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A Generalized SSAR Model and Predictive Distribution with an Application to VaR

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Abstract

The asymmetrical movements between the downward and upward phases of the sample paths of time series have been sometimes observed. By generalizing the SSAR (simultaneous switching autoregressive) models, we introduce a class of nonlinear time series models having the asymmetrical sample paths in the upward and downward phases. We show that the class of generalized SSAR models is useful for estimating the asymmetrical predictive distribution given the present and past information. Applications to the prediction based on the predictive median and the estimation of the VaR (value at risk) in financial risk management are discussed.

Key Words

Asymmetrical Sample Paths, Generalized SSAR Model, Transformation Models, Predictive Distribution, Predictive Median, Value at Risk (VaR).

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1. Introduction

In the past decades several nonlinear time series models have been proposed by statisticians and some of them have been used in practical applications. Among the nonlinear phenomena observed in actual time series data we shall forcus on the asymmetrical movements of time series in the upward phase and the downward phase, which have been observed by a number of time series researchers and economists. In particular it has been argued that major financial series including stock prices display some kind of asymmetrical movements in the upward and downward phases. There have been several attempts to deal with this type of phenomena in nonlinear time series analysis.

In this paper we shall propose a class of nonlinear time series models, which is a generalization of the SSAR (simultaneous switching autoregressive) models, and develop a new approach to deal with the asymmetrical sample paths of time series. Earlier, we have introduced the stationary and nonstationary SSAR time series models and discussed their statistical properties in some detail (Kunitomo and Sato (1996, 1999, 2000a, 2000b), Sato and Kunitomo (1996)). The SSAR models have been developed for applications in econometric analyses including the disequilibrium econometric models and the time series models with adjustments in financial markets. Although the SSAR models have been discussed in econometric applications, there are some interesting new aspects for nonlinear time series modelling. Thus we are trying to extend the SSAR models to a class of GSSAR models in time sereis analysis and discuss some possible applications. Since the most important application of time series models is prediction, we shall discuss some related applications based on the predictive distribution of the GSSAR models. In particular we shall propose a new estimation method of the percentile points of the predictive distribution such as the median when it is not necessarily symmetrical and the resulting volatility function of the time series can be also asymmetrical. It is essentially the same problem as the estimation of the Value-at-Risk (VaR) in recent financial risk management.

There can be other approaches in the estimation problem of asymmetrical volatility function and VaR. In particular some non-linear statistical time series models along the line of Nelson (1991) and Harvey and Shephard (1993) in the econometric or financial analysis of the asymmetrical volatility functions have been known and there have been many related studies already appeared, which are closely related to the application we shall investigate in this paper. However, our approach to the problem of estimating asymmetrical volatility function and measuring VaR is a simple but different one from the exisiting literatures in this respect.

In Section 2, we shall introduce a generalized univariate SSAR (GSSAR) model and discuss some examples. Then in Section 3 we shall investigate some properties of the GSSAR models including the geometric ergodicity and the estimation problems of the GSSAR models. In Section 4, we shall discuss the predictive distribution in the GSSAR models and the estimation problem of percentiles of the predictive distribution such as the median. As an application we shall discuss the prediction based on the median and the estimation of VaR in the financial risk management and give a real case study on the interest rates futures market in Japan. Then some concluding remarks will be given in Section 5. The proofs of some theoretical results in Section 3 are given in Appendix.

2. A Generalized SSAR model

We consider a class of the generalized SSAR (simultaneous switching autoregressive) models. Let $\{y_t, t = 0, \underline{+}1, \cdots\}$ be the univariate time series satisfying

$$\Delta y_t = G_\sigma(r_0 + \sum_{j=1}^{p} r_j y_{t-j} + v_t\sqrt{h_t}) \ , \tag{2.1}$$

where $\Delta y_t = y_t - y_{t-1}$, $G_\sigma(\cdot)$ is a continuous function, r_j $(j = 0, 1, \cdots, p)$ are unknown parameters, v_t are i.i.d random variables with $E[v_t] = 0, E[v_t^2] = 1$, and h_t (≥ 1) is the volatility function.

We assume that

(i) $\{v_t\}$ are i.i.d random variables with a continuous symmetric density function $f(v)$ (and its distribution function $F(v)$), which is positive almost everywhere in $\mathbf{R}$,

and

(ii) h_t are the $\mathcal{F}_{t-1}-$measurable functions [1] and $\mathcal{F}_{t-1}$ is the $\sigma-$field generated by the random variables $\{y_s, s \leq t-1\}$.

In this paper we further assume that

(iii) $G_\sigma(x)$ is a strictly increasing function satisfying

$$\lim_{x\to\infty} \frac{G_\sigma(x)}{x} = \sigma_1 \ > 0 \ , \tag{2.2}$$

and

$$\lim_{x\to-\infty} \frac{G_\sigma(x)}{x} = \sigma_2 \ > 0 \ , \tag{2.3}$$

where σ is the vector of unknown transformation parameters including σ_i $(i = 1, 2)$ appeared in (2.2) and (2.3).

In the above formulation (2.1) is slightly different from many nonlinear time series models including the threshold autoregressive (TAR) models developed by Tong (1990). We shall be mainly interested in the time series movements which can be quite different in the upward phase ($\Delta y_t \geq 0$) and the downward phase ($\Delta y_t < 0$). Then (2.1) can give a simple but rich way to represent the time series modelling with these two phases. Because the transformation function $G_\sigma(\cdot)$ has some unknown parameters and the random noise v_t at t has not been realized at time $t-1$, the phase (the upward phase or downward phase, for instance) at time t is not determined in advance at time $t-1$. Also we shall be mainly interested in the case when the transformation function $G_\sigma(\cdot)$ in (2.1) is not necessarily differentiable.

In this section we illustrate the distinctive features of the class of the GSSAR models and give some examples whose reduced forms [2] are written as (2.1). When $\sigma = \sigma_1 = \sigma_2$ and $G_\sigma(x) = \sigma x$, then (2.1) is the AR(p) model when we take the volatility function $h_t = 1$ $(a.s.)$. Except this special case, the sample paths of the GSSAR models are asymmetrical in the upward and downward phases. The class of the GSSAR models is a generalization of the SSAR (simultaneous switching autoregressive) models developed

[1] It is possible to deal with the more general case on the volatility function such as the stochastic volatility models. See Harvey and Shephard (1993), for instance. However, then the estimation procedure becomes more complicated than the methods in Section 3.2.

[2] We use the distinction between the structural forms and the reduced forms of the SSAR models. This terminology has been standard in the traditional econometric literatures.

by the present authors. (Kunitomo and Sato (1996, 1999, 2000a, 2000b) and Sato and Kunitomo (1996).) They have discussed some multivariate versions of the SSAR time series models, but we shall discuss univariate examples in this paper. In the following Example 1 and Example 2 we take $h_t = 1$ only for the resulting expository simplicities. The first example is the simplest case, but it highlights the distinctive features of the class of the SSAR models.

