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The Folk Theorem with Private Monitoring

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Abstract

This paper investigates infinitely repeated prisoner-dilemma games, where the discount factor is less than but close to 1. We assume that monitoring is imperfect and private, and players' private signal structures satisfy the conditional independence. We require almost no conditions concerning the accuracy of private signals. We assume that there exist no public signals and no public randomization devices, and players cannot communicate and use only pure strategies. It is shown that the Folk Theorem holds in that every individually rational feasible payoff vector can be approximated by a sequential equilibrium payoff vector. Moreover, the Folk Theorem holds even if each player has no knowledge of her opponent's private signal structure.

Keywords: Repeated Prisoner-Dilemma Games, Private Monitoring, Conditional Independence, Folk Theorem, Limited Knowledge.

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1. Introduction

This paper investigates infinitely repeated prisoner-dilemma games, where the discount factor is less than but close to 1. We assume that players not only imperfectly but also *privately* monitor their opponents' actions. Players cannot observe their opponents' actions directly, but can only observe their own private signals that are drawn according to a density function over closed intervals conditional upon the action profile played. There are no public signals.

We assume that players' private signals are *conditionally independent*, i.e., players can obtain *no* information on what their opponents have observed by observing their own private signals. We show that the *Folk Theorem* holds in that every individually rational feasible payoff vector can be sustained by a sequential equilibrium in the limit of the discount factor. This result is permissive, because we require almost *no* conditions concerning the accuracy of players' private signals.

The study of repeated games with private monitoring is relatively new. Most earlier works in this area have assumed that monitoring is either perfect or public and have investigated only perfect public equilibria. It is well known that under a mild condition, every individually rational feasible payoff vector can be sustained by a perfect public equilibrium in the limit of the discount factor, provided that monitoring is imperfect but public.[1] Perfect public equilibrium requires that the past histories relevant to future play are common knowledge in every period. This common knowledge property makes equilibrium analyses tractable, because players' future play can always be described as a Nash equilibrium.[2]

As the signal is *not* public in the present paper, the Folk Theorem is not immediate and the problem is more delicate. When monitoring is only private, it is inevitable that an equilibrium sustaining implicit collusion depends on players' private histories, and therefore, the past histories relevant to future play are not common knowledge. This makes equilibrium analyses much more difficult, especially in the discounting case, because players' future play cannot be described as a Nash equilibrium. Even when a player is certain that a particular opponent has deviated, the other players will typically not share this certainty, and they will be unable to coordinate on an equilibrium that punishes the deviant in the continuation game. Nevertheless a more complicated argument establishes the Folk Theorem.

Hence, we have the Folk Theorem with completely public signals on the one hand, and we have the Folk Theorem even with completely private signals on the other hand.

To the best of my knowledge, Radner (1986) is the first paper on repeated games with private monitoring. Radner assumed no discounting, and showed that every individually rational feasible payoff vector can be sustained by a Nash equilibrium.[3] The two papers

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[1] See Fudenberg, Levine, and Maskin (1994). For the survey, see Pearce (1992).

[2] With imperfect public monitoring, Obara (1999) and Kandori and Obara (2000) investigated the role of private strategies that depends on the chosen actions as well as the observed public signals. They showed that efficiency can be drastically improved by using private strategies.

[3] See also Lehrer (1989) for the study of repeated games with no discounting and with private monitoring. Fudenberg and Levine (1991) investigated infinitely repeated games with

by Matsushima (1990a, 1990b) appear to be the first to investigate the discounting case. Matsushima (1990a) provided an Anti-Folk Theorem, showing that it is impossible to sustain implicit collusion by pure strategy Nash equilibria when private signals are conditionally independent and Nash equilibria are restricted to be independent of payoff-irrelevant private histories. The present paper establishes the converse result: the Folk Theorem *holds* when we use pure strategy Nash equilibria that can depend on payoff-irrelevant private histories.

Matsushima (1990b) conjectured that a Folk Theorem type result could be obtained even with private monitoring and discounting when players can communicate by making publicly observable announcements. Subsequently, Kandori and Matsushima (1998) and Compte (1998) proved the Folk Theorem with communication. Communication synthetically generates public signals and consequently it is possible to conduct the dynamic analysis in terms of perfect public equilibria as in the paper by Fudenberg, Levine and Maskin (1994) on the Folk Theorem with imperfect public monitoring. The present paper assumes that players make *no* publicly observable announcements.

Interest in repeated games with private monitoring and no communication has been stimulated by a number of recent papers, including Sekiguchi (1997), Bhaskar (1999), Piccione (1998), and Ely and Valimaki (1999). Sekiguchi (1997) investigated a restricted class of prisoner-dilemma games on the assumption that monitoring was *almost perfect* and that players' private signals were conditionally independent. Sekiguchi was the first to show that an efficient payoff vector can be approximated by a mixed strategy Nash equilibrium payoff vector even if players cannot communicate. By using public randomization devices, Bhaskar and Obara (2000) extended Sekiguchi's result to more general games.

Piccione (1998) and Ely and Valimaki (1999) also considered repeated prisoner-dilemma games when the discount factor is close to 1, and provided their respective Folk Theorems. Both papers constructed mixed strategy equilibria in which each player is indifferent between the right action and the wrong action irrespective of her opponent's possible future strategy. Piccione used dynamic programming techniques over infinite state spaces, while Ely and Valimaki used two-state Markov strategies. Both papers investigated only the almost-perfect monitoring case, and most of their arguments rely heavily on this assumption.[4]

Mailath and Morris (1998) investigate the robustness of perfect public equilibria when monitoring is almost public, i.e., each player can always discern accurately which private signal her opponent has observed by observing her own private signal. The present paper does *not* assume that monitoring is almost public.

In consequence, this paper has many substantial points of departure from the earlier literature. We assume that there exist no public signals, players make no publicly observed announcements, and there exist no public randomization devices. We do not require that monitoring is either almost perfect or almost public. Hence, the present paper can be regarded as the first work to provide affirmative answers to the possibility of implicit collusion with discounting when monitoring is *truly imperfect* and *truly private*.

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discounting and with private monitoring in terms of epsilon-equilibria.

[4] In the last section of his paper, Piccione provides an example in which implicit collusion is possible even if players' private observation errors are not infinitesimal.

As such, this paper may offer important economic implications within the field of industrial organization. In the real economy, communication between rival firms' executives is restricted by Anti-Trust Law, on the assumption that such communication enhances the possibility of a self-enforcing cartel agreement.[5] Moreover, in reality, firms usually cannot directly observe the prices or quantities of rival firms and the aggregate level of consumer demand is stochastic. Instead, each firm's only information about its opponents' actions within any particular period, is its own realized sales level and, therefore, each firm cannot know what its opponents have observed. These circumstances tend to promote the occurrence of price wars, as each firm cannot know whether a fall in its own sales is due to a fall in demand or a secret price cut by a rival firm. In this way, it has been widely believed that a cartel agreement is most likely to be breached when each firm's monitoring of its opponents' actions is truly private.[6] In contrast, the present paper shows that collusive behavior is possible even if communication is prohibited and each firm obtains no public information on the prices or quantities of its rivals.

The technical aspect of the present paper is closely related to Piccione (1998) and Ely and Valimaki (1999), particularly the latter. The paper is also related to Matsushima (1999), which investigated the impact of multimarket contact on implicit collusion in the imperfect public monitoring case and provided the efficiency result by using the idea of a *review strategy* equilibrium. Our equilibrium construction may be viewed as extending the equilibrium construction of Ely and Valimaki combined with that of Matsushima to general private signal structures.

Furthermore, we consider the situation in which players have limited knowledge of their private signal structures as follows. Each player knows her own private signal structure, but does *not* know her opponent's private signal structure. Hence, each player's strategy depends on her own private signal structure, but is independent of her opponent's private signal structure. We clarify whether the Folk Theorem can be achieved by using only players' strategies that depend only on their own private signal structures.

Each player behaves according to a mapping that assigns a strategy for this player to each possible conditional density function over her own private signal. Their mappings are assumed to be common knowledge, but each player does not know which strategy in the range of the opponent's mapping is actually played. It is also assumed common knowledge that players' private signal structures satisfy the conditional independence.

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[5] See such industrial organization textbooks as Scherer and Ross (1990) and Tirole (1988). Matsushima (1990b), Kandori and Matsushima (1998), Compte (1998), and Aoyagi (2000) provided a justification of why communication is so important for the self-enforcement of a cartel agreement.

[6] Stigler (1964) is closely related. Moreover, Green and Porter (1984) investigated repeated quantity-setting oligopoly when the market demand is stochastic and firms cannot observe the quantities of their rival firms. They assumed that firms can publicly observe the market-clearing price. In contrast, the present paper assumes that there exist *no* publicly observable signals such as the market-clearing price.

We require that every pair of strategies in the ranges of their mappings are sequential equilibria and approximately sustains the same payoff vector.

We establish the *Folk Theorem* even with the above informational constraint. That is, for every individually rational feasible payoff vector, there exists a profile of mappings assigning each possible private signal structure a sequential equilibrium that approximately sustains this payoff vector.

The organization of the paper is as follows. Section 2 defines the model. Section 3 presents the Folk Theorem. Section 4 provides the outline of its proof. Section 5 provides the complete proof. Section 6 considers the situation in which players have limited knowledge of their private signal structures. Section 7 concludes.

2. The Model

An infinitely repeated prisoner-dilemma game $\Gamma(\delta) = ((A_i, u_i, \Omega_i)_{i=1,2}, \delta, p)$ is defined as follows. In every period $t \geq 1$, players 1 and 2 play a prisoner-dilemma game $(A_i, u_i)_{i=1,2}$. We denote $j \neq i$, i.e., $j = 1$ when $i = 2$, and $j = 2$ when $i = 1$. Player $i's$ set of actions is given by $A_i = \{c_i, d_i\}$. Let $A \equiv A_1 \times A_2$. Player $i's$ instantaneous payoff function is given by $u_i : A \to R$. We assume that $u_i(c) = 1$, $u_i(d) = 0$, $u_i(d / c_j) = 1 + x_i > 1$, and $u_i(c / d_j) = -y_i < 0$, where we denote $c \equiv (c_1, c_1)$ and $d \equiv (d_1, d_2)$. We assume that $x_1 + x_2 \leq y_1 + y_2$, i.e., the payoff vector (1,1) is efficient. The feasible set of payoff vectors $V \subset R^2$ is defined as the convex hull of the set $\{(1,1), (0,0), (1 + x_1, -y_2), (-y_1, 1 + x_2)\}$. The discount factor is denoted by $\delta \in [0,1)$.

