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Plurality Mechanisms, Virtual Implementation, and Condorcet-Decisiveness

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Abstract

We investigate implementation of social choice functions as mappings from states to lotteries under complete information. We assume that for every agent, any pair of distinct states induces distinct preferences. A social choice function is called Condorcet-decisive if it always enforces the Condorcet winner among its range. We introduce plurality mechanisms, where each agent makes a single announcement and the lottery associated with the opinion announced by the largest number of agents is enforced. We show that a social choice function is virtually implementable via plurality mechanisms combined with constrained random dictatorship, if and only if it is Condorcet-decisive.

Keywords: Plurality Mechanisms, Constrained Random Dictatorship, Domain Restrictions, Pure Strategy Nash Equilibrium, Virtual Implementation, Condorcet-Decisiveness.

Journal of Economic Literature Classification Numbers: C72, D71, D78, H41.

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1. Introduction

This paper investigates implementation of a social choice function in pure strategy Nash equilibrium under complete information. A social choice function is defined as a mapping from states to lotteries. We restrict the set of states in that for every agent, any pair of distinct states induces distinct preferences. Associated with a social choice function, we introduce the *plurality mechanism*, in which each agent simultaneously makes a single announcement about the state, and the central planner enforces the lottery that the social choice function assigns to the opinion announced by the *largest* number of agents. We investigate how implementation works when we confine our attention to a *variant* of the plurality mechanism.

We must note that when there exist three or more agents, every inconstant social choice function is never implementable via the plurality mechanism, because every message profile making all agents announce the same but incorrect state is always a Nash equilibrium. Based on this, we define a *virtual* plurality mechanism as its modified version in ways that with a small probability, each agent is randomly selected as a *constrained dictator* and the planner chooses the lottery that maximizes this agent's utility among a restricted subset of lotteries when the state that she announces is correct.

A social choice function is said to be *virtually implementable* if at every state, for every $\varepsilon \in (0,1)$ close to zero, the honest message profile is the unique pure strategy Nash equilibrium in the game defined by the state and the virtual plurality mechanism, where the probability of the central planner following constrained random dictatorship is as small as ε. The purpose of the paper is to characterize the set of social choice functions that are virtually implementable even if virtual plurality mechanisms are only constructible.

A social choice function is said to be *Condorcet-decisive* if it always enforces the *Condorcet winner* among its range, i.e., if at every state, there exist a *majority* of agents who prefer the lottery that the social choice function assigns to this state to any other lottery in its range. The main results are as follows. *A social choice function is virtually implementable if it is Condorcet-decisive. When the number of agents is odd, a social choice function is virtually implementable only if it is Condorcet-decisive. Moreover, when the number of agents is even and the set of states is enough inclusive, a social choice function is virtually implementable only if it is Condorcet-decisive.*

Suppose that the central planner knows that there exists the Condorcet winner among the set of candidate lotteries, but she does not know which lottery is the correct Condorcet winner. Then, the social choice function assigning the Condorcet winner at every state is undoubtedly one of the most plausible from the normative viewpoints. Our results imply that this social choice function is acceptable not only from the normative viewpoints, but also from the positive viewpoints, because it is the only virtually implementable social choice function via virtual plurality mechanisms.

On the other hand, suppose that the central planner does not know whether there exists the Condorcet winner or not. Then, how to determine the social choice function would be problematic from the normative viewpoints, because the Condorcet criterion alone cannot solve it. Our results imply that it is problematic not only from the normative viewpoints, but also from the positive viewpoints, because every inconstant social choice function is never virtually implementable via virtual plurality mechanisms, and therefore, the central planner has to design more complicated and artificial mechanisms.