Example 1 : SSAR(1)

Kunitomo and Sato (1996) have introduced a simple stationary simultaneous switching autoregressive (SSAR) time series model by considering a class of multivariate disequilibrium econometric models. For the expository purpose let $\{y_t\}$ be a sequence of scalar time series satisfying

$$y_t = \begin{cases} a_1 y_{t-1} + \sigma_1 v_t & \text{if} \quad y_t \geq y_{t-1} \\ b_1 y_{t-1} + \sigma_2 v_t & \text{if} \quad y_t < y_{t-1} \end{cases} , \tag{2.4}$$

where a_1, b_1, σ_i $(\sigma_i > 0; i = 1, 2)$ are unknown parameters, and $\{v_t\}$ are a sequence of i.i.d. random variables with $E(v_t) = 0$ and $E(v_t^2) = 1$. This is in the structural form of time series model with the econometric terminology. By imposing the coherency condition given by

$$\frac{1 - a_1}{\sigma_1} = \frac{1 - b_1}{\sigma_2} = -r_1 ,$$

the time series model can be rewritten as

$$y_t = \begin{cases} a_1 y_{t-1} + \sigma_1 v_t & \text{if} \quad v_t \geq -r_1 y_{t-1} \\ b_1 y_{t-1} + \sigma_2 v_t & \text{if} \quad v_t < -r_1 y_{t-1} \end{cases} . \tag{2.5}$$

It is the reduced form of a structural time series model (2.4) in the econometric terminology and its Markovian representation is given by

$$\Delta y_t = [\sigma_1 I(v_t + r_1 y_{t-1} \geq 0) + \sigma_2 I(v_t + r_1 y_{t-1} < 0)][r_1 y_{t-1} + v_t] , \tag{2.6}$$

where $I(\cdot)$ is the indicator function. When $\sigma_1 = \sigma_2 = \sigma$, then the present SSAR(1) model becomes the standard $AR(1)$ model by re-parametrizing $a_1 = b_1 = 1 + \sigma r_1$.

Example 2 : SSAR(p)

The univariate SSAR(p) model in the structural form, which includes the SSAR(1) model as a special case, has been given by

$$y_t = \begin{cases} a_0 + \sum_{j=1}^{p} a_j y_{t-j} + \sigma_1 v_t & \text{if} \quad y_t \geq y_{t-1} \\ b_0 + \sum_{j=1}^{p} b_j y_{t-j} + \sigma_2 v_t & \text{if} \quad y_t < y_{t-1} \end{cases} . \tag{2.7}$$

By defining the set of parameters $a_0 = r_0\sigma_1, a_1 = 1 + r_1\sigma_1, a_j = r_j\sigma_1$ $(j = 2, \cdots, p)$ and $b_0 = r_0\sigma_2, a_2 = 1 + r_1\sigma_2, a_j = r_j\sigma_2$ $(j = 2, \cdots, p)$, the reduced form of the time series model is given by

$$\Delta y_t = \sigma(t)(r_0 + \sum_{j=1}^{p} r_j y_{t-j} + v_t) \ , \tag{2.8}$$

where $\sigma(t) = \sigma_1 I(\Delta y_t \geq 0) + \sigma_2 I(\Delta y_t < 0)$.

Example 3 : GSSAR(1)

A more general form of (univariate) SSAR(p) model with multiple state spaces can be defined by the function

$$G_\sigma(x) = \begin{cases} \sigma_1 x & \text{if} \quad x \geq c_1 \\ \sigma_0 x & \text{if} \quad c_2 \leq x < c_1 \\ \sigma_2 x & \text{if} \quad x < c_2 \ , \end{cases} \tag{2.9}$$

where σ_i and c_i $(i = 1, 2, 3)$ are real constants. In this case there are three phases on the state space depending on the values of Δy_t . It is mathematically trivial to extend the GSSAR(p) model with $m+1$ phases by dividing the state space into $m+1$ phases as the threshold AR models already introduced by Tong (1990).
When the threshold parameters $c_i (i = 1, 2, 3)$ in Example 3 are unknown, however, the statistical estimation problem of these parameters becomes non-standard.

We now give two examples which are related to the application we shall discuss in the area of financial time series analysis. We need some non-linear as well as non-stationary time series models because we can often observe these features in actual financial time series.

Example 4 : Asymmetrical ARCH(r) model

Let $y_t = \log P_t$ and P_t is the stock price at time t. Then without considering the stock dividends the stock return process is given by $R_t = \Delta y_t = \log(P_t / P_{t-1})$ and the asymmetrical volatility model for the return process can be represented by the GSSAR model

$$\Delta y_t = G_\sigma(v_t \sqrt{h_t}) \ , \tag{2.10}$$

where

$$h_t = 1 + \sum_{j=1}^{r} \alpha_j h_{t-j} v_{t-j}^2 \ , \tag{2.11}$$

and α_j $(j = 1, \cdots, r)$ are unknown parameters with $\alpha_j \geq 0$ and $\sum_{j=1}^{r} \alpha_j < 1$. When $G_\sigma(x) = \sigma x$, then the resulting GSSAR model is the same as the standatrd ARCH model developed by Engle (1982).

Example 5 : SSIAR(p)

We should mention that Kunitomo and Sato (1999, 2000b) have introduced a univariate nonstationary SSAR model. Let y_t be an I(1) time series process and satisfy

$$\Delta y_t = \sigma(t) \left\{ a_0 + \sum_{i=1}^{p} a_i(t-i) \sigma(t-i)^{-1} \Delta y_{t-i} + \sqrt{h_t} v_t \right\} \ , \tag{2.12}$$

where $\sigma(t) = \sigma_1 I(\Delta y_t \geq 0) + \sigma_2 I(\Delta y_t < 0)$, $a_1(t-1) = 1 + r_1 \sigma(t-1)$, $a_i(t-i) = r_i \sigma(t-i)$ $(i = 2, \cdots, p)$, and the volatility function h_t is given by (2.11). If we write the transformation function as $G_\sigma(x) = (\sigma_1 I(x \ge 0) + \sigma_2 I(x < 0))x$, we have

$$\Delta y_t = G_\sigma \left\{ r_0 + \sum_{i=1}^{p} r_i \Delta y_{t-i} + G_\sigma^{-1}(\Delta y_{t-1}) + \sqrt{h_t} v_t \right\} , \tag{2.13}$$

where $G_\sigma^{-1}(x) = (\sigma_1^{-1} I(x \ge 0) + \sigma_2^{-1} I(x < 0))x$. This representation for $\{y_t\}$ is similar to (2.1), but it is slightly more complicated. This non-linear nonstationary time series model was originally derived by considering adjustments in financial markets and it can be written as a Markovian representation form by using the state space representation for $\{y_t\}$.

Let $(p + r) \times 1$ state vector $\boldsymbol{x}_t$ be given by

$$\boldsymbol{x}_t = \begin{pmatrix} \Delta y_t \\ \vdots \\ \Delta y_{t-p+1} \\ u_t^2 \\ \vdots \\ u_{t-r+1}^2 \end{pmatrix} , \tag{2.14}$$

where we denote $u_t^2 = v_t^2 h_t$ and the volatility function h_t is given by (2.11). Then the state space for $\boldsymbol{x_t}$ is $\boldsymbol{R}^p \times \boldsymbol{R}_+^r$ and we can form a Markovian representation for $\{\boldsymbol{x}_t\}$ as

$$\boldsymbol{x}_t = \boldsymbol{H}_\sigma(\boldsymbol{x}_{t-1}, v_t) , \tag{2.15}$$

where

$$\boldsymbol{H}_\sigma(\boldsymbol{x}_{t-1}, v_t) = \begin{pmatrix} G_\sigma(r_0 + \sum_{i=1}^{p} r_i \Delta y_{t-i} + G_\sigma^{-1}(\Delta y_{t-1}) + v_t \sqrt{h_t}) \\ \Delta y_{t-1} \\ \vdots \\ \Delta y_{t-p+1} \\ v_t^2 h_t \\ u_{t-1}^2 \\ \vdots \\ u_{t-r+1}^2 \end{pmatrix} .$$

We notice that the last example is just one non-stationary SSAR model having a Markovian representation and there can be other extensions. The important point is the fact that if we relax the differentiability of the transformation function $G_\sigma(\cdot)$, then there can be many non-linear time series models which may be useful for practical applications.