At the end of every period, player i observes her own *private* signal ω_i. The set of player $i's$ private signals is defined as $\Omega_i \equiv [0,1]$. Let $\Omega \equiv \Omega_1 \times \Omega_2$. A signal profile $\omega \equiv (\omega_1, \omega_2) \in \Omega$ is determined according to a conditional density function $p(\omega | a)$. Let $p_i(\omega_i | a) \equiv \int_{\omega_j \in \Omega_j} p(\omega | a) d\omega_j$. We assume that player $i's$ private signal structure has full support, i.e., that $p_i(\omega_i | a) > 0$ for all $a \in A$ and all $\omega_i \in \Omega_i$.[7] We may regard $u_i(a)$ as the expected value defined by

$$u_i(a) \equiv \int_{\omega_i \in \Omega_i} \pi_i(\omega_i, a_i) p_i(\omega_i | a) d\omega_i ,$$

where $\pi_i(\omega_i, a_i)$ is the realized instantaneous payoff for player i when player i chooses action a_i and observes her own private signal ω_i. For every subset $W_i \subset \Omega_i$, let $p_i(W_i | a) \equiv \int_{\omega_i \in W_i} p_i(\omega_i | a) d\omega_i$.

An example is the model of a price-setting duopoly. Actions c_i and d_i are regarded as the choices of high price $\lambda_i(c_i)$ and low price $\lambda_i(d_i)$, respectively, for firm $i's$ commodity, where $\lambda_i(c_i) > \lambda_i(d_i) \geq 0$. Firm $i's$ sales when private signal ω_i is observed is given by $q_i(\omega_i) \geq 0$. The realized instantaneous profit for firm i is given by $\pi_i(\omega_i, a_i) = \lambda_i(a_i) q_i(\omega_i) - C_i(q_i(\omega_i))$, where $C_i(q_i) \geq 0$ is firm $i's$ total cost of production.

We assume that $p_i(\omega_i | a)$ is *continuous with respect to* $\omega_i \in \Omega_i$. From this continuity, it follows that for every $\xi \in [0,1]$, we can choose subsets $\Omega_i^*(\xi) \subset \Omega_i$ and $\Omega_i^{**}(\xi) \subset \Omega_i$ satisfying that

$$\xi = p_i(\Omega_i^*(\xi) | c) = \frac{p_i(\Omega_i^*(\xi) \cap \Omega_i^* | c)}{p_i(\Omega_i^* | c)}$$

$$= p_i(\Omega_i^*(\xi) | c / d_j) = \frac{p_i(\Omega_i^*(\xi) \cap \Omega_i^* | c / d_j)}{p_i(\Omega_i^* | c / d_j)}$$

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[7] We assume it only for simplicity. We can derive the same results without this assumption.

$$= \frac{p_i(\Omega_i^{**}(\xi) \cap \Omega_i^{**} \mid d)}{p_i(\Omega_i^{**} \mid d)} = \frac{p_i(\Omega_i^{**}(\xi) \cap \Omega_i^{**} \mid d / c_j)}{p_i(\Omega_i^{**} \mid d / c_j)}.$$

The probability of $\Omega_i^*(\xi)$ is the same between c and c / d_j, and is equal to ξ. The probability of $\Omega_i^*(\xi)$ conditional on Ω_i^* is equal to ξ, irrespective of c and c / d_j. The probability of $\Omega_i^{**}(\xi)$ conditional on Ω_i^{**} is the same between d and d / c_j, and is equal to ξ.

We assume that players' private signal structures satisfy *the minimal information requirement* in that for each $i \in \{1,2\}$,

$$p_i(\cdot|a) \neq p_i(\cdot|a') \text{ for all } a \in A \text{ and all } a' \in A / \{a\}.$$

From the minimal information requirement, it follows that we can choose subsets $\Omega_i^* \subset \Omega_i$, $\Omega_i^{**} \subset \Omega_i$, and $\Omega_i^+ \subset \Omega_i$ satisfying that

$$p_i(\Omega_i^* \mid c) < p_i(\Omega_i^* \mid c / d_j),$$

$$p_i(\Omega_i^{**} \mid d) < p_i(\Omega_i^{**} \mid d / c_j),$$

and

$$p_i(\Omega_i^+ \mid d / c_j) < p_i(\Omega_i^+ \mid d).$$

We assume also that players' private signal structures satisfy *the conditional independence* in that

$$p(\omega|a) = p_1(\omega_1|a) p_2(\omega_2|a) \text{ for all } a \in A \text{ and all } \omega \in \Omega.$$

A private history for player i up to period t is denoted by $h_i^t \equiv (a_i(\tau), \omega_i(\tau))_{\tau=1}^t \in (A_i \times \Omega_i)^t$, where $a_i(\tau) \in A_i$ is the action chosen by player i in period τ, and $\omega_i(\tau) \in \Omega_i$ is the private signal observed by player i in period τ. The null history for player i is denoted by h_i^0. Let $h^t \equiv (h_1^t, h_2^t)$. The set of all private histories for player i is denoted by H_i. A (pure) *strategy for player* i is defined as a function $s_i : H_i \to A_i$. The set of strategies for player i is denoted by S_i. Let $S \equiv S_1 \times S_2$. Player i's normalized long-run payoff induced by a strategy profile $s \in S$ after period t when her private history up to period $t-1$ is h_i^{t-1} is denoted by

$$v_i(\delta, s, h_i^{t-1}) \equiv (1-\delta) E[\sum_{t=1}^{\infty} \delta^{t-1} u_i(a(t)) \mid s, h_i^{t-1}].$$

A strategy profile $s \in S$ is said to be a *sequential equilibrium* in $\Gamma(\delta)$ if for each $i = 1,2$, every $s_i' \in S_i$, every $t = 1,2,...$, and every $h_i^{t-1} \in H_i$,

$$v_i(\delta, s, h_i^{t-1}) \geq v_i(\delta, s / s_i', h_i^{t-1}).$$

Player i's normalized long-run payoff induced by a strategy profile $s \in S$ is denoted by $v_i(\delta, s) \equiv v_i(\delta, s, h_i^0)$. Let $v(\delta, s) \equiv (v_1(\delta, s), v_2(\delta, s))$.

Definition 1: A payoff vector $v = (v_1, v_2) \in R^2$ is *sustainable* if for every $\varepsilon > 0$, and every infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ satisfying $\lim_{m \to +\infty} \delta^m = 1$, there exists an infinite sequence of strategy profiles $(s^m)_{m=1}^{\infty}$ such that for every large enough m, s^m is a sequential equilibrium in $\Gamma(\delta^m)$, and for each $i \in \{1,2\}$,

$$v_i - \varepsilon \leq \lim_{m \to +\infty} v_i(\delta^m, s^m) \leq v_i + \varepsilon .$$

A strategy profile $s \in S$ is said to be a *Nash equilibrium* in $\Gamma(\delta)$ if for each $i = 1,2$, and every $s_i' \in S_i$,

$$v_i(\delta, s) \geq v_i(\delta, s / s_i') .$$

Since each player's private signal structure has full support, the set of Nash equilibrium payoff vectors is equivalent to the set of sequential equilibrium payoff vectors. Note that the set of sustainable payoff vectors is compact.

3. The Folk Theorem

A feasible payoff vector $v \in V$ is said to be *individually rational* if it is more than or equal to the minimax payoff vector, i.e., $v \geq (0,0)$. Let

$$z^{[1]} \equiv (0, \frac{1+y_1+x_2}{1+y_1}) \text{ and } z^{[2]} \equiv (\frac{1+y_2+x_1}{1+y_2}, 0).$$

Note that the set of all individually rational feasible payoff vectors is equivalent to the convex hull of the set $\{(1,1),(0,0),z^{[1]},z^{[2]}\}$. The Folk Theorem is provided as follows.

Folk Theorem: *Suppose that players' private signal structures satisfy the minimal information requirement and the conditional independence. Then, every individually rational feasible payoff vector is sustainable.*

This theorem is in contrast to the Anti-Folk Theorem provided by Matsushima (1990a). Matsushima showed that the repetition of the one-shot Nash equilibrium is the only Nash equilibrium if players' private signals are conditionally independent and only pure strategies are permitted, which are restricted to be *independent of payoff-irrelevant histories*. A strategy profile s is said to be independent of payoff-irrelevant histories if for each $i = 1,2$, every $t = 1,2,\ldots$, every $h_i^t \in H_i$, and every $h_i'^t \in H_i$,

$$s_i|_{h_i^t} = s_i|_{h_i'^t} \text{ whenever } p_i(h_j^t | s, h_i^t) = p_i(h_j^t | s, h_i'^t) \text{ for all } h_j^t \in H_j,$$

where $p_i(h_j^t | s, h_i^t)$ is the probability anticipated by player i that the opponent j observes private history $h_j^t \in H_j$ when player i observes private history $h_i^t \in H_i$, provided that both players behave according to $s \in S$. The independence of payoff-irrelevant histories implies that whenever a player anticipates the opponent's future strategy in the same way then she plays the same strategy. In contrast to Matsushima (1990a), the present paper establishes the Folk Theorem when players' private signals are conditionally independent and only pure strategies are permitted, but which *depend* on payoff-irrelevant histories.[8]

Section 5 provides the complete proof of the Folk Theorem. The proof will be divided into three steps, i.e., Steps 1, 2, and 3. Step 1 shows that the payoff vectors (1,1), (1,0), (0,1) and (0,0) are sustainable, where (1,1) is an efficient payoff vector. Since Step 1 includes the logical core of the theorem, it would be helpful to provide the outline of Step 1 in the next section.

[8] Bhaskar (2000) investigated the robustness of equilibria to payoff perturbation in repeated games. He pointed out that when the payoff perturbation does not violate the additive separability of utility functions and players' private signal structures satisfy the conditional independence, the equilibria constructed in the present paper are not robust to this payoff perturbation. As Bhaskar mentioned by himself, this result crucially depends on the assumption that the additive separability is common knowledge among the players even with payoff perturbation. In the paper we put ourselves on the strong stand that the additive separability assumption only schematizes and simplifies the phenomena under investigation, in which the payoff functions may not be exactly additive-separable.

4. Efficient Sustainability: Outline

This section provides the outline of Step 1 that shows why (1,1), (1,0), (0,1) and (0,0) are sustainable.

Fix an integer $T > 0$ arbitrarily, which is chosen sufficiently large. Fix a discount factor $\delta \in (0,1)$ arbitrarily, which is chosen close to 1. Fix an integer $r_i^*(T) \in \{1,...,T\}$, and two real numbers $\bar{\xi}_i \in (0,1)$ and $\underline{\xi}_i \in (0,1)$ arbitrarily, which will be specified later.

