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Effects of Stochastic Interest Rates and Volatility on Contingent Claims

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Abstract

We investigate the effects of the stochastic interest rates and the volatility of the underlying asset price on the contingent claim prices including futures and options prices. The futures price can be decomposed into the forward price and an additional term, and the options price can be decomposed into the Black-Scholes formula and several additional terms by applying the asymptotic expansion approach in the small disturbance asymptotics developed by Kunitomo and Takahashi (1995, 1998, 2001, 2003a, 2003b). The technical method is based on a new application of the *Malliavin-Watanabe Calculus* or the *Watanabe-Yoshida Theory* on Malliavin Calculus in stochastic analysis. We illustrate our new formulae and their numerical accuracy by using the modified CIR processes for the short term interest rates and stochastic volatility. We discuss implications of our results for financial economics.

Key Words

Stochastic Interest Rate, Stochastic Volatility, Contingent Claims, Futures, Options, Asymptotic Expansion Approach, Malliavin-Watanabe Calculus, Near Completeness.

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1 Introduction

In the past two decades a considerable number of studies have been devoted to the generalizations of the standard theory of financial contingent claims in finance. Because the option theory originally developed by Black and Scholes (1973) assumed that the underlying asset price follows the geometrical Brownian Motion and there is a constant risk free interest rate among several other assumptions, there have been some attempts to relax these aspects of the standard Black-Scholes theory.

One of the important characteristics of many asset prices is the phenomenon that the volatility of asset returns does not seem to be constant and change randomly over time. This empirical observation has led to the direction that the volatility should be incorporated into the analysis as a state variable. In the options pricing models, the effects of stochastic volatility have been investigated by Hull and White (1987, 1988), Johnson and Shanno (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991), Heston (1993), and Ball and Roma (1994). Another important aspect in the analysis of contingent claims has been the fact that the spot interest rate is the fundamental economic variable in the economy and it cannot be treated as a constant. The feature that interest rates are stochastic has been incorporated into the modeling of the contingent claims. Merton (1973), Rabinovitch (1989), Turnbull and Milne (1991), and Amin and Jarrow (1992) are the representative researches in this respect. Recently Kim and Kunitomo (1999) also have developed the option pricing theory in the direction closely related to the present study.

However, the attempts to construct the stochastic model for the contingent claims that allow the randomness of both the volatility and interest rates have not been made in abundance. See Amin and Ng (1993), Bakshi and Chen (1997), Baily and Stulz (1989), Helmer and Longstaff (1991), and Scott (1997) for this line of investigations. Due to the fact that we need to treat both the stochastic volatility and interest rate processes at the same time, the existing valuation methods in the past become complicated in the general case and their results can be hardly analytical except some special cases of the underlying stochastic processes. For instance, the closed-form expressions obtained by Bakshi and Chen (1997), and Scott (1997) contain the Fourier inversion formula even in the simple cases. The methods based on the inversion formulae have been further developed by Bakshi and Madan (2000), and Duffie, Pan, and Singleton (2000).

In this paper, we shall develop a general framework of the analysis of the European type contingent claims including the futures price, the forward price, and the options prices that incorporate both the stochastic volatility and stochastic interest rates. Our method is based on the asymptotic expansion approach called *the small disturbance asymptotics* in which we consider a sequence of stochastic processes when the diffusion parameters of some stochastic processes are small under the probability measure Q, which is equivalent to the historical probability measure P. The small disturbance asymptotics under Q has been recently developed by Kunitomo and Takahashi (1995, 1998, 2001, 2003a, 2003b, 2004), Takahashi (1999), Yoshida (1992), Takahashi and Yoshida (2001a,b). It has been justified by the *Malliavin-Watanabe Calculus* mathematically in the rigorous manner, which is an infinite dimensional analysis of Wiener functionals and the generalized Wiener functionals on the abstract Wiener space, and it can be also called *the Watanabe-Yoshida theory* on Malliavin Calculus. (See Watanabe (1987), Chapter V of Ikeda and Watanabe (1989), Shigekawa (1998), and Yoshida

(1992) for its basics.) The details of this theory, the mathematical notations, and applications to derivative pricing problems have been explained by Kunitomo and Takahashi (1998, 2001, 2003a). Our formulation of stochastic processes in this paper differs from that of Kunitomo and Takahashi (1998, 2003a) mathematically because the limiting stochastic processes are the solution of stochastic differential equation and we need the non-degeneracy condition for the partial Malliavin covariance of certain random variables. In this sense this paper gives a new application and a real example of Malliavin-Watanabe Calculus to financial derivative analysis.

As we shall show in this paper, our approach has the advantage that we have the explicit formula for the theoretical values of contingent claims which can be decomposed into the leading term plus several additional terms when both the stochastic volatility and the spot interest rate follow the general class of continuous processes. The leading term is the well-known formula when both volatility and interest rate are constant and hence it allows us to investigate the effects of the stochastic volatility and stochastic interest rates on the contingent claims explicitly in the analytical way.

When the volatility function is stochastic, the underlying financial market is incomplete. We shall adopt an analogous argument on the market completion by Romano and Touzi (1997) and it is possible to give the financial (or economic) meaning of our results. Also we shall show that it is straightforward to incorporate more complicated stochastic models into our framework including the HJM term structure of interest rates model.

In Section 2 we give the general framework of our analysis when the spot interest rate and the volatility of security price follow diffusion processes. Then in Section 3 we present the main results on the futures price, the forward price, and the European options prices. Section 4 provides a numerical example with some modified CIR type interest rate and volatility processes and gives some evidence on the numerical accuracy of our formulae. Section 5 gives the discussions on the related issues including an interpretation of our framework, the risk premium functionals, the change of measure problem, and an extension to the HJM type term structure of interest rates. Also some concluding remarks are given in Section 5.3. Section 6 is the mathematical appendix for some proofs omitted in Sections 2 and 3.

2 Asymptotic Behavior of the Underlying Asset Price Process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, Q)$ be a complete filtered probability space [1] with the probability measure Q. We consider a continuous time economy where some securities and bonds are traded in the interval $[0,T]$ $(T < +\infty)$ without any friction and we assume that there does neither exist any default risk nor any transaction costs associated with bonds and securities. Let $S_t^{(\epsilon,\delta)}$ $(0 < t \leq T)$ be the price of the underlying security at t with two parameters $0 < \epsilon, \delta \leq 1$. In this section we focus on the situation that this security pays no dividends and the price process follows the stochastic differential equation

$$S_t^{(\epsilon,\delta)} = S_0 + \int_0^t r_s^{(\epsilon)} S_s^{(\epsilon,\delta)} ds + \int_0^t \sigma_s^{(\delta)} S_s^{(\epsilon,\delta)} dW_{1s} \ , \tag{2.1}$$

[1] We use the standard arguments on the completion of the original probability space without any explicit exposition in this paper.

where $r_t^{(\epsilon)}$ is the instantaneous spot interest rate at t with the parameter $0 < \epsilon \leq 1$, $\sigma_s^{(\delta)}$ is the instantaneous volatility at t with the parameter $0 < \delta \leq 1$, and W_{1t} is the Brownian motion under Q . The non-negative stochastic process $\sigma_s^{(\delta)}$ follows the stochastic differential equation :

$$\sigma_s^{(\delta)} = \sigma_0 + \int_0^s \mu_\sigma(\sigma_u^{(\delta)}, u, \delta) du + \delta \int_0^s w_\sigma(\sigma_u^{(\delta)}, u) dW_{2u} , \tag{2.2}$$

where W_{2t} is the second Brownian motion under Q. We note that when the volatility function is not a traded asset, the markets for the contingent claims on the underlying asset $\{S_t^{(\epsilon,\delta)}\}$ could be incomplete.

For the interest rate processes, we assume that there exists a locally riskless money market and the money market account (accumulation factor) is given by $M_t^{(\epsilon)} = \exp(\int_0^t r_s^{(\epsilon)} ds)$. We also consider the situation when there also exist the bond markets in the economy and let $P^{(\epsilon)}(s,t)$ $(0 \leq s \leq t)$ be the discount bond price at s with the maturity date t . We assume that the non-negative (instantaneous) spot interest rate process $r_t^{(\epsilon)}$, which is consistent with the money market and the discount bond markets, follows the stochastic differential equation :

$$r_s^{(\epsilon)} = r_0 + \int_0^s \mu_r(r_u^{(\epsilon)}, u, \epsilon) du + \epsilon \int_0^s w_r(r_u^{(\epsilon)}, u) dW_{3u} , \tag{2.3}$$

where W_{3t} is the third Brownian motion under Q.
As the simplest case we have the situation when all discount bond prices $P^{(\epsilon)}(s,t)$ $(0 \leq s \leq t \leq T)$ are solely determined by the single factor $\{r_t^{(\epsilon)}\}$. As the more general case, we shall discuss the HJM term structure of interest rates model in which the spot interest rate is not necessarily Markovian. (See (5.73) and (5.74).)

In (2.1)-(2.3) we consider the situation when three Brownian motions under Q are correlated and their instantaneous correlations are given by

$$\mathbf{E}^Q \left[\begin{array}{cc} dW_t & dW_t' \end{array} \right] = \begin{pmatrix} 1 & \rho_\sigma & \rho_r \\ \rho_\sigma & 1 & \rho_{\sigma r} \\ \rho_r & \rho_{r\sigma} & 1 \end{pmatrix} dt , \tag{2.4}$$

where we denote $dW_t = (dW_{1t}, dW_{2t}, dW_{3t})'$.
In Sections 2 and 3 we treat both the interest rate process and the volatility function of the asset returns as Markovian processes under the probability measure Q. The form of the stochastic differential equations in (2.1)-(2.3) should be interpreted as the representation under the probability measure Q, (alternatively we write $Q(\sigma^{(\delta)}, r^{(\epsilon)})$), which is equivalent to the historical probability measure P for the observed price process and the interest rate process. The probability measure Q is a martingale measure in the sense that the discounted security price process

$$Y_T^{(\epsilon,\delta)} = S_T^{(\epsilon,\delta)} \exp\left(-\int_t^T r_s^{(\epsilon)} ds\right) \tag{2.5}$$

is a martingale under Q given the volatility process and the interest rate process. We shall further discuss the related interpretation in Section 5.1, which is similar to the one by Romano and Touzi (1997).

Let $G(S_T^{(\epsilon,\delta)})$ be the non-negative payoff function of the European contingent claim at T on the underlying security $\{S_t^{(\epsilon,\delta)}\}$. The contingent claims are regarded as Wiener functionals in the Wiener space which are not necessarily smooth in the standard mathematical sense. In this paper we try to analyze the theoretical values of the contingent claims based on the payoff which is a function of time T security price. The immediate examples of this type are the forward contract, the futures contract, and the standard European options contracts. For our purpose, we need to define the theoretical values of this type of contingent claims [2] .

Definition 2.1 *: The theoretical value of the European contingent claim with the terminal payoff function $G(S_T^{(\epsilon,\delta)})$ at time t is defined by*

$$G_t^{(\epsilon,\delta)} = \mathbf{E}^Q[G(S_T^{(\epsilon,\delta)}) \exp\left(-\int_t^T r_s^{(\epsilon)} ds\right) |\mathcal{F}_t] \tag{2.6}$$

provided that the expected value is finite, where the expectation operator is taken with respect to a probability measure Q given the σ-field $\mathcal{F}_t$.

We shall analyze the effects of the stochastic volatility and the stochastic interest rates on the theoretical value of the contingent claims when both ϵ and δ are small. In order to develop the asymptotic expansion approach when both δ and ϵ are small, we need to have some regularity conditions for that the solutions of (2.1)-(2.3) are well-behaved and the stochastic expansions of the stochastic processes $\{r_t^{(\varepsilon)}\}$ and $\{\sigma_t^{(\delta)}\}$ can be allowed.

Assumption I : (i) Given $0 < \epsilon, \delta \leq 1$ the drift functions $\mu_r(r_t^{(\epsilon)}, t, \epsilon)$, $\mu_\sigma(\sigma_t^{(\delta)}, t, \delta)$ $(\mathbf{R}\times[0,T]\times[0,1) \mapsto \mathbf{R}$) and the diffusion functions $w_r(r_t^{(\epsilon)}, t), w_\sigma(\sigma_t^{(\delta)}, t)$ $(\mathbf{R}\times[0,T] \mapsto \mathbf{R}$) are progressively measurable such that $r_t^{(\epsilon)}$ and $\sigma_t^{(\delta)}$ are $\mathcal{F}_t$-measurable. (ii) For the (finite) terminal period $T > 0$ the stochastic processes $r_t^{(\epsilon)}$ and $\sigma_t^{(\delta)}$ satisfy the conditions $0 < \int_0^T (\sigma_s^{(\delta)})^2 ds \ < +\infty$ $(a.s.)$ and

$$\mathbf{E}^Q[e^{2\int_0^T \ r_s^{(\epsilon)} ds} + e^{6\int_0^T (\sigma_s^{(\delta)})^2 ds}] \ < +\infty \ . \tag{2.7}$$

[2] It has been well recognized that, in the present situaton, the market for the contingent claims is incomplete unless the volatility is not a traded asset or not perfectly correlated with any other traded asset. This implies that we cannot find a self-financing portfolio that replicates the contingent claim and consequently leads to a unique price for it. Mathematically, the question of completeness is linked with the uniqueness of the equivalent martingale measure Q and the representaion property as (2.6). Then there could be a range of prices for contingent claim that are arbitrage-free which has been systematically investigated by Karatzas and Shereve (1998), for instance. Provided that there are no arbitrage opportunities, we have the relation: $G_t^{(\epsilon,\delta)} \in [G_t^L, G_t^U]$, where $G_t^L = \inf_Q E^Q[G(S_T^{(\epsilon,\delta)}) \exp(-\int_t^T r_s^{(\epsilon)})|\mathcal{F}_t]$ and $G_t^U = \sup_Q E^Q[G(S_T^{(\epsilon,\delta)}) \exp(-\int_t^T r_s^{(\epsilon)})|\mathcal{F}_t]$. To choose an equivalent martingale measure in a meaningful way is the subject of ongoing research. One candidate is to adopt the general equilibrium approach including Cox, Ingersoll and Ross (1985), and Bakshi and Chen (1997). By specifying the risk-preferences of the investors and the state processes of the economy, it is possible to obtain a conditional expectation representation under the corresponding equivalent martingale measure Q. The second strategy involves finding a selection principle to reduce the class of all possible measures Q to a subclass within which a unique measure can be found such as the minimal martingale measure by Heath, Platen and Schweizer (2001), for instance. As the third one the market completion by means of traded contingent claims à la Romano and Touzi (1997) will be put forth in Section 5.1 in more depth. With these possibilities in mind which are consistent with our formulation, we continue our discussion by fixing a martingale measure Q in Sections 2-4.