3. Some Statistical Properties

3.1 Geometric Ergodicity

The first important property of a statistical time series model is whether it is ergodic or not. For the Markovian time series models, the geometric ergodicity and the related concepts have been developed in the nonlinear time series analysis. For the precise definitions of related concepts including irreducibility, aperiodicity, small set, and ergodicity of the Markov chains with the general state space, see Nummelin (1984) or Tong (1990).

We consider the structural form of the GSSAR model given by (2.1) with (2.11). Then we have the next result on the geometric ergodicity for the class of GSSAR model. The proof is given in Appendix.

Theorem 3.1 *In the structural form of the GSSAR model given by (2.1) with (2.11) we also assume $E[v_t^2] < \infty$. Then a set of sufficient conditions for the geometric ergodicity are given by*

$$\rho_1 = \max\{\sum_{i=1}^{p} |a_i|, \sum_{i=1}^{p} |b_i|\} < 1 \tag{3.1}$$

and

$$\rho_2 = \max\{\sigma_1, \sigma_2, 1\} \sum_{i=1}^{r} \alpha_i < 1 \ , \tag{3.2}$$

where $a_1 = 1 + r_1\sigma_1, b_1 = 1 + r_1\sigma_2$, $a_i = r_i\sigma_1, b_i = r_i\sigma_2$ $(i = 1, \cdots, p)$, and $\alpha_i (\geq 0)$ $(i = 1, \cdots, r)$.

We notice that the conditions given by (3.1) and (3.2) are quite strong and they are not the necessary conditions. It has been known that when $p = 1$ with a constant term and $h_t = 1$ $(a.s.)$,

$$a_1 < 1, b_1 < 1, a_1 b_1 < 1 \tag{3.3}$$

are the necessary and sufficient conditions for the geometric ergodicity with the assumption of $E[|v_t|] < \infty$. Furthermore, Kunitomo (1999) has investigated the ergodic regions for the 2nd order threshold (TAR(2)) models and the SSAR(2) models and found that they are quite complex even in some special cases. Thus the necessary and sufficient conditions are still open problems in the GSSAR models when $p \geq 1$ and $r \geq 1$. For the SSIAR(p) models we discussed in **Example 5**, Kunitomo and Sato (2000b) have given a set of sufficient conditions on the geometric ergodicity.

We next consider the conditions for the existence of moments of the process $\{y_t\}$ when it is geometrically ergodic. For this purpose we have the following sufficient conditions which are quite strong. The proof is similar to the previous one and it is omitted.

Theorem 3.2 *In the structural form of the GSSAR model given by (2.1) with (2.11) we also assume the condition (3.2) and $E[v_t^2] < \infty$. Then a set of sufficient conditions for the existence of 2nd order moments of $\{y_t\}$ are given by*

$$\rho_3 = \max\{\sum_{i=1}^{p} |a_i| + \sigma_1, \sum_{i=1}^{p} |b_i| + \sigma_2\} < 1 \ . \tag{3.4}$$

Again we notice that the condition given by (3.2) and (3.4) are quite strong. It has been known that when $p = 1$ with a constant term and $h_t = 1$ $(a.s.)$,

$$a_1 < 1, b_1 < 1, a_1 b_1 < 1 \tag{3.5}$$

and $E[|v_t|^k] < \infty$ $(k \geq 1)$ are the sufficient conditions for the existence of the k-th order moments of $\{y_t\}$.

3.2 Estimation

In the nonlinear time series analysis the least squares method has been often used to estimate the unknown parameters of the time series models in the form of (2.1). Even if the function $G_\sigma(\cdot)$ is known except σ, there is a serious bias problem in the least squares estimation of the structural forms due to the nonlinear transformation involved. We shall discuss two estimation methods of the unknown parameters in (2.1).

Instead of the least squares estimation method, Sato and Kunitomo (1996) have proposed to use the maximum likelihood (ML) estimation for the SSAR models. By generalizing their arguments to the GSSAR models, the maximum likelihood (ML) estimator for the GSSAR models under the Gaussian disturbances is defined by maximizing the conditional log-likelihood function for y_t $(t = 1, \cdots, T)$:

$$\begin{aligned} \log L_T(\boldsymbol{\theta}) \quad &= \quad -\frac{1}{2}\sum_{t=1}^{T}\log(2\pi h_t(\alpha)) + \frac{1}{2}\sum_{t=1}^{T}\log|\frac{dG_\sigma^{-1}}{d\Delta y_t}| + \\ & \quad -\frac{1}{2h_t(\alpha)}\sum_{t=1}^{T}[G_\sigma^{-1}(\Delta y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t-j}]^2 \ , \end{aligned} \tag{3.6}$$

where the initial conditions y_t $(-\max\{p, r\}+1 \leq t \leq 0)$ are given. In the above notation $\boldsymbol{\alpha} = (\alpha_i)$ $(i = 1, \cdots, r)$ and we denote the vector of whole structural parameters as

$$\boldsymbol{\theta} = (r_0, r_1, \cdots, r_p, \alpha_1, \cdots, \alpha_r, \sigma)$$

when we use (2.11) as the volatility function.

As for the asymptotic properties of the ML estimation method when the underlying process is (geometrically) ergodic, under a set of regularity conditions and the Gaussian disturbances the ML estimator $\hat{\boldsymbol{\theta}}_{ML}$ of unknown parameter $\boldsymbol{\theta}$ is consistent and asymptotically normally distributed as

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}_0\right) \xrightarrow{d} N\left[0,\ I(\boldsymbol{\theta}_0)^{-1}\right] \quad , \tag{3.7}$$

where

$$I(\boldsymbol{\theta}_0) = \lim_{T\to\infty}\frac{1}{T}E\left[-\frac{\partial^2 \log L_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right] \ , \tag{3.8}$$

which is assumed to be a non-singular matrix and $\boldsymbol{\theta}_0$ is the vector of true parameters.

Because the estimation problem for the GSSAR models is quite similar to the estimation problem of the structural equations in the nonlinear simultaneous equations, an alternative estimation method under the non-Gaussian disturbance terms would be the nonlinear instrumental variables (IV) estimation. Actually it is a special case of the Generalized Method of Moments (GMM) proposed by Hansen (1982) (see Hamilton

(1994), for instance). Given the initial conditions and the observations for y_t, one type of the instrumental variables (IV) estimators is defined by minimizing the criterion function

$$\boldsymbol{Q}_T(\boldsymbol{\theta}) = \boldsymbol{F}_T(\boldsymbol{\theta})' \boldsymbol{H}_T^{-1} \boldsymbol{F}_T(\boldsymbol{\theta}) \ , \tag{3.9}$$

where

$$\begin{aligned} \boldsymbol{F}_T(\boldsymbol{\theta}) &= \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} v_t(\boldsymbol{\theta}) \\ v_t^2(\boldsymbol{\theta}) - 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ y_{t-1} \\ \vdots \\ y_{t-q} \end{pmatrix} \ , \qquad (3.10) \\ \boldsymbol{H}_T &= \boldsymbol{W} \otimes \frac{1}{T}\sum_{t=2}^{T} \begin{pmatrix} 1 \\ y_{t-1} \\ \vdots \\ y_{t-q} \end{pmatrix} \begin{pmatrix} 1 \\ y_{t-1} \\ \vdots \\ y_{t-q} \end{pmatrix}' \ , \end{aligned}$$

$\boldsymbol{W}$ is a 2×2 nonsingular matrix and $v_t(\boldsymbol{\theta}) = [G_\sigma^{-1}(\Delta y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t-j}]/h_t(\boldsymbol{\alpha})$. In the above notation the number of instrumental variables is denoted by q which satisfies the condition $q \geq \max\{p, r\} + 1$.
As for the asymptotic properties of the ML estimation method when the underlying process is (geometrically) ergodic, under a set of regularity conditions and fairly general distributions for the disturbances the nonlinear instrumental variables estimator $\hat{\boldsymbol{\theta}}_{IV}$ of unknown parameter $\boldsymbol{\theta}$ is consistent and asymptotically normally distributed. The asymptotic variance-covariance matrix of the instrumental variables method has been given in Kunitomo and Sato (2000a).