Several authors in the implementation literature have constructed their respective mechanisms where each agent has redundant, *slack* messages that she never announces as long as playing equilibrium behavior.[1] By announcing such a slack message, each agent could deviate from any unwanted message profile. As they have commonly indicated, it might be inevitable to make mechanisms so complicated and artificial in order to implement a wide variety of social choice functions. Such complexity and artificiality are, however, serious obstacles from the practical viewpoints.[2] In contrast, we confine our attention to *direct* mechanisms that are particularly simple and plausible in that each agent has *no* slack messages and the central planner follows such intuitive decision-making procedures as plurality and constrained dictatorship. And then, we prove that there still exist social choice functions that are not only virtually implementable, but also normatively quite appealing.[3]

The use of constrained random dictatorship was originated by Matsushima (1988) and cultivated by Abreu and Sen (1990). These works, together with Abreu and Matsushima (1992), showed that the combination of constrained random dictatorship with the addition of slack messages is a powerful way of eliminating unwanted equilibria.[4] In contrast, the present paper clarifies the possibility that the use of constrained random dictatorship alone could eliminate all unwanted equilibria without any assistance of slack messages, where the restriction on the set of states will play a

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[1] See, for example, the survey by Moore (1992) and its references.

[2] It has been discussed whether a social choice function is implementable when mechanisms have slack messages but are restricted to be finite or bounded. See Jackson (1992).

[3] The recent works by Matsushima (2002a, 2002b) investigated implementation where the central planner can design direct mechanisms that are *not* plurality-based.

[4] Abreu and Matsushima (1992) required the uniqueness of mixed strategy equilibria.

significant role.

We must distinguish the *positive* use of plurality to specify mechanisms with its *normative* use to specify social choice functions. It is well known in the public economics literature that a plurality-based social choice function is not necessarily acceptable, because it is sometimes inconsistent with the Condorcet criterion.[5] On the other hand, the paper shows that a plurality-based mechanism is a powerful and efficient way of finding out the Condorcet winner. Suppose that there exists the Condorcet winner at every state, that each agent plays honestly, and that the size of the set of states is given by K. Then, by $K-1$ times repeatedly operating pair-wise majority rules, the central planner can eventually find out the correct Condorcet winner. However, once the central planner specifies the procedure as a multi-stage game form, each agent may have incentive to behave dishonestly. Moreover, when K is big, it must take so long time to finalize the procedure.[6] In contrast to this, a virtual plurality mechanism has nice properties that each agent has incentive to behave honestly and a single round of direct revelation is enough for finding out the Condorcet winner.

The organization of the paper is as follows. Section 2 provides the model. Section 3 shows that Condorcet-decisiveness is sufficient for virtual implementation via virtual plurality mechanisms. Section 4 shows that it is necessary. Section 5 discusses alternative definitions of plurality mechanism.

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[5] See, for example, Mueller (1989).

[6] Herrero and Srivastava (1992) investigated implementation via multi-stage mechanisms by using the solution concept of backward induction, and many social choice functions, including Condorcet-decisive social choice functions, are implementable.

2. The Model

Let $N = \{1,...,n\}$, Ω, A and Δ denote the finite set of agents, the finite set of states, the finite set of pure alternatives and the set of lotteries over A, respectively, where $n \geq 2$. Agent $i's$ preference over lotteries is given by $u_i : \Delta \times \Omega \to R$ and satisfies the expected utility hypothesis. A social choice function is defined by a mapping from states to lotteries $f : \Omega \to \Delta$, whose range is denoted by $\Delta(f)$. We define a *direct mechanism* by (M, g) where $M_i = \Omega$ is the set of agent $i's$ messages, $M = \prod_{i \in N} M_i$, and $g : M \to \Delta$. The *honest message rule* for each agent i is defined by $\mu_i : \Omega \to \Omega$ where

$$\mu_i(\omega) = \omega \quad \text{for all } \omega \in \Omega .$$

We denote $\mu = (\mu_i)_{i \in N}$ and $\mu(\omega) = (\mu_i(\omega))_{i \in N}$. A direct mechanism (M, g) will be simply denoted by g. For every $\omega \in \Omega$, (g, ω) defines a game. A message profile $m \in M$ is said to be a (pure strategy) *Nash equilibrium* in (g, ω) if

$$u_i(g(m), \omega) \geq u_i(g(m / m_i'), \omega) \quad \text{for all } i \in N \text{ and all } m_i' \in M_i .$$