Assumption II : (i) The drift functions are continuously twice differentiable and their first and second derivatives are bounded uniformly in ϵ and δ . The diffusion functions are continuously differentiable and their first derivatives are bounded uniformly in ϵ and δ. (ii) For any $0 < t \leq T$ there exist unique solutions $\{r_t\}$ and $\{\sigma_t\}$ for the ordinary differential equations

$$r_t = r_0 + \int_0^t \mu_r(r_s, s, 0)ds \ , \tag{2.8}$$

and

$$\sigma_t = \sigma_0 + \int_0^t \mu_\sigma(\sigma_s, s, 0)ds \ . \tag{2.9}$$

(iii) Suppose δ is a function [3] of ϵ with the notation $\delta = \delta(\epsilon)$. There exists a positive constant c ($0 < c < \infty$) such that

$$\lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} = c \ . \tag{2.10}$$

The uniquness and existence for the stochastic differential equations of (2.2) and (2.3) follow from Assumption II-(i). A set of standard conditions for them as the Lipschitz type conditions and the growth condition on the drift functions and diffusion functions for $\{r_t^{(\epsilon)}\}$ and $\{\sigma_t^{(\delta)}\}$ have been discussed by Chapter IV of Ikeda and Watanabe (1989) or Nagai (1999), for instance. We need the integrability condition given by Assumption I-(ii) for the security price process $S_t^{(\epsilon,\delta)}$. Also we need some smoothness conditions of underlying stochastic processes with respect to the parameters ϵ and δ.

By applying Ito's lemma, the solution of (2.1) can be expressed as

$$S_t^{(\epsilon,\delta)} = S_0 \exp\left\{ \int_0^t \sigma_s^{(\delta)} dW_{1s} - \frac{1}{2}\int_0^t \left(\sigma_s^{(\delta)}\right)^2 ds + \int_0^t r_s^{(\epsilon)} ds \right\} \ . \tag{2.11}$$

Under Assumption I-(ii) the first two parts of (2.11) consist an exponential martingale and also the security price $S_t^{(\epsilon,\delta)}$ is square-integrable.

In the rest of this section, we shall investigate the asymptotic behavior of the security price process in the situation [4] when $\epsilon \downarrow 0$ and $\delta \downarrow 0$ under Assumption II. It should be noted that Assumption II can be relaxed with some additional complications. We shall derive the explicit form of $S_t^{(\epsilon,\delta)}$ for any $0 < t \leq T$ in this small disturbance asymptotic approach.

Let

$$A_r^{(\epsilon)}(t) = \frac{1}{\epsilon}[r_t^{(\epsilon)} - r_t] \ , \tag{2.12}$$

[3] It is certainly possible to analyze other situations including the cases when $c = 0$ or $c = \infty$ by our method with some complications. We shall use the notation δ for $\delta(\epsilon)$ for the simplicity without making any confusions.

[4] Recently, Sircar and Papanicolau (1999) and Fouque et. al. (2001) have used "the small randomness of volatility", which is related to but different from the small disturbance asymptotics developed by Kunitomo and Takahashi (1995, 1998, 2001, 2003a, 2003b), Takahashi (1995, 1999), Kim and Kunitomo (1999).

where $r_t = r_t^{(0)}$ is the solution satisfying the ordinary differential equation (2.8). By substituting $r_t^{(\epsilon)} = r_t + \epsilon A_r^{(\epsilon)}(t)$ into (2.3), we have

$$
\begin{aligned}
\epsilon A_r^{(\epsilon)}(t) &= \int_0^t \Big\{ [\mu_r(r_s + \epsilon A_r^{(\epsilon)}(s), s, \epsilon) - \mu_r(r_s, s, \epsilon)] + [\mu_r(r_s, s, \epsilon) - \mu_r(r_s, s, 0)] \Big\} ds \\
&\quad + \epsilon \int_0^t w_r(r_s + \epsilon A_r^{(\epsilon)}(s), s) dW_{3s} \,. \qquad (2.13)
\end{aligned}
$$

Then by using Assumption II-(i), we can find positive constants c_1 and c_2 such that for any t

$$
|A_r^{(\epsilon)}(t)| \leq \int_0^t [c_1 |A_r^{(\epsilon)}(s)| + c_2] ds + | \int_0^t w_r(r_s + \epsilon A_r^{(\epsilon)}(s), s) dW_{3s}| \,. \qquad (2.14)
$$

Also by using the standard arguments in stochastic analysis, the martingale inequality, and the Gronwall inequality, we can find positive constants c_3 and c_4 such that

$$
\mathbf{E}^Q[|A_r^{(\epsilon)}(t)|^2] \leq c_3 e^{c_4 t} \,, \mathbf{E}^Q[\sup_{0 \leq t \leq T} |A_r^{(\epsilon)}(t)|^2] < +\infty \qquad (2.15)
$$

uniformly with respect to ϵ . Hence we confirm the convergence in probability that $r_t^{(\epsilon)} \to r_t$ uniformly with respect to t as $\epsilon \downarrow 0$.
Let

$$
B_r^{(\epsilon)}(t) = \frac{1}{\epsilon^2} [r_t^{(\epsilon)} - r_t - \epsilon A_r(t)] \,, \qquad (2.16)
$$

where [5] $A_r(t) = p \lim_{\epsilon \downarrow 0} A_r^{(\epsilon)}(t)$. Then by substituting $r_t^{(\epsilon)} = r_t + \epsilon A_r(t) + \epsilon^2 B_r^{(\epsilon)}(t)$ into (2.3), we can use a similar argument recursively to lead that $\mathbf{E}^Q[|B_r^{(\epsilon)}(t)|^2]$ is bounded uniformly with respect to t and ϵ and we have the uniform convergence of $A_r^{(\epsilon)}(t)$ to $A_r(t)$ with respect to t as $\epsilon \downarrow 0$ in probability. We need similar arguments on the existence and convergence of random variables $A_r(t)$, which we have omitted. By using the above arguments under Assumption II, the stochastic expansion of the instantaneous interest rate $r_t^{(\epsilon)}$ can be expressed by

$$
r_t^{(\epsilon)} = r_t + \epsilon \; A_r(t) + R_1 \qquad (2.17)
$$

as $\epsilon \downarrow 0$, where the remainder term R_1 is in the order $o_p(\epsilon)$. Then by using (2.13) and convergence arguments of its each terms, $A_r(t)$ can be regarded as the solution of the stochastic differential equation :

$$
A_r(t) = \int_0^t [\partial \mu_r(r_s, s, 0) A_r(s) + \partial^{\epsilon} \mu_r(r_s, s, 0)] ds + \int_0^t w_r(r_s, s) dW_{3s} \,, \qquad (2.18)
$$

where we denote

$$
\partial \mu_r(r_s, s, 0) = \left. \frac{\partial \mu_r(r_s^{(\epsilon)}, s, \epsilon)}{\partial r_s^{(\epsilon)}} \right|_{r_s^{(\epsilon)} = r_s, \epsilon = 0} \,, \qquad (2.19)
$$

and

$$
\partial^{\epsilon} \mu_r(r_s, s, 0) = \left. \frac{\partial \mu_r(r_s^{(\epsilon)}, s, \epsilon)}{\partial \epsilon} \right|_{\epsilon = 0} \,. \qquad (2.20)
$$

[5] By using (2.13) and (2.18) it is possible to show $\mathbf{E}[\sup_{0<t\leq T} \|A_r^{(\epsilon)}(t) - A_r(t)\|] \to 0$ by using the standard but quite similar arguments as in Mathematical Appendices of Kunitomo and Takahashi (2003a).

In order to have a concise representation for $A_r(t)$, let Y_t^r be the solution of $dY_t^r = \partial\mu_r(r_t,t,0)Y_t^r dt$ with the initial condition $Y_0^r = 1$. Then (2.18) can be solved as

$$A_r(t) = \int_0^t Y_t^r (Y_s^r)^{-1}[w_r(r_s,s)dW_{3s} + \partial^\epsilon \mu_r(r_s,s,0)ds]\ . \tag{2.21}$$

Similarly, under Assumption II we can expand the integral equation (2.2) with respect to δ . By using the same argument as $r_t^{(\epsilon)}$, the stochastic expansion of the stochastic volatility $\sigma_t^{(\delta)}$ can be also expressed by

$$\sigma_t^{(\delta)} = \sigma_t + \delta\ A_\sigma(t) + R_2 \tag{2.22}$$

as $\delta \downarrow 0$, where the leading term σ_t is the solution of the ordinary differential equation (2.9), the second term is given by $A_\sigma(t) = p\lim_{\delta\downarrow 0} A_\sigma^{(\delta)}(t)$ with $A_\sigma^{(\delta)}(t) (= [\sigma_t^{(\delta)} - \sigma_t]/\delta)$, and the remainder term R_2 is of the order $o_p(\delta)$. Let Y_t^σ be the solution of $dY_t^\sigma = \partial\mu_\sigma(\sigma_t,t,0)Y_t^\sigma dt$ with the initial condition $Y_0^\sigma = 1$. Then because $A_\sigma(t)$ is the solution of the corresponding stochastic differential equation as (2.18) for $\{\sigma_t^{(\delta)}\}$, we can express $A_\sigma(t)$ as

$$A_\sigma(t) = \int_0^t Y_t^\sigma (Y_s^\sigma)^{-1}[w_\sigma(\sigma_s,s)dW_{2s} + \partial^\delta \mu_\sigma(\sigma_s,s,0)ds]\ , \tag{2.23}$$

where $\partial w_\sigma(\sigma_s,s,0)$ and $\partial^\delta\mu_\sigma(\sigma_s,s,0)$ are defined in the same ways as (2.19) and (2.20).

In order to state the asymptotic behavior of the security price process $S_t^{(\epsilon,\delta)}$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, we need new mathematical device of the *Malliavin-Watanabe Calculus* recently developed in stochastic analysis. Let the H-differentiation be defined by $DF_h(\mathbf{w}) = \lim_{\epsilon\to 0}(1/\epsilon)[F(\mathbf{w}+\epsilon\mathbf{h}) - F(\mathbf{w})]$ for a Wiener functional $F(\mathbf{w})$ and $\mathbf{h} \in \mathbf{M}$, where $\mathbf{M}$ is the Cameron-Martin subspace of continuous functions with square-integrable derivatives in the Wiener space $\mathbf{W}$. Then the Malliavin covariance is given by $\sigma_{MC}(F) =< DF(\mathbf{w}), DF(\mathbf{w}) >_H$, where $< \cdot >_H$ is the inner product of $\mathbf{M}$ space. We summarize the first result on the asymptotic behavior of the security price process $S_t^{(\epsilon,\delta)}$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ in the next lemma. The proof is given in Section 6.1.

Lemma 2.1 *: (i) Under Assumptions I and II,*

$$\sup_{0\le s\le t\le T} |S_s^{(\epsilon,\delta)} - S_s| \to 0\ (a.s.), \tag{2.24}$$

as $\epsilon \downarrow 0$ *and* $\delta \downarrow 0$, *where* S_t *is the solution of the stochastic differential equation*

$$dS_t = r_t S_t dt + \sigma_t S_t dW_{1t}\ , \tag{2.25}$$

and $\{r_t\}$ *and* $\{\sigma_t\}$ *are the solutions of the ordinary differential equations (2.8) and (2.9).*

(ii) Let $\sigma_{MC}(S_t^{(\epsilon,\delta)})$ *be the Malliavin covariance of the stochastic process* $S_t^{(\epsilon,\delta)}$ *and* $S_t^{(\epsilon,\delta)} \in \mathbf{D}_{2,1}(\mathbf{R})$ [6] *. Then we have*

$$\sup_{0\le s\le t\le T} |\sigma_{MC}(S_s^{(\epsilon,\delta)}) - S_s^2 \int_0^s \sigma_u^2 du| \to 0\ (a.s.) \tag{2.26}$$

[6] The definitions of the H-differentiation and the space $\mathbf{D}_{2,1}$ have been adopted from Chapter V of Ikeda and Watanabe (1989) as the standard reference. See Kunitomo and Takahashi (1998, 2003a) for the more details. Also we note that Yoshida (2003) has recently developed the general theory of conditional expansions in the Malliavin Calculus and our formulation could be interpreted as a special case of his Example 4.

as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ under Assumptions I and II.

Next we shall derive the asymptotic expansion of the security price process $S_t^{(\epsilon,\delta)}$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$. For this purpose we insert (2.17) and (2.22) into (2.11). By evaluating the probability order calculations, we obtain the expressions for the first term of the exponential part in (2.11) as

$$\int_0^t \left[r_s^{(\epsilon)} - \frac{1}{2}\left(\sigma_s^{(\delta)}\right)^2 \right] ds = \int_0^t \left(r_s - \frac{\sigma_s^2}{2} \right) ds + \epsilon \int_0^t A_r(s) ds - \delta \int_0^t \sigma_s A_\sigma(s) ds + R_3$$

and for the second term of the exponential part in (2.11) as

$$\int_0^t \sigma_s^{(\delta)} dW_{1s} = \int_0^t \sigma_s dW_{1s} + \delta \int_0^t A_\sigma(s) dW_{1s} + R_4 \ , \tag{2.27}$$

where R_i $(i=3,4)$ are the remaining terms of the higher orders.
If we set the leading term as

$$X_{1t} = \int_0^t \sigma_s dW_{1s} \ , \tag{2.28}$$

we can write (2.11) as

$$\begin{aligned} S_t^{(\epsilon,\delta)} &= S_0 \exp\left\{ \left[X_{1t} + \int_0^t \left(r_s - \frac{\sigma_s^2}{2} \right) ds \right] \right. \\ &\quad \left. + \epsilon \int_0^t A_r(s) ds + \delta \left[\int_0^t A_\sigma(s) dW_{1s} - \int_0^t \sigma_s A_\sigma(s) ds \right] + R_5 \right\} , \end{aligned} \tag{2.29}$$

and

$$\exp\left[\int_0^T r_t^{(\epsilon)} dt \right] = \exp\left\{ \int_0^T r_s ds + \epsilon \int_0^T A_r(s) ds + R_6 \right\} , \tag{2.30}$$

where R_5 and R_6 are the remaining terms of higher orders.
Then we can obtain a stochastic expansion of the price process of the security at time t with respect to ϵ and δ which can be summarized in the next lemma.