4. Applications

4.1 Predictive Distribution and VaR

One important application of the GSSAR models is the problem of estimating the percentile points of the predictive distribution, which is defined by the conditional distribution function given the past information. It is the same as the marginal distribution if the observations are realizations of i.i.d. random variables. We notice that the predictive distribution under the GSSAR models can be asymmetrical even if we have the Gaussian disturbances.

From (2.1), the conditional probability distribution at $t+1$ given $\mathcal{F}_t$ is given by

$$\begin{aligned} H_{t+1|t}(y) &= P(y_{t+1} \leq y | \mathcal{F}_t) \\ &= P(v_{t+1}\sqrt{h_{t+1}} \leq G_\sigma^{-1}(y - y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t+1-j} | \mathcal{F}_t) \\ &= F(\frac{G_\sigma^{-1}(y - y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t+1-j}}{\sqrt{h_{t+1}}}) \ , \qquad (4.1) \end{aligned}$$

where $F(\cdot)$ is the distribution function of $\{v_t\}$. Then by using a recursive argument for any $h \geq 1$ the $k-$period ahead predictive distribution given the information $\mathcal{F}_t$ is given by

$$H_{t+k|t}(y) = E[F(\frac{G_\sigma^{-1}(y - y_{t+k-1}) - r_0 - \sum_{j=1}^{p} r_j y_{t+k-j}}{\sqrt{h_{t+k}}})|\mathcal{F}_t] \ , \tag{4.2}$$

where $E[\ \cdot\ |\mathcal{F}_t]$ is the conditional expectation operator with respect to the random variables $v_{t+k-1}, \cdots, v_{t+1}$.

There are some applications based on the predictive distribution of the GSSAR models. First for the $100(1-\gamma)\%$ prediction interval we consider

$$\begin{aligned} 1-\gamma &= P(y_L \le y_{t+1} \le y_U|\mathcal{F}_t) \\ &= P\{\frac{G_\sigma^{-1}(y_L - y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t+1-j}}{\sqrt{h_{t+1}}} \le v_{t+1} \\ &\qquad \le \frac{G_\sigma^{-1}(y_U - y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t+1-j}}{\sqrt{h_{t+1}}}|\mathcal{F}_t\} \ . \end{aligned} \tag{4.3}$$

Then the $100(1-\gamma)\%$ prediction interval can be estimated by

$$[y_t + G_{\hat{\sigma}}\{-z(\frac{\gamma}{2})\sqrt{h_{t+1}(\hat{\alpha})} + \hat{r}_0 + \sum_{j=1}^{p} \hat{r}_j y_{t+1-j})\}, y_t + G_{\hat{\sigma}}\{z(\frac{\gamma}{2})\sqrt{h_{t+1}(\hat{\alpha})} + \hat{r}_0 + \sum_{j=1}^{p} \hat{r}_j y_{t+1-j})\}],$$

where $z(\frac{\gamma}{2})$ is the upper $\frac{\gamma}{2}$ percentile point of the distribution function $F(v)$, $\hat{r}_j$ $(j = 0, 1, \cdots, p)$ are the estimates of the parameters r_j $(j = 0, 1, \cdots, p)$, $\hat{\sigma}$ is the vector for the estimates of σ_i $(i = 1, 2)$, and $\hat{\alpha}$ is the vector for the estimates of α_i $(i = 1, \cdots, r)$. In particular we can use the above relation for estimating the median of the predictive distribution. By solving

$$\frac{1}{2} = P(v_{t+1} \le \frac{G_\sigma^{-1}(y_M - y_t) - r_0 - \sum_{j=1}^{p} r_j y_{t+1-j}}{\sqrt{h_{t+1}}}|\mathcal{F}_t) \ , \tag{4.4}$$

and using the assumption of the symmetry for the density function $f(v)$, we have an estimate of the median of the predictive distribution as

$$\hat{y}_{t+1|t}^{med} = y_t + G_{\hat{\sigma}}(\hat{r}_0 + \sum_{j=1}^{p} \hat{r}_j y_{t+1-j}) \ . \tag{4.5}$$

We note that the median of the predictive distribution, which is called the predictive median, is independent of the distribution function $F(\cdot)$ of disturbances under the assumption of symmetrical density. On the other hand, the mean of the predictive distribution is generally a complicated function of both the transformation function $G_\sigma(\cdot)$ and the distribution function $F(\cdot)$ of disturbances. Hence we expect that (4.5) gives a robust prediction method when the marginal distribution of time series is not necessarily symmetric.

Since the predictive distribution can be asymmetrical, there is a direct application in estimating the Value-at-risk (VaR) in the financial management problem. The VaR value has been usually defined by $(-1)\times$ (the lower γ percentile point) when the return process is a sequence of i.i.d random variables. However, it is has been often observed that the return processes have autocorrelations as well as asymmetrical volatilities in actual financial data analyses. When there are some autocorrelations it may be natural to define the VaR value by

$$1-\gamma = P(y_t + G_{\hat{\sigma}}[-z(\gamma)\sqrt{h_{t+1}(\hat{\alpha})} + \hat{r}_0 + \sum_{j=1}^{p} \hat{r}_j y_{t+1-j}],\ +\infty)\ , \tag{4.6}$$

where $z(\gamma)$ is the upper γ percentile point of the distribution function $F(v)$. In this way it is quite easy to calculate the VaR value even if there are autocorrelations and the return process has the asymetrical volatility function at the same time.

4.2 Simulation Results

In this subsection we report the results of our simulations on the prediction based on the predictive median and the VaR value in the financial risk management. For the prediction based on the median, we simulate a set of random numbers from the SSAR(1)-ARCH(1) model. Then we estimate the median of the 1-period ahead predictive distribution by fitting both the AR(1)-ARCH(1) model and the SSAR(1)-ARCH(1) model. We have calculated the absolute sums of prediction errors

$$M_n = \frac{1}{n}\sum_{i=1}^{n} |y_{t+1}(i) - \hat{y}_{t+1|t}^{med}(i)|\ , \tag{4.7}$$

where $y_{t+1}(i)$ and $\hat{y}_{t+1|t}^{med}(i)$ are the i-th simulated values of y_{t+1} and $\hat{y}_{t+1|t}^{med}$. We denote $M_n(AR)$ and $M_n(SSAR)$ as the average sum of absolute errors by the AR fitting and the SSAR fitting, respectively. In Table 4.1 we report the ratio

$$R_n = \frac{M_n(SSAR)}{M_n(AR)}\ . \tag{4.8}$$

In our procedure of the SSAR fitting we first estimate the unknown parameters of (2.1) by using the maximum likelihood methods and use them to estimate the percentile points of the one period ahead predictive distributions as we have explained in Section 4.1. The number of data in each simulation is 500, which may be reasonable in actual applications because the VaR values are usually calculated from the daily time series data.

From Table 4.1 we find that we have achieved a substantial improvement in one-step ahead predition when the time series exhibits some asymmetry in the upward-phase and the downward-phase. Also we should note that the conditional expectation of the one-step ahead value $E[y_{t+1}|\mathcal{F}_t]$ gives a more complicated predictor.