The domain of f is restricted as follows. We assume that *for every agent* $i \in N$, *any pair of distinct states induces distinct preference orderings over* Δ, i.e., for every $\omega \in \Omega$, and every $\omega' \in \Omega / \{\omega\}$, there exist $\alpha \in \Delta$ and $\alpha' \in \Delta / \{\alpha\}$ such that

$$u_i(\alpha, \omega) > u_i(\alpha', \omega) \quad \text{and} \quad u_i(\alpha, \omega') < u_i(\alpha', \omega') .$$

Hence, by knowing a single agent's preference ordering, the central planner can know all other agents' preference orderings. We also assume that for every agent $i \in N$, neither state induces complete indifference over all lotteries, i.e., for every $\omega \in \Omega$, there exist $a \in A$ and $a' \in A / \{a\}$ such that $u_i(a, \omega) \neq u_i(a', \omega)$.

For every $\omega \in \Omega$, and every $k \in \{1,...,|A|\}$, let $\gamma_i(\omega, k) \in A$ denote the pure alternative that agent i prefers in the k-th place among A. We define $l_i : \Omega \to \Delta$ by

$$l_i(\omega)(\gamma_i(\omega, k)) \equiv \frac{|A| - k + 1}{\sum_{\rho=1}^{|A|} \rho} \quad \text{for all } \omega \in \Omega \text{ and all } k \in \{1,...,|A|\}.$$

Since $l_i(\omega)(\gamma_i(\omega, k))$ is decreasing with respect to $k \in \{1,...,|A|\}$, it follows from the assumptions that for every $\omega \in \Omega$, $\alpha = l_i(\omega)$ maximizes $u_i(\alpha, \omega)$ among the range of l_i, i.e.,

$$u_i(l_i(\omega), \omega) > u_i(l_i(\omega'), \omega) \quad \text{for all } \omega \in \Omega \text{ and all } \omega' \in \Omega / \{\omega\} .$$

We define the *random-dictator mechanism* $y : M^n \to \Delta$ by

$$y(m) = \frac{1}{n}\sum_{i \in N} l_i(m_i).$$

According to y, with probability $\frac{1}{n}$, each agent is selected as a constrained dictator and can choose any lottery in the range of l_i, where she can maximize her utility by telling the truth.

For every $m \in M$, and every $\omega \in \Omega$, let $n(m,\omega) \in \{0,...,n\}$ denote the number of agents $i \in N$ who announce $m_i = \omega$, i.e., tell the truth. We define the *plurality mechanism* $z : M^n \to \Delta$ in ways that for every $i \in N$, and every $m \in M$,

$$z(m) = f(m_i) \text{ if } n(m,m_i) \geq n(m,m_j) \text{ for all } j \in N \text{ and}$$

$$n(m,m_i) > n(m,m_j) \text{ for all } j < i.$$

Note that when there are multiple plurality opinions, the opinion that is announced by the lowest agent is selected among them. Section 5 will discuss an alternative definition of plurality mechanism in this respect.

For every $\varepsilon \in (0,1)$, we specify a direct mechanism g^ε as a combination of the plurality mechanism and the random-dictator mechanism in that for every $m \in M$,

$$g^\varepsilon(m) = (1-\varepsilon)z(m) + \varepsilon y(m).$$

Note that for every $\omega \in \Omega$,

$$g^\varepsilon(\mu(\omega)) = (1-\varepsilon) f(\omega) + \varepsilon y(m),$$

and therefore, $g^\varepsilon(\mu(\omega))$ is $\varepsilon -$ *close to* $f(\omega)$ in that

$$\left| g^\varepsilon(\mu(\omega))(a) - f(\omega)(a) \right| \leq \varepsilon \text{ for all } a \in A.$$

A social choice function f is said to be *virtually implementable* if for every $\varepsilon \in (0,1)$, and every $\omega \in \Omega$, the honest message profile $\mu(\omega)$ is the unique Nash equilibrium in (g^ε, ω).