Consider the following Markov strategies for player i that consist of $2T$ states, i.e., states (c_i,τ) and states (d_i,τ) for $\tau = 1,...,T$. When player $i's$ state is (a_i,τ), player i chooses the action a_i. When player $i's$ state is (a_i,τ) and $\tau < T$, player $i's$ state in the next period will be $(a_i,\tau+1)$. When player $i's$ state is (c_i,T) and player i has observed at most $r_i^*(T)$ private signals that belong to Ω_i^* during the last T periods, player $i's$ state in the next period will be $(c_i,1)$. When player $i's$ state is (c_i,T) and player i has observed at least $r_i^*(T)+1$ private signals that belong to Ω_i^*, player $i's$ state in the next period will be $(c_i,1)$ with probability $1-\bar{\xi}_i$, and $(d_i,1)$ with probability $\bar{\xi}_i$. When player $i's$ state is (d_i,T) and all the private signals that player i has observed during the last T periods belong to Ω_i^{**}, player $i's$ state in the next period will be $(c_i,1)$. When player $i's$ state is (d_i,T) and player i has observed at least one private signal that does not belong to Ω_i^{**}, player $i's$ state in the next period will be $(c_i,1)$ with probability $1-\underline{\xi}_i$, and $(d_i,1)$ with probability $\underline{\xi}_i$. We define two strategies for player i, $\bar{s}_i$ and $\underline{s}_i$, as the Markov strategies that start with states $(c_i,1)$ and $(d_i,1)$, respectively.

The T periods starting with state $(c_i,1)$ (state $(d_i,1)$) is regarded as *the cooperative phase* (*the punishment phase*, respectively). The strategies $\bar{s}_i$ and $\underline{s}_i$ are regarded as the *review strategies* starting with the cooperative phase and the punishment phase, respectively.[9] We will say that the T times repeated play *passes the review of player* i if either player i have observed at most $r_i^*(T)$ private signals that belong to Ω_i^* during the cooperative phase or all the private signals that she have observed during the punishment phase belong to Ω_i^{**}. Otherwise we will say that the T times repeated play *fails the review of player* i. When the T times repeated play passes the review of player i, player i will certainly play the cooperative behavior according to the strategy $\bar{s}_i$ from the next period. When the T times repeated play fails the review, she will play the

[9] The idea of review strategy was originated by Radner (1985) and explored by Abreu, Milgrom and Pearce (1991), Matsushima (1999), Kandori and Matsushima (1998), and Compte (1998). These papers made future punishment triggered either by bad histories of the public signals during the review phase, or by bad messages announced at the last stage of the review phase. In contrast to these works, the present paper assumes the non-existence of such public signals or messages.

punishment behavior according to the strategy $\underline{s}_i$, instead of $\bar{s}_i$, with a positive probability.

By applying the idea of the construction addressed by Ely and Valimaki (1999), accompanied with an extended version of the Law of Large Numbers, we can show that there exist $\bar{\xi}_i$, $\underline{\xi}_i$, and $r_i^*(T)$ for each $i \in \{1,2\}$ such that the payoff vectors $v(\delta,\bar{s})$, $v(\delta,\bar{s}/\underline{s}_1)$, $v(\delta,\bar{s}/\underline{s}_2)$, and $v(\delta,\underline{s})$ approximate $(1,1)$, $(1,0)$, $(0,1)$ and $(0,0)$, respectively, and that for each $i \in \{1,2\}$,

$$v_i(\delta,\bar{s}) = v_i(\delta,\bar{s}/\underline{s}_i) \text{ and } v_i(\delta,\underline{s}) = v_i(\delta,\underline{s}/\bar{s}_i)\,. \tag{1}$$

Ely and Valimaki investigated the Markov strategies only in the case of $T = 1$. They assumed that monitoring is almost perfect, and showed the existence of such $\bar{\xi}_i$, $\underline{\xi}_i$ and $r_i^*(T)$. In the case of $T = 1$, equalities (1) imply that all strategies for player i *are the best replies to* $\bar{s}$, that is,

$$v_i(\delta,\bar{s}) = v_i(\delta,\bar{s}/s_i) \text{ for all } s_i \in S_i\,.$$

Hence, it follows that equalities (1) are sufficient for $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$ and $\bar{s}/\underline{s}_2$ to be Nash equilibria, and, therefore, simply by constructing the Markov strategies with $T = 1$, Ely and Valimaki proved that $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$ are sustainable when monitoring is almost perfect.[10] We can generalize their result to the case that there exists a private signal for each player i such that the likelihood ratio between the action profiles c and c/d_j is almost zero. With this almost zero likelihood ratio condition, by making the punishment phase triggered by the occurrence of the neighborhood of this signal, we can prove that even if monitoring is not almost perfect, $(1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$ are all sustainable.[11]

We can apply the idea of Ely and Valimaki or its generalization to the Markov strategies with sufficiently large T. By choosing $r_i^*(T)$ more than $Tp_i(\Omega_i^*|c)$ but less than T, we can make the likelihood ratio between the T times repeated choice of the action profile c and the T times repeated choice of the action profile c/d_j that player i observes at least $r_i^*(T)+1$ private signals that belong to Φ_i^*, almost zero. For example, suppose that $r_i^*(T) = T-1$. Then, the likelihood ratio is equal to $\left(\frac{p_i(\Omega_i^*|c)}{p_i(\Omega_i^*|c/d_j)}\right)^T$, which is close to zero when T is sufficiently large. Hence, by regarding the T times repeated game as the component game, we can prove the existence of such $\bar{\xi}_i$, $\underline{\xi}_i$, and $r_i^*(T)$.

However, in contrast to the case of $T = 1$, equalities (1) does not imply that $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$, and $\bar{s}/\underline{s}_2$ are Nash equilibria in the case of $T > 1$. We denote by $\hat{S}_i \subset S_i$ the set of strategies for player i satisfying that player i chooses the same action whenever in the same review phase, i.e., for every $t = 1,2...$, every $t' = 1,2...$, every $h_i^{t-1} \in H_i$, and every

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[10] Obara (1999) independently explored the same idea of construction as Ely and Valimaki in the study of private strategies with imperfect public monitoring.

[11] See Sections 3 and 4 of the earlier version of the paper (Matsushima (2000)).

$h_i^{t'-1} \in H_i$,

$$s_i(h_i^{t-1}) = s_i(h_i^{t'-1}) \text{ if } kT+1 \le t \le t' \le (k+1)T \text{ for some integer } k.$$

Note that $\bar{s}_i \in \hat{S}_i$ and $\underline{s}_i \in \hat{S}_i$. Equalities (1) imply only that

$$v_i(\delta, \bar{s}) = v_i(\delta, \bar{s}/s_i) \text{ and } v_i(\delta, \underline{s}) = v_i(\delta, \underline{s}/s_i) \text{ for all } s_i \in \hat{S}_i.$$

Hence, in order to complete the proof that $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$, and $\bar{s}/\underline{s}_2$ are Nash equilibria, we have to check also that

$$v_i(\delta, \bar{s}) \ge v_i(\delta, \bar{s}/s_i) \text{ and } v_i(\delta, \underline{s}) \ge v_i(\delta, \underline{s}/s_i) \text{ for all } s_i \notin \hat{S}_i.$$

Note from the definitions of $\bar{s}_i$ and $\underline{s}_i$, and the full-support assumption, that all we have to check is that

$$v_i(\delta, \bar{s}) \ge v_i(\delta, \bar{s}/s_i) \text{ for all } s_i \notin \hat{S}_i.$$

The conditional independence guarantees that during the first T periods, the past history of the private signals for player i provides player i with *no* information about the opponent $j's$ future play, as long as the opponent $j's$ strategy belongs to $\hat{S}_j$. This, together with equalities (1), implies that if $\bar{s}_i$ is not the best reply to $\bar{s}_j$, then there exist $s_i \in S_i$ and $(a_i(1),...,a_i(T)) \in A_i^T$ such that

$$\text{either } v_i(\delta, \bar{s}/s_i) > v_i(\delta, \bar{s}) \text{ or } v_i(\delta, \underline{s}/s_i) > v_i(\delta, \underline{s}),$$

and that the strategy s_i suggests player i to choose the action $a_i(t)$ in each period $t \in \{1,...,T\}$ and to play the strategy $\bar{s}_i|_{h_i^T}$ after period $T+1$, i.e.,

$$s_i(h_i^{t-1}) = a(t) \text{ for all } t = 1,...,T \text{ and all } h_i^{t-1} \in H_i,$$

and

$$s_i|_{h_i^T} = \bar{s}_i|_{h_i^T} \text{ for all } h_i^T \in H_i.$$

Since the action d_i is dominant for player i in the component game, we can assume without loss of generality that there exists $\tau \in \{1,...,T-1\}$ such that

$$a_i(t) = d_i \text{ for all } t \in \{1,...,\tau\}, \text{ and } a_i(t) = c_i \text{ for all } t \in \{\tau+1,...,T\}.$$

For every $\tau \in \{0,...,T\}$, we define a strategy for player i, $\bar{s}_{i,\tau}$, by

$$\bar{s}_{i,\tau}|_{h_i^T} = \bar{s}_i|_{h_i^T} \text{ for all } h_i^T \in H_i,$$

$$\bar{s}_{i,\tau}(h_i^{t-1}) = d_i \text{ for all } t \in \{1,...,\tau\},$$

and

$$\bar{s}_{i,\tau}(h_i^{t-1}) = c_i \text{ for all } t \in \{\tau+1,...,T\}.$$

Note that $\bar{s}_i = \bar{s}_{i,0}$, and

$$v_i(\delta, \bar{s}) = v_i(\delta, \bar{s}/\bar{s}_{i,0}) = v_i(\delta, \bar{s}/\bar{s}_{i,T}) = v_i(\delta, \bar{s}/\underline{s}_i). \qquad (2)$$

All we have to check is that for every $\tau \in \{0,...,T\}$,

$$v_i(\delta,\bar{s}) \geq v_i(\delta,\bar{s}/\bar{s}_{i,\tau}) \text{ and } v_i(\delta,\underline{s}) \geq v_i(\delta,\underline{s}/\bar{s}_{i,\tau}).$$[12]

With no substantial difficulty, we can show that $v_i(\delta,\underline{s}) \geq v_i(\delta,\underline{s}/\bar{s}_{i,\tau})$ for all $\tau \in \{0,...,T\}$.[13] However, it is not immediate to check whether it holds that $v_i(\delta,\bar{s}) \geq v_i(\delta,\bar{s}/\bar{s}_{i,\tau})$ for all $\tau \in \{0,...,T\}$. For example, suppose that $r_j^*(T) = T-1$ again. Consider the situation in which players behave according to the strategy profile $\bar{s}/\bar{s}_{i,\tau}$. The probability that all the private signals that player j observes during the first T periods belong to Ω_j^* is equal to $p_j(\Omega_j^* \mid c/d_i)^\tau p_j(\Omega_j^* \mid c)^{T-\tau}$, and therefore, the probability that player j will play the punishment behavior from period $T+1$ is equal to

$$\bar{\xi}_i^* p_j(\Omega_j^* \mid c/d_i)^\tau p_j(\Omega_j^* \mid c)^{T-\tau}.$$

Note that this probability is strictly convex with respect to τ , which implies that the payoff difference $v_i(\delta,\bar{s}/\bar{s}_{i,\tau}) - v_i(\delta,\bar{s}/\bar{s}_{i,\tau-1})$ is decreasing with respect to τ . This, together with equalities (2), implies that $v_i(\delta,\bar{s}/\bar{s}_{i,\tau}) > v_i(\delta,\bar{s})$ for all $\tau \in \{1,...,T-1\}$. Hence, it follows that $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$, and $\bar{s}/\underline{s}_2$ are not Nash equilibria when $r_j^*(T) = T-1$.

