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Mechanism Design with Side Payments: Individual Rationality and Iterative Dominance[+]

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Abstract

We investigate the collective decision problem with incomplete information and side payments. We show that for generic prior distributions, there exists a direct mechanism associated with the social choice function that satisfies budget balancing, incentive compatibility, and interim individual rationality. We consider the possibility of a risk-averse principal's extracting the full surplus in agency problems with adverse selection. We also show that for generic prior distributions, there exists a modified direct mechanism associated with the virtual social choice function, which satisfies budget balancing and interim individual rationality, such that truth telling is the unique triple iteratively undominated message rule profile.

Journal of Economic Literature Classification Numbers: C70, D44, D60, D71, D78, D82

Key Words: Incentive Compatibility, Budget Balancing, Interim Individual Rationality, Iterative Dominance, Full Surplus Extraction, Risk-Averse Principal.

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1. Introduction

This paper investigates the collective decision problem with incomplete information where players' utilities are quasi-linear. Each player receives her private signal, and makes a public announcement about what is her private signal. Based on players' announcements, and according to the mechanism that players have constructed and agreed to enforce in advance, players collectively choose an alternative and make budget balancing side payments. Players' utilities may depend not only on her private signal but also on the other players' private signals, i.e., the private signal structure may satisfy the interdependent value assumption. Moreover, their private signals may have information about payoff-irrelevant factors such as their interim outside values.

Since players' private signals are not contractible, each player may misrepresent her private signal without being punished for lying. Moreover, after receiving her private signal implying that her interim outside value is higher than the interim expected utility that she can obtain in the collective decision, each player may have incentive not to participate in the collective decision. Hence, it would be very important for players to construct a mechanism with budget balancing that satisfies incentive compatibility in that truth telling is a Bayesian Nash equilibrium, and also satisfies interim individual rationality in that by participating in the collective decision after receiving her private signal, each player can always obtain the interim expected utility that is larger than or equals her interim outside value.[1] More ideally, truth telling should be described by the unique equilibrium behavior, and we should reach the uniqueness by operating only a small number of iterative removals of undominated strategies.[2]

The first purpose of this paper is to show a sufficient condition on the prior distribution, under which, irrespective of the social choice function, there exists a direct mechanism that satisfies budget balancing, incentive compatibility, and interim individual rationality. We show that such a direct mechanism exists for generic prior distributions, when there exist three or more players, their private signals are correlated, and the size of the set of each player's private signals is approximately set.

Myerson and Satterthwaite (1983) showed an impossibility result that in a model of bilateral trading with independent private signals, private values, and zero interim outside values, there exist no such mechanisms. In contrast, this paper shows the possibility result when the number of players is three or more and the private signals are correlated. Fudenberg, Levine, and Maskin (1995) showed that with three or more players and correlated signals, there exists a mechanism satisfying incentive compatibility, interim individual rationality, and a weaker version of budget balancing

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1 There exist many works on mechanism design with side payments, showing their respective existence theorems of budget balancing mechanisms with incentive compatibility that do not necessarily satisfy interim individual rationality. See D'Aspremont and Gerard-Varet (1979), Arrow (1979), D'Aspremont, Crèmer, and Gerard-Varet (1990, 2002), Crèmer and Riordan (1985), Rob (1989), Aoyagi (1998), Chung (1999), and others.

2 There exist a huge volume of works on unique implementation with incomplete information. See the survey by Palfrey (1992). Most of the works assumed that no side payments or only small side payments are allowed, and therefore, required a social choice function to be incentive compatible with no side payments as being necessary for its implementability. On the other hand, this paper allows large side payments, and does not require incentive compatibility of the social choice function without side payments.

that the expected value of the sum of side payments is non-positive. In contrast, this paper fully requires budget balancing that the sum of side payments is always zero.