Lemma 2.2 : *Under Assumptions I and II, an asymptotic expansion of the price process of the security $S_t^{(\epsilon,\delta)}$ at any particular time point t as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ is given by*

$$\begin{aligned} S_t^{(\epsilon,\delta)} &= S_0 \exp\left\{ [X_{1t} + \int_0^t \left(r_s - \frac{\sigma_s^2}{2} \right) ds] \right. \\ &+ \epsilon [\frac{\Sigma_{12}^{(r)}(t)}{\sigma(t)^2} X_{1t} + \lambda_r(t)] + \delta \Sigma_{12}^{(\sigma)}(t) \left[\frac{{X_{1t}}^2}{\sigma(t)^4} - \frac{X_{1t}}{\sigma(t)^2} - \frac{1}{\sigma(t)^2} \right] \\ &+ \left. \delta \lambda_\sigma(t) [\frac{X_{1t}}{\sigma(t)^2} - 1] + [\epsilon z_{1t} + \delta (z_{3t} - z_{2t}) + R_5] \right\} , \end{aligned} \tag{2.31}$$

where z_{it} $(i=1,2,3)$ are the random variables with $E^Q[z_{it}|X_{1t}] = 0, E^Q[z_{it}^2] < \infty$ $(i = 1,2,3)$,

$$\sigma(t)^2 = Var(X_{1t}) = \int_0^t \sigma_s^2 ds \ , \tag{2.32}$$

$$\Sigma_{12}^{(r)}(t) = \rho_r \int_0^t \left(\int_u^t Y_s^r ds \right) (Y_u^r)^{-1} w_r(r_u, u) \sigma_u du \ , \tag{2.33}$$

$$\lambda_r(t) = \int_0^t \left(\int_u^t Y_s^r ds \right) (Y_u^r)^{-1} \partial^\epsilon \mu_r(r_u, u, 0) du \ , \tag{2.34}$$

$$\Sigma_{12}^{(\sigma)}(t) = \rho_\sigma \int_0^t \left(\int_u^t \sigma_s Y_s^\sigma ds \right) (Y_u^\sigma)^{-1} w_\sigma(\sigma_u, u) \sigma_u du \ , \tag{2.35}$$

$$\lambda_\sigma(t) = \int_0^t \left(\int_u^t \sigma_s Y_s^\sigma ds \right) (Y_u^\sigma)^{-1} \partial^\delta \mu_\sigma(\sigma_u, u, 0) du \ , \tag{2.36}$$

and R_5 *is the remainder term of the order* $o_p(\epsilon, \delta)$.

Let $X_t^{(\epsilon,\delta)} = \log(S_t^{(\epsilon,\delta)})$ and $X_{1t}^{(\epsilon,\delta)} = \int_0^t \sigma_s^{(\delta)} W_{1s}$. Then Assumption I implies the boundedness of some expectations such that $\mathbf{E}^Q[|X_t^{(\epsilon,\delta)}|^2] < +\infty$ and $\mathbf{E}^Q[\exp(X_t^{(\epsilon,\delta)}))] < +\infty$ by using the Cauchy=Schwartz inequality and *Theorem 5.3* of Ikeda=Watanabe (1989) for exponential martingales. The remaining proof of Lemma 2.2 is given in Section 6.2. We should note that we have the non-degeneracy of the random variable X_{1t} for any $0 < t \leq T$ as the key condition from our Assumption I-(ii). Let also $\sigma_{MC}(X_{1t}^{(\epsilon,\delta)})$ be the Malliavin covariance of $X_{1t}^{(\epsilon,\delta)}$, which is *the partial Malliavin covariance* of the three dimensional process in (2.1)-(2.3). Then the integrated volatility function $\sigma(t)^2$ can be interepreted as the limit of the corresponding partial Malliavin covariance $\sigma_{MC}(X_{1t}^{(\epsilon,\delta)})$ as $\epsilon \downarrow 0$.

By evaluating inequalities and other related arguments, it is possible to take the expectation operator and its asymptotic expansion in terms of ϵ and δ in (2.6) with respect to the probability measure Q when

$$G(S_T^{(\epsilon,\delta)}) \leq \max\{K_1, S_T^{(\epsilon,\delta)}\} \tag{2.37}$$

for some constant K_1 . Because the derivative securities such as the forward contracts, the futures contracts, and the standard option contracts in the next section satisfy this condition, we can take the expectation operators and their asymptotic expansions formally in our discussions.

3 Futures, Forward, and Options Prices

3.1 Futures and Forward Prices

First, we consider the theoretical values of the futures contract written on the security $S_t^{(\epsilon,\delta)}$ that matures at time T. The standard financial theory asserts that the futures price at time zero, F_0, is determined by

$$F_0 = \mathbf{E}^Q \left[S_T^{(\epsilon,\delta)} \right] \ , \tag{3.38}$$

where the expectation is taken with respect to the probability measure Q. (See Chapter 8 of Duffie (1996) for the simple arbitrage free arguments for the forward contract and the futures contract, for instance.) By applying Lemma 2.1 and using the moment relations that for $X_{1T} \sim N(0, \sigma(T)^2)$ we have $\mathbf{E}[\exp(X_{1T})] = \exp(\sigma(T)^2/2)$, $\mathbf{E}[X \exp(X_{1T})] = \sigma(T)^2 \exp(\sigma(T)^2/2)$, and $\mathbf{E}[X_{1T}^2 \exp(X_{1T})] = \sigma(T)^2[\sigma(T)^2 + 1] \exp(\sigma(T)^2/2)$. Then we immediately obtain the following result.

Theorem 3.1 *: Under Assumptions I and II, an asymptotic expansion of the theoretical value of the futures contract at time zero with the delivery time T, F_0, is given by*

$$F_0 = S_0 \exp\left(\int_0^T r_s ds\right) \left\{1 + \epsilon\, \Sigma_{12}^{(r)}(T) + \epsilon\, \lambda_r(T)\right\} + o(\epsilon, \delta) \tag{3.39}$$

as $\epsilon \downarrow$ and $\delta \downarrow 0$, where $\Sigma_{12}^{(r)}(T)$ and $\lambda_r(T)$ are given by (2.33) and (2.34), respectively.

We should notice that there is no effect due to the stochastic volatility in (3.39) up to the order of $o(\epsilon, \delta)$ although there might be some effects on the higher order terms. From this result, we can predict that for the futures price the effect of volatility is considerably smaller than the effect of stochastic interest rate on the futures price. Next we consider the theoretical value of the forward contract written on the security $S_t^{(\epsilon,\delta)}$ that matures at time T. The standard financial theory asserts that the forward price at time zero, f_0, is determined by

$$f_0 = \frac{\mathbf{E}^Q\left[\exp\left(-\int_0^T r_s^{(\epsilon)} ds\right) S_T^{(\epsilon,\delta)}\right]}{\mathbf{E}^Q\left[\exp\left(-\int_0^T r_s^{(\epsilon)} ds\right)\right]}\ . \tag{3.40}$$

By using the fact $\mathbf{E}[\exp(aX)] = \exp(\frac{a^2}{2}\Sigma)$ when $X \sim N[0,\Sigma]$ for any constant a, the denominator of (3.40) can be written as

$$\mathbf{E}^Q\left[\exp\left(-\int_0^T r_s^{(\epsilon)} ds\right)\right] = \exp\left(-\int_0^T r_s ds - \epsilon\, \lambda_r(T)\right) \times [1 + R_6]\ ,$$

where R_6 is the remainder term of the order $o_p(\epsilon)$. Then we have the following result on the forward price.

Theorem 3.2 *: Under Assumptions I and II, an asymptotic expansion of the theoretical value of the forward contract at time zero with the delivery time T, f_0, is given by*

$$f_0 = S_0 \exp\left(\int_0^T r_s ds\right) \times \{1 + \epsilon\, \lambda_r(T)\} + o(\epsilon, \delta) \tag{3.41}$$

as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, where $\lambda_r(T)$ is given by (2.34).

Let the first term in (3.39) be F_D and set the coefficient of ϵ in (3.39) to be F_r. Then Theorem 3.1 asserts that the futures price can be decomposed into the futures price under constant interest rate and the adjustment terms as

$$F_0 = F_C + [F_D - F_C] + \epsilon\, F_r + o(\epsilon, \delta) \tag{3.42}$$

where F_C is the futures price under constant interest rate r_0 and volatility σ_0. The second term in (3.42) represents the adjustment value induced by the deterministic interest rate. The third term is the adjustment value induced by incorporating the stochastic interest rate. Because Y_s^r in $\Sigma_{12}^{(r)}(T)$ takes only positive values, we have the following result.

Corollary 3.1 *: Under Assumptions I and II, we have the relations depending on ρ_r such that*

$$\lim_{\epsilon,\delta\downarrow 0} \frac{1}{\epsilon}\left[F_0 - f_0\right] > 0 \quad if \quad \rho_r > 0 \,, \tag{3.43}$$

$$\lim_{\epsilon,\delta\downarrow 0} \frac{1}{\epsilon}\left[F_0 - f_0\right] = 0 \quad if \quad \rho_r = 0 \,, \tag{3.44}$$

$$\lim_{\epsilon,\delta\downarrow 0} \frac{1}{\epsilon}\left[F_0 - f_0\right] < 0 \quad if \quad \rho_r < 0 \,. \tag{3.45}$$

There have been some discussions on the relation between the futures price and the forward price when the short term interest rate is stochastic. First, when the interest rate is independent of the underlying asset, the futures price is equal to the forward price, which is also the direct result from (3.40). Second, when the interest rate has positive (negative) correlation with the underlying asset, the futures price is greater (smaller) than forward price. This result has been also presented in Equation (26) of Cox, Ingersoll, and Ross (1981). From our analysis, we have found that the relations between the futures price and the forward price mentioned above hold even when the volatility of underlying asset is stochastic in the small disturbance asymptotics sense.

3.2 Options

We consider the theoretical value of the European call options contract written on the security $S_t^{(\epsilon,\delta)}$ that matures at time T. For a given exercise price K at the expiry date T, the theoretical price of such options at the initial date, V_0, can be given by

$$V_0 = \mathbf{E}^Q\left[\left[Z_T^{(\epsilon,\delta)}\right]^+\right] \tag{3.46}$$

under the probability measure Q, where $[\cdot]^+$ denotes the function $\max[0,\cdot]$ and

$$Z_T^{(\epsilon,\delta)} = \exp\left(-\int_0^T r_t^{(\epsilon)} ds\right)\left[S_T^{(\epsilon,\delta)} - K\right] \,.$$

By substituting (2.31) into $Z_T^{(\epsilon,\delta)}$ and using (2.30) on the discount factor, we can obtain the expression for $Z_T^{(\epsilon,\delta)}$ as

$$Z_T^{(\epsilon,\delta)} = Z_0 + \delta Z_1^{\delta} + \epsilon Z_1^{\epsilon} + \delta(z_{3T}^* - z_{2T}^*) + \epsilon z_{1T}^* + R_7 \;\;, \tag{3.47}$$

where

$$\begin{aligned}
Z_0 &= S_0 \exp\left(X_{1T} - \frac{1}{2}\sigma^2(T)\right) - K\exp\left(-\int_0^T r_t dt\right), \\
Z_1^{\delta} &= S_0 \exp\left(X_{1T} - \frac{1}{2}\sigma(T)^2\right) \times \left[\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2}\left(\frac{X_{1T}^2}{\sigma(T)^2} - X_{1T} - 1\right) + \lambda_\sigma(T)\left(\frac{X_{1T}}{\sigma(T)^2} - 1\right)\right], \\
Z_1^{\epsilon} &= K\exp\left(-\int_0^T r_t dt\right) \times \left\{\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} X_{1T} + \lambda_r(T)\right\},
\end{aligned}$$

and R_7 is the remainder term of the order $o_p(\epsilon,\delta)$. In the above expression the random variables z_{iT}^* $(i=1,2,3)$ have been defined by $z_{1T}^* = z_{1T} K \exp\left(-\int_0^T r_t dt\right)$ and $z_{iT}^* = z_{iT} S_0 \exp(X_{1T} - \frac{1}{2}\sigma(T)^2)$ $(i=2,3)$. Then we need to evaluate

$$
\begin{aligned}
V_0 &= \mathbf{E}^Q[Z_0 I(S_T^{(\epsilon,\delta)} - K)] + \mathbf{E}^Q[Z_1^{(\epsilon,\delta)} I(S_T^{(\epsilon,\delta)} - K)] \\
&\quad + \mathbf{E}^Q[Z_1^{(\epsilon,\delta)} I(S_T^{(\epsilon,\delta)} - K)] + \mathbf{E}^Q[R_7 I(S_T^{(\epsilon,\delta)} - K)] \ ,
\end{aligned} \tag{3.48}
$$

where $I(\,\cdot\,)$ is the indicator function of $[0,+\infty)$. By the result of lengthy derivations as outlined in Section 6.3, we finally have obtained the theoretical value of the European call options contract as the next theorem.

Theorem 3.3 *: Under Assumptions I and II, an asymptotic expansion of the theoretical value of the European call option with maturity T when the interest rate and volatility are stochastic, V_0, is given by*

$$
\begin{aligned}
V_0 &= \left[S_0 \Phi(d_1) - K \exp\left(-\int_0^T r_t dt\right) \Phi(d_2) \right] \\
&+ \epsilon \left[\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)} S_0 \phi(d_1) + \lambda_r(T) \Phi(d_2) K \exp\left(-\int_0^T r_t dt\right) \right] \\
&+ \delta \left[-\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} d_2 + \frac{\lambda_\sigma(T)}{\sigma(T)} \right] S_0 \phi(d_1) \ + o(\epsilon,\delta)
\end{aligned} \tag{3.49}
$$

as $\epsilon, \delta \downarrow 0$, where $\Phi(\cdot)$ is the distribution function of the standard normal variable, $\phi(\cdot)$ is its density function, $d_2 = d_1 - \sigma(T)$,

$$
d_1 = \frac{1}{\sigma(T)} \left[\log \frac{S_0}{K} + \int_0^T \left(r_s + \frac{1}{2}\sigma_s^2 \right) ds \right] , \tag{3.50}
$$

and also $\Sigma_{12}^{(r)}(T)$,$\Sigma_{12}^{(\sigma)}(T)$, $\lambda_r(T)$ and $\lambda_\sigma(T)$ are defined by (2.33), (2.35), (2.34), and (2.36), respectively.