In spite of it, we can show, by choosing $r_j^*(T)$ more than but close to $Tp_j(\Omega_j^* \mid c)$, that $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$, and $\bar{s}/\underline{s}_2$ are all Nash equilibria, as follows. We can choose $r_j^*(T)$ satisfying that the probability that player j observes exactly $r_j^*(T)$ private signals that belong to Ω_j^* during the first T periods is larger than

$$\frac{1}{p_j(\Omega_j^* \mid c/d_i) - p_j(\Omega_j^* \mid c)},$$

when players behave according to the strategy profile $\bar{s}$. This implies that the increase of the probability that the opponent j will play the punishment behavior, i.e., play the strategy $\underline{s}_j$, from period $T+1$ is sufficiently large when player i changes the strategy $\bar{s}_i$ into the strategy $\bar{s}_{i,\tau}$. This holds, irrespective of the number of periods τ in which player i chooses the action d_i against the suggestion by the strategy $\bar{s}_i$ during the first T periods. Hence, it follows that $v_i(\delta,\bar{s}) \geq v_i(\delta,\bar{s}/\bar{s}_{i,\tau-1})$ for all $\tau \in \{1,...,T-1\}$, and therefore, that $\bar{s}$, $\underline{s}$, $\bar{s}/\underline{s}_1$, and $\bar{s}/\underline{s}_2$ are all Nash equilibria.

The idea of the specification of $r_i^*(T)$ is closely related to Matsushima (1999). Matsushima investigated infinitely repeated games with discounting where monitoring is imperfect and public. In every period, players play T different prisoner-dilemma games at one time, and observe T different public signals. Matsushima showed that by choosing the number T sufficiently large, we could make the efficient payoff vector sustainable under imperfect public monitoring whenever it is sustainable under perfect monitoring.

[12] Kandori and Matsushima (1998) constructed their own review strategy equilibria, in which players make informative communication at long intervals. They assumed the conditional independence, which plays the similar role to that in the present paper in simplifying the analysis.

[13] The derivation of these inequalities crucially depends on the specification that the threshold for the review in the punishment phase is set equal to $T-1$. See the complete proof in the next section.

Hence, the efficient payoff vector is sustainable even if the discount factor is far less than unity and monitoring in each prisoner-dilemma game is truly imperfect. Matsushima constructed the trigger strategy profile where the future punishment is triggered by the fact that the number of the prisoner-dilemma games in which the bad signal has been observed at one time is more than the threshold $r_i^*(T)$. Matsushima proved the efficient sustainability by specifying this threshold $r_i^*(T)$ in the same way as in Step 1. Here, the incentive constraint that a player i has no strict incentive to choose d_i in all games at one time, is binding. This binding constraint corresponds to the inequalities that $v_i(\delta, \bar{s}) = v_i(\delta, \bar{s} / s_i)$ for all $s_i \in \hat{S}_i$ in the present paper. Hence, it follows that in the end, the present paper and Matsushima (1999) resolve themselves into the same existence problem of the threshold $r_i^*(T)$.

5. The Proof of the Folk Theorem

The proof of the Folk Theorem is divided into three steps.

Step 1: We show that (1,1), (1,0), (0,1) and (0,0) are sustainable. Consider the situation in which players T times repeatedly play the prisoner-dilemma game. For every $r \in \{0,\ldots,T\}$, and every $\tau \in \{0,...,T\}$, we denote by $f_i^*(r,T,\tau)$ the probability that player i observes exactly r private signals that belong to Ω_i^* during the T periods when she chooses the action c_i in all the periods and the opponent j chooses the action d_j in the first r periods and chooses the action c_j in the last $T-r$ periods. Let $F_i^*(r,T,\tau) \equiv \sum_{r'=0}^{r} f_i^*(r',T,\tau)$.

Lemma 1: *For every positive real number* $z > 0$*, there exists an infinite sequence of positive integers* $(r_i^*(T))_{T=1}^{\infty}$ *satisfying that*

$$\lim_{T\to\infty} F_i^*(r_i^*(T),T,0) = 1, \tag{3}$$

$$\lim_{T\to\infty} \frac{r_i^*(T)}{T} = p_i(\Omega_i^* \mid c), \tag{4}$$

and

$$\lim_{T\to\infty} Tf_i^*(r_i^*(T),T,0) \geq z. \tag{5}$$

Proof: It follows from the Law of Large Numbers that for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \sum_{r:\left|\frac{r}{T}-p_i(\Omega_i^*|c)\right|<\varepsilon} f_i^*(r,T,0) = 1.$$

This implies that there exists an infinite sequence of positive real numbers $(\varepsilon(T))_{T=0}^{\infty}$ such that

$$\lim_{T\to\infty} \varepsilon(T) = 0, \text{ and } \lim_{T\to\infty} \sum_{r:\left|\frac{r}{T}-p_i(\Omega_i^*|c)\right|<\varepsilon(T)} f_i^*(r,T,0) = 1.$$

Note that for every large enough T, there exists r in the neighborhood of $Tp_i(\Omega_i^* \mid c)$ such that $Tf_i^*(r,T,0) > z$. Hence, we can choose $(r_i^*(T))_{T=1}^{\infty}$ satisfying that

$$\left|\frac{r_i^*(T)}{T} - p_i(\Omega_i^* \mid c)\right| < \varepsilon(T),$$

$$Tf_i^*(r_i^*(T),T,0) \geq z,$$

and for every r satisfying that $T\{p_i(\Omega_i^* \mid c) + \varepsilon(T)\} > r > r_i^*(T)$,

$$Tf_i^*(r,T,0) \leq z.$$

Note that

$$\lim_{T\to\infty}\{p_i(\Omega_i^*\mid c)+\varepsilon(T)-\frac{r_i^*(T)}{T}\}=0,$$

and therefore,

$$\lim_{T\to\infty} F_i^*(r_i^*(T),T,0)$$
$$\geq \lim_{T\to\infty}\sum_{r:\left|\frac{r}{T}-p_i(\Omega_i^*|c)\right|<\varepsilon(T)} f_i^*(r,T,0)-\lim_{T\to\infty}\sum_{r:\,p_i(\Omega_i^*|c)+\varepsilon(T)>\frac{r}{T}>\frac{r_i^*(T)}{T}} f_i^*(r,T,0)$$
$$\geq 1-z\lim_{T\to\infty}\{p_i(\Omega_i^*\mid c)+\varepsilon(T)-\frac{r_i^*(T)}{T}\}=1.$$

Q.E.D.

Let

$$z>\frac{1}{p_i(\Omega_i^*\mid c/d_j)-p_i(\Omega_i^*\mid c)}, \tag{6}$$

and choose $(r_i^*(T))_{T=1}^{\infty}$ satisfying the properties in Lemma 1. Note from equality (4) and the Law of Large Numbers that

$$\lim_{T\to\infty} F_i^*(r_i^*(T),T,T)=0. \tag{7}$$

For every $r\in\{0,...,T\}$, we denote by $f_i^{**}(T,\tau)$ the probability that all the private signals for player i observed during the T periods belong to Ω_i^{**} when player i chooses the action d_i in all the periods and the opponent j chooses the action d_j in the first r periods and chooses the action c_j in the last $T-r$ periods. Note that

$$\lim_{T\to\infty}\frac{f_i^{**}(T,T)}{f_i^{**}(T,0)}=\lim_{T\to\infty}(\frac{p_i(\Omega_i^{**}\mid d)}{p_i(\Omega_i^{**}\mid d/c_j)})^T=0. \tag{8}$$

Fix an infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to\infty}\delta^m=1$. We choose an infinite sequence of positive integers $(T^m)_{m=1}^{\infty}$ satisfying that

$$\lim_{m\to\infty}T^m=\infty,\ \gamma^m\equiv(\delta^m)^{T^m},\ \lim_{m\to\infty}\gamma^m=1,$$

and for each $i=1,2$,

$$\lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}f_i^{**}(T^m,0)>y_j.$$

From equalities (3), (7), and (8), we can choose an infinite sequence $(\bar{v}^m,\underline{v}^m,(\bar{\xi}_i^m,\underline{\xi}_i^m)_{i=1,2})_{m=1}^{\infty}$ satisfying that $\bar{\xi}_i^m\in[0,1]$ and $\underline{\xi}_i^m\in[0,1]$ for all $m=1,2,...$,

$$\lim_{m\to\infty}\bar{v}^m=(1,1),$$

$$\lim_{m\to\infty}\underline{v}^m=(0,0),$$

and for each $i=1,2$, and every large enough m,

$$\bar{v}_j^m=1-\frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m\{1-F_i^*(r_i^*(T^m),T^m,0)\}(\bar{v}_j^m-\underline{v}_j^m)$$

$$= 1 + x_j - \frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m\{1 - F_i^*(r_i^*(T^m), T^m, T^m)\}(\bar{v}_j^m - \underline{v}_j^m)\,, \qquad (9)$$

and

$$\underline{v}_j^m = \frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m, T^m)(\bar{v}_j^m - \underline{v}_j^m)$$

$$= -y_j + \frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m, 0)(\bar{v}_j^m - \underline{v}_j^m)\,. \qquad (10)$$

For every $m = 1, 2, \ldots$, we define $\Phi_i^{*m} \subset \Omega_i^{T^m}$ and $\Phi_i^{**m} \subset \Omega_i^{T^m}$ by

$$\Phi_i^{*m} \equiv \{\,(\omega_i(1), \ldots, \omega_i(T^m)) \in \Omega_i^{T^m} : \text{either } \omega_i(t) \in \Omega_i^* \text{ for at most } r_i^*(T^m) \text{ periods, or } \omega_i(T^m) \notin \Omega_i^*(\bar{\xi}_i^m)\,\},$$

and

$$\Phi_i^{**m} \equiv \{\,(\omega_i(1), \ldots, \omega_i(T^m)) \in \Omega_i^{T^m} : \text{either } \omega_i(t) \in \Omega_i^{**} \text{ for all } t \in \{1, \ldots, T^m\}\,, \text{ or } \omega_i(T^m) \notin \Omega_i^{**}(\underline{\xi}_i^m)\,\}.$$