3. Sufficiency

A lottery $\alpha \in \Delta$ is said to be the *Condorcet winner among* a subset of lotteries $\Psi \subset \Delta$ at the state $\omega \in \Omega$, if $\alpha \in \Psi$, and for every $\alpha' \in \Psi / \{\alpha\}$,

$$\#\{i \in N : u_i(\alpha,\omega) \geq u_i(\alpha',\omega)\} \geq \frac{n+1}{2}. \tag{1}$$

Inequalities (1) imply that for every other lottery $\alpha' \in \Psi / \{\alpha\}$, there exist a majority of agents who weakly prefer α to α'. A social choice function f is said to be *Condorcet-decisive* if for every $\omega \in \Omega$, $f(\omega)$ is the Condorcet winner among $\Delta(f)$ at ω.

The following proposition states that Condorcet-decisiveness is sufficient for virtual implementation.

Proposition 1: *A Condorcet-decisive social choice function f is virtually implementable.*

Proof: Fix $\varepsilon \in (0,1)$ and $\omega \in \Omega$ arbitrarily. We will show that $\mu(\omega)$ is a Nash equilibrium in $(g^{\varepsilon}, \omega)$. Suppose $n \geq 3$. Then, for every $i \in N / \{1\}$, and every $m_i \neq \omega$, it follows that $z(\mu(\omega)/m_i) = z(\mu(\omega)) = f(\omega)$, and therefore,

$$u_i(g^{\varepsilon}(\mu(\omega)),\omega) - u_i(g^{\varepsilon}(\mu(\omega)/m_i)),\omega) = \frac{\varepsilon}{n}\{u_i(l_i(\omega),\omega) - u_i(l_i(m_i),\omega)\} > 0.$$

Suppose $n = 2$. Then, for every $i \in N / \{1\}$, and every $m_i \neq \omega$, it follows that either $z(\mu(\omega)/m_i) = z(\mu(\omega)) = f(\omega)$ or $z(\mu(\omega)/m_i) = f(m_i)$. Inequalities (1) and $n = 2$ imply that every agent $i \in N$ weakly prefers $f(\omega)$ to $f(m_i)$, and therefore,

$$u_i(g^{\varepsilon}(\mu(\omega)),\omega) - u_i(g^{\varepsilon}(\mu(\omega)/m_i)),\omega) > 0 \text{ for all } i \in N.$$

Hence, it follows that $\mu(\omega)$ is a Nash equilibrium in $(g^{\varepsilon}, \omega)$.

Let $\varepsilon \in (0,1)$ be sufficiently close to zero. Fix $m \neq \mu(\omega)$ arbitrarily. Note that there exists $i \in N$ such that $n(m, m_i) \geq n(m, m_j)$ for all $j \in N$, and $n(m, m_i) > n(m, m_j)$ for all $j < i$. Note that $z(m) = f(m_i)$. We will show that m is not a Nash equilibrium in $(g^{\varepsilon}, \omega)$.

Suppose $m_i = \omega$. For every $j \in N$, if $m_j \neq \omega$, then it follows that

$z(m/\mu_j(\omega)) = z(m) = f(\omega)$, and therefore, $u_j(g^\varepsilon(m/\mu_j(\omega))),\omega) - u_j(g^\varepsilon(m),\omega) > 0$. This implies that m is not a Nash equilibrium in (g^ε,ω).

Suppose that $m_i \neq \omega$, and m is a Nash equilibrium in (g^ε,ω). Fix $j \in N$ arbitrarily. If $m_j = m_i \neq \omega$, then, it must hold that

$$u_j(g^\varepsilon(m/\mu_j(\omega))),\omega) - u_j(g^\varepsilon(m),\omega) \leq 0,$$

and therefore,

$$u_j(z(m/\mu_j(\omega)),\omega) < u_j(f(m_i),\omega), \text{ and} \tag{2}$$

$$z(m/\mu_j(\omega)) \neq z(m) = f(m_i), \tag{3}$$

because $u_j(l_j(\omega),\omega) - u_j(l_j(m_i),\omega) > 0$. If $m_j \notin \{m_i,\omega\}$, then it must hold that $z(m/\mu_j(\omega)) \in \{f(\omega), f(m_i)\}$, and

