assumption (19) is justified when the impacts of own population are much stronger than those of other populations, i.e., $|u_{ii}| \gg |u_{ik}|$ for all $i \neq k$. For example, urban costs and benefits such as congestion and product variety are usually closely related to population size within a region only but not to the populations of other regions. This implies that the change in population share x_i on the utility levels in other regions u_j is zero for all $j(\neq i)$.

Since assumption (19) implies the symmetry assumption $u_{ij} = u_{ji}$, Theorem 2 applies. That is, insofar as the market equilibrium is unique, it is the socially optimum allocation of population for any marginal utility u_{ii} and for any PDA dynamic.

Corollary 1 *Suppose the intraregional market interactions* u_{ii} *are much larger than the interregional ones* u_{ij} $(j \neq i)$*. Then, for any PDA dynamic, the socially optimum allocation of population is automatically attained by the market mechanism if the equilibrium is unique.*

Since assumption (19) also implies assumption (10), we know from Theorem 1 that the ESS conditions coincide with the asymptotic stability conditions for any PDA dynamic. The necessary and sufficient conditions for ESS are the negative definiteness of U_E, which are $(-1)^i U_E[1 \cdots i] > 0$ for all $i = 2, \ldots, n$.

Without loss of generality, let

$$u_{11} \geq u_{22} \geq \cdots \geq u_{nn}. \tag{20}$$

We have the following nearly necessary and sufficient conditions for the stability of spatial equilibrium under the assumption (19).

Proposition 2 *Any PDA dynamic (3) is asymptotically stable and is ESS if (i) or (ii) holds.*
(i) $u_{11} \leq 0$ *and* $u_{22} < 0$*,*
(ii) $u_{11} > 0 > u_{22}$ *and* $\sum_i 1/u_{ii} > 0$*.*
Any PDA dynamic (3) is neither asymptotically stable nor ESS if (iii) or (iv) holds.
(iii) $u_{11} > 0 > u_{22}$ *and* $\sum_i 1/u_{ii} < 0$*,*
(iv) $u_{11} > 0$ *and* $u_{22} \geq 0$*.*

A proof is in the Appendix. As in Theorem 1, Proposition 2 provides nearly unifying conditions that are both sufficient and necessary.

Stability condition (i) in Proposition 2 says that if an increasing population always lowers its utility level and a decreasing population always raises its utility level, no individual has an incentive to migrate, and hence it is stable. In other words, congestion diseconomies within all regions ensure the stability of interior equilibrium.

On the other hand, instability condition (iv) says that if a small migration between positively sloped regions takes place due to a disturbance, then it is unstable. Since the utility level increases in the region experiencing in-migration while it decreases in a region experiencing out-migration, it generates further migration between the regions, leading to instability.[7] That is, the existence of sufficiently large agglomeration economies within at least two regions destroys the stability of interior equilibrium.

If there is only one positive slope of the utility function, conditions (ii) and (iii) apply. For $n = 2$, suppose a small (net) migration were to occur from region 2 with $u_{22} < 0$ to region 1 with $u_{11} > 0$, then the utility levels in both regions would increase. If $u_{11}^{-1} + u_{22}^{-1} > 0$ as in case (ii), then

$$\frac{\Delta x_1}{\Delta u_1} + \frac{\Delta x_2}{\Delta u_2} = \frac{\Delta x_1}{\Delta u_1} - \frac{\Delta x_1}{\Delta u_2} > 0, \tag{21}$$

and hence the increase in region 2's utility is higher. This necessarily generates the reverse migration from region 1 to region 2, restoring the original equilibrium $\boldsymbol{x}^*$. On the other hand, if $u_{11}^{-1} + u_{22}^{-1} < 0$ as in case (iii), region 1's utility is higher, which is unstable.