Let the first term in (3.49) be BS_D and we set the coefficients of ϵ and δ to be BS_r and BS_σ, respectively. Then Theorem 3.3 asserts that the European stock call option price can be decomposed into the original Black-Scholes price and the adjustment terms as

$$
V_0 = BS + [BS_D - BS] + \epsilon\, BS_r + \delta\, BS_\sigma + o(\epsilon,\delta) \tag{3.51}
$$

where BS stands for the original Black-Scholes option price under the assumptions of constant interest rate and volatility. The second term in (3.51) represents the adjustment value induced by the deterministic interest rate which in itself relies on the assumed interest rate and volatility model. The third term and the fourth term are the adjustment values induced by the stochastic interest rate and the stochastic volatility, respectively. When the interest rate is stochastic and the volatility is constant, for instance, the valuation problem in this case has been investigated by Kim and Kunitomo (1999). Also when $r_s = r_0$ and $\epsilon = 0$, the situation corresponds to those for many stochastic volatility models, which have been studied by several researchers. Hence our results include many previous studies as special cases in the sense of the small disturbance asymptotics.

By using the similar procedure for the valuation of European call options, we can derive the theoretical value for the European put options whose payoff function is given by $[K - S_T^{(\epsilon,\delta)}]^+$ at the maturity time T. By deriving the asymptotic expansion of the random variable $[-Z_T^{(\epsilon,\delta)}]^+$, we have the next result.

Theorem 3.4 : *Under Assumptions I and II, an asymptotic expansion of the theoretical value of the European put option with maturity T when the interest rate and volatility are stochastic, V_0^*, is given by*

$$
\begin{aligned}
V_0^* &= \left[K \exp\left(-\int_0^T r_t dt\right)\Phi(-d_2) - S_0\Phi(-d_1)\right] \\
&+ \epsilon\,[\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)} S_0\phi(d_1) - \lambda_r(T)\Phi(-d_2)K\exp\left(-\int_0^T r_t dt\right)] \\
&+ \delta\,[-\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} d_2 + \frac{\lambda_\sigma(T)}{\sigma(T)}]S_0\phi(d_1) \ + o(\epsilon,\delta)
\end{aligned}
\tag{3.52}
$$

as $\epsilon, \delta \downarrow 0$, where $\Phi(\cdot)$ is the distribution function of standard normal variable, $\phi(\cdot)$ is its density function, and d_i $(i = 1, 2)$, $\Sigma_{12}^{(r)}(T)$, $\Sigma_{12}^{(\sigma)}(T)$, $\sigma(T)$, $\lambda_r(T)$, and $\lambda_\sigma(T)$ are the same as in Theorem 3.3 .

4 A Numerical Example and Related Analysis

4.1 A CIR Type Interest Rates and Volatilitiy

In this section, we give some numerical examples to illustrate our theoretical results in Section 3. For this purpose we assume that the spot interest rate and volatility processes are of the CIR square-root process originally proposed by Cox, Ingersoll, and Ross (1985b) for the spot interest rate model. Let the state variables $(Z_{1t}^{(\delta)}, Z_{2t}^{(\epsilon)})$ follow

$$
dZ_{1t}^{(\delta)} = \kappa_\sigma(\bar{\sigma} + \delta\,\nu_\sigma - Z_{1t}^{(\delta)})dt + \delta\sqrt{Z_{1t}^{(\delta)}}dW_{2t}
\tag{4.53}
$$

and

$$
dZ_{2t}^{(\epsilon)} = \kappa_r(\bar{r} + \epsilon\,\nu_r - Z_{2t}^{(\epsilon)})dt + \epsilon\sqrt{Z_{2t}^{(\epsilon)}}\,dW_{3t}
\tag{4.54}
$$

under the probability measure Q . We take a sufficiently large K_2 and set $r_t^{(\epsilon)} = Z_{1t}^{(\epsilon)}$ $(r_t^{(\epsilon)} < K_2)$, $\sigma_t^{(\delta)} = Z_{2t}^{(\delta)}$ $(\sigma_t^{(\delta)} < K_2)$, and $r_t^{(\epsilon)} = K_2, \sigma_t^{(\delta)} = K_2$ otherwise.

We notice that (4.54) is the special case of (2.3) when we take $\mu_r(r_s^{(\epsilon)}, s, \epsilon) = \kappa_r(\bar{r} + \epsilon\ \nu_r - r_s^{(\epsilon)})$ and $w_r(r_s^{(\epsilon)}, s) = \sqrt{r_s^{(\epsilon)}}$, where κ_r, $\bar{r}$, and ϵ are positive constants and $K_2 = +\infty$. Also (4.53) is also a special case of (2.2). The assumption of the CIR type processes for two state variables could be justified by the reason that both the nominal short term interest rates and the volatility functions take non-negative values with the mean-reversion properties. [7] .

In our example it is possible to give some simple formulae for the valuation problems. The solution of the ordinary differential equation (2.8) for the spot interest rate is given by $r_t = r_0 e^{-\kappa_r t} + \bar{r}(1-e^{-\kappa_r t})$ and for the volatility process the solution of the differential equation (2.9) is provided by $\sigma_t = \sigma_0 e^{-\kappa_\sigma t} + \bar{\sigma}(1-e^{-\kappa_\sigma t})$. Then the variance function $\sigma(T)^2$ can be calculated from $\int_0^T \sigma_s^2 ds$ and it is given by

$$\sigma(T)^2 = \frac{(\bar{\sigma}-\sigma_0)\left\{4\bar{\sigma}e^{\kappa_\sigma T}\left(1-e^{\kappa_\sigma T}\right)+(\bar{\sigma}-\sigma_0)\left(e^{2\kappa_\sigma T}-1\right)\right\}}{2e^{2\kappa_\sigma T}\kappa_\sigma} + \bar{\sigma}^2 T \ .$$

In the standard Black-Scholes model, $\sigma(T)$ reduces to $\sigma_0\sqrt{T}$, which corresponds to the case when $\bar{\sigma} = \sigma_0$. Also we have that $Y_t^r = e^{-\kappa_r t}$ and $Y_t^\sigma = e^{-\kappa_\sigma t}$. By using the definition of $\Sigma_{12}^{(r)}(T)$ in (2.33), the integration operation gives the following expression:

$$\begin{aligned}
&\Sigma_{12}^{(r)}(T) \\
= \ & \frac{\rho_r}{\kappa_r}\int_0^T \left(1-e^{-\kappa_r(T-u)}\right)\left(e^{-\kappa_r u}(r_0-\bar{r})+\bar{r}\right)^{\frac{1}{2}}\left(\sigma_0 e^{-\kappa_\sigma u}+\bar{\sigma}(1-e^{-\kappa_\sigma u})\right)du \\
= \ & \frac{\rho_r}{\kappa_r}\left[\frac{2\sqrt{\bar{r}}\bar{\sigma}\left((1+2e^{\kappa_r T})\sqrt{r_0}-3\,e^{\frac{\kappa_r T}{2}}\sqrt{r_0-\bar{r}(1-e^{\kappa_r T})}\right)+\bar{\sigma}\left(\bar{r}(1+2e^{\kappa_r T})-r_0\right)a_r}{2e^{\kappa_r T}\kappa_r\sqrt{\bar{r}}}\right. \\
&\left. +\int_0^T \frac{\bar{\sigma}-\sigma_0}{e^{\kappa_\sigma u}}\left(e^{-\kappa_r(T-u)}-1\right)\left(e^{-\kappa_r u}(\bar{r}(e^{\kappa_r u}-1)+r_0)\right)^{\frac{1}{2}}du\right],
\end{aligned}$$

where we have used the notation a_r defined by

$$a_r = \log\left[\frac{\bar{r}(2e^{\kappa_r T}-1)+r_0+2\,e^{\frac{\kappa_r T}{2}}\sqrt{\bar{r}^2(e^{\kappa_r T}-1)+\bar{r}r_0}}{(\sqrt{r_0}+\sqrt{\bar{r}})^2}\right].$$

Also by using the definition of $\Sigma_{12}^{(\sigma)}(T)$ in (2.35), we can calculate the integration explicitly as

$$\begin{aligned}
\Sigma_{12}^{(\sigma)}(T) &= \frac{\rho_\sigma}{\kappa_\sigma}\int_0^T\left[\bar{\sigma}\left(1-e^{-\kappa_\sigma(T-u)}\right)+\frac{\bar{\sigma}-\sigma_0}{2}e^{-\kappa_\sigma u}\left(e^{-2\kappa_\sigma(T-u)}-1\right)\right] \\
&\quad\times\left[\sigma_0 e^{-\kappa_\sigma u}+\bar{\sigma}\left(1-e^{-\kappa_\sigma u}\right)\right]^{\frac{3}{2}}du \\
&= \rho_\sigma\frac{\gamma_1+\gamma_2+\gamma_3\cdot a_\sigma}{60e^{2\kappa_\sigma T}\kappa_\sigma^2},
\end{aligned}$$

[7] It may be important to note that the smoothness conditions in Assumption II are not satisfied in the sense that the diffusion functions $w_r(r,t)$ and $w_\sigma(\sigma,t)$ are not differentiable at the origin and the non-negative processes are not bounded from the above in the present case. As Section 4 of Kunitomo and Takahashi (2001) have indicated, however, we can have the corresponding smoothed versions of the modified CIR type stochastic processes because there is only one point where the differentiability breaks down. We have used the state space representation on $Z_{1t}^{(\epsilon)}$ and $Z_{2t}^{(\delta)}$ and truncation because we need the integrability condition of (2.7). However, these modifications do not have any problem for practical applications because it is possible to show that the probabilities of hitting the origin and/or the boundary K_2 are $o(\epsilon,\delta)$ and we can ignore the possibility of explosive solutions when the initial state variables are positive without these conditions

where γ_i (i=1,2,3) and a_σ are similarly defined by

$$\begin{aligned}
\gamma_1 &= 2\sqrt{\frac{\bar\sigma(e^{\kappa_\sigma T}-1)+\sigma_0}{e^{\kappa_\sigma T}}}\left(\bar\sigma^2(24-13e^{\kappa_\sigma T}-116e^{2\kappa_\sigma T})\right.\\
&\left.+\bar\sigma\sigma_0(13e^{\kappa_\sigma T}-48)+24\sigma_0^2\right),\\
\gamma_2 &= -2\sqrt{\sigma_0}\left(\bar\sigma^2(45-90e^{\kappa_\sigma T}-60e^{2\kappa_\sigma T})+\bar\sigma\sigma_0(60e^{\kappa_\sigma T}-20e^{2\kappa_\sigma T}-75)\right.\\
&\left.+\sigma_0^2(30-6e^{2\kappa_\sigma T})\right),\\
\gamma_3 &= 15\sqrt{\bar\sigma}\left(\bar\sigma^2(6e^{\kappa_\sigma T}+4e^{2\kappa_\sigma T}-3)+6\bar\sigma\sigma_0(1-e^{\kappa_\sigma T})-3\sigma_0^2\right),
\end{aligned}$$

and

$$a_\sigma = \log\left[\frac{\bar\sigma(2e^{\kappa_\sigma T}-1)+\sigma_0+2e^{\frac{\kappa_\sigma T}{2}}\sqrt{\bar\sigma^2(e^{\kappa_\sigma T}-1)+\bar\sigma\sigma_0}}{(\sqrt{\bar\sigma}+\sqrt{\sigma_0})^2}\right].$$

The standard notation in the Black-Scholes formula d_1 becomes in the present case

$$d_1 = \frac{1}{\sigma(T)}\left(\log\frac{S_0}{K}+\bar r T+\frac{1}{\kappa_r}(r_0-\bar r)(1-e^{-\kappa_r T})+\frac{\sigma(T)^2}{2}\right),$$

and $d_2 = d_1 - \sigma(T)$.
Finally, we can simplify the leading term in the discount factor considerably and it is given by

$$\exp\left(-\int_0^T r_t\,dt\right) = \exp\left(-\frac{r_0-\bar r}{\kappa_r}\left(1-e^{-\kappa_r T}\right)-\bar r\,T\right).$$

4.2 Numerical Accuracy

We report some results on the numerical accuracy of our formulae by using the CIR interest rates and volatility example. We take $\nu_\sigma = \nu_r = 0$ and hence $\lambda_r(T) = \lambda_\sigma(T) = 0$ for the resulting simplicity. Since we are interested in the case when the covariances between three state variables are not zeros, we give a set of numerical values for the cases of $\rho_r, \rho_\sigma = -0.5, 0.5$. Among many cases of our numerical examples we shall report only one case. It is the case when the initial interest rate is in the downward phase and we set $r_0 = 0.11$, $\bar r = 0.08$, $\kappa_r = 2.0$, and $\epsilon = 0.1$. For the stochastic process on the volatility function, it is the upward phase case and we set $\sigma_0 = 0.2$, $\bar\sigma = 0.3$, $\kappa_\sigma = 4.0$, and $\delta = 0.1$. Also we take that the time to maturity is assumed to be one year ($T = 1$). For the futures price, we set $S_0 = 100$. For the call option case, we set $K = 100$ and $S_0 = 90, 100, 110$ to incorporate moneyness of options.

Each table in this sub-section corresponds to the numerical value of the approximations up to $o(\epsilon, \delta)$ based on the asymptotic expansions in Theorem 3.1 and Theorem 3.3 by ignoring higher order terms. For the call option case, the option value under the original Black-Scholes model has been also given for the comparative purpose. As the benchmark, we also provide the Monte Carlo simulation results in the first row of each table. The number of simulated sample paths is $10,000$ and the time interval is 250 . As the discretization method of sample paths, we have adopted the Euler-Maruyama approximation. All results are the mean of 200 simulation trials.

Table 1 describes the numerical accuracy of our formula for the futures price. As mentioned in Section 3, it should be noted that in our analysis the forward price under the stochastic interest and the stochastic volatility differs from the futures price slightly. In addition, the parameters of stochastic volatility are not appeared in our formula which is in the order of $o(\epsilon, \delta)$. We can observe in Table 1 that our formula is numerically very close to the simulation results. For example, when $\rho_r = \rho_\sigma = -0.5$, the pricing bias is 0.006 yen, which is equal to only 0.005 % of the true value.

Table 2 describes the numerical example for the European call option for at-the-money case. When $\kappa_r = \kappa_\sigma = 0$ and $\epsilon = \delta = 0$, the case corresponds to the original Black-Scholes economy with constant risk free interest rate and volatility. The Black-Scholes value in this case BS is 13.868. We can observe in Table 2 that our option pricing formula is very close to the simulation results. For example, when $\rho_r = -0.5$ and $\rho_\sigma = 0.5$, the discrepancy between them is 0.013 yen, which is only 0.085%. Table 3 and Table 4 also give the numerical results with the same set of parameter values for the in-the-money and the out-of-the-money cases, respectively.