$$u_j(g^\varepsilon(m/\mu_j(\omega))),\omega) - u_j(g^\varepsilon(m),\omega) \leq 0,$$

and therefore,

$$u_j(f(\omega),\omega) < u_j(f(m_i),\omega), \text{ and} \tag{4}$$

$$z(m/\mu_j(\omega)) = f(\omega), \tag{5}$$

because $u_j(l_j(\omega),\omega) - u_j(l_j(m_i),\omega) > 0$. Moreover, inequality (3) and equality (5) imply that if $m_j = m_i$, then it must hold that $z(m/\mu_j(\omega)) = f(\omega)$, and therefore,

$$u_j(f(\omega),\omega) < u_j(f(m_i),\omega),$$

because of inequality (2). However, inequalities (4) and $n(m,\omega) < \frac{n+1}{2}$ imply that there exists $j \in N$ such that $m_j = m_i$ and $u_j(f(\omega),\omega) \geq u_j(f(m_i),\omega)$. This is a contradiction.

From the above arguments, we have proved that no message profile $m \neq \mu(\omega)$ is a Nash equilibrium in $(g^{\varepsilon}, \omega)$.

Q.E.D.

4. Necessity

The following proposition states that when the number of agents is odd, Condorcet-decisiveness is necessary for virtual implementation.

Proposition 2: *If* n *is odd and a social choice function* f *is virtually implementable, then it is Condorcet-decisive.*

Proof: Suppose that f is not Condorcet-decisive. Then, there exist $\omega \in \Omega$ and $\omega' \in \Omega / \{\omega\}$ such that

$$\#\{i \in N : u_i(f(\omega'),\omega) > u_i(f(\omega),\omega)\} \geq \frac{n+1}{2}.$$

Let $m \in M$ be the message profile satisfying that $m_i \in \{\omega, \omega'\}$ for all $i \in N$, $n(m,\omega) = \frac{n-1}{2}$, $n(m,\omega') = \frac{n+1}{2}$, and

$$u_i(f(\omega'),\omega) > u_i(f(\omega),\omega) \text{ if } m_i = \omega'. \quad (6)$$

Note that $z(m) = f(\omega') \neq f(\omega)$. For every $i \in N$, if $m_i = \omega$, then, for every $m_i' \in M_i$, it follows that $z(m/m_i') = z(m) = f(\omega')$, and therefore,

$$u_i(g^\varepsilon(m/m_i')),\omega) - u_i(g^\varepsilon(m),\omega) = \frac{\varepsilon}{n}\{u_i(l_i(m_i'),\omega) - u_i(l_i(\omega),\omega)\} < 0.$$

Suppose that there exist no $i \in N$ and no $\hat{m}_i \notin \{\omega, \omega'\}$ such that $m_i = \omega'$, $z(m/\hat{m}_i) = z(m) = f(\omega')$, and $u_i(l_i(\hat{m}_i),\omega) - u_i(l_i(\omega'),\omega) > 0$. For every $i \in N$, if $m_i = \omega'$, then it follows that $z(m/\mu_i(\omega)) = f(\omega)$, $z(m/m_i') \in \{f(\omega), f(\omega')\}$ for all $m_i \notin \{\omega, \omega'\}$, and therefore,

$$u_i(g^\varepsilon(m/m_i')),\omega) - u_i(g^\varepsilon(m),\omega) \leq 0,$$

because of inequality (6). Hence, m is a Nash equilibrium in (g^ε, ω).