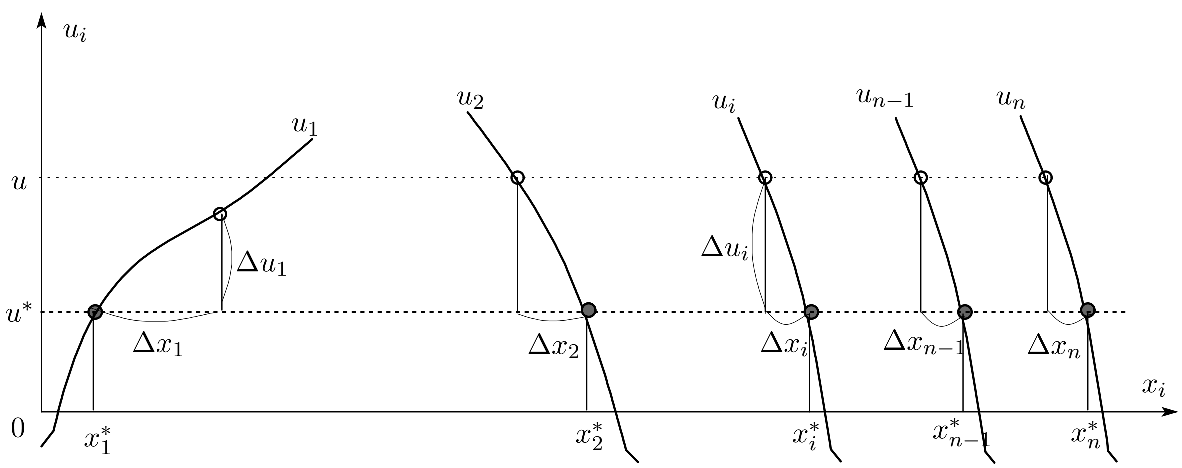

Figure 1: Stability in a multiregional system

A similar principle holds for $n \geq 3$. Suppose a small (net) migration from regions $i = 2, \ldots, n$ with $u_{ii} < 0$ to region 1 with $u_{11} > 0$ takes place, while keeping the fixed total population as

$$\Delta x_1 = -\sum_{i=2}^{n} \Delta x_i > 0. \tag{22}$$

[7]It should be mentioned that Konishi, Le Breton and Weber (1997a) showed the existence of strong Nash equilibrium related to (i) in Proposition 2 in the presence of negative externality called "partial rivalry," and Konishi, Le Breton and Weber (1997b) showed the nonexistence of equilibrium related to (iv) in Proposition 2 in the presence of positive externality.

The change in the utility level in each region is illustrated in the diagram of Figure 1. Now, from (22), the stability condition is rewritten as

$$\frac{\Delta x_1}{\Delta u_1} + \sum_{i=2}^{n} \frac{\Delta x_i}{\Delta u_i} = \sum_{i=2}^{n} \Delta x_i \left(\frac{1}{\Delta u_i} - \frac{1}{\Delta u_1} \right) > 0. \tag{23}$$

If each Δu_i were to be the same for all $i = 2, \dots, n$ as in Figure 1, then (23) would be reduced to (21). Since the utility increase in region 1 is smaller than that in other regions ($\Delta u_1 < \Delta u_i$), the stability of the multiregional system is guaranteed.

8 Conclusion

In this paper, we have considered the spatial distribution of economic activities in a multiregional dynamical system assuming that other variables, such as prices and quantities, are solved as a function of the distribution in general equilibrium.

With regular conditions on utility functions ($u_{ij} + u_{jk} + u_{ki} = u_{ik} + u_{kj} + u_{ji}$), we have proven that asymptotic stability conditions and ESS conditions coincide in any PDA dynamic. In this case, the stability of spatial equilibrium is ensured only by computing the signs of the principal minors of the payoff matrix U, without computing all the eigenvalues of the dynamical system FU (Theorem 1).

Imposing the symmetric assumption on the marginal utility functions ($u_{ij} = u_{ji}$), we have shown that the market outcome of population distribution (equilibrium configuration) coincides with globally efficient allocation of population (social optimum configuration) (Theorem 2).

Imposing a further assumption of negligible interregional externalities ($u_{ij} = 0, \forall i \neq j$), we have derived simple stability conditions of spatial interior equilibrium (Proposition 2). Due to the simple form of the conditions, we were able to interpret them and explain how the multiregional system becomes stable in several ways. We have shown that strong positive externalities due to agglomeration economies destroy the interior equilibrium of the system whereas strong negative externalities due to agglomeration diseconomies, such as congestion, stabilize the interior equilibrium.