As a corollary of the above result, we show that a player can extract the full surplus even with the constraints of incentive compatibility, interim individual rationality, and budget balancing. Crèmer and McLean (1985, 1988) showed that a principal can extract the full surplus in a multi-agent problem, where there exist three or more agents who receive their private signal, but the principal receive no private signals. In contrast, this paper shows that the full surplus extraction is possible even when all players including the principal receive their private signals. McAfee and Reny (1992) showed that in the model of bilateral trading a la Myerson and Satterthwaite, with continuums of signals and correlated signals, there exists a mechanism with budget balancing and incentive compatibility such that the seller can virtually extract the full surplus. Here, the maximal value of side payments diverges infinity as the agents' total rent approaches to zero. In contrast, this paper shows that the full surplus extraction is exactly possible even with bounded transfers.

As an application, we consider agency problems in which a risk-averse uninformed principal hires multiple risk-neutral agents with private information. We show that either with two agents or with independent private signals, it might be impossible for the principal to extract the full surplus. On the other hand, we show that with three or more agents and correlated signals, it is generically possible.

Although most previous works on agency problems with adverse selection concentrated on the case of risk-neutral principals, the study of risk-averse principals might have high potential ability to explain real economic phenomena. As its example, we consider auctions with a risk-averse seller and multiple risk-neutral buyers, showing the possibility of the seller extracting the full surplus without harming incentive compatibility and interim individual rationality of the buyers.

The second purpose of this paper is to show sufficient conditions under which there exist a mechanism with budget balancing, incentive compatibility, and interim individual rationality such that truth telling is the unique iteratively undominated strategy profile, where we need only two or three rounds of iterative removal of undominated strategies. In particular, with private values and with four or more players, even if we restrict our attention to direct mechanisms, unique implementation in terms of twice iterative dominance might be generically possible. Moreover, we show that even without any substantial restriction on utility functions such as private values, unique and virtual implementation in terms of triple iterative dominance may be generically possible by using slightly modified direct mechanisms. That is, unique and virtual implementation is generically possible when players' interim preferences are not common knowledge.

Abreu and Matsushima (1992) showed the possibility of uniquely and virtually implementing social choice functions in terms of iterative dominance. Abreu and Matsushima used only small side payments, constructed mechanisms that are much more complicated than direct mechanisms, and needed so many rounds of iterative removal in order to reach the unique strategy profile. Several experimental reports, however, indicate that real individuals are boundedly rational so that they may stop calculating the unique profile after practicing only second or third iterative removals. Hence, the use of only a small number of iterative removals would be an important restriction in practice. From the above viewpoint of bounded rationality, the present paper constructed only simple mechanisms and uses only two or three rounds of iterative removal in order to reach the unique profile.

The organization of this paper is as follows. Section 2 defines the model. Section 3 investigates direct mechanisms satisfying budget balancing, incentive compatibility, and interim individual rationality. Both Sections 4 and 5 investigate the possibility of the full surplus extraction. Section 4 considers the agency problem with a risk-averse principal and multiple risk neutral agents. Section 5 considers the partnership problem with risk-neutral players. Section 6 shows a sufficient condition on the prior distribution under which the existence of mechanisms with interim individual rationality is trivial. Section 7 investigates the possibility of uniquely implementing the social choice function via direct mechanisms in terms of iterative dominance. Sections 8 and 9 investigate the possibility of uniquely and virtually implementing the social choice function in terms of iterative dominance. Section 10 concludes.

2. The Model

Let $N \equiv \{1,2,...,n\}$ denote the finite set of *players*. We will assume that $n \geq 3$ except in Proposition 1 of this section, in Proposition 3 of Section 3, and in the first part of Proposition 4 of Section 4. Each player $i \in N$ receives her *private signal* ω_i, and Ω_i denotes the *finite* set of private signals for player i. Let $\Omega \equiv \prod_{i \in N} \Omega_i$, $\Omega_{-i} \equiv \prod_{j \in N/\{i\}} \Omega_j$, and $\Omega_{-i-j} \equiv \prod_{h \in N/\{i,j\}} \Omega_h$. A private signal profile $\omega \equiv (\omega_i)_{i \in N} \in \Omega$ is randomly drawn according to a *common* prior distribution $p : \Omega \to [0,1]$. We assume that p has *full* support, i.e., $p(\omega) > 0$ for all $\omega \in \Omega$. The conditional probabilities are denoted by