We specify an infinite sequence of two strategy profiles $(\bar{s}^m, \underline{s}^m)_{m=1}^{\infty}$ as follows. For every $t = 1, \ldots, T^m$, and every $h_i^{t-1} \in H_i$,

$$\bar{s}_i^m(h_i^{t-1}) = c_i\,, \text{ and } \underline{s}_i^m(h_i^{t-1}) = d_i\,.$$

Recursively, for every $k = 1, 2\ldots$, every $t \in \{kT^m + 1, \ldots, kT^m + \tau\}$, and every $h_i^{t-1} \in H_i$,

$$\bar{s}_i^m(h_i^{t-1}) = c_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \in \Phi_i^* \text{ and } \bar{s}_i^m(h_i^{t-\tau-1}) = c_i\,,$$

$$\bar{s}_i^m(h_i^{t-1}) = c_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \in \Phi_i^{**} \text{ and } \bar{s}_i^m(h_i^{t-\tau-1}) = d_i\,,$$

$$\bar{s}_i^m(h_i^{t-1}) = d_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \notin \Phi_i^* \text{ and } \bar{s}_i^m(h_i^{t-\tau-1}) = c_i\,,$$

$$\bar{s}_i^m(h_i^{t-1}) = d_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \notin \Phi_i^{**} \text{ and } \bar{s}_i^m(h_i^{t-\tau-1}) = d_i\,,$$

$$\underline{s}_i^m(h_i^{t-1}) = c_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \in \Phi_i^* \text{ and } \underline{s}_i^m(h_i^{t-\tau-1}) = c_i\,,$$

$$\underline{s}_i^m(h_i^{t-1}) = c_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \in \Phi_i^{**} \text{ and } \underline{s}_i^m(h_i^{t-\tau-1}) = d_i\,,$$

$$\underline{s}_i^m(h_i^{t-1}) = d_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \notin \Phi_i^* \text{ and } \underline{s}_i^m(h_i^{t-\tau-1}) = c_i\,,$$

and

$$\underline{s}_i^m(h_i^{t-1}) = d_i \text{ if } (\omega_i(t-\tau-T^m), \ldots, \omega_i(t-\tau-1)) \notin \Phi_i^{**} \text{ and } \underline{s}_i^m(h_i^{t-\tau-1}) = d_i\,.$$

Note that

$$\bar{s}_i^m|_{h_i^{T^m}} = \bar{s}_i^m \text{ if } (\omega_i(1),\dots,\omega_i(T^m)) \in \Phi_i^{*m},$$

$$\bar{s}_i^m|_{h_i^{T^m}} = \underline{s}_i^m \text{ if } (\omega_i(1),\dots,\omega_i(T^m)) \notin \Phi_i^{*m},$$

$$\underline{s}_i^m|_{h_i^{T^m}} = \bar{s}_i^m \text{ if } (\omega_i(1),\dots,\omega_i(T^m)) \in \Phi_i^{**m},$$

and

$$\underline{s}_i^m|_{h_i^{T^m}} = \underline{s}_i^m \text{ if } (\omega_i(1),...,\omega_i(T^m)) \notin \Phi_i^{**m}.$$

Equalities (9) and (10) imply that

$$v_j(\delta^m, \bar{s}^m) = v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = \bar{v}_j^m, \tag{11}$$

$$v_j(\delta^m, \underline{s}^m) = v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = \underline{v}_j^m, \tag{12}$$

and, therefore,

$$\lim_{m\to\infty} v_j(\delta^m, \bar{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \bar{s}^m / \underline{s}_j^m) = 1,$$

and

$$\lim_{m\to\infty} v_j(\delta^m, \underline{s}^m) = \lim_{m\to\infty} v_j(\delta^m, \underline{s}^m / \bar{s}_j^m) = 0.$$

We show below that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m / \underline{s}_j^m$, and $\underline{s}^m / \bar{s}_j^m$ are Nash equilibria for every large enough m. For every $\tau \in \{0,...,T^m\}$, and each $i \in \{1,2\}$, we define a strategy for player i, $\bar{s}_{i,\tau}^m$, by

$$\bar{s}_{i,\tau}^m|_{h_i^{T^m}} = \bar{s}_i^m|_{h_i^{T^m}} \text{ for all } h_i^{T^m} \in H_i,$$

and

$$\bar{s}_{i,\tau}^m(h_i^{t-1}) = d_i \text{ for all } t \in \{1,...,\tau\}, \text{ and } \bar{s}_{i,\tau}^m(h_i^{t-1}) = c_i \text{ for all } t \in \{\tau+1,...,T^m\}.$$

Note that $\bar{s}_i^m = \bar{s}_{i,0}^m$, and

$$v_i(\delta^m, \bar{s}^m) = v_i(\delta^m, \bar{s}^m / \bar{s}_{i,0}^m) = v_i(\delta^m, \bar{s}^m / \bar{s}_{i,T}^m) = v_i(\delta^m, \bar{s}^m / \underline{s}_i^m).$$

From the conditional independence, all we have to show is that

$$v_j(\delta^m, \bar{s}^m) \geq v_j(\delta^m, \bar{s}^m / \bar{s}_{j,\tau}^m) \text{ and } v_j(\delta^m, \underline{s}^m) \geq v_j(\delta^m, \underline{s}^m / \bar{s}_{j,\tau}^m) \text{ for all } \tau \in \{0,...,T^m\}.$$

First, we show that $v_j(\delta^m, \bar{s}^m) \geq v_j(\delta^m, \bar{s}^m / \bar{s}_{j,\tau}^m)$ for all $\tau \in \{0,...,T^m\}$. The following lemma is the same as Lemma 2 in Matsushima (1999) and, therefore, we will omit its proof.

Lemma 2: *For every $T = 1,2,...$, and every $r \in \{0,...,T\}$, there exists an integer $\tau_i^*(T,r) \in \{0,...,T\}$ such that*

$$f_i^*(r,T,\tau) \geq f_i^*(r,T,\tau-1) \text{ if } \tau \leq \tau^*(T,r),$$

and

$$f_i^*(r,T,\tau) \leq f_i^*(r,T,\tau-1) \text{ if } \tau > \tau_i^*(T,r).$$

The difference of the probabilities that the event Φ_i^{*m} does not occur in period T^m between the strategy profiles $\bar{s}/\bar{s}_{j,\tau}^m$ and $\bar{s}/\bar{s}_{j,\tau-1}^m$ is equal to

$$\bar{\xi}_i^m \{p_i(\Omega_i^* \mid c/d_j) - p_i(\Omega_i^* \mid c)\}\; f_i^*(r_i^*(T^m), T^m - 1, \tau - 1),$$

which implies that

$$\frac{1}{1-\delta^m}\{v_j(\delta^m, \bar{s}^m/\bar{s}_{j,\tau}^m) - v_j(\delta^m, \bar{s}^m/\bar{s}_{j,\tau-1}^m)\}$$
$$= x_j - \bar{\xi}_i^m\{\; p_i(\Omega_i^* \mid c/d_j)$$
$$- p_i(\Omega_i^* \mid c)\}\; f_i^*(r_i^*(T^m), T^m - 1, \tau - 1)\,\frac{\gamma^m}{1-\delta^m}\,(\bar{v}_j^m - \underline{v}_j^m).$$

Lemma 2 implies that this payoff difference is non-increasing with respect to τ if $1 \le \tau \le \tau_i^*(T^m, r_i^*(T^m))$, but it is non-decreasing if $\tau_i^*(T^m, r_i^*(T^m)) < \tau \le T^m$. This, together with the equality $v_j(\delta^m, \bar{s}^m/\bar{s}_{j,0}^m) = v_j(\delta^m, \bar{s}^m/\bar{s}_{j,T}^m)$, implies that

$$v_j(\delta^m, \bar{s}^m/\bar{s}_{j,\tau}^m) \le v_j(\delta^m, \bar{s}^m) \text{ for all } \tau \in \{0,...,T^m\}$$
$$\text{if } v_j(\delta^m, \bar{s}^m/\bar{s}_{j,1}^m) \le v_j(\delta^m, \bar{s}^m/\bar{s}_{j,0}^m).$$

From equality (7), the latter equality of (9), $\lim_{m\to\infty}\gamma^m = \lim_{m\to\infty}(\delta^m)^{T^m} = 1$, $\lim_{m\to\infty}\bar{v}^m = (1,1)$, and $\lim_{m\to\infty}\underline{v}^m = (0,0)$, it follows that

$$\lim_{m\to\infty}\frac{\gamma^m}{(1-\delta^m)T^m}\bar{\xi}_i^m = \lim_{m\to\infty}\frac{\gamma^m}{1-\gamma^m}\bar{\xi}_i^m \lim_{m\to\infty}\frac{\sum_{t=0}^{T^m-1}(\delta^m)^t}{T^m} = x_j,$$

and, therefore,

$$\lim_{m\to\infty}\frac{1}{1-\delta^m}\{v_j(\delta^m, \bar{s}^m/\bar{s}_{j,1}^m) - v_j(\delta^m, \bar{s}^m/\bar{s}_{j,0}^m)\}$$
$$= x_j - x_j\{\; p_i(\Omega_i^* \mid c/d_j) - p_i(\Omega_i^* \mid c)\}\lim_{T\to\infty} T f_i^*(r_i^*(T), T-1, 0).$$

Note that

$$f_i^*(r, T-1, 0) = \frac{(T-1)!}{r!(T-1-r)!}\,p_i(\Omega_i^* \mid c)^r\{1 - p_i(\Omega_i^* \mid c)\}^{T-1-r}$$
$$= (1 - \frac{r}{T})\{1 - p(\Omega_i^* \mid c)\}^{-1} f_i^*(r, T, 0).$$

Hence, it follows from Lemma 1 and inequality (6) that

$$\lim_{m\to\infty}\frac{1}{1-\delta^m}\{v_j(\delta^m, \bar{s}^m/\bar{s}_{j,1}^m) - v_j(\delta^m, \bar{s}^m/\bar{s}_{j,0}^m)\}$$
$$= x_j - x_j\{\; p_i(\Omega_i^* \mid c/d_j) - p_i(\Omega_i^* \mid c)\}\lim_{T\to\infty} T f_i^*(r_i^*(T), T-1, 0)$$
$$\le x_j - x_j\{\; p_i(\Omega_i^* \mid c/d_j) - p_i(\Omega_i^* \mid c)\} z < 0,$$

and therefore, we have proved that $v_j(\delta^m, \bar{s}^m) \ge v_j(\delta^m, \bar{s}^m/\bar{s}_{j,\tau}^m)$ for all $\tau \in \{0,...,T^m\}$.