Table 4.1 (Ratio of Prediction Errors)

	$a_1 = 0.8$ $b_1 = 0.8$	$a_1 = 0.2$ $b_1 = 0.2$	$a_1 = 0.0$ $b_1 = 0.0$	$a_1 = 0.8$ $b_1 = 0.2$	$a_1 = 0.8$ $b_1 = -0.8$	$a_1 = 0.2$ $b_1 = -0.2$
$\alpha_1 = 0.0$	1.00074	1.00019	1.00055	0.92106	0.83358	0.98891
$\alpha_1 = 0.4$	0.99998	1.00039	1.00005	0.89849	0.74736	0.98629

For the application of VaR, the GSSAR model we have used in our simulations is the stationary SSAR(1)-ARCH(1) model and we also have used the nonstationary SSARI(1)-ARCH(1) model because the stochastic processes for financial applications should have the nonstationarity as well as the asymmetrical sample paths. In both cases as the volatility function we take

$$h_t = 1 + \alpha_1 h_{t-1} v_{t-1}^2 \ , \tag{4.9}$$

where $0 \leq \alpha_1 < 1$. In our simulations we specify a set of the parameter values, but from the simulated data sets we try to estimate the one percentile point as if it were not known.

Table 4.2 (SSAR: Symmetric Case)

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
a_1	0.8	0.8	0.2	0.2	0	0
b_1	0.8	0.8	0.2	0.2	0	0
α_1	0	0.4	0	0.4	0	0.4
AR(1%)	1.010%	1.05%	1.01%	1.03%	0.96%	1.19%
SS(1%)	0.997%	1.09%	1.05%	1.04%	0.97%	1.20%
BIAS(AR)	1.20E-03	1.18E-03	6.87E-03	6.27E-03	7.02E-03	6.59E-03
BIAS(SS)	1.20E-03	1.41E-03	7.47E-03	7.07E-03	7.90E-03	7.65E-03
MSE(AR)	3.50E-04	9.58E-04	5.42E-03	1.56E-02	8.44E-03	2.50E-02
MSE(SS)	4.57E-04	1.14E-03	6.73E-03	1.84E-02	1.03E-02	3.07E-02
TRUE	9.72E-03	9.63E-03	9.70E-03	9.73E-03	9.13E-03	1.12E-02
STD	3.35E-01	4.29E-01	8.18E-01	1.05E+00	9.96E-01	1.30E+00

In Tables 4.2-4.5 AR(1%) and SS(1%) are the average percentages of the data below the (true) lower one percentile point. BIAS and MSE of the estimated VaR values are calculated as the averages of the simulation results.

Table 4.3 (SSAR: Asymmetrical Case)

	Case 7	Case 8	Case 9	Case 10	Case 11	Case 12
a_1	0.8	0.8	0.8	0.8	0.2	0.2
b_1	0.2	0.2	-0.8	-0.8	-0.2	-0.2
α_1	0	0.4	0	0.4	0	0.4
AR(1%)	4.09%	3.48%	4.66%	3.91%	1.95%	1.71%
SS(1%)	1.09%	1.04%	1.17%	1.09%	1.08%	1.09%
BIAS(AR)	3.94E-01	4.09E-01	7.22E-01	6.49E-01	2.22E-01	2.27E-01
BIAS(SS)	6.22E-03	8.86E-03	1.70E-02	1.63E-02	1.02E-02	7.33E-03
MSE(AR)	2.16E-01	4.87E-01	1.50E-00	4.85E+00	1.13E-01	3.14E-01
MSE(SS)	7.64E-03	2.06E-02	3.75E-02	9.97E-02	1.50E-02	4.52E-02
TRUE	1.00E-02	1.01E-02	1.08E-02	1.00E-02	1.04E-02	1.02E-02
STD	8.30E-01	1.07E+00	1.39E-00	1.78E+00	1.02E+00	1.31E+00

Table 4.2 and Table 4.3 summarize the simulation results when the underlying processes are the AR(1)-ARCH(1) model and the SSAR(1)-ARCH(1) model. When the transformation function $G_\sigma(\cdot)$ is linear, the VaR values in both models are quite similar. Hence we can estimate the one percentile point consistently both by the AR fitting and the GSSAR fitting. However, when the transformation function is not linear, the situation can be rather drastically different. Although the estimation of the GSSAR modelling is reasonable, the estimation results based on the standard AR model with or without ARCH fitting are badly biased.

Table 4.4 and Table 4.5 summarize the simulation results when the underlying processes are the IAR(1)-ARCH(1) model and the SSIAR(1)-ARCH(1) model. The results of simulations in the non-stationary time series models are quite similar to the results from the AR(1)-ARCH(1) model and the SSAR-ARCH model in Tables 4.2 and 4.3. Hence our simulation results suggest that our method using the non-stationary SSAR models gives a reasonable way to calculate the VaR values even if the predictive distribution of the return process is asymmetrical. The standard practive in estimating VaR values in financial risk management can be biased considerably if the underlying process has some asymmetry.

Table 4.4 (SSIAR: Symmetric Case)

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
a_1	0.8	0.8	0.2	0.2	0	0
b_1	0.8	0.8	0.2	0.2	0	0
α_1	0	0.4	0	0.4	0	0.4
AR(1%)	1.05%	0.94%	1.08%	1.05%	0.98%	1.16%
SS(1%)	1.08%	1.00%	1.11%	1.05%	0.98%	1.12%
BIAS(AR)	2.07E-03	2.19E-03	8.26E-03	0.97E-03	9.99E-03	1.11E-02
BIAS(SS)	2.29E-03	2.54E-03	8.71E-03	0.92E-03	1.05E-02	1.12E-02
MSE(AR)	5.13E-04	1.39E-03	8.03E-03	2.34E-02	1.24E-02	3.65E-02
MSE(SS)	6.71E-04	1.66E-03	1.00E-03	2.67E-02	1.57E-02	4.10E-02
TRUE	9.53E-03	0.91E-03	1.00E-02	1.00E-02	0.92E-03	1.03E-02
STD	3.36E-01	4.40E-01	8.15E-01	1.05E+00	1.00E+00	1.30E+00

Table 4.5 (SSIAR: Asymmetrical Case)

	Case 7	Case 8	Case 9	Case 10	Case 11	Case 12
a_1	0.8	0.8	0.8	0.8	0.2	0.2
b_1	0.2	0.2	-0.8	-0.8	-0.2	-0.2
α_1	0	0.4	0	0.4	0	0.4
AR(1%)	4.19%	4.12%	4.98%	4.65%	1.95%	2.18%
SS(1%)	1.11%	1.09%	1.09%	1.15%	0.96%	1.25%
BIAS(AR)	2.70E-01	3.61E-01	6.04E-01	7.92E-01	3.03E-01	3.55E-01
BIAS(SS)	1.04E-02	7.91E-03	2.04E-02	2.59E-02	1.13E-02	1.65E-02
MSE(AR)	3.18E-01	4.71E-01	1.99E+00	3.21E+00	1.33E-01	2.03E-01
MSE(SS)	1.08E-02	2.72E-02	5.97E-02	1.50E-02	2.31E-02	6.11E-02
TRUE	9.67E-03	9.70E-03	9.97E-03	9.87E-03	9.07E-03	1.18E-02
STD	5.16E-01	6.74E-01	7.61E-01	9.61E-01	9.78E-01	1.28E+00

4.3 A Continuous Diffusion Process

In (2.1) we have introduced the GSSAR model in the framework of the time series analysis with discrete time. One problem in the SSAR models can be in the fact that it is not straightforward to derive the corresponding continuous time stochastic processes. Since there could be some points where the differentiability $G_\sigma(\cdot)$ could breaks down in the SSAR models, we need to take their smoothed versions. For an illustration, we take the time interval Δt in the GSSAR(1) model (replace $t-\Delta t$ for $t-1$) and consider the situation $\Delta t \downarrow 0$. If we can approximate the transformation function by its smoothed version $G_{\sigma,\epsilon,\Delta t}(\cdot)$, then as the limit we could have the diffusion equation

$$y_t = y_0 + \int_0^t g_{1,\epsilon}(y_s, s)ds + \int_0^t g_{2,\epsilon}(y_s, s)dB_s \ , \tag{4.10}$$

where B_t stands for the standard Brownian motion and ϵ is a parameter. One possibility for the drift function and the diffusion function in the stochastic differential equation may be given by

$$g_1(y_t, t) = G_{\sigma,\epsilon,0}(r_0^* + r_1^* y_t) + \frac{1}{2} G''_{\sigma,\epsilon,0}(r_0^* + r_1^* y_t) h_t^* \ , \tag{4.11}$$

$$g_2(y_t, t) = G'_{\sigma,\epsilon,0}(r_0^* + r_1^* y_t)\sqrt{h_t^*} \tag{4.12}$$

where r_0^* and r_1^* are coefficients, h_t^* is the continuous analogue of the volatility function in the discrete time. If there are several points where the transformation function $G_\sigma(\cdot)$ is not differentiable, we need further approximation arguments.