$$p_i(\omega_i) \equiv \sum_{\omega_{-i} \in \Omega_{-i}} p(\omega),$$

$$p_i(\omega_{-i} | \omega_i) \equiv \frac{p(\omega)}{p_i(\omega_i)},$$

$$p_i^j(\omega_j | \omega_i) \equiv \sum_{\omega_{-i-j} \in \Omega_{-i-j}} p_i(\omega_{-i} | \omega_i),$$

$$p_{ij}(\omega_{-i-j} | \omega_i) \equiv \sum_{\omega_j \in \Omega_j} p_i(\omega_{-i} | \omega_i),$$

and

$$p_{ij}(\omega_{-i-j} | \omega_i, \omega_j) \equiv \frac{p_i(\omega_{-i} | \omega_i)}{p_i^j(\omega_j | \omega_i)},$$

where $\omega_{-i} \equiv (\omega_j)_{j \in N/\{i\}} \in \Omega_{-i}$ and $\omega_{-i-j} \equiv (\omega_h)_{h \in N/\{i,j\}} \in \Omega_{-i-j}$. For every subset $D_j \subset \Omega_j$, we denote

$$p_i^j(D_j | \omega_i) \equiv \sum_{\omega_j \in D_j} p_i^j(\omega_j | \omega_i).$$

Let A denote the set of *alternatives*. Let Δ denote the set of *simple lotteries* on A. We assume that each player $i's$ utility is *quasi-linear* in that when the private signal profile and the alternative are $\omega \in \Omega$ and $a \in A$ respectively and player i receives the side payment $t_i \in R$, her utility is given by

$$u_i(a, \omega) + t_i.$$

We assume the *expected utility* hypothesis. For every simple lottery $\alpha \in \Delta$, we will write $u_i(\alpha, \omega) = \sum_{a \in \Gamma} u_i(a, \omega)\alpha(a)$, where Γ is the support of the lottery α that is countable. A *social choice function* is defined by $f : \Omega \to A$, where $f(\omega)$ is regarded as the alternative that is desirable when the private signal profile is $\omega \in \Omega$.

For every $i \in N$, we denote by M_i the set of messages for player i. Let $M \equiv \prod_{i \in N} M_i$. A *message rule for player* i is defined as a function $\phi_i : \Omega_i \to M_i$. Let Φ_i denote the set of message rules for player i. Let $\Phi \equiv \prod_{i \in N} \Phi_i$ and $\phi \equiv (\phi_i)_{i \in N} \in \Phi$.

Fix the set of message profiles M arbitrarily. A *mechanism* is defined by (g, x), where $g: M \to \Delta$ is an outcome function, $x_i : M \to R$ is a side payment function for player i, and $x = (x_i)_{i\in N}$ is a side payment function that is *budget balancing* in that

$$\sum_{i\in N} x_i(\omega) = 0 \text{ for all } \omega \in \Omega .$$

When all players announce a message profile $m = (m_i)_{i\in N} \in M$, the resultant lottery and side payment for each player i are $g(m)$ and $x_i(m)$ respectively. When $g(m)$ is a degenerate lottery, it will be regarded as a pure alternative.

Fix a positive integer k arbitrarily. Let $\Phi_i^0 = \Phi_i$. For every positive integer h, we define $\Phi_i^h \subset \Phi_i$ as the set of message rules for player i, $\phi_i \in \Phi_i^{h-1}$, satisfying that there exists no $\phi_i' \in \Phi_i^{h-1} / \{\phi_i\}$ such that for every $\phi_{-i} \in \Phi_{-i}^{h-1}$,

$$\sum_{\omega\in\Omega} \{u_i(g(\phi_{-i}(\omega_{-i}), \phi_i'(\omega_i)), \omega) + x_i(\phi_{-i}(\omega_{-i}), \phi_i'(\omega_i))\} p(\omega)$$
$$> \sum_{\omega\in\Omega} \{u_i(g(\phi(\omega)), \omega) + x_i(\phi(\omega))\} p(\omega) .$$