Next, we show that $v_j(\delta^m, \underline{s}^m) \ge v_j(\delta^m, \underline{s}^m/\bar{s}_{j,\tau}^m)$ for all $\tau \in \{0,...,T^m\}$. Note that

$$f_i^{**}(T^m, \tau) = p_i(\Omega_i^{**} \mid d)^\tau (p_i(\Omega_i^{**} \mid d/c_j))^{T^m - \tau} = q^{T^m - \tau} f_i^{**}(T^m, T^m),$$

where

$$q \equiv \frac{p_i(\Omega_i^{**} \mid d / c_j)}{p_i(\Omega_i^{**} \mid d)} > 1 .$$

Hence, it follows from equalities (10) that

$$\begin{aligned} & v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \\ & = \frac{1-\delta^m}{1-\gamma^m}(-y_j)\sum_{t=\tau+1}^{T^m}(\delta^m)^{t-1} + \frac{\gamma^m}{1-\gamma^m}\underline{\xi}_i^m f_i^{**}(T^m,\tau)(\bar{v}_j^m - \underline{v}_j^m) \\ & = \frac{1-\delta^m}{1-\gamma^m}(-y_j)\sum_{t=\tau+1}^{T^m}(\delta^m)^{t-1} + q^{T^m-\tau}\underline{v}_j^m , \end{aligned}$$

and, therefore,

$$\begin{aligned} & v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau-1}) - v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \\ & = (\frac{1}{\delta^m})^{T^m-\tau}(\frac{(\delta^m)^{T^m-1}-\gamma^m}{1-\gamma^m})(-y_j) + q^{T^m-\tau}(q-1)\underline{v}_j^m . \end{aligned} \tag{13}$$

Given that m is large enough, we can assume that

$$1 < \frac{1}{\delta^m} < q ,$$

which, together with equality (13), inequality $\frac{(\delta^m)^{T^m-1}-\gamma^m}{1-\gamma^m}(-y_j) < 0$, and inequality $(q-1)\underline{v}_j^m > 0$, implies that for every $\tau \in \{1,...,T^m\}$,

$$\begin{aligned} & v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau-1}) \geq v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \\ & \qquad \text{if } v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \geq v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau+1}) , \end{aligned}$$

and

$$\begin{aligned} & v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \leq v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau+1}) \\ & \qquad \text{if } v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau-1}) \leq v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) . \end{aligned}$$

These inequalities, together with equality $v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,0}) = v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,T^m})$, imply that

$$v_j(\delta^m, \underline{s}^m) \geq v_j(\delta^m, \underline{s}^m / \bar{s}^m_{j,\tau}) \text{ for all } \tau \in \{1,...,T^m\} .$$

From these observations, we have proved that $\bar{s}^m$, $\underline{s}^m$, $\bar{s}^m / \underline{s}_j^m$ and $\underline{s}^m / \bar{s}_j^m$ are Nash equilibria for every large enough m. Hence, (1,1), (1,0), (0,1) and (0,0) are sustainable.

Step 2: We show that $z^{[1]}$ and $z^{[2]}$ are sustainable. Consider $z^{[1]}$ only. We can prove that $z^{[2]}$ is sustainable in the same way.

Consider the situation in which players M times repeatedly play the prisoner-dilemma game. For every $r \in \{0,...,M\}$, and every $\tau \in \{0,...,M\}$, we denote by $f_2^+(r,M,\tau)$ the probability that player 2 observes exactly r private signals that belong to Ω_2^+ during the M periods when she chooses the action d_2 in all the periods, and the opponent 1 chooses the action d_1 in the first τ periods and the action c_1 in the last $M-\tau$ periods. Let $F_2^+(r,M,\tau) \equiv \sum_{r'=0}^{r} f_2^+(r',M,\tau)$. We choose an infinite sequence of positive integers $(r_2^+(M))_{M=1}^{\infty}$ satisfying that

$$\lim_{M\to\infty} F_2^+(r_2^+(M),M,0) = 1, \tag{14}$$

$$\lim_{M\to\infty} \frac{r_2^+(M)}{M} = p_2(\Omega_2^+ \mid d/c_1), \tag{15}$$

and

$$\lim_{M\to\infty} Mf_2^+(r_2^+(M),M,0) > \frac{1+y_1}{y_1\{p_2(\Omega_2^+ \mid d) - p_2(\Omega_2^+ \mid d/c_1)\}}. \tag{16}$$

In the same way as in Lemma 1, such a sequence $(r_2^+(M))_{M=1}^{\infty}$ exists. Equality (15), together with the Law of Large Numbers, implies that

$$\lim_{M\to\infty} F_2^+(r_2^+(M),M,M) = 0. \tag{17}$$

We choose a positive real number $b>0$ arbitrarily, which is less than but close to $\frac{1}{1+y_1}$, satisfying that

$$\lim_{M\to\infty} Mf_2^+(r_2^+(M),M,0) > \frac{by_1}{(1-b)^2\{p_2(\Omega_2^+ \mid d) - p_2(\Omega_2^+ \mid d/c_1)\}}.$$

Let

$$v^* \equiv b(-y_1, 1+x_2) + (1-b)(1,1).$$

Note that v^* approximates $z^{[1]}$, and

$$v_1^* > z_1^{[1]} = 0.$$

Fix an infinite sequence of discount factors $(\delta^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty} \delta^m = 1$. Choose an infinite sequence of positive integers $(M^m)_{m=1}^{\infty}$ satisfying that

$$\lim_{m\to\infty} M^m = \infty, \ \chi^m \equiv (\delta^m)^{M^m},$$

and

$$\lim_{m\to\infty} \chi^m = 1-b. \tag{18}$$

For every $m = 1,2,\ldots$, we define

$$\Phi_2^{+m} \equiv \{(\omega_2(1),\ldots,\omega_2(M^m)) \in \Omega_2^{M^m} : \omega_2(t) \in \Omega_2^+ \text{ for at most } r_2^+(M^m) \text{ periods}\}.$$

Let $(\bar{s}^m, \underline{s}^m)_{m=1}^{\infty}$ be the infinite sequence of the two strategy profiles specified in Step 1.

We specify an infinite sequence of strategy profiles $(\hat{s}^m)_{m=1}^{\infty}$ by

$$\hat{s}^m(h_2^{t-1}) = (c_1, d_2) \text{ if } 1 \le t \le M^m,$$

$$\hat{s}_1^m|_{h_1^{T^m}} = \bar{s}_1^m \text{ for all } h_1^{T^m} \in H_1,$$

for every $h_2^{T^m} \in H_2$,

$$\hat{s}_2^m|_{h_2^{T^m}} = \bar{s}_2^m \text{ if } (\omega_2(1), \ldots, \omega_2(M^m)) \in \Phi_2^+,$$

and

$$\hat{s}_2^m|_{h_2^{T^m}} = \underline{s}_2^m \text{ if } (\omega_2(1), \ldots, \omega_2(M^m)) \notin \Phi_2^+.$$

According to the strategy $\hat{s}^m$, players choose the action profile (c_1, d_2) in the first M^m periods. From period $M^m + 1$, player 1 certainly plays the strategy $\bar{s}_1^m$, whereas player 2 plays the strategy $\bar{s}_2^m$ (the strategy $\underline{s}_2^m$) if the M times repeated play passes the review of player 1 (fails the review of player 1, respectively). Note that

$$v_1(\delta^m, \hat{s}^m) = (1-\chi^m)(-y_1) + \chi^m[F_2^+(r_2^+(M^m), M^m, M^m)\bar{v}_1^m$$
$$+\{1 - F_2^+(r_2^+(M^m), M^m, M^m)\}\underline{v}_1^m],$$

and

$$v_2(\delta^m, \hat{s}^m) = (1-\chi^m)(1+x_2) + \chi^m \bar{v}_2^m.$$

Note from equalities (14), (17), and (18) that

$$\lim_{m\to\infty} v(\delta^m, \hat{s}^m) = b(-y_1, 1+x_2) + (1-b)(1,1) = v^*.$$

Hence, $v(\delta^m, \hat{s}^m)$ approximates $z^{[1]}$ for every large enough m.

We show below that $\hat{s}^m$ is a Nash equilibrium for every large enough m. Step 1 has proved that $(\hat{s}_1^m|_{h_1^{M^m}}, \hat{s}_2^m|_{h_2^{M^m}})$ is a Nash equilibrium for every h^{M^m} and every large enough m. Since players' private signal structures satisfy the conditional independence and the action d_2 is dominant for player 2 in the component game, it follows that the repeated choice of the action d_2 during the first M^m periods is the best reply for player 2. Hence, all we have to check is that the repeated choice of the action c_1 during the first M^m periods is the best reply for player 1 for every large enough m.

For every $\tau \in \{0, \ldots, M^m\}$, we define a strategy for player 1, $\hat{s}_{1,\tau}^m$, by

$$\hat{s}_{1,\tau}^m|_{h_1^{M^m}} = \hat{s}_1^m|_{h_1^{M^m}} \text{ for all } h_1^{M^m} \in H_1,$$

and

$$\hat{s}_{1,\tau}^m(h_1^{t-1}) = d_1 \text{ for all } t \in \{1, \ldots, \tau\}, \text{ and } \hat{s}_{1,\tau}^m(h_1^{t-1}) = c_1 \text{ for all } t \in \{\tau+1, \ldots, M^m\}.$$

Note that $\hat{s}_1^m = \hat{s}_{1,0}^m$. From the conditional independence, all we have to check is that

$$v_1(\delta^m, \hat{s}^m) \ge v_1(\delta^m, \hat{s}^m / \hat{s}_{1,\tau}^m) \text{ for all } \tau \in \{0, \ldots, M^m\}.$$

The difference of the probabilities that the event Φ_2^{+m} does not occur in period M^m between strategies $\hat{s}^m / \hat{s}_{1,\tau}^m$ and $\hat{s}^m / \hat{s}_{1,\tau-1}^m$ is equal to

$$\{p_2(\Omega_2^+ | d) - p_2(\Omega_2^+ | d / c_1)\} \; f_2^+(r_2^+(M^m), M^m - 1, \tau - 1),$$

which implies that

$$\frac{1}{1-\delta^m}\{v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,\tau})-v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,\tau-1})\}=y_1-\{p_2(\Omega_2^+\mid d)$$

$$-p_2(\Omega_2^+\mid d/c_1)\}\ f_2^+(r_2^+(M^m),M^m-1,\tau-1)\ \frac{\chi^m}{1-\delta^m}\ (\bar{v}_1^m-\underline{v}_1^m)\,.$$

In the same way as in Lemma 2, it follows that for every $M=1,2,...$, and every $r\in\{0,...,M\}$, there exists an integer $\tau_2^+(M,r)\in\{0,...,M\}$ such that

$$f_2^+(r,T,\tau)\geq f_2^+(r,T,\tau-1)\ \text{ if }\ \tau\leq\tau_2^+(T,r)\,,$$

and

$$f_2^+(r,T,\tau)\leq f_2^+(r,T,\tau-1)\ \text{ if }\ \tau>\tau_2^+(T,r)\,.$$

Hence, this payoff difference is non-increasing with respect to τ if $\tau\leq\tau_2^+(T,r)$, but it is non-decreasing if $\tau>\tau_2^+(T,r)$. This implies that if there exists $r\in\{1,...,M^m\}$ such that $v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,\tau})>v_1(\delta^m,\hat{s}^m)$, then

either $v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,1})>v_1(\delta^m,\hat{s}^m)$ or $v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,M^m})>v_1(\delta^m,\hat{s}^m)$.