4.3 Term Structure of Implied Volatilities

In our setting (2.1)-(2.3), the implied volatility $\sigma^*(T)$ is the same as the implied average volatility $\bar{\sigma}(T)$ when $\epsilon = \delta = 0$, where we denote $\bar{\sigma}(T)^2 = (1/T)\int_0^T \sigma_t^2 dt$. In the general case, however, by using *Theorem 3.3* the implied volatility and the implied average volatility can be defined by the solutions of the nonlinear equation

$$
\begin{aligned}
\frac{V_0}{S_0} &= \Phi[\frac{-\log k + Tr_0 + \frac{T}{2}\sigma^*(T)^2}{\sigma^*(T)\sqrt{T}}] - k\exp(-Tr_0)\,\Phi[\frac{-\log k + Tr_0 - \frac{T}{2}\sigma^*(T)^2}{\sigma^*(T)\sqrt{T}}] \\
&= [\Phi(d_1) - k\exp(-T\bar{r}(T))\,\Phi(d_2)] \\
&\quad + \epsilon[\frac{\Sigma_{12}^{(r)}(T)}{\sqrt{T}\bar{\sigma}(T)} S_0\phi(d_1) + \lambda_r(T)\Phi(d_2)k\exp(-T\bar{r}(T))] \\
&\quad + \delta[-\frac{\Sigma_{12}^{(\sigma)}(T)}{T\bar{\sigma}(T)^2} d_2 + \frac{\lambda_\sigma(T)}{\sqrt{T}\bar{\sigma}(T)}]\phi(d_1) \; + o(\epsilon, \delta),
\end{aligned} \tag{4.55}
$$

where $k = K/S_0$, $d_2 = d_1 - \sqrt{T}\bar{\sigma}(T)$, $d_1 = \left[-\log k + T\bar{r}(T) + T\frac{1}{2}\bar{\sigma}(T)^2\right]/[\bar{\sigma}(T)\sqrt{T}]$, and we denote the average interest rate $\bar{r}(T) = (1/T)\int_0^T r_t dt$.
Here we have interpreted the implied volatility as the volatility calculated from the Black-Scholes formula by using call options market prices. The implied average volatility is a reasonable volatility index if the actual price of the call options in market is equal to the theoretical value in *Definition 2.1.* Furthermore, by ignoring higher order terms the above equation can be solved with respect to $\sigma^*(T)$ and it can be written as

$$
\sigma^*(T) = H^{(\epsilon,\delta)}(k, \bar{r}(T), \bar{\sigma}(T), T, \Sigma_{12}^{(r)}(T), \Sigma_{12}^{(\sigma)}, \lambda_r(T), \lambda_\sigma(T)) \; . \tag{4.56}
$$

Hence it is analytically possible to investigate the various shapes on the implied volatility $\sigma^*(T)$ or the implied average volatility as the functions of the variables k and T including the well-known phenomena of *volatility smile.*

5 Discussion

5.1 Risk Premium, Complete Market, and Near Completeness

We have interpreted the stochastic processes in (2.1)-(2.4) under the measure Q, which can be different from the probability measure P governing the observable stochastic processes. When we consider the measure P for the observable underlying asset price and the bond prices with the spot interest rate, we could write the stochastic differential equation under P as

$$r_s^{(\epsilon)} = r_0 + \int_0^s \mu_r^*(r_u^{(\epsilon)}, u)du + \epsilon \int_0^s w_r(r_u^{(\epsilon)}, u)dB_{3u}^* \ , \tag{5.57}$$

where $\mu_r^*(\cdot, u)$ is the drift function, $w_r(\cdot, u)$ is the diffusion function, and $B_{3u}^*(u)$ is the Brownian motion for the spot interest rate under P. Also the stochastic differential equation of the volatility function for the underlying asset could be written as

$$\sigma_t^{(\delta)} = \sigma_0 + \int_0^t \mu_\sigma^*(\sigma_u^{(\delta)}, u)du + \delta \int_0^t w_\sigma(\sigma_u^{(\delta)}, u)[c_{22}dB_{2u}^* + c_{23}dB_{3u}^*] \ , \tag{5.58}$$

where $\mu_\sigma^*(\cdot, u)$ is the drift function, $w_\sigma(\cdot, u)$ is the diffusion function, B_{2u}^* is the Brownian motion for the volatility function which is independent of B_{3u}^* under P, and c_{2i} $(i = 2, 3)$ are constants with the normalization condition $E^P[c_{22}B_{2t}^* + c_{23}B_{3t}^*]^2 = t$. The stochastic differential equation for the underlying asset price under P could be written as

$$\begin{aligned} S_t^{(\delta)} &= S_0 + \int_0^t \mu_S^*(S_u^{(\delta)}, u)S_u^{(\delta)}du \\ &\quad + \int_0^t \sigma_u^{(\delta)} S_u^{(\delta)}[c_{11}dB_{1u}^* + c_{12}dB_{2u}^* + c_{13}dB_{3u}^*] \ , \end{aligned} \tag{5.59}$$

where $\mu_S^*(\cdot, u)$ is the drift function, B_{1u}^* is the Brownian motion for the security market under P, which is independent of B_{iu}^* $(i = 2, 3)$, and c_{1i} $(i = 1, 2, 3)$ are constants with the normalization condition $E^P[c_{11}B_{1t}^* + c_{12}B_{2t}^* + c_{13}B_{3t}^*]^2 = t$. We note that the asset price $S_t^{(\delta)}$ has only one parameter δ under the probability P in this setting.

In financial economics it has been often considered when there exist risk premium functionals associated with the Brownian motions. In this section we denote $\lambda_i^{(\epsilon,\delta)}(u)$ as the risk premium processes for the Brownian motions B_{iu}^* $(i = 1, 2, 3)$ under the probability measure P. For the simplicity of our discussion on the risk premium functionals we first consider the single factor model in the bond market and let $P^{(\epsilon)}(t, T)$ $(= P^{(\epsilon)}(r_t^{(\epsilon)}, t, T))$ be the discount bond price with the maturity period T satisfying

$$dP^{(\epsilon)}(t, T) = \mu_P^{(\epsilon)}(t, T)P^{(\epsilon)}(t, T)dt + \nu^{(\epsilon)}(t, T)P^{(\epsilon)}(t, T)dB_{3t}^* \tag{5.60}$$

where $\mu_P^{(\epsilon)}(t, T)$ and $\nu^{(\epsilon)}(t, T)$ are the drift term and the volatility function of the bond price, respectively. By using the standard argument of the no-arbitrage condition the risk premium functional $\lambda_3^{(\epsilon)}(t)$ can be determined by

$$\nu^{(\epsilon)}(t, T)\lambda_3^{(\epsilon,\delta)}(t) = \mu_P^{(\epsilon)}(t, T) - r_t^{(\epsilon)} \ ,$$

provided that $\nu^{(\epsilon)}(t,T) > 0$ for any $\epsilon > 0$ and the consistency condition $\lim_{\epsilon\downarrow 0} \nu^{(\epsilon)}(t,T) = 0$. For the stochastic volatility let us assume that there exists a traded contingent claim whose price can be written as $G_t^{(\epsilon,\delta)} (= G(S_t^{(\delta)}, \sigma^{(\delta)}, r_t^{(\epsilon)}, t))$ at t. Then this price process can be re-written as

$$
\begin{aligned}
(5.61) \qquad dG_t^{(\epsilon,\delta)} &= \mu_G^{(\epsilon,\delta)}(t) G_t^{(\epsilon,\delta)} dt \\
&+ \sigma_G^{(\epsilon,\delta)}(t) G_t^{(\epsilon,\delta)} [c_{11}^*(t) dB_{1t}^* + c_{12}^*(t) dB_{2t}^* + c_{13}^*(t) dB_{3t}^*(t)],
\end{aligned}
$$

where $\mu_G^{(\epsilon,\delta)}(t)$ and $\sigma_G^{(\epsilon)}(t)$ are the drift term and the volatility function of the contingent claim price. Together with the stochastic process for $S_t^{(\delta)}$, again by using the standard arguments of the no-arbitrage conditions, the risk premium functional $\lambda_i^{(\epsilon,\delta)}(t)$ $(i = 1,2,3)$ can be determined by

$$
\begin{pmatrix} c_{11}\sigma_t^{(\delta)} & c_{12}\sigma_t^{(\delta)} & c_{13}\sigma_t^{(\delta)} \\ c_{11}^*(t)\sigma_G^{(\epsilon,\delta)}(t) & c_{12}^*(t)\sigma_G^{(\epsilon,\delta)}(t) & c_{13}^*(t)\sigma_G^{(\epsilon,\delta)}(t) \\ 0 & 0 & \nu^{(\epsilon)}(t,T) \end{pmatrix} \begin{pmatrix} \lambda_{1t}^{(\epsilon,\delta)} \\ \lambda_{2t}^{(\epsilon,\delta)} \\ \lambda_{3t}^{(\epsilon,\delta)} \end{pmatrix} = \begin{pmatrix} \mu_S^{(\delta)}(t) - r_t^{(\epsilon)} \\ \mu_G^{(\epsilon,\delta)}(t) - r_t^{(\epsilon)} \\ \mu_P^{(\epsilon)}(t,T) - r_t^{(\epsilon)} \end{pmatrix} .
$$

(5.62)

Then the risk premium functionals can be uniquely given by solving the above equations if we have the non-degeneracy conditions $\nu^{(\epsilon)}(t,T) > 0, \sigma_G^{(\epsilon,\delta)}(t) > 0, \sigma_t^{(\delta)} > 0$ and

$$
(5.63) \qquad c_{11}{c_{12}}^*(t) - c_{11}^*(t)c_{12} > 0\ , c_{11} > 0\ .
$$

for $0 < t \leq T$. Our method here is a kind of spanning problem and market completion by financial contingent claims and this problem has been systematically investigated by Romano and Touzi (1997) in a slightly different formulation.

Furthermore, in order to use the change of measures procedure, we need to assume the Novikov condition for the transformation of measures

$$
(5.64) \qquad \mathbf{E}^P[\exp(\frac{1}{2}\int_0^T \sum_{i=1}^3 \Big((\lambda_i^{(\epsilon,\delta)})^2(s)ds\Big)] < +\infty \ \ (t > 0)\ .
$$

Then we can take the Brownian Motions under the martingale measure Q, which is equivalent to the original measure P for (5.57)-(5.59) by using the *Maruyama-Girsanov* transformation with

$$
(5.65) \qquad B_{it} = B_{it}^* + \int_0^t \lambda_i^{(\epsilon,\delta)}(s)ds \quad (i = 1,2,3)\ .
$$

We note that the expectation in (5.64) is taken with respect to the probability measure P. In this formulation of the risk premium functionals they are dependent on the underlying stochastic processes in a complicated way in the general case. Because the stochastic volatility and the stochastic interest rate become non-stochastic in the limit, we may assume the condition that the risk premium functionals are *locally deterministic* in the small disturbance asymptotics sense.

Assumption III : The risk premium functionals $\lambda_i^{(\epsilon,\delta)}(t)$ $(i = 1,2,3)$ are bounded and satisfy (5.63)-(5.64) and as $\epsilon \downarrow 0$ and $\delta \downarrow 0$,

$$
(5.66) \qquad \lambda_i^{(\epsilon,\delta)}(t) = \lambda_i(t) + O_p(\epsilon,\delta) \quad (i = 2,3)\ ,
$$

where $\lambda_i(t)$ $(i = 2, 3)$ are the deterministic functions of time.

This is a restrictive assumption and it is possible to relax this condition, but the following analysis will become more complex considerably. By redefining $W_{1t} = c_{11}B_{1t} + c_{12}B_{2t} + c_{13}B_{3t}$, $W_{2t} = c_{22}B_{2t} + c_{23}B_{3t}$, and $W_{3t} = B_{3t}$ such that we have the correlation structure of (2.4) with c_{ij} $(i \le j)$, we have the security price process and the interest rate processes with (2.1)-(2.4). The drift condition in the form of (2.1) is the standard formulation under the assumption of the existence of a locally riskless money market. By ignoring the higher order terms, under the Assumption III we can write the drift function of $r_t^{(\epsilon)}$ as

$$\mu_r(r_u^{(\epsilon)}, u, \epsilon) = \mu_r^*(r_u^{(\epsilon)}, u) - \epsilon w_r(r_u^{(\epsilon)}, u)\lambda_3(u) ,$$

and in the present case $A_r(t)$ of (2.18) should be the solution of the stochastic integral equation,

$$(5.67) \qquad A_r(t) = \int_0^t \partial\mu_r^*(r_s, s)A_r(s)ds + \int_0^t w_r(r_s, s)[dW_{3s} - \lambda_3(s)ds] .$$

Then it can be solved as

$$(5.68) \qquad A_r(t) = \int_0^t Y_t^r (Y_s^r)^{-1} w_r(r_s, s)[dW_{3s} - \lambda_3(s)ds] .$$

Similarly, we can write the drift function of $\sigma_t^{(\delta)}$ as

$$\mu_\sigma(\sigma_u^{(\delta)}, u, \delta) = \mu_\sigma^*(\sigma_u^{(\delta)}, u) - \delta w_\sigma(\sigma_u^{(\delta)}, u)[c_{22}\lambda_2(u) + c_{23}\lambda_3(u)] ,$$

and we have the corresponding equation for (2.23) as

$$(5.69) \qquad A_\sigma(t) = \int_0^t Y_t^\sigma (Y_s^\sigma)^{-1} w_\sigma(\sigma_s, s)[dW_{2s} - (c_{22}\lambda_2(s) + c_{23}\lambda_3(s))ds] .$$

Also in the present case we have

$$(5.70) \qquad \lambda_r(T) = (-1)\int_0^T \left(\int_u^T Y_s^r ds\right)(Y_u^r)^{-1} w_r(r_u, u)\lambda_3(u)du ,$$

and

$$(5.71)\ \lambda_\sigma(T) = (-1)\int_0^T \left(\int_u^T \sigma_s Y_s^\sigma ds\right)(Y_u^\sigma)^{-1} w_\sigma(\sigma_u, u)[c_{22}\lambda_2(u) + c_{23}\lambda_3(u)]du .$$

In this way we can obtain the corresponding expressions in Sections 2 and 3 under the probability measure Q. The correlation coefficients among the underlying Brownian motions are invariant with respect to the change of measures that we have discussed. In order to use the formulae in practice we have derived, however, we have to estimate the values of the parameters appeared. We can ignore the extra terms discussed in this subsection if the risk premum functions are small and in the order of $o_p(1)$.

We should mention that for the underlying asset prices and their derivatives there have been different approaches to determine the prices of contingent claims at actual financial markets, i.e., the (economic) general equilibrium prices, which have been extensively investigated. (See Cox, Ingersoll, Ross (1985a,b) for instance.) In the general equilibrium framework the risk functional $\lambda_i^{(\epsilon,\delta)}(t)$ $(i = 1,2,3)$ in (5.62) should be determined as the function of individual preferences. Then the resulting relation between the drift functions of the underlying asset prices and the risk premium functionals could be in the form of (5.62).