4.4 A Case Study of Interest Rates Futures

In this subsection we shall report an empirical application using a set of financial time series data on interest rates futures at Tokyo. In our empirical data analysis we have used the time series data on the interest rates futures whose underlying assets are the Government Bonds (Kokusai) with 10 years maturity. They have been the most popular interest rates futures traded in Japan. The data sets are the closing daily data from January 1997 to April 2001 and all price data were transformed into their logarithms. Because the data exhibit the non-stationarity in their levels as well as some asymmetrical movements, we have used the non-stationary SSAR model with the ARCH effect given by

$$\Delta y_t = G_\sigma \left\{ r_0 + r_1 \Delta y_{t-1} + G_\sigma^{-1}(\Delta y_{t-1}) + \sqrt{h_t} v_t \right\} , \tag{4.13}$$

where the volatility function h_t is given by (2.11) with $r = 1$. (It is the SSIAR(1)-ARCH(1) model.) We have shown the data set in Figure (b) and the estimation results on coefficients in Figures (a), respectively. From our data sets we have estimated the SSIAR(1)-ARCH(1) models sequentially and examined the corresponding VaR values calculated from their one period ahead predictive distributions.

In the Japanese Government Bond market there was a significant price decline at the end of 1998 because of the Bond market crisis occurred. As a result the estimated coefficients $a = 1+r_1\sigma_1(= a_1)$ and $b = 1+r_1\sigma_2(= b_1)$ were not stable during that period and we have observed a significant increase in the volatility coefficients σ_i $(i = 1, 2)$. After this specific period, the volatility function becomes stable since the begining of 1999. The estimated coefficent of the volatility functions are shown in Figure (e).

Then as we have explained in Section 4.1, we can estimate the VaR values from the one period ahead predictive distributions sequentially. In Figures (c) and (d) we have presented the estimated VaR values (one percentile and ten percentiles) by using the SSARI modelling and the standard AR modelling without ARCH disturbances. In Figures (g) and (h) we have presented the estimated VaR values (one percentile and ten percentiles) by using the SSARI modelling and the AR modelling with ARCH disturbances. (For the comparative purpose the estimated percentile values by using the standard VaR method was presented in Figure (f) [3] .)

There are some interesting observations from Figures. When we use the ARCH models, the estimated VaR values often tend to fluctuate over time while we have stable VaR values when we do not use the ARCH models. It may be partly because we have used only one period ahead predictive distributions. It is possible to use the multiperiod predictive distributions, but then we need to specify the prediction horizon. As we can see from our Figures we have some evidence that the results from our approach is slightly better than the results from the standard time series modelling.

5. Conclusions

In this paper we have introduced a class of the GSSAR models in the nonlinear time sereis analysis. This class is an extension of the SSAR models developed by the authors and it is suitable for the estimation of asymmetric predictive distributions. In particular it is quite useful for estimating the VaR value in financial risk management when there are autocorrelations and the sample paths in the upward and downward phases can be asymmetrical. On the other hand, the traditional methods of estimating the VaR values are badly biased when we use the linear time series models including the standard ARCH modelling.

It should be also noted that the GSSAR modelling is a simple way to handle the nonlinear phenomena we have discussed. The model selection procedure can be developed straightforwardly within the GSSAR modelling along the line developed by

[3] The standard method of estimatining the VaR values has been explained in JP Morgan (1996), for instance.

Akaike (1973). It seems that other time series models known for financial applications, for instance, become often very complicated and it is quite difficult to judge whether a particular model is appropriate for practical applications under different data sets.

Finally there can be several problems remain to be considered. We have some multivariate extensions of the SSAR models, but the multivariate models with the volatility functions are obvious extensions in the GSSAR modelling. More importantly, the non-parametric or semi-parametric estimation problem of the transformation functions in the non-linear time series analysis should be investigated.

Mathematical Appendix

In this appendix, we give the proof of Theorem 3.1 in Section 3. In order to give the proof we need one important result on the ergodicity of the Markov chain with the general state space. (See Appendix of Tong (1990), for instance.) Let $\mathbf{x}_t$ be the m-dimensional Markov chain with the general state space $\mathbf{R}^m$.

Lemma A.1 *: Let $\{\boldsymbol{x}_t\}$ be ϕ-irreducible and aperiodic Markov chain. Suppose that there exists a compact set $\boldsymbol{C}$, a non-negative measurable function $g(\cdot)$, which is continuous, and constants $r > 1, \gamma > 0,$ and $K > 0$ such that for $\boldsymbol{x} \in \boldsymbol{R}^m$*

$$E[rg(\boldsymbol{x}_t)|\boldsymbol{x}_{t-1} = \boldsymbol{x}] < g(\boldsymbol{x}) - \gamma \quad (\boldsymbol{x} \notin \boldsymbol{C}) \ , \tag{A.1}$$

and

$$E[g(\boldsymbol{x}_t)|\boldsymbol{x}_{t-1} = \boldsymbol{x}] < K \quad (\boldsymbol{x} \in \boldsymbol{C}) \ . \tag{A.2}$$

Then $\{\boldsymbol{x}_t\}$ is geometrically ergodic.

When a Markov chain $\{\mathbf{x_t}\}$ is ergodic, we can define the stationary distribution of the process $\{\mathbf{x_t}\}$. We consider the ergodic (Markovian) situation when a probability measure $\pi(\cdot)$ satisfies

$$\pi(\boldsymbol{A}) = \int_{\boldsymbol{x}} P(\boldsymbol{x}, \boldsymbol{A})\pi(d\boldsymbol{x}) \tag{A.3}$$

for any $\boldsymbol{A} \in \mathcal{B}(\boldsymbol{R}^m)$, where $P(\boldsymbol{x}, \boldsymbol{A})$ is the transition probability given $\boldsymbol{x}$. Then if we take the initial distribution as the same as $\pi(\cdot)$, the process $\{\boldsymbol{x}_t\}$ is strictly stationary.

Proof of Theorem 3.1 : When $p > 0$ and $r > 0$, we define a $(p + r) \times 1$ state vector $\boldsymbol{x}_t$ by

$$\boldsymbol{x}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \\ u_t^2 \\ \cdots \\ u_{t-r+1}^2 \end{pmatrix} , \tag{A.4}$$

where $u_t = v_t\sqrt{h_t}$. We then consider the Markovian representation for $\{\boldsymbol{x}_t\}$ when the state space for $\boldsymbol{x_t}$ is $\boldsymbol{R}^p \times \boldsymbol{R}_+^r$. By using (2.1) and (2.11), we have the Markovian representation

$$\boldsymbol{x}_t = \boldsymbol{H}_\sigma(\boldsymbol{x}_{t-1}, v_t) \ , \tag{A.5}$$

where

$$\boldsymbol{H}_\sigma(\boldsymbol{x}_{t-1}, v_t) = \begin{pmatrix} y_{t-1} + G_\sigma(r_0 + \sum_{i=1}^{p} r_i y_{t-i} + v_t\sqrt{h_t}) \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \\ v_t^2 h_t \\ u_{t-1}^2 \\ \cdots \\ u_{t-r+1}^2 \end{pmatrix} .$$

We use the criterion function

$$g(\boldsymbol{x}) = 1 + \max_{j=1,\cdots,p+r} |x_{(j)}|\rho_j \ , \tag{A.6}$$

for $\boldsymbol{x} = (x_{(i)})$ and some $\rho_1 > \cdots > \rho_p > 0$ and $\rho_{p+1} > \cdots > \rho_{p+r} > 0$.