Suppose that there exist such $i \in N$ and $\hat{m}_i \notin \{\omega, \omega'\}$. Without loss of generality, we can choose $\hat{m}_i \notin \{\omega, \omega'\}$ satisfying that

$$u_i(l_i(\hat{m}_i),\omega) - u_i(l_i(\omega''),\omega) \geq 0 \text{ for all } \omega'' \neq \omega. \quad (7)$$

We will show that $m/\hat{m}_i$ is a Nash equilibrium in (g^ε, ω). Note that $z(m/\hat{m}_i) = f(\omega')$, and for every $j \in N$,

$$z(m/(\hat{m}_i, m_j')) = z(m/\hat{m}_i) = f(\omega') \text{ for all } m_j' \in M_j \text{ if } m_j' = \omega,$$

and

$$z(m/(\hat{m}_i, m'_j)) = f(\omega) \text{ for all } m'_j \neq \omega' \text{ if } m_j = \omega'.$$

Hence, it follows from inequality (6) that

$$u_j(g^\varepsilon(m/m'_j)),\omega) - u_j(g^\varepsilon(m),\omega) \leq 0 \text{ for all } j \neq i \text{ and all } m'_j \in M_j.$$

The supposition implies that for every $m'_i \neq \omega$, $z(m/m'_i) = z(m/\hat{m}_i) = f(\omega')$, and therefore,

$$u_i(g^\varepsilon(m/m'_i),\omega) - u_i(g^\varepsilon(m/\hat{m}_i),\omega)$$

$$= \frac{\varepsilon}{n}\{u_i(l_i(m'_i),\omega) - u_i(l_i(\hat{m}_i),\omega)\} \leq 0,$$

because of inequality (7). Moreover, note that $z(m/\mu_i(\omega)) = f(\omega)$, and therefore,

$$u_i(g^\varepsilon(m/\mu_i(\omega)),\omega) - u_i(g^\varepsilon(m/\hat{m}_i),\omega)$$

$$= (1-\varepsilon)\{u_i(f(\omega),\omega) - u_i(f(m_i),\omega)\}$$

$$+ \frac{\varepsilon}{n}\{u_i(l_i(m'_i),\omega) - u_i(l_i(m_i),\omega)\} \leq 0,$$

because $\varepsilon \in (0,1)$ is sufficiently close to zero and $u_i(f(\omega),\omega) - u_i(f(m_i),\omega) < 0$. From the above arguments, we have proved that $m/\hat{m}_i$ is a Nash equilibrium in (g^ε, ω).

Q.E.D.

The following proposition states that when the set of states is enough inclusive, Condorcet-decisiveness is necessary for virtual implementation even if the number of agents is even.

Proposition 3: *A social choice function f is not virtually implementable if n is even and there exist $\omega \in \Omega$ and $\omega' \in \Omega/\{\omega\}$ such that*

$$\#\{i \in N : u_i(f(\omega'),\omega) > u_i(f(\omega),\omega)\} \geq \frac{n+1}{2},$$

and

$$u_1(f(\omega'),\omega) < u_1(f(\omega),\omega).$$

Proof: Let $\varepsilon \in (0,1)$ be sufficiently close to zero. Let $m \in M$ be the message profile satisfying that $m_i \in \{\omega, \omega'\}$ for all $i \in N$, $n(m,\omega) = \frac{n}{2}$, $n(m,\omega') = \frac{n}{2}$, and

$$u_i(f(\omega'),\omega) > u_i(f(\omega),\omega) \quad \text{if} \quad m_i = \omega'. \tag{8}$$

Note that $m_1 = \omega'$, and therefore, $z(m) = f(\omega')$. For every $i \in N$, if $m_i = \omega$, then, for every $m_i' \in M_i$, it follows that $z(m/m_i') = z(m) = f(\omega')$, and therefore,

$$u_i(g^\varepsilon(m/m_i'),\omega) - u_i(g^\varepsilon(m),\omega) = \frac{\varepsilon}{n}\{u_i(l_i(m_i'),\omega) - u_i(l_i(\omega),\omega)\} < 0.$$

Suppose that $n \geq 4$. For every $i \in N$, if $m_i = \omega'$, then, for every $m_i' \in M_i/\{\omega'\}$, it follows that $z(m/m_i') = f(\omega)$, and therefore,

$$u_i(g^\varepsilon(m/m_i'),\omega) - u_i(g^\varepsilon(m),\omega)$$

$$= (1-\varepsilon)\{u_i(f(\omega),\omega) - u_i(f(m_i),\omega)\}$$

$$+ \frac{\varepsilon}{n}\{u_i(l_i(m_i'),\omega) - u_i(l_i(m_i),\omega)\} \leq 0,$$

because ε is sufficiently close to zero and inequality (8) holds.