When $\epsilon \downarrow 0$ and $\delta \downarrow 0$, the limiting case is the complete market for the underlying asset $S_t^{(\delta)}$ and its derivatives in any case because $\lim_{\epsilon \to 0} r_t^{(\epsilon)} = r_t$ and $\lim_{\delta \to 0} \sigma_t^{(\delta)} = \sigma_t$ in the sense of probability. This is the case regardless whether there exists any additional bond markets or contingent claim markets depending on the interest rates and the volatility of underlying asset price. However, when $\delta > 0$, the market is incomplete in the case when there does not exists an additional financial market on the volatility of the underlying asset. Thus in this case the security market with the price processes developed in Section 2 is incomplete, but not far from the complete market as the limiting case. We can call the situation which we have been considering *the near complete market* in this respect.

5.2 The HJM Case

In our formulation of the spot interest rate process, we have assumed (2.3) or (5.57), which is a Markovian type continuous process. There has been another type of the term structure of interest rate processes originally developed by Heath, Jarrow, and Morton (1992). In the HJM framework the instantaneous spot interest rate is not necessarily a Markovian continuous process.

Let also the price of discount bond $P^{(\epsilon)}(s,t)$ be continuously differentiable with respect to t and $P^{(\epsilon)}(s,t) > 0$ for $0 \leq s \leq t \leq T$ and $0 < \epsilon \leq 1$. The instantaneous forward rate at s for future date t $(0 \leq s \leq t \leq T)$ is defined by

$$f^{(\epsilon)}(s,t) = -\frac{\partial \log P^{(\epsilon)}(s,t)}{\partial t} \ . \tag{5.72}$$

In the HJM term structure model of interest rates we assume that the bond market is complete and under the equivalent martingale measure, say Q^*, a family of forward rate processes $f^{(\varepsilon)}(s,t)$ follow the stochastic integral equations

$$\begin{aligned} f^{(\varepsilon)}(s,t) = f(0,t) \quad &+ \quad \varepsilon^2 \int_0^s \left[\sum_{i=1}^n \omega_{f,i}(f^{(\varepsilon)}(v,t),v,t) \int_v^t \omega_{f,i}(f^{(\varepsilon)}(v,y),v,y)dy \right] dv \\ &+ \quad \varepsilon \int_0^s \sum_{i=1}^n \omega_{f,i}(f^{(\varepsilon)}(v,t),v,t) dW_{3,i}(v) \ , \end{aligned} \tag{5.73}$$

where $f(0,t)$ $(= f^{(0)}(0,t))$ are assumed to be observable and the non-random initial forward rates, $W_{3,i}(v)$ $(i = 1,\cdots,n)$ are n independent Brownian motions, and $\omega_{f,i}(f^{(\varepsilon)}(v,t),v,t)$ $(i = 1,\cdots,n)$ are the diffusion functions. This formulation has been adopted by Kunitomo and Takahashi (1995, 2001), which have investigated the valuation problems of interest rate based contingent claims in detail. When $f^{(\varepsilon)}(s,t)$ is continuous at $s = t$ for $0 \leq s \leq t \leq T$, the spot interest rate at t can be defined by

$$r^{(\varepsilon)}(s) = \lim_{t \downarrow s} f^{(\varepsilon)}(s,t) \ . \tag{5.74}$$

Kunitomo and Takahashi (1995, 1998, 2001) have derived the asymptotic expansion of the discount factor when $\varepsilon \downarrow 0$ under a set of assumptions on the boundedness and smoothness of volatility functions. The equation (3.21) of Kunitomo and Takahashi (2001) implies

$$e^{-\int_0^T r^{(\varepsilon)}(s)ds} = P(0,T)[1 - \varepsilon \int_0^T A(s,s)ds] + o_p(\varepsilon) \ , \tag{5.75}$$

where

$$P(0,T) = \exp[-\int_0^T f(0,u)du] \ , \tag{5.76}$$

and

$$\int_0^T A(t,t)dt = \int_0^T \sum_{i=1}^n \int_v^T \omega_{f,i}(f^{(0)}(v,t),v,t)dt\, dW_{3,i}(v) \ . \tag{5.77}$$

Then it is straightforward to modify our analysis developed in Sections 2 and 3. The essential point is to replace the formula for $\Sigma_{12}^{(r)}$ by

$$\Sigma_{12}^{(r)}(T) = \int_0^T \sigma_v \left[\sum_{i=1}^n \rho_{f,i} \int_v^T \omega_{f,i}(f^{(0)}(v,s),v,s)ds \right] dv \ , \tag{5.78}$$

where $\rho_{f,i}$ are the correlation coefficients between dW_{1t} and $dW_{3,i}(t)$ $(i = 1, \cdots, n)$. Then it is possible to obtain the corresponding results on the futures, the forward, and the options prices to Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4 when the forward interest rate processes follow the stochastic integral equation given by (5.73) and the security returns exhibit the stochastic volatility.

5.3 Concluding Remarks

We have investigated the value of contingent claims when both the interest rate and volatility are stochastic. We have found that the European options value can be decomposed into the original Black-Scholes option price and the adjustment terms, which reflect the effects of the randomness of interest rates and volatility in the underlying stochastic processes. Similarly, the futures price can be decomposed into the futures price under constant interest rate and some adjustment terms. We also have illustrated the numerical examples and examined the accuracy of our formulae. Since our formulae are relatively simple and quite accurate, they shall be useful for practical applications.

The framework we have developed includes many situations as special cases and we have the standard Black-Scholes economy as the limiting case as both $\epsilon \downarrow 0$ and $\delta \downarrow 0$. It makes possible to examine the effects of the stochastic volatility and the stochastic interest rate in a unified way. As we have illustrated by examples of the futures and options in Section 3, our method can be also applicable to analyze other European type contingent claims.

Since the asymptotic expansion method we have developed has a solid mathematical basis called the *Malliavin-Watanabe Calculus* or *the Watanabe-Yoshida theory* on the Malliavin Calculus in stochastic analysis, the formulae in this paper are not *ad-hoc* approximations and they have rigorous mathematical basis. Also they are numerically accurate even for practical applications as we have illustrated in Section 4. In addition to these aspects we should mention that we have derived the asymptotic expansions for the futures price and options price in the order of $o_p(\epsilon, \delta)$, but it is certainly possible to obtain the approximations up to any higher order terms by using the asymptotic expansion approach. However, the calculations we need will become rapidly more demanding than we have derived in Section 2.

Finally, we should mention that the theoretical results reported in this paper can be utilized in the empirical study of actual devivatives markets. Kim (2002), for instance, has shown some preliminary results, but they are quite involved and the full details shall be reported in another occasion.

6 Mathematical Appendix

In this appendix we give the detailed proofs of Lemmas and Theorems stated in the previous sections.

6.1 Proof of Lemma 2.1

[**i**] Let $X_t^{(\epsilon,\delta)} = \log(S_t^{(\epsilon,\delta)})$ and $X_t = \log(S_t)$. By using (2.1) and (2.25), we represent the difference of two stochastic processes as

$$
\begin{aligned}
(6.79) \qquad & X_t^{(\epsilon,\delta)} - X_t \\
= & \int_0^t [(r_s^{(\epsilon)} - r_s) - \frac{1}{2}(\sigma_s^{(\delta)2} - \sigma_s^2)]ds + \int_0^t [\sigma_s^{(\delta)} - \sigma_s S_s]dW_{1s} \\
= & \int_0^t \{\epsilon A_r^{(\epsilon)}(s) - \frac{1}{2}[(\sigma_s + \delta A_\sigma^{(\delta)}(s))^2 - \sigma_s^2)\}ds + \int_0^t [\delta A_\sigma^{(\delta)}(s)]dW_{1s} \ .
\end{aligned}
$$

Then by using Assumption II, for any $0 \le t \le T$ there exists a positive constant M_1 such that

$$
\begin{aligned}
(6.80) \qquad & |X_t^{(\epsilon,\delta)} - X_t| \\
\le & M_1 \max(\epsilon, \delta) \int_0^t |A_r^{(\epsilon)}(s) + A_\sigma^{(\delta)}(s) + A_\sigma^{(\delta)2}(s)|ds + \delta | \int_0^t A_\sigma^{(\delta)}(s) dW_{1s}| \ ,
\end{aligned}
$$

where $A_r^{(\epsilon)}(t)$ is given by (2.12) and $A_\sigma^{(\delta)}(t)(= [\sigma_t^{(\delta)} - \sigma_t]/\delta)$, respectively. Hence there exist a positive constant M_2 such that

$$
(6.81) \qquad E[\sup_{0 \le s \le t} |X_s^{(\epsilon,\delta)} - X_s|^2] \le \varepsilon^2 M_2 \ .
$$

By using the Borel-Cantelli Lemma and the standard argument in stochastic analysis (see Chapter IV of Ikeda and Watanabe (1989)), we have the first part of our results under Assumptions I and II as $\epsilon \downarrow 0$.

[**ii**] We denote 3×1 vector $\mathbf{h} = (h_i) \in \mathbf{H}$ be the square-integrable continuous functions in $[0, T]$. By differentiating (2.3) and (2.2) in the direction of $\mathbf{h}$, we have the stochastic differential equations

$$
\begin{aligned}
(6.82) \qquad D_h \sigma_t^{(\delta)} = & \int_0^t \partial \mu_\sigma(\sigma_s^{(\delta)}, s, \delta) D_h \sigma_s^{(\delta)} ds \\
& + \delta \int_0^t \partial w_\sigma(\sigma_s^{(\delta)}, s) D_h \sigma_s^{(\delta)} dW_{2s} + \delta \int_0^t w_\sigma(\sigma_s^{(\delta)}, s) dh_{2s} \ ,
\end{aligned}
$$

and

$$D_h r_t^{(\epsilon)} = \int_0^t \partial\mu_r(r_s^{(\epsilon)}, s, \epsilon) D_h r_s^{(\epsilon)} ds + \epsilon \int_0^t \partial w_r(r_s^{(\epsilon)}, s) D_h r_s^{(\epsilon)} dW_{3s} + \epsilon \int_0^t w_r(r_s^{(\epsilon)}, s) dh_{3s} \ . \tag{6.83}$$

Then we can represent the solution of these equations as

$$D_h \sigma_t^{(\delta)} = \int_0^t Y_\sigma^{(\delta)}(t) Y_\sigma^{(\delta)}(s)^{-1} \delta \ w_\sigma(\sigma_s^{(\delta)}, s) \dot{h_{2s}} ds \ , \tag{6.84}$$

and

$$D_h r_t^{(\epsilon)} = \int_0^t Y_r^{(\epsilon)}(t) Y_r^{(\epsilon)}(s)^{-1} \epsilon \ w_r(r_s^{(\epsilon)}, s) \dot{h_{3s}} ds \ , \tag{6.85}$$

where $Y_\sigma^{(\delta)}(t)$ and $Y_r^{(\epsilon)}(t)$ are the solutions of

$$Y_\sigma^{(\delta)}(t) = \int_0^t \partial\mu_\sigma(\sigma_s^{(\delta)}, s, \delta) Y_\sigma^{(\delta)}(s) ds + \delta \int_0^t \partial w_\sigma(\sigma_s^{(\delta)}, s) Y_\sigma^{(\delta)}(s) dW_{2s} \tag{6.86}$$

and

$$Y_r^{(\epsilon)}(t) = \int_0^t \partial\mu_r(r_s^{(\epsilon)}, s, \epsilon) Y_r^{(\epsilon)}(s) ds + \epsilon \int_0^t \partial w_r(r_s^{(\epsilon)}, s) Y_r^{(\epsilon)}(s) dW_{3s} \ . \tag{6.87}$$

Also by differentiating (2.1) with respect to **h**, we have the stochastic differential equation as

$$D_h S_t^{(\epsilon,\delta)} = \int_0^t r_s^{(\epsilon)} D_h S_s^{(\epsilon,\delta)} ds + \int_0^t \sigma_s^{(\delta)} D_h S_s^{(\epsilon,\delta)} dW_{1s} + \int_0^t \sum_{i=1}^3 g_i^{(\epsilon,\delta)}(s) ds \ , \tag{6.88}$$

where we denote $g_1^{(\epsilon,\delta)}(s) = \sigma_s^{(\delta)} S_s^{(\epsilon,\delta)} \dot{h_{1s}}$, $g_2^{(\epsilon,\delta)}(s) = (D_h \sigma_s^{(\delta)}) S_s^{(\epsilon,\delta)}$, and $g_3^{(\epsilon,\delta)}(s) = (D_h r_s^{(\epsilon)}) S_s^{(\epsilon,\delta)}$.
By applying the Fubini-type theorem to (6.82) and (6.83), we can further represent

$$\int_0^t g_2^{(\epsilon,\delta)}(s) ds = \int_0^t \delta \int_s^t Y_\sigma^{(\delta)}(u) S_u^{(\epsilon,\delta)} du Y_\sigma^{(\delta)}(s)^{-1} w_\sigma(\sigma_s^{(\delta)}, s, \delta) \dot{h_{2s}} ds \ , \tag{6.89}$$

and

$$\int_0^t g_3^{(\epsilon,\delta)}(s) ds = \int_0^t \epsilon \int_s^t Y_r^{(\epsilon)}(u) S_u^{(\epsilon,\delta)} du Y_r^{(\epsilon)}(s)^{-1} w_r(r_s^{(\epsilon)}, s, \epsilon) \dot{h_{3s}} ds \ , \tag{6.90}$$

respectively. Then we can represent the H-derivative for the solution of security price equation as

$$D_h S_t^{(\epsilon,\delta)} = \int_0^t Y_S^{(\epsilon,\delta)}(t) Y_S^{(\epsilon,\delta)}(s)^{-1} [\sum_{i=1}^3 g_i^{(\epsilon,\delta)}(s)] ds \ , \tag{6.91}$$

where $Y_S^{(\epsilon,\delta)}(t)$ is the solution of the stochastic differential equation

$$dY_S^{(\epsilon,\delta)}(t) = r_s^{(\epsilon)} Y_S^{(\epsilon,\delta)}(t) dt + \sigma_s^{(\epsilon,\delta)} Y_S^{(\epsilon,\delta)}(t) dW_{1t} \ . \tag{6.92}$$

Hence we can represent the Malliavin-covariance of $S_t^{(\epsilon,\delta)}$ as

$$\sigma_{MC}(S_t^{(\epsilon,\delta)}) = \sum_{i=1}^3 \int_0^t Y_S^{(\epsilon,\delta)}(t)^2 Y_S^{(\epsilon,\delta)}(s)^{-2} (g_i^{(\epsilon,\delta)}(s))^2 ds \ . \tag{6.93}$$