First, we take a sufficiently large M and we consider the first p components of the vector $\boldsymbol{x}_t$ such that $\|\boldsymbol{x}_{t-1}\| > M$. Then we have

$$E[\max_{1\leq j\leq p} \rho_j |y_{t+1-j}||\boldsymbol{x}_{t-1} = \boldsymbol{x}] \tag{A.7}$$

$$\leq \ c_1 + \sigma\rho_1 \sum_{j=1}^{r} \alpha_j u_{t-j}^2 \tag{A.8}$$

$$+ \max\{(\rho_1 \sum_{j=1}^{p} |a_j||y_{t-j}|, \rho_1 \sum_{j=1}^{p} |b_j||y_{t-j}|, \max_{2\leq j\leq p} \rho_j |y_{t+1-j}|\} \ ,$$

where c_1 is a positive constant, $\sigma = \max\{\sigma_1, \sigma_2\}$ and we have the last inequality because $h_t \geq 1$ and $\sqrt{h_t} \leq h_t$. Under the conditions in our assumptions there exist $0 \leq \theta_1 < 1$ and $\rho_1 > \rho_2 > \cdots > \rho_p > 0$ such that $\theta_1 > \rho_{j+1}/\rho_j$ $(j = 1, \cdots, p-1)$ and

$$\max\{|a_1| + \sum_{j=2}^{p} |a_j| \frac{\rho_1}{\rho_j}, |b_1| + \sum_{j=2}^{p} |b_j| \frac{\rho_1}{\rho_j}\} < \theta_1 < 1 \ . \tag{A.9}$$

Hence we have an equality

$$E[\max_{1\leq j\leq p} \rho_j |y_{t+1-j}||\boldsymbol{x}_{t-1} = \boldsymbol{x}] \tag{A.10}$$

$$\leq \ c_1 + \sigma\rho_1 \sum_{j=1}^{r} \alpha_j u_{t-j}^2 + \theta_1 \max_{1\leq j\leq p} \{\rho_j |y_{t-j}|\} \ .$$

Next, we consider the last r components of $\boldsymbol{x}_t$ on the volatility function. Because of our conditions on the ARCH terms, there exist $0 \leq \theta_2 < 1$ and $\rho_{p+1} > \rho_{p+2} > \cdots > \rho_{p+r} > 0$, $\theta_2 > \rho_{j+1}/\rho_j$ $(j = p, \cdots, p+r-1)$ such that

$$\max\{\sum_{j=1}^{r} \alpha_j \frac{\rho_{p+1}}{\rho_{p+j}}, \sigma \sum_{j=p}^{r} \alpha_j \frac{\rho_{p+1}}{\rho_{p+j}}\} < \theta_2 < 1 \ . \tag{A.11}$$

Then we can take

$$
\begin{aligned}
E[\rho_{p+1}u_t^2|\boldsymbol{x}_{t-1}=\boldsymbol{x}] &\leq c_2+\rho_{p+1}\sum_{j=1}^{r}\alpha_j u_{t-j}^2 \\
&< c_2+\sum_{j=1}^{r}(\alpha_j\frac{\rho_{p+1}}{\rho_{p+j}})\rho_{p+i}u_{t-j}^2 \qquad \text{(A.12)} \\
&\leq c_2+\theta_2\max_{1\leq i\leq r}\{\rho_{p+i}x_{(p+i)}\}\ .
\end{aligned}
$$

Then by using (A.9) and (A.11), we have

$$
E[g(\boldsymbol{x}_t)|\boldsymbol{x}_{t-1}=\boldsymbol{x}]<c_3+\theta_3 g(\boldsymbol{x})\ , \qquad \text{(A.13)}
$$

where $\theta_3 = \max\{\theta_1,\theta_2\} < 1$ and c_3 is a positive constant. Hence we can take a sufficiently large M such that for $\|\boldsymbol{x}\| > M$ the conditions in (A.1) and (A.2) are satisfied.

The rest of our proof is similar to the one for Theorem 2.3 of Kunitomo and Sato (2000b). Since the Markov chain for $\boldsymbol{x}_t$ is aperiodic and $\phi-$irreducible due to our assumptions, we can apply Lemma A.1 and establish that $\{\boldsymbol{x}_t\}$ is geometrically ergodic. Finally, we need to consider the case when $r=0$ separately, but the result is obvious from our derivations in this case and the details are omitted. *Q.E.D.*

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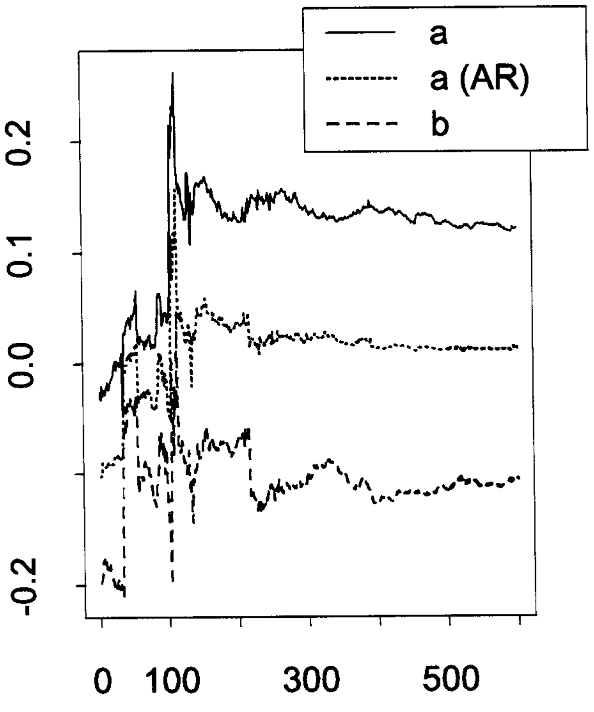
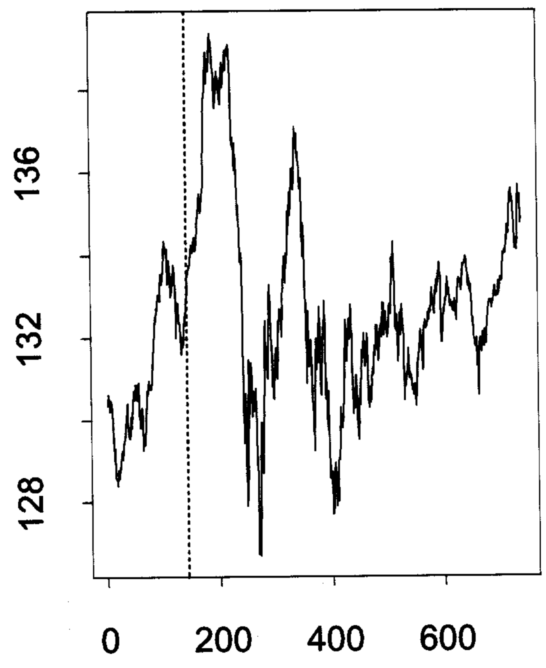
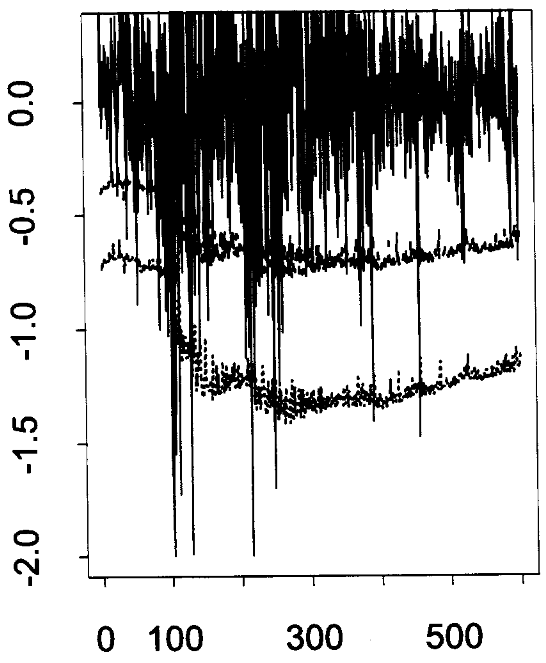