From equality (18), it follows that

$$\lim_{m\to\infty}\frac{\chi^m}{(1-\delta^m)M^m}=\lim_{m\to\infty}\frac{\chi^m}{1-\chi^m}\ \lim_{m\to\infty}\frac{\sum_{t=0}^{M^m-1}(\delta^m)^t}{M^m}=\frac{(1-b)^2}{b}\,,$$

which, together with $\lim_{m\to\infty}\bar{v}^m=(1,1)$, and $\lim_{m\to\infty}\underline{v}^m=(0,0)$, implies that

$$\lim_{m\to\infty}\frac{1}{1-\delta^m}\{v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,1})-v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,0})\}=y_1-\{p_2(\Omega_2^+\mid d)$$

$$-p_2(\Omega_2^+\mid d/c_1)\}\ \frac{(1-b)^2}{b}\lim_{M\to\infty}Mf_2^+(r_2^+(M),M-1,0)\,.$$

Note that

$$f_2^+(r,M-1,0)$$

$$=\frac{(M-1)!}{r!(M-1-r)!}\,p_2(\Omega_2^+\mid d/c_1)^r\{1-p_i(\Omega_2^+\mid d/c_1)\}^{M-1-r}$$

$$=(1-\frac{r}{M})\{1-p(\Omega_2^+\mid d/c_1)\}^{-1}\,f_2^+(r,M,0)\,.$$

Hence, it follows from equality (15) and inequality (16) that

$$\lim_{m\to\infty}\frac{1}{1-\delta^m}\{v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,1})-v_1(\delta^m,\hat{s}^m/\hat{s}^m_{1,0})\}$$

$$=y_1-\{p_2(\Omega_2^+\mid d)-p_2(\Omega_2^+\mid d/c_1)\}\ \frac{(1-b)^2}{b}\lim_{M\to\infty}Mf_2^+(r_2^+(M),M,0)$$

$$<y_1-\frac{(1-b)^2(1+y_1)}{by_1}\,,$$

which is less than zero, because b is less than $\frac{1}{1+y_1}$. Hence, it follows that $v_1(\delta^m, \hat{s}^m / \hat{s}^m_{1,1}) < v_1(\delta^m, \hat{s}^m)$ for every large enough m.

Note from $\lim_{m\to\infty} \underline{v}^m = (0,0)$ and equality (17) that

$$\lim_{m\to\infty} v_1(\delta^m, \hat{s}^m / \hat{s}^m_{M^m})) = \lim_{m\to\infty} \chi^m \{F_2^+(r_2^+(M^m), M^m, M^m)\bar{v}_1^m$$
$$+(1-F_2^+(r_2^+(M^m), M^m, M^m))\underline{v}_1^m\} = 0 .$$

Since $\lim_{m\to\infty} v_1(\delta^m, \hat{s}^m) = v_1^* > 0$, it follows that $v_1(\delta^m, \hat{s}^m / \hat{s}^m_{1,M^m}) < v_1(\delta^m, \hat{s}^m)$ for every large enough m.

Hence, we have proved that $z^{[1]}$ is sustainable. Similarly, $z^{[2]}$ is sustainable too.

Step 3: Fix a positive integer K and K individually rational feasible payoff vectors $v^{\{1\}},\dots,v^{\{K\}}$, arbitrarily, where $v^{\{k\}}\in\{(1,1),(1,0),(0,1),z^{[1]},z^{[2]}\}$ for all $k\in\{1,\dots,K\}$. We show that $\frac{\sum_{k=1}^{K}v^{[k]}}{K}$ is sustainable. Fix $(\delta^m)_{m=1}^{\infty}$ arbitrarily, which satisfies $\lim_{m\to+\infty}\delta^m=1$. Fix $\varepsilon>0$ arbitrarily. For every $k\in\{1,...,K\}$, let $(s^{\{k,m\}})_{m=1}^{\infty}$ be an infinite sequence of strategy profiles satisfying that for every large enough m, $s^{\{k,m\}}$ is a Nash equilibrium, and that for each $i\in\{1,2\}$,

$$v_i^{\{k\}}-\varepsilon\le\lim_{m\to\infty}v_i(\delta^m,s^{\{k,m\}})\le v_i^{\{k\}}+\varepsilon .$$

We specify an infinite sequence of strategy profiles $(\overset{=m}{s})_{m=1}^{\infty}$ satisfying that for every $i\in\{1,2\}$, and every $k\in\{1,\dots,K\}$,

$$\overset{=m}{s_i}(h_i^{k-1})=s_i^{\{k,m\}}(h_i^0),$$

and for every $t\ge K+1$,

$$\overset{=m}{s_i}(h_i^{t-1})=s_i^{\{k,m\}}(\tilde{h}_i^{\tilde{t}})\ \text{ if } t=K\tilde{t}+k \text{ and}$$
$$(\tilde{a}_i(\tau),\tilde{\omega}_i(\tau))=(a_i(K\tau+k),\omega_i(K\tau+k))\ \text{ for all } \tau\in\{1,...,\tilde{t}\}.$$

Note that for each $i\in\{1,2\}$,

$$\lim_{m\to\infty}v_i((\delta^m)^{\frac{1}{K}},\overset{=m}{s})=\lim_{m\to\infty}\frac{\sum_{k=1}^{K}(\delta^m)^{\frac{k-1}{K}}v_i(\delta^m,s^{\{k,m\}})}{\sum_{k=1}^{K}(\delta^m)^{\frac{k-1}{K}}}$$
$$\in[\frac{\sum_{k=1}^{K}v_i^{\{k\}}}{K}-\varepsilon,\frac{\sum_{k=1}^{K}v_i^{\{k\}}}{K}+\varepsilon].$$

Since $s^{\{k,m\}}$ is a Nash equilibrium in $\Gamma(\delta^m)$ for every large enough m, it follows that $\overset{=m}{s}$ is a Nash equilibrium in $\Gamma((\delta^m)^{\frac{1}{K}})$ for every large enough m. Hence, $\frac{\sum_{k=1}^{K}v^{[k]}}{K}$ is sustainable.

Since (1,1), (0,0), $z^{[1]}$, and $z^{[2]}$ are sustainable and the set of individually rational feasible payoff vectors is equivalent to the convex hull of the set $\{(0,0),(1,1),z^{[1]},z^{[2]}\}$, we have proved that every individually rational feasible payoff vector is sustainable.

Hence, we have completed the proof of the Folk Theorem.

6. Limited Knowledge

This section investigates the situation in which players have limited knowledge of their private signal structures. Each player i knows her own monitoring ability p_i, but does not know her opponent's monitoring ability p_j. Hence, player i must behave according to a strategy that does not depend on p_j.

For each $i = 1,2$, fix an arbitrary compact and nonempty subset P_i^* of conditional density functions on player $i's$ private signal, which satisfy the minimal information constraint. Let $P^* \equiv P_1^* \times P_2^*$. Each player i only knows which element of P_i^* is the correct conditional density function for her own private signal. It is common knowledge that the correct conditional density function belongs to P^*. It is also common knowledge that players' private signal structures satisfy the conditional independence. A mapping that assigns each element of P_i^* a strategy for player i is denoted by $\rho_i : P_i^* \to S_i$. Let $\rho \equiv (\rho_1, \rho_2)$, and $\rho(p) \equiv (\rho_1(p_1), \rho_2(p_2))$. Player i behaves according to the strategy $\rho_i(p_i) \in S_i$ irrespective of her opponent's monitoring ability $p_j \in P_j^*$.

The following proposition states that the Folk Theorem holds for every $p \in P^*$ even if players have no knowledge of their opponents' signal structures.

Proposition: *For every individually rational payoff vector* $v \in V$ *, every* $(\delta^m)_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty} \delta^m = 1$*, and every* $\varepsilon > 0$*, there exists* $(\rho^m)_{m=1}^{\infty}$ *such that for every* $p \in P^*$*, and every large enough* m*,* $\rho^m(p)$ *is a sequential equilibrium in* $\Gamma(\delta^m)$*, and for each* $i = 1,2$,

$$v_i - \varepsilon < \lim_{m\to\infty} v_i(\delta^m, \rho^m(p)) < v_i + \varepsilon .$$

Proof: Fix $(\delta^m)_{m=1}^{\infty}$ arbitrarily, where $\lim_{m\to\infty} \delta^m = 1$. Note from the compactness of P_i^* that there exist $\bar{e}_i > 0$, $\underline{e}_i > 0$, $\Psi_i^* : P_i^* \to 2^{\Omega_i}$, $\Psi_i^{**} : P_i^* \to 2^{\Omega_i}$, and $\Psi_i^+ : P_i^* \to 2^{\Omega_i}$ such that $\bar{e}_i > \underline{e}_i$, and for every $p_i \in P_i^*$,

$$\underline{e}_i = p_i(\Psi_i^*(p_i) \,|\, c) = p_i(\Psi_i^{**}(p_i) \,|\, d) = p_i(\Psi_i^+(p_i) \,|\, d/c_j),$$

and

$$\bar{e}_i = p_i(\Psi_i^*(p_i) \,|\, c/d_j) = p_i(\Psi_i^{**}(p_i) \,|\, d/c_j) = p_i(\Psi_i^+(p_i) \,|\, d).$$

For every $p_i \in P_i^*$, we set the associated sets Ω_i^*, Ω_i^{**}, and Ω_i^+ in the proof of the Folk Theorem equivalent to $\Psi_i^*(p_i)$, $\Psi_i^{**}(p_i)$, and $\Psi_i^+(p_i)$, respectively. Hence, we can choose $(r_i^*(T))_{T=1}^{\infty}$, $(T^m)_{m=1}^{\infty}$, $(M^m)_{m=1}^{\infty}$, and $(\bar{v}^m, \underline{v}^m, (\bar{\xi}_i^m, \underline{\xi}_i^m)_{i=1,2})_{m=1}^{\infty}$ independently of $p_j \in P_j^*$, which were introduced in the proof of the Folk Theorem. This implies that we can choose $\bar{s}_i^m$, $\underline{s}_i^m$ and $\hat{s}_i^m$ independently of $p_j \in P_j^*$. We denote $\bar{s}_i^{m,p_i}$, $\underline{s}_i^{m,p_i}$ and $\hat{s}_i^{m,p_i}$, instead of $\bar{s}_i^m$, $\underline{s}_i^m$ and $\hat{s}_i^m$, respectively.