We notice that as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, $\sigma_t^{(\delta)} \to \sigma_t$, $r_t^{(\epsilon)} \to r_t$, and $Y_S^{(\epsilon,\delta)}(t) \to S_t$ by using similar but tedious standard arguments. Then we have the convergence results such that as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ $g_1^{(\epsilon,\delta)}(t) \to S_t\sigma_t \dot{h}_{1s}$ and $g_i^{(\epsilon,\delta)}(t) \to 0$ $(i = 2, 3)$. Hence we have the desired result. (**Q.E.D.**)

6.2 Proof of Lemma 2.2

By using the stochastic expansions, we shall focus on the stochastic approximation based on X_{1t} as the leading term with respect to ϵ and δ in (2.29). By the Fubini-type theorem [8] for stochastic integrals, we notice that

$$\begin{aligned}
\int_0^t A_r(s)ds &= \int_0^t \left(\int_u^t Y_s^r ds\right) (Y_u^r)^{-1}[w_r(r_u, u)dW_{3u} + \partial^\epsilon \mu_r(r_u, u, 0)du] \, , \\
\int_0^t \sigma_s A_\sigma(s)ds &= \int_0^t \left(\int_u^t \sigma_s Y_s^\sigma ds\right) (Y_u^\sigma)^{-1}[w_\sigma(\sigma_u, u)dW_{2u} + \partial^\delta \mu_\sigma(\sigma_u, u, 0)du] \, .
\end{aligned}$$

Then by using the Gaussian property of W_t, we have the representation $dW_{2t} = \rho_\sigma dW_{1t} + dW_{2t}^*$ and $dW_{3t} = \rho_r dW_{1t} + dW_{3t}^*$, where we can construct the random variables W_{2t}^* and W_{3t}^* being independent of W_{1t} under Q. The conditional expectation given X_{1t} as the leading term is determined by

$$\mathbf{E}^Q\left[\int_0^t A_r(s)ds \Big| \int_0^t \sigma_s dW_{1s} = x\right] = \frac{\Sigma_{12}^{(r)}(t)}{\sigma(t)^2} x + \lambda_r(t) \, , \tag{6.94}$$

where we denote

$$\Sigma_{12}^{(r)}(t) = Cov\left(\int_0^t A_r(s)ds, \, X_{1t}\right) \, . \tag{6.95}$$

Then from (2.21) we confirm the relations (2.33) and (2.34). By the same token, the second conditional expectation given X_{1t} as the leading term is given by

$$\mathbf{E}^Q\left[\int_0^t \sigma_s A_\sigma(s)ds \Big| \int_0^t \sigma_s dW_{1s} = x\right] = \frac{\Sigma_{12}^{(\sigma)}(t)}{\sigma(t)^2} x + \lambda_\sigma(t) \, , \tag{6.96}$$

where

$$\Sigma_{12}^{(\sigma)}(t) = Cov\left(\int_0^t \sigma_s A_\sigma(s)ds, \, X_{1t}\right) \, . \tag{6.97}$$

Also by using the independent Gaussian random variables, we have the expression that the third conditional expectation given X_{1t} is determined by

$$\begin{aligned}
&\mathbf{E}^Q\left[\int_0^t A_\sigma(s)dW_{1s} \Big| \int_0^t \sigma_s dW_{1s} = x\right] \qquad (6.98)\\
&= \mathbf{E}^Q\Bigg[\int_0^t \int_0^s Y_s^\sigma (Y_u^\sigma)^{-1} w_\sigma(\sigma_u, u)dW_{2u}dW_{1s} \\
&\quad + \int_0^t \int_0^s Y_s^\sigma \, (Y_u^\sigma)^{-1} \, \partial^\delta \mu_\sigma(\sigma_u, u, 0)du \, dW_{1s} \Big| \int_0^t \sigma_s dW_{1s} = x\Bigg] \\
&= \Sigma_{12}^{(\sigma)}(t)\left[\frac{x^2}{\sigma(t)^4} - \frac{1}{\sigma(t)^2}\right] + \frac{\lambda_\sigma(t)}{\sigma(t)^2} \, x \, .
\end{aligned}$$

[8] We can use a simplified (but slightly extended in a sense) vesrion of *Lemma 4.1* of Ikeda and Watanabe (1989) for the present purpose.

We note that the second equation in (6.98) can be obtained by applying *Lemma 6.1* in Kunitomo and Takahashi (2001) (or *Lemma 6.4* of Kunitomo and Takahashi (2003b) for the details). From the above relatively tedious expositions, we sum up the expressions of three random variables as

$$\int_0^t A_r(s)ds = \frac{\Sigma_{12}^{(r)}(t)}{\sigma^2(t)} X_{1t} + \lambda_r(t) + z_{1t} \ , \tag{6.99}$$

$$\int_0^t \sigma_s A_\sigma(s)ds = \frac{\Sigma_{12}^{(\sigma)}(t)}{\sigma^2(t)} X_{1t} + \lambda_\sigma(t) + z_{2t} \ , \tag{6.100}$$

$$\int_0^t A_\sigma(s)dW_{1s} = \Sigma_{12}^{(\sigma)}(t)\left[\frac{X_{1t}^2}{\sigma(t)^4} - \frac{1}{\sigma(t)^2}\right] + \frac{\lambda_\sigma(t)}{\sigma(t)^2} X_{1t} + z_{3t} \ , \tag{6.101}$$

where z_{it} $(i = 1,2,3)$ are the random variables being uncorrelated with X_{1t} and we have the relations $\mathbf{E}^Q\left[z_{it}\,\middle|\,\int_0^t \sigma_s dW_{1s} = x\right] = 0$ $(i = 1,2,3)$ and $\mathbf{E}^Q[z_{it}^2] < +\infty$ by our constructions.
By substituting (6.99)-(6.101) into (2.29), we have the result. We need the order calculations and their justifications as we have illustrated for $r_t^{(\epsilon)}$. It is straightforward to do them as we have indicated at the end of Section 2 and we have omitted them. (**Q.E.D.**)

6.3 Proof of Theorem 3.3

Our derivation of the formula in Theorem 3.3 consists of three steps. The rigorous justifications of our forml calculations can be given as we have mentioned at the end of Section 2.
[**i**] Since the payoff function in the valuation problem of option contracts is not differentiable in the usual sense, we need to evaluate some expectations with the indicator function. First we prepare a useful lemma.

Lemma 6.1 *: Under the assumptions, the probability of the event* $\{S_T^{(\epsilon,\delta)} - K \geq 0\}$ *can be expressed as*

$$\begin{aligned} &\Pr\left(S_T^{(\epsilon,\delta)} - K \geq 0\right) \\ &= \Pr\left(X_{1T} \geq C_0 + \epsilon C_{12}^r + \delta\, C_{12}^\sigma - \epsilon\, z_{1T} - \delta(z_{3T} - z_{2T})\right) + R_8 \ , \end{aligned} \tag{6.102}$$

where R_8 *is the remainder terms of the order* $o(\epsilon,\delta)$, $C_{12}^r = -C_0\Sigma_{12}^{(r)}(T)/\sigma^2(T) - \lambda_r(T)$, *and*

$$C_0 = \log\frac{K}{S_0} - \int_0^T \left(r_s - \frac{\sigma_s^2}{2}\right) ds \ , \tag{6.103}$$

$$C_{12}^\sigma = \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2}(1 + C_0) - \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^4} C_0^2 - \lambda_\sigma(T)\left[\frac{C_0}{\sigma(T)^2} - 1\right] . \tag{6.104}$$

Proof of Lemma 6.1 :
By using Lemma 2.2 for the event $\{S_T^{(\epsilon,\delta)} - K \geq 0\}$, the probability we need to evaluate is given by

$$\Pr\left\{ X_{1T}\left[1 + \epsilon\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} - \delta\left(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} - \frac{\lambda_r(T)}{\sigma(T)^2}\right) + \delta\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^4} X_{1T}\right]\right.$$

$$\geq \log\frac{K}{S_0} - \int_0^T\left(r_s - \frac{\sigma_s^2}{2}\right)ds - \epsilon\lambda_r(T) + \delta(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} + \lambda_\sigma(T)) - \epsilon z_{1T} - \delta(z_{3T} - z_{2T}) + R_5\Big\} \tag{6.105}$$

where R_5 is the remainder term of the order $o_p(\epsilon,\delta)$ in (2.31).
When $\Sigma_{12}^{(\sigma)}(T) \neq 0$ [9] , we can evaluate

$$\begin{aligned} & \Pr\left(1 + \epsilon\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} - \delta(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} - \frac{\lambda_r(T)}{\sigma(T)^2}) + \delta\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^4}X_{1T} < 0\right) \\ \leq\ & \Pr\left(\delta|\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^4}X_{1T}| > |1 + \epsilon\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} - \delta(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} - \frac{\lambda_r(T)}{\sigma(T)^2})|\right) + o_p(\delta^2) \\ =\ & O(\delta^2) \end{aligned} \tag{6.106}$$

by using the Gaussianity of the random variable X_{1T} and the Tchebichev inequality.
We note that when $\Sigma_{12}^{(\sigma)}(T) = 0$ we can take ϵ and δ sufficiently small such that the first parenthesis of left-hand side of (6.105) is positive. Hence by using (6.105) we can evaluate the conditional probability as

$$\begin{aligned} & \Pr\Bigg(X_{1T} \geq \left[C_0 - \epsilon\lambda_r(T) + \delta(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} + \lambda_\sigma(T)) - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})\right] \\ & \qquad \times \left[1 + \epsilon\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} - \delta(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} - \frac{\lambda_r(T)}{\sigma(T)^2}) + \delta\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^4}X_{1T}\right]^{-1}\Bigg) \\ =\ & \Pr\left(X_{1T} \geq C_0 + \epsilon C_{12}^r + \delta\, C_{12}^\sigma - \epsilon\, z_{1T} - \delta(z_{3T} - z_{2T}) + R_9\right), \end{aligned} \tag{6.107}$$

where we have used the notations in the present Lemma and the remainder term R_9. Finally we notice that the remainder term R_9 is R_5 times some polynomial function of X_{1T} which is the Gaussian random variable. Because we have $\mathbf{E}[R_5^2] < \infty$ and $R_5 = o_p(\epsilon,\delta)$, we have (6.102). **(Q.E.D.)**

[**ii**] The next step is to evaluate the expectation operators. We define
$\Lambda_1 = \mathbf{E}^Q[Z_0 I(S_T^{(\epsilon,\delta)} - K)]$, $\Lambda_2^\delta = \mathbf{E}^Q[Z_1^\delta I(S_T^{(\epsilon,\delta)} - K)]$, and $\Lambda_2^\epsilon = \mathbf{E}^Q[Z_1^\epsilon I(S_T^{(\epsilon,\delta)} - K)]$, where $I(\cdot)$ is the indicator function of $[0,+\infty)$. By recalling the definition of $Z_T^{(\epsilon,\delta)}$ and using the inequality (6.102), Λ_1 is represented as

$$\begin{aligned} \Lambda_1 &= \mathbf{E}^Q\left[\int_{x \geq C_0 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})} S_0\exp\left(x - \frac{1}{2}\sigma(T)^2\right)\phi_{\sigma(T)^2}(x)dx\right] \\ & -K\exp\left(-\int_0^T r_t dt\right)\mathbf{E}^Q\left[\int_{x \geq C_0 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})} \phi_{\sigma(T)^2}(x)dx\right], \end{aligned} \tag{6.108}$$

where $\phi_{\sigma(T)^2}(x)$ is the density function of the normal random variable $X_{1T} = x$ with zero mean and the variance $\sigma(T)^2$. By transforming the random variable from x to

[9] We note that $\Sigma_{12}^{(\sigma)}(T)$ has been defined by (2.35), which is a deterministic function of several quantities involved. It is one of convenient consequences of the asymptotic expansion approach we have adopted in this paper.

$y_1 = [x - \sigma(T)^2]/\sigma(T)$ and $y_2 = x/\sigma(T)$, we rewrite

$$
\begin{aligned}
\Lambda_1 &= S_0 \mathbf{E}^Q \left[\int_{y_1 \geq \frac{1}{\sigma(T)}[C_0 - \sigma(T)^2 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})]} \phi(y_1) dy_1 \right] \\
&\quad - K \left[\exp\left(- \int_0^T r_t dt \right) \right] \mathbf{E}^Q \left[\int_{y_2 \geq \frac{1}{\sigma(T)}[C_0 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})]} \phi(y_2) dy_2 \right] .
\end{aligned}
\tag{6.109}
$$

By construction the random variables z_{iT} $(i = 1, 2, 3)$ are uncorrelated to the random variable X_{1T} so that they are uncorrelated with y_1 and y_2. Then by taking the expectations with respect to y_1 and y_2 and using the distribution function of the standard normal random variables $\Phi[\cdot]$, we have

$$
\begin{aligned}
\Lambda_1 &= \left[S_0 \Phi\left(\sigma(T) - \frac{C_0}{\sigma(T)} \right) - K \exp\left(- \int_0^T r_t dt \right) \Phi\left(- \frac{C_0}{\sigma(T)} \right) \right] \\
&\quad + \left[S_0 \phi\left(\sigma(T) - \frac{C_0}{\sigma(T)} \right) - K \exp\left(- \int_0^T r_t dt \right) \phi\left(- \frac{C_0}{\sigma(T)} \right) \right] \mathbf{E}^Q \left[C(\epsilon, \delta) \right] ,
\end{aligned}
$$

where $C(\epsilon, \delta) = [-\epsilon C_{12}^r - \delta C_{12}^\sigma + \epsilon z_{1T} + \delta(z_{3T} - z_{2T})]/\sigma(T)$. By using the notation $d_1 = \sigma(T) - C_0/\sigma(T)$,$d_2 = d_1 - \sigma(T)$, and rearranging each terms, we have

$$
\begin{aligned}
\Lambda_1 &= \left[S_0 \Phi(d_1) - K \exp\left(- \int_0^T r_t dt \right) \Phi(d_2) \right] \\
&\quad + \left[S_0 \phi(d_1) - K \exp\left(- \int_0^T r_t dt \right) \phi(d_2) \right] \times \mathbf{E}^Q \left[C(\epsilon, \delta) \right] .
\end{aligned}
\tag{6.110}
$$