A message rule profile $\phi \in \Phi$ is said to be *k times iteratively undominated* in (p, g, x) if $\phi \in \Phi^k$. We will use the concepts of iterative dominance only in the latter part of this paper, i.e., in Sections 6, 7, 8, and 9, where we will consider only the case of $k \in \{2,3\}$. In all sections except Sections 6, 8, and 9, we consider only *direct mechanisms* where

$$M_i = \Omega_i \text{ for all } i \in N ,$$

and

$$g = f .$$

Hence, a direct mechanism is denoted by (f, x), where

$$x_i : \Omega \to R \text{ for all } i \in N .$$

We denote by $\phi^* \in \Phi$ the *honest message rule profile* in a direct mechanism where

$$\phi^*(\omega) = \omega \text{ for all } \omega \in \Omega .$$

For every $i \in N$, and every $\omega_i \in \Omega_i$, we denote by $U_i^*(\omega_i) \in R$ the *interim outside value* that player i can obtain when she observes her private signal ω_i and decides not to participate in the collective decision. We introduce the following requirement on x.

Interim Individual Rationality (IIR): For every $i \in N$, and every $\omega_i \in \Omega_i$,

$$\sum_{\omega_{-i}\in\Omega_{-i}} \{u_i(f(\omega)), \omega) + x_i(\omega)\} p_i(\omega_{-i} \mid \omega_i) \geq U_i^*(\omega_i) .$$

IIR requires that when all players announce messages honestly in the direct mechanism (f, x), the resultant interim expected utility for each player must be larger than or equal her interim outside value $U_i^*(\omega_i)$, and therefore, each player has incentive to participate in the collective decision irrespective of her private signal. We assume that

$$(1) \qquad \sum_{i\in N, \omega_i\in\Omega_i} \{ \sum_{\omega_{-i}\in\Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} \mid \omega_i) - U_i^*(\omega_i)\} p_i(\omega_i) \geq 0 ,$$

which implies that the sum of players' ex ante expected utilities is greater than or equals the sum of players' ex ante expected outside values. The following proposition states that this assumption is necessary and sufficient for the existence of a budget balancing side payment function satisfying IIR. We denote $\mu_i : \Omega_i \to R_+ \cup \{0\}$, $\mu = (\mu_i)_{i\in N}$, and $\lambda : \Omega \to R$.

Proposition 1: *Suppose $n \geq 2$. Then, there exists a budget balancing side payment function x that satisfies IIR, if and only if the inequality (1) holds.*

Proof: From the theorem in Fan (1956), it follows that there exists a budget balancing side payment function x that satisfies IIR, if and only if for every (μ, λ), whenever

(2) $$\lambda(\omega) = p_i(\omega_{-i} | \omega_i)\mu_i(\omega_i) \text{ for all } i \in N \text{ and all } \omega \in \Omega,$$

then

(3) $$\sum_{i\in N, \omega_i \in \Omega_i} \{ \sum_{\omega_{-i}\in\Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} | \omega_i) - U_i^*(\omega_i)\}\mu_i(\omega_i) \geq 0.$$

Suppose that (μ, λ) satisfies the equalities (2). Suppose that there exist $i \in N$, $j \in N/\{i\}$, $\omega_i \in \Omega_i$, and $k \geq 0$ such that $\sum_{\omega_j\in\Omega_j} \mu_j(\omega_j) = k$ and $\mu_i(\omega_i) > kp_i(\omega_i)$. Then, for every $\omega_{-i} \in \Omega_{-i}$,

$$p_i(\omega_{-i} | \omega_i)\{\mu_i(\omega_i) - kp_i(\omega_i)\} = p_j(\omega_{-j} | \omega_j)\{\mu_j(\omega_j) - kp_j(\omega_j)\} > 0.$$

This implies that $\mu_j(\omega_j) > kp_j(\omega_j)$ for all $\omega_j \in \Omega_j$, and therefore, $\sum_{\omega_j\in\Omega_j} \mu_j(\omega_j) > k$, which is a contradiction. Hence, whenever (μ, λ) satisfies the equalities (2), then there exists $k \geq 0$ such that $\mu_i(\omega_i) = kp_i(\omega_i)$ for all $i \in N$ and all $\omega_i \in \Omega_i$. From the inequality (3), it follows that

$$\sum_{i\in N, \omega_i \in \Omega_i} \{ \sum_{\omega_{-i}\in\Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} | \omega_i) - U_i^*(\omega_i)\}\mu_i(\omega_i)$$
$$= k \sum_{i\in N, \omega_i \in \Omega_i} \{ \sum_{\omega_{-i}\in\Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} | \omega_i) - U_i^*(\omega_i)\}p_i(\omega_i) \geq 0,$$

which implies the inequality (1).