We specify $(\bar{\rho}_i^m)_{m=1}^{\infty}$ as follows. For every $p_i \in P_i^*$, every $t = 1,2,...$, and every

$h_i^{t-1} \in H_i$,

$$\bar{\rho}_i^m(p_i)|_{h_i^{t-1}} = \bar{s}_i^{m,p_i}|_{h_i^{t-1}} \text{ if } a_i(\tau) = \bar{s}_i^{m,p_i}(h_i^{\tau-1}) \text{ for all } \tau \in \{1,...,t-1\},$$

and

$$\bar{\rho}_i^m(p_i)|_{h_i^{t-1}} \text{ maximizes player } i's \text{ long-run payoff for } \bar{s}_j^{m,p_j} \text{ if}$$
$$a_i(\tau) \neq \bar{s}_i^{m,p_i}(h_i^{\tau-1}) \text{ for some } \tau \in \{1,...,t-1\}.$$

The specifications of Ω_i^* and Ω_i^{**} in the above way guarantee that such a $(\bar{\rho}_i^m)_{m=1}^\infty$ can be specified independently of $p_j \in P_j^*$. Here, we must note that it is necessary to make the values $\bar{\xi}_i^* = \bar{\xi}_i^{**} = \bar{\varepsilon}_i$ and $\underline{\xi}_i^* = \underline{\xi}_i^{**} = \underline{\varepsilon}_i$ independent of $p_i \in P_i^*$. If not, then the best reply for player i when the history h_i^{t-1} is off the equilibrium path, i.e., when $a_i(\tau) \neq \bar{s}_i^{m,p_i}(h_i^{\tau-1})$ for some $\tau \in \{1,...,t-1\}$, might depend on the opponent's strategy, i.e., on the opponent's private signal structure $p_j \in P_j^*$. It follows from the specification of $(\bar{\rho}_j^m)_{m=1}^\infty$ based on $(\bar{s}_j^{m,p_j})_{m=1}^\infty$ that $\bar{\rho}_i^m(p_i)|_{h_i^{t-1}}$ maximizes player $i's$ long-run payoff for $\bar{\rho}_j^m(p_j)$ if $a_i(\tau) \neq \bar{s}_i^{m,p_i}(h_i^{\tau-1})$ for some $\tau \in \{1,...,t-1\}$. In the same way as $(\bar{\rho}_i^m)_{m=1}^\infty$, we can specify $(\underline{\rho}_i^m)_{m=1}^\infty$ and $(\hat{\rho}_i^m)_{m=1}^\infty$, based on $\underline{s}_i^{m,p_i}$ and $\hat{s}_i^{m,p_i}$, respectively.

The proof of the Folk Theorem implies that for every $p \in P^*$, and every large enough m, $(\bar{\rho}_1^m(p_1), \bar{\rho}_2^m(p_2))$, $(\bar{\rho}_1^m(p_1), \underline{\rho}_2^m(p_2))$, $(\underline{\rho}_1^m(p_1), \bar{\rho}_2^m(p_2))$, $(\underline{\rho}_1^m(p_1), \rho_2^m(p_2))$, and $(\hat{\rho}_1^m(p_1), \hat{\rho}_2^m(p_2))$ are Nash equilibria and, therefore, sequential equilibria, and that these strategy profiles approximately sustain (1,1), (1,0), (0,1), (0,0), and $z^{[1]}$, respectively. In the same way as the above arguments, we can prove also that there exists $(\tilde{\rho}_i^m)_{m=1}^\infty$ such that for every $p \in P^*$, and every large enough m, $(\tilde{\rho}_1^m(p_1), \tilde{\rho}_2^m(p_2))$ is a sequential equilibrium and approximately sustains $z^{[2]}$.

Fix a positive integer K and K individually rational feasible payoff vectors $v^{\{1\}},\dots,v^{\{K\}}$, arbitrarily, where $v^{\{k\}} \in \{(1,1),(1,0),(0,1),z^{[1]},z^{[2]}\}$ for all $k \in \{1,\dots,K\}$. Let $(\rho^{\{k,m\}})_{m=1}^\infty$ denote the function satisfying that for every $p \in P^*$ and every large enough m, $\rho^{\{k,m\}}(p)$ is a sequential equilibrium and approximately sustains $v^{\{k\}}$. We specify $(\rho^m)_{m=1}^\infty$ satisfying that for every $i \in \{1,2\}$, every $p_i \in P_i^*$, and every $k \in \{1,\dots,K\}$,

$$\rho_i^m(p_i)(h_i^{k-1}) = \rho_i^{\{k,m\}}(p_i)(h_i^0),$$

and for every $t \geq K+1$,

$$\rho_i^m(p_i)(h_i^{t-1}) = \rho_i^{\{k,m\}}(p_i)(\tilde{h}_i^{\tilde{t}}) \text{ if } t = K\tilde{t} + k \text{ and}$$
$$(\tilde{a}_i(\tau), \tilde{\omega}_i(\tau)) = (a_i(K\tau + k), \omega_i(K\tau + k)) \quad \text{for all} \quad \tau = 1,\dots,\tilde{t}.$$

Note that for every $p \in P^*$, and every large enough m, $\rho^m(p)$ is a sequential equilibrium in $\Gamma((\delta^m)^{\frac{1}{K}})$, and that for each $i \in \{1,2\}$,

$$\lim_{m\to\infty} v_i((\delta^m)^{\frac{1}{K}}, \rho^m(p)) = \lim_{m\to\infty} \frac{\sum_{k=1}^{K} (\delta^m)^{\frac{k-1}{K}} v_i(\delta^m, \rho^{\{k,m\}}(p))}{\sum_{k=1}^{K} (\delta^m)^{\frac{k-1}{K}}}$$

$$\in [\frac{\sum_{k=1}^{K} v_i^{\{k\}}}{K} - \varepsilon, \frac{\sum_{k=1}^{K} v_i^{\{k\}}}{K} + \varepsilon].$$

Hence, we have proved that $\frac{\sum_{k=1}^{K} v^{[k]}}{K}$ is sustainable.

Since the set of individually rational feasible payoff vectors is equivalent to the convex hull of the set $\{(1,1),(1,0),(0,1),z^{[1]},z^{[2]}\}$, we have proved this proposition.

Q.E.D.

7. Conclusion

The present paper investigated infinitely repeated prisoner-dilemma games with discounting, where players are sufficiently patient and monitoring is imperfect and private. We provided the Folk Theorem when players' private signal structures satisfy the conditional independence, and showed that the Folk Theorem holds even if players have no knowledge of their opponents' private signal structures. In the paper we required no conditions concerning the accuracy of private signals except the minimal information requirement.

Whether the Folk Theorem holds even without conditional independence is an open question. The conditional independence simplified the way to check whether the review strategy profiles constructed in the paper are sequential equilibria. Hence, all we have to do was to show that there exists no strategy preferred to the review strategy that is the same as the review strategy after the first review phase and does not depend on private signal histories during the first review phase. Without conditional independence the problem is more complicated, because there may still exist a strategy preferred to the review strategy that does depend on private signal histories even during the first review phase. When the private signal history observed by a player in the middle of the review phase implies that with high probability the opponent has already received many bad signals and recognized that the review was failed, the player may have no incentive to choose the cooperative action in the remainder of the review phase.

In the same way as the present paper, we may be able to establish the Folk Theorem even without conditional independence, if for each $i \in \{1,2\}$ there exist subsets $\Omega_i^* \subset \Omega_i$, $\Omega_i^{**} \subset \Omega_i$, and $\Omega_i^+ \subset \Omega_i$ satisfying that for every $\omega_j \in \Omega_j$, and every $\omega_j' \in \Omega_j$,

$$p_i(\Omega_i^* \mid c, \omega_j) < p_i(\Omega_i^* \mid c/d_j, \omega_j'),$$
$$p_i(\Omega_i^{**} \mid d, \omega_j) < p_i(\Omega_i^{**} \mid d/c_j, \omega_j'),$$

and

$$p_i(\Omega_i^+ \mid d/c_j, \omega_j) < p_i(\Omega_i^+ \mid d, \omega_j'),$$

where $p_i(W_i \mid a, \omega_j)$ is the probability that the event $W_i \subset \Omega_i$ occurs when players choose $a \in A$ and player j observes $\omega_j \in \Omega_j$. When such subsets do not exist, we may need to explore a further extended form of review strategies, probably combined with a device of punishment and reward on hyperplanes a la Matsushima (1989) and Fudenberg, Levine, and Maskin (1994).

The present paper considered only repeated prisoner-dilemma games. We can extend our efficiency result to a class of games with more than two actions as follows. Suppose that a player i has an action d_i' other than the actions c_i and d_i, and there exist $\alpha \in [0,1]$ and $\alpha' \in [0,1]$ such that

$$u_i(c/d_i') \le \alpha u_i(c) + (1-\alpha) u_i(c/d_i),$$
$$p_j(\Omega_j^* \mid c/d_i') > \alpha p_j(\Omega_j^* \mid c) + (1-\alpha) p_j(\Omega_j^* \mid c/d_i) > p_j(\Omega_j^* \mid c),$$
$$u_i(d/d_i') \le \alpha' u_i(d) + (1-\alpha') u_i(d/c_i),$$

and

$$p_j(\Omega_j^{**} \mid d/d_i') < \alpha' p_j(\Omega_j^{**} \mid d) + (1-\alpha') p_j(\Omega_j^{**} \mid d/c_i)$$

$$< p_j(\Omega_j^{**} \mid d / c_i).$$

Since the action d_i' is worse than a mixture of the actions c_i and d_i, player i have no incentive to choose d_i' when her opponent plays the strategy constructed in the paper.

The more intensive study of private monitoring in general repeated games with more than two actions and more than two players,[14] and also in general stochastic games, should be expected to start in the near future.

[14] For the three or more player case with almost perfect monitoring, see, for example, Bhaskar and Obara (2000), and Ely and Valimaki (1999).

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