Next, we try to evaluate the second term Λ_2^δ . By recalling the definition of Z_1^δ and using the inequality (6.103), it can be written by

$$
\begin{aligned}
\Lambda_2^\delta &= \mathbf{E}^Q \Bigg\{ \int_{x \geq C_0 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})} S_0 \exp\left(x - \frac{\sigma(T)^2}{2} \right) \\
&\quad \times \left[\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} \left(\frac{x^2}{\sigma^2(T)} - x - 1 \right) + \lambda_\sigma(T) \left(\frac{x}{\sigma(T)^2} - 1 \right) \right] \phi_{\sigma(T)^2}(x) dx \Bigg\} .
\end{aligned}
\tag{6.111}
$$

By tranforming from x to $y_1 = [x - \sigma(T)^2]/\sigma(T)$ and using the Gaussian property such as $\partial\phi(y_1)/\partial y_1 = -y_1 \phi(y_1)$ and $\partial(y_1 \phi(y_1))/\partial y_1 = (1 - y_1^2)\phi(y_1)$, we can express Λ_2^δ as

$$
\begin{aligned}
\Lambda_2^\delta &= \mathbf{E}^Q \Bigg\{ \int_{y_1 \geq \frac{1}{\sigma(T)}[C_0 - \sigma(T)^2 + \epsilon C_{12}^r + \delta C_{12}^\sigma - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})]} S_0 \\
&\quad \times \left[\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)} y_1 + \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} (y_1^2 - 1) + \frac{\lambda_\sigma(T)}{\sigma(T)} y_1 \right] \phi(y_1) dy_1 \Bigg\} \\
&= S_0 \mathbf{E}^Q \left\{ \left(\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)} + \frac{\lambda_\sigma(T)}{\sigma(T)} \right) \phi(y_*) - \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} y_* \phi(y_*) \right\} ,
\end{aligned}
\tag{6.112}
$$

where

$$
y_* = \frac{1}{\sigma(T)} [-C_0 + \sigma(T)^2 - \epsilon C_{12}^r - \delta C_{12}^\sigma + \epsilon z_{1T} + \delta(z_{3T} - z_{2T})] .
$$

Then by expanding the Gaussian density function and using the notation $d_1 = \sigma(T) - \frac{C_0}{\sigma(T)}$, we have

$$
\begin{aligned}
\Lambda_2^{\delta} &= S_0 \mathbf{E}^Q \Bigg\{ \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)} \left[\phi(d_1) + \phi^{'}(d_1) C(\epsilon) + \phi^{'}(d_1) C(\delta) \right] \\
&\quad - \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} \left[y_* [\phi(d_1) + \phi^{'}(d_1) C(\epsilon) + \phi^{'}(d_1) C(\delta)] \right] \Bigg\} , \qquad (6.113)
\end{aligned}
$$

where $C(\epsilon) = [-\epsilon(C_{12}^r - z_{1T})]/\sigma(T)$ and $C(\delta) = [-\delta(C_{12}^{\sigma} - (z_{3T} - z_{2T}))]/\sigma(T)$. By ignoring all the terms involving ϵ and δ because we are concerned with the order of $o(\epsilon, \delta)$, we have

$$
\Lambda_2^{\delta} = S_0 \Bigg\{ [\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)} + \frac{\lambda_{\sigma}(T)}{\sigma(T)}] \phi(d_1) - \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2} d_1 \phi(d_1) \Bigg\} + R_{10} , \qquad (6.114)
$$

where R_{10} is the remainder term of the order $o(\epsilon, \delta)$.
Now we shall evaluate the term Λ_2^{ϵ} . By using the definition of $Z_1^{(\epsilon,\delta)}$ and the inequality (6.102), we can express Λ_2^{ϵ} as

$$
\begin{aligned}
\Lambda_2^{\epsilon} &= K \exp\left(- \int_0^T r_t dt \right) \\
&\quad \times \mathbf{E}^Q \Bigg\{ \int_{x \geq C_0 + \epsilon C_{12}^r + \delta C_{12}^{\sigma} - \epsilon z_{1T} - \delta(z_{3T} - z_{2T})} [\frac{\Sigma_{12}^{(r)}(T)}{\Sigma_{11}(T)} x + \lambda_r(T)] \phi_{\sigma(T)^2}(x) dx \Bigg\} .
\end{aligned}
$$

By transforming $y_2 = x/\sigma(T)$ and using the Gaussian property, we can obtain

$$
\Lambda_2^{\epsilon} = K \exp\left(- \int_0^T r_t dt \right) \{ \frac{\Sigma_{12}^{(r)}(T)}{\Sigma_{11}(T)} \sigma(T) \mathbf{E}^Q [\phi(y^*)] + \lambda_r(T) \mathbf{E}^Q [\Phi(y^*)] \} ,
$$

where $y^* = [-C_0 - \epsilon C_{12}^r - \delta C_{12}^{\delta} + \epsilon z_{1T} + \delta(z_{3T} - z_{2T})]/\sigma(T)$. By using the notation $d_2 = -C_0/\sigma(T)$, we have

$$
\Lambda_2^{\epsilon} = K \exp\left(- \int_0^T r_t dt \right) \left[\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2} \sigma(T) \phi(d_2) + \lambda_r(T) \Phi(d_2) \right] + R_{11}, \qquad (6.115)
$$

where R_{11} is the remaining term of the order $o(\epsilon, \delta)$.

[**iii**] : As the third step, by taking the conditional expectations and ignoring higher order terms, the first term Λ_1 is given by

$$
\begin{aligned}
\Lambda_1 &= \left[S_0 \Phi(d_1) - K \exp\left(- \int_0^T r_t dt \right) \Phi(d_2) \right] \\
&\quad + \left[S_0 \phi(d_1) - K \exp\left(- \int_0^T r_t dt \right) \phi(d_2) \right] [-\epsilon \frac{C_{12}^r}{\sigma(T)} - \delta \frac{C_{12}^{\sigma}}{\sigma(T)}] , \qquad (6.116)
\end{aligned}
$$

where $d_2 = -C_0/\sigma(T) = d_1 - \sigma(T)$, and $\Phi(\cdot)$ and $\phi(\cdot)$ are the distribution function and the density function of the standard normal random variable, respectively. Then by collecting three terms and ignoring higher order terms we have the expression as

$$
\Lambda_1 + \delta \Lambda_2^{\delta} + \epsilon \Lambda_2^{\epsilon} = \left[S_0 \Phi(d_1) - K \exp\left(- \int_0^T r_t dt \right) \Phi(d_2) \right] + \epsilon \Psi^{\epsilon} + \delta \Psi^{\delta}, \qquad (6.117)
$$

where

$$\begin{aligned}\Psi^{\epsilon} &= -\frac{C_{12}^{r}}{\sigma_(T)}\left[S_0\phi(d_1) - K\exp\left(-\int_0^T r_t dt\right)\phi(d_2)\right] \\ &\quad + K\ \exp\left(-\int_0^T r_t dt\right)\left[\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)}\phi(d_2) + \lambda_r(T)\Phi(d_2)\right]\end{aligned}$$

and

$$\begin{aligned}\Psi^{\delta} &= -\frac{C_{12}^{\sigma}}{\sigma_(T)}\left[S_0\phi(d_1) - K\exp\left(-\int_0^T r_t dt\right)\phi(d_2)\right] \\ &\quad + S_0\left\{\left[\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)} + \frac{\lambda_\sigma(T)}{\sigma(T)^2}\right]\phi(d_1) - \frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2}\, d_1\phi(d_1)\right\}.\end{aligned}$$

After simple manipulations of three terms, we can derive the formula in Theorem 3.3. For the sake of completeness, we give a key lemma which makes our final results in compact forms. The proof is a result of direct calculations.

Lemma 6.2 *: With the notations we have used in this section and Section 3 we have the equality*

$$S_0\phi(d_1) - K\exp\left(-\int_0^T r_t dt\right)\phi(d_2) = 0\ . \tag{6.118}$$

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Table 1: Futures Price under Downward Stochastic Interest Rate and Upward Volatility

We set $r_0 = 0.11 > \bar{r} = 0.08$, $\kappa_r = 2.0$, $\epsilon = 0.1$ for the interest rate process and $\sigma_0 = 0.20 < \bar{\sigma} = 0.30$, $\kappa_\sigma = 4.0$, $\delta = 0.1$ for the volatility process. Furthermore, we assume $S_0 = 100$ and $T = 1.0$. F_0 denotes the futures price at initial time and f_0 represents the forward price and $F_0 = F_C + [F_D - F_C] + \epsilon\, F_r + o(\epsilon, \delta) = f_0 + \epsilon\, F_r + o(\epsilon, \delta)$, where F_C is the futures price under constant interest rate and volatility. F_D is the futures price under the deterministic interest rate and it is equal to f_0. ϵF_r captures the effect of randomness of the interest rate and it is equal to $F_0 - f_0$. Simul. represents the Monte Carlo simulation result for F_0 and Diff. is the difference between the simulation result and F_0. The values in parentheses represent $|Diff.|/Simul.$ as the percentage

	ρ_σ=ρ_r=-0.5	ρ_σ=-0.5,ρ_r=0.5	ρ_σ=0.5,ρ_r=-0.5	ρ_σ=ρ_r=0.5
Simul.	109.621	109.876	109.621	109.879
F_C	111.628	111.628	111.628	111.628
$F_D - F_C$	-1.885	-1.885	-1.885	-1.885
f_0	109.743	109.743	109.743	109.743
ϵF_r	-0.128	0.128	-0.128	0.128
F_0	109.615	109.871	109.615	109.871
$Diff.$	0.006	0.005	0.006	0.008
	(0.005%)	(0.007%)	(0.005%)	(0.007%)

Table 2: European Stock Call Option Value under Downward Stochastic Interest Rate and Upward Volatility: At-the-money Case

We set $r_0 = 0.11 > \bar{r} = 0.08$, $\kappa_r = 2.0$, $\epsilon = 0.1$ for the interest rate process and $\sigma_0 = 0.20 < \bar{\sigma} = 0.30$, $\kappa_\sigma = 4.0$, $\delta = 0.1$ for the volatility process. Furthermore, we assume $S_0 = K = 100$ and $T = 1.0$. BS is the original Black-Scholes value and BS_i $(i = D, r, \sigma)$ are the same as in (3.51). V_0 stands for the option value under the stochastic volatility and interest rate $BS + [BS_D - BS] + \epsilon BS_r + \delta BS_\sigma$. SR_{VC} is the option value under stochastic interest rate and constant volatility $BS + [BS_D - BS] + \epsilon BS_r$ when $\bar{\sigma} = \sigma_0$ and $\delta = 0$. SV_{RC} is the option value under constant interest rate and stochastic volatility $BS + [BS_D - BS] + \delta BS_\sigma$ when $\bar{r} = r_0$ and $\epsilon = 0$. SR_{VD} is the option value under the stochastic interest rate and deterministic volatility $BS + [BS_D - BS] + \epsilon BS_r$. SV_{RD} is the option value under the deterministic interest rate and stochastic volatility $BS + [BS_D - BS] + \epsilon BS_\sigma$. Simul. represents the Monte Carlo simulation result for V_0 and Diff. is the difference between simulation result and V_0.

	ρ_σ=ρ_r=-0.5	ρ_σ=-0.5,ρ_r=0.5	ρ_σ=0.5,ρ_r=-0.5	ρ_σ=ρ_r=0.5
Simul.	15.435	15.730	15.363	15.667
BS	13.868	13.868	13.868	13.868
BS_D-BS	1.667	1.667	1.667	1.667
ϵBS_r	-0.150	0.150	-0.150	0.150
δBS_σ	0.035	0.035	-0.035	-0.035
SR_{VC}	12.778	12.933	12.778	12.933
SV_{RC}	16.488	16.488	16.399	16.399
SR_{VD}	15.385	15.686	15.385	15.686
SV_{RD}	15.570	15.570	15.500	15.500
V_0	15.420	15.721	15.350	15.651
$Diff.$	0.015	0.009	0.013	0.016
	(0.097%)	(0.057%)	(0.085%)	(0.102%)

Table 3: European Stock Call Option Value under Downward Stochastic Interest Rate and Upward Volatility: In-the-money Case

We set $r_0 = 0.11 > \bar{r} = 0.08$, $\kappa_r = 2.0$, $\epsilon = 0.1$ for the interest rate process and $\sigma_0 = 0.20 < \bar{\sigma} = 0.30$, $\kappa_\sigma = 4.0$, $\delta = 0.1$ for the volatility process. Furthermore, we assume $S_0 = 110 > K = 100$ and $T = 1.0$. Other symbols are the same as those in Table 2.

	ρ_σ=ρ_r=-0.5	ρ_σ=-0.5,ρ_r=0.5	ρ_σ=0.5,ρ_r=-0.5	ρ_σ=ρ_r=0.5
Simul.	22.913	23.170	22.741	23.010
BS	21.984	21.984	21.984	21.984
BS_D-BS	0.961	0.961	0.961	0.961
ϵBS_r	-0.132	0.132	-0.132	0.132
δBS_σ	0.084	0.084	-0.084	-0.084
SR_{VC}	20.677	20.794	20.677	20.794
SV_{RC}	24.137	24.137	23.958	23.958
SR_{VD}	22.812	23.077	22.812	23.077
SV_{RD}	23.029	23.029	22.860	22.860
V_0	22.897	23.162	22.728	22.993
Diff.	0.016	0.008	0.013	0.017
	(0.070%)	(0.035%)	(0.057%)	(0.074%)

Table 4: European Stock Call Option Value under Downward Stochastic Interest Rate and Upward Volatility: Out-of-the-money Case

We set $r_0 = 0.11 > \bar{r} = 0.08$, $\kappa_r = 2.0$, $\epsilon = 0.1$ for the interest rate process and $\sigma_0 = 0.20 < \bar{\sigma} = 0.30$, $\kappa_\sigma = 4.0$, $\delta = 0.1$ for the volatility process. Furthermore, we assume $S_0 = 90 < K = 100$ and $T = 1.0$. Other symbols are the same as those in Table 2.

	ρ_σ=ρ_r=-0.5	ρ_σ=-0.5,ρ_r=0.5	ρ_σ=0.5,ρ_r=-0.5	ρ_σ=ρ_r=0.5
Simul.	9.241	9.541	9.309	9.609
BS	7.363	7.363	7.363	7.363
BS_D-BS	2.049	2.049	2.049	2.049
ϵBS_r	-0.151	0.151	-0.151	0.151
δBS_σ	-0.032	-0.032	0.032	0.032
SR_{VC}	6.584	6.749	6.584	6.749
SV_{RC}	10.066	10.066	10.109	10.109
SR_{VD}	9.261	9.562	9.261	9.562
SV_{RD}	9.379	9.379	9.444	9.444
V_0	9.228	9.530	9.293	9.595
Diff.	0.013	0.011	0.016	0.014
	(0.141%)	(0.115%)	(0.172%)	(0.146%)