Q.E.D.

In the same way as in the proof of Proposition 1, we can check that even if we can choose any set of message profiles M, the inequality (1) is still necessary and sufficient for the existence of an indirect mechanism (g, x) and a message rule profile $\phi \in \Phi$ satisfying that

$$g(\phi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega,$$

and satisfying that when players announce messages according to the message rule profile ϕ, the resultant interim expected utility for each player $i \in N$ is larger than or equals her interim outside value, i.e.,

$$\sum_{\omega_{-i}\in\Omega_{-i}} \{u_i(g(\phi(\omega))), \omega) + x_i(\phi(\omega))\} p_i(\omega_{-i} | \omega_i) \geq U_i^*(\omega_i) \text{ for all } \omega_i \in \Omega_i.$$

We introduce the following requirement on x.

Incentive Compatibility (IC): For every $i \in N$, every $\omega_i \in \Omega_i$, and every $m_i \in \Omega_i$,

$$\sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega), \omega) + x_i(\omega)\} p_i(\omega_{-i} \mid \omega_i)$$
$$\geq \sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega / m_i), \omega) + x_i(\omega / m_i)\} p_i(\omega_{-i} \mid \omega_i) .$$

IC implies that the honest message rule profile ϕ^* is a Bayesian Nash equilibrium in the direct mechanism (f, x).

3. Interim Individual Rationality and Incentive Compatibility

We introduce the following two conditions on p.

Condition 1: For every $(\omega_1,\omega_2)\in\Omega_1\times\Omega_2$, the collection of probability distributions on Ω_{-1-2} given by

$$\{p_{12}(\cdot\mid\omega_1',\omega_2')\mid(\omega_1',\omega_2')\in\Omega_1\times\Omega_2,\ \textit{either}\ \omega_1'=\omega_1\ \textit{or}\ \omega_2'=\omega_2\}$$

is linearly independent in that for every $(w(\omega_1'))_{\omega_1'\in\Omega_1}\in R^{|\Omega_1|}$, and every $(w(\omega_2'))_{\omega_2'\in\Omega_2/\{\omega_2\}}\in R^{|\Omega_2|-1}$, whenever

(4) $$\sum_{\omega_1'\in\Omega_1}w(\omega_1')p_{12}(\omega_{-1-2}\mid\omega_1',\omega_2)+\sum_{\omega_2'\in\Omega_2/\{\omega_2\}}w(\omega_2')p_{12}(\omega_{-1-2}\mid\omega_1,\omega_2')=0$$

for all $\omega_{-1-2}\in\Omega_{-1-2}$,

then

(5) $$w(\omega_1')=0\ \text{ for all }\ \omega_1'\in\Omega_1,\text{ and }\ w(\omega_2')=0\ \text{ for all }\ \omega_2'\in\Omega_2/\{\omega_2\}.$$

Condition 2: For every $i\in N/\{1,2\}$, the collection of probability distributions on Ω_{-i} given by

$$\{p_i(\cdot\mid\omega_i)\mid\omega_i\in\Omega_i\}$$

is linearly independent in that for every $(w(\omega_i))_{\omega_i\in\Omega_i}\in R^{|\Omega_i|}$, whenever

(6) $$\sum_{\omega_i\in\Omega_i}w(\omega_i)p_i(\omega_{-i}\mid\omega_i)=0\ \text{ for all }\ \omega_{-i}\in\Omega_{-i},$$

then

(7) $$w(\omega_i)=0\ \text{ for all }\ \omega_i\in\Omega_i.$$

We must note that if $|\Omega_{-1-2}|\geq|\Omega_1|+|\Omega_2|-1$, Condition 1 holds for generic prior distributions. We must note that if $|\Omega_{-i}|\geq|\Omega_i|$ for all $i\in N/\{1,2\}$, Condition 2 holds for generic prior distributions. The following theorem states that if $|\Omega_{-1-2}|\geq|\Omega_1|+|\Omega_2|-1$, and $|\Omega_{-i}|\geq|\Omega_i|$ for all $i\in N/\{1,2\}$, for generic prior distributions there exists a budget-balancing side payment function x that satisfies IC and IIR, irrespective of the social choice function f.