So far we have obtained the stability conditions of spatial equilibrium, and Ginsburgh *et al.* (1985) showed the existence of spatial equilibrium. However, not everything has been elucidated on the nature of spatial equilibrium. In particular, there is no guarantee that a spatial stable equilibrium always exists, although its existence seems quite likely. In fact, our preliminary analysis indicates that the spatial stable equilibrium always exists under a class of utility functions.

Appendix

Proof of Lemma 2

(i)⇒(ii). This is simply because (ii) is a special case of (i).

(ii)⇒(iii). Obviously,

$$0 = U_A[ijk] - U_E[ijk] = (u_{ij} + u_{jk} + u_{ki} - u_{ji} - u_{kj} - u_{ik})^2,$$

holds for any different combination of $i, j, k = 1, \dots, n$.

(iii)⇒(iv). Let

$$a_j = u_{1j} - u_{j1}, \quad i, j = 1, \dots, n.$$

Then,

$$a_j - a_i = u_{1j} + u_{i1} - u_{1i} - u_{j1} = u_{j1} + u_{1i} + u_{ij} - u_{ji} - u_{1i} - u_{j1} = u_{ij} - u_{ji}.$$

(iv)⇒(i). We only prove that $U_E[1 \cdots i] = U_A[1 \cdots i]$ holds for $i = 3, \dots, n$ below. Other cases can be proven by suitably renumbering the regions.

$$\begin{aligned}
U_E[1\cdots i] &= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & u_{11} & (u_{12}+u_{21})/2 & \cdots & (u_{1i}+u_{i1})/2 \\ 1 & (u_{12}+u_{21})/2 & u_{22} & \cdots & (u_{2i}+u_{i2})/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (u_{1i}+u_{i1})/2 & (u_{2i}+u_{i2})/2 & \cdots & u_{ii} \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & u_{11} & u_{12}-(a_1-a_2)/2 & \cdots & u_{1i}-(a_1-a_i)/2 \\ 1 & u_{21}+(a_1-a_2)/2 & u_{22} & \cdots & u_{2i}-(a_2-a_i)/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{i1}+(a_1-a_i)/2 & u_{i2}+(a_2-a_i)/2 & \cdots & u_{ii} \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & u_{11}+a_1/2 & u_{12}+a_2/2 & \cdots & u_{1i}+a_i/2 \\ 1 & u_{21}+a_1/2 & u_{22}+a_2/2 & \cdots & u_{2i}+a_i/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{i1}+a_1/2 & u_{i2}+a_2/2 & \cdots & u_{ii}+a_i/2 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & u_{11} & u_{12} & \cdots & u_{1i} \\ 1 & u_{21} & u_{22} & \cdots & u_{2i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_{i1} & u_{i2} & \cdots & u_{ii} \end{vmatrix} \\
&= U_A[1\cdots i].
\end{aligned}$$

□

Proof of Proposition 2

Since the asymptotic stability conditions for any F and the ESS conditions coincide under assumption (19) by Theorem 1, it is sufficient to check the condition (9). From (20), the LHS of (9) is rewritten as

$$(-1)^m U_E[1\cdots m] = \begin{cases} 0 & \text{if two or more } u_{ii} \text{ are zeroes,} \\ (-1)^{m-1}\prod_{i\neq k}^{m} u_{ii} & \text{if only } u_{kk} \text{ is zero,} \\ (-1)^{m-1}\prod_{i=1}^{m} u_{ii}\sum_{j=1}^{m}\frac{1}{u_{jj}} & \text{otherwise.} \end{cases} \tag{24}$$

(i) If $u_{11} \neq 0$, then it is obvious from the last line in (24). If $u_{11} = 0$, then

$$(-1)^m U_E[1 \cdots m] = \prod_{i=2}^{n} (-u_{ii}) > 0,$$

which satisfies (9) with strict inequality.

(ii) Obvious from the last line in (24).

(iii) Since $(-1)^n U_E[1 \cdots n] = u_{11} \prod_{i=2}^{n}(-u_{ii}) \sum_{j=1}^{n} 1/u_{jj} < 0$, this violates condition (9) for $m = n$.

(iv) Since $(-1)^2 U_E[12] = -u_{11} - u_{22} < 0$, this violates (9) for $m = 2$. □

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