Suppose that $n = 2$. We can choose $\omega' \in \Omega/\{\omega\}$ satisfying that

$$(1-\varepsilon)\{u_1(f(\omega'),\omega) - u_1(f(\omega''),\omega)\}$$

$$+ \frac{\varepsilon}{n}\{u_1(l_1(\omega'),\omega) - u_1(l_1(\omega''),\omega)\} \geq 0 \quad \text{for all} \quad \omega'' \in \Omega,$$

because ε is sufficiently close to zero. For every $m_1' \in M_1$, it follows that $z(m/m_1') = f(m_1')$, and therefore,

$$u_1(g^\varepsilon(m/m_1'),\omega) - u_1(g^\varepsilon(m),\omega) = (1-\varepsilon)\{u_1(f(m_1'),\omega) - u_1(f(\omega'),\omega)\}$$

$$+ \frac{\varepsilon}{n}\{u_1(l_1(m_1'),\omega) - u_1(l_1(\omega'),\omega)\} \geq 0.$$

From the above arguments, we have proved that m is a Nash equilibrium in (g^ε,ω).

Q.E.D.

5. Alternative Definition

The paper has defined the plurality mechanism in ways that when there exist multiple plurality opinions, the opinion that is announced by the lowest agent is selected among them. The positive result of Proposition 1 crucially depends on this deterministic selection. An alternative definition is that the selection is *stochastic* in that when the number of the plurality opinions is as large as $r \in \{1,..., \min[|\Omega|, n]\}$, each of the plurality opinions is randomly selected with positive probability $\frac{1}{r}$. Based on this, the plurality mechanism z is redefined by

$$z(m) = \frac{1}{|\Omega(m)|} \sum_{\omega \in \Omega(m)} f(\omega) \quad \text{for all} \quad m \in M ,$$

where $\Omega(m) \subset \Omega$ denotes the set of all plurality opinions associated with the message profile m, i.e., the set of states ω satisfying that

$$n(m,\omega) \geq n(m,\omega') \quad \text{for all} \quad \omega' \in \Omega .$$

Virtual plurality mechanisms are also redefined on the basis of this redefined plurality mechanism.

We can show that even Condorcet-decisive social choice functions are not necessarily virtually implementable. Suppose that $n \geq 4$, n is even, and a social choice function f satisfies that there exist $\omega \in \Omega$, $\omega' \in \Omega$ and $\omega'' \in \Omega / \{\omega'\}$ such that

$$\#\{i \in N : u_i(f(\omega'),\omega) > u_i(f(\omega''),\omega)\} = \frac{n}{2},$$

and

$$\#\{i \in N : u_i(f(\omega''),\omega) > u_i(f(\omega'),\omega)\} = \frac{n}{2}.$$

Let $m \in M$ be the message profile satisfying that for every $i \in N$,

$$m_i = \omega' \quad \text{if} \quad u_i(f(\omega'),\omega) > u_i(f(\omega''),\omega),$$

and

$$m_i = \omega'' \quad \text{if} \quad u_i(f(\omega''),\omega) > u_i(f(\omega'),\omega).$$

Note that

$$z(m) = \frac{1}{2}\{f(\omega') + f(\omega'')\} \neq f(\omega).$$

Suppose that ε is sufficiently close to zero. For every $i \in N$, and every $m_i' \in M_i / \{m_i\}$, since

$$z(m/m_i') = f(\omega'') \ \text{ if } \ m_i = \omega',$$

and

$$z(m/m_i') = f(\omega') \ \text{ if } \ m_i = \omega'',$$

it follows that

$$u_i(g^{\varepsilon}(m/m_i')),\omega) - u_i(g^{\varepsilon}(m),\omega) < 0,$$

which implies that m is a Nash equilibrium in (g^{ε},ω). Hence, it follows that the social choice function f is not virtually implementable via redefined virtual plurality mechanisms.

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