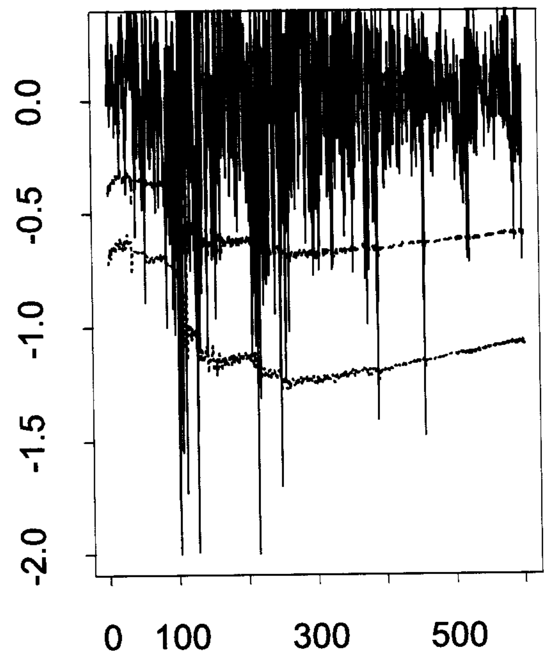
(a) estimated coefficients



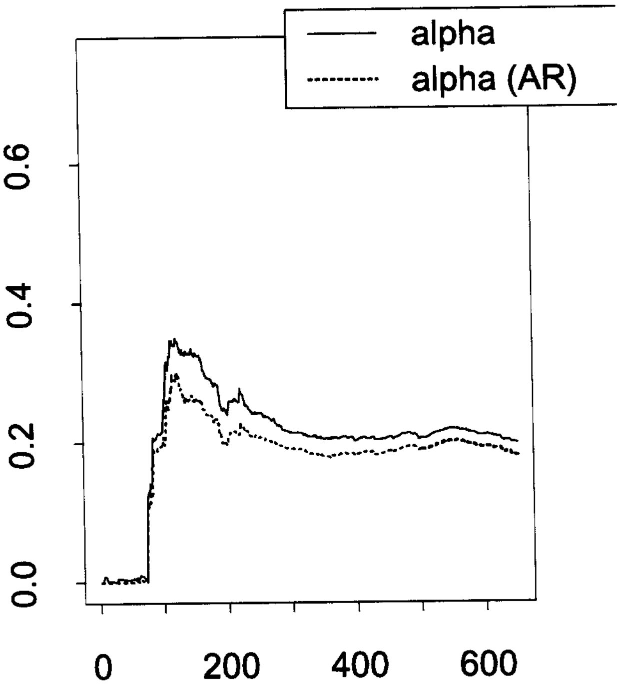
(b) observations



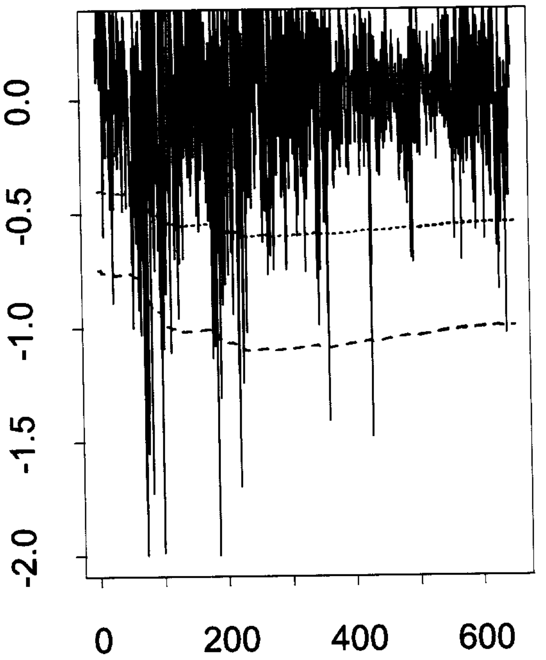
(c) SSIAR model (10%,1%)



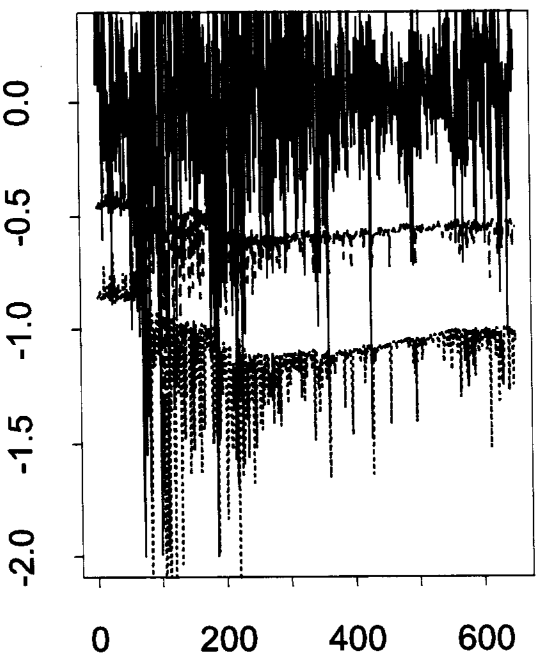
(d) AR model (10%,1%)



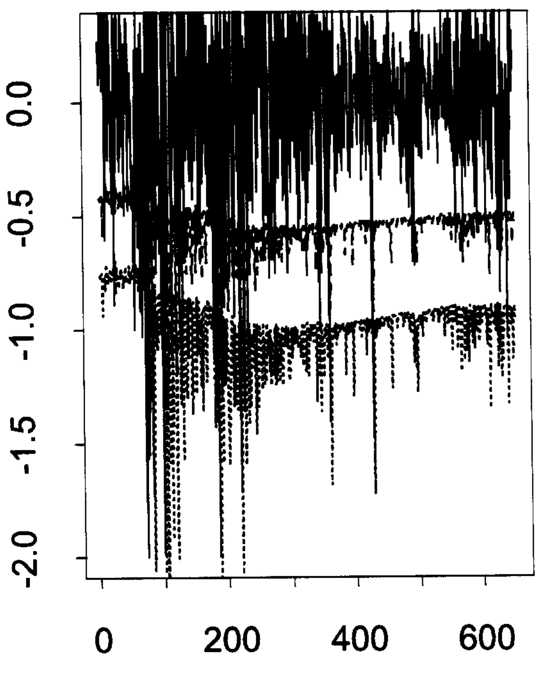
(e) estimated ARCH coefficients



(f) iid-Normal model (10%,1%)



(g) SSIAR-ARCH model (10%,1%)



(h) AR-ARCH model (10%,1%)