Theorem 2: *Suppose that p satisfies Conditions 1 and 2. Then, there exists a budget-balancing side payment function x that satisfies IC and IIR.*

Proof: We denote $\alpha_i:\Omega_i^2\to R_+\cup\{0\}$ and $\alpha=(\alpha_i)_{i\in N}$. From the theorem in Fan (1956), it follows that there exists a budget-balancing side payment function x that satisfies IC and IIR, if and only if for every (α,μ,λ), whenever

(8) $$\lambda(\omega)=p_i(\omega_{-i}\mid\omega_i)\sum_{\omega_i'\in\Omega_i}\alpha_i(\omega_i,\omega_i')-\sum_{\omega_i'\in\Omega_i}p_i(\omega_{-i}\mid\omega_i')\alpha_i(\omega_i',\omega_i)$$

$$+\,p_i(\omega_{-i}\mid\omega_i)\mu_i(\omega_i)\ \text{ for all }\ i\in N\ \text{ and all }\ \omega\in\Omega,$$

then

(9)
$$\sum_{\substack{i\in N\\ \omega\in\Omega}}\sum_{\omega_i'\in\Omega_i}\{u_i(f(\omega),\omega)-u_i(f(\omega/\omega_i'),\omega)\}p_i(\omega_{-i}\mid\omega_i)\alpha_i(\omega_i,\omega_i')$$
$$+\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_{-i}\in\Omega_{-i}}u_i(f(\omega),\omega)p_i(\omega_{-i}\mid\omega_i)-U_i^*(\omega_i)\}\mu_i(\omega_i)\geq 0\,.$$

Fix (α,μ,λ) arbitrarily. Suppose that the equalities (8) hold. Fix $(\omega_1,\omega_2)\in\Omega_1\times\Omega_2$ arbitrarily. Let

$$w(\omega_1')=p_1^2(\omega_2\mid\omega_1)\alpha_1(\omega_1',\omega_1)\quad\text{for all}\quad \omega_1'\in\Omega_1/\{\omega_1\}\,,$$
$$w(\omega_2')=-p_2^1(\omega_1\mid\omega_2)\alpha_2(\omega_2',\omega_2)\quad\text{for all}\quad \omega_2'\in\Omega_2/\{\omega_2\}\,,$$

and

$$w(\omega_1)=p_1^2(\omega_2\mid\omega_1)\{\mu_1(\omega_1)+\sum_{\omega_1'\in\Omega_1/\{\omega_1\}}\alpha_1(\omega_1,\omega_1')\}$$
$$-\,p_2^1(\omega_1\mid\omega_2)\{\mu_2(\omega_2)+\sum_{\omega_2'\in\Omega_2/\{\omega_2\}}\alpha_2(\omega_2,\omega_2')\}\,.$$

Note that the equalities (4) hold. It follows from Condition 1 that the equalities (5) hold, and therefore,

$$\alpha_1(\omega_1',\omega_1)=0\quad\text{for all}\quad \omega_1'\in\Omega_1/\{\omega_1\}\,,$$
$$\alpha_2(\omega_2',\omega_2)=0\quad\text{for all}\quad \omega_2'\in\Omega_2/\{\omega_2\}\,,$$

and

$$p_1^2(\omega_2\mid\omega_1)\{\mu_1(\omega_1)+\sum_{\omega_1'\in\Omega_1/\{\omega_1\}}\alpha_1(\omega_1,\omega_1')\}$$
$$-\,p_2^1(\omega_1\mid\omega_2)\{\mu_2(\omega_2)+\sum_{\omega_2'\in\Omega_2/\{\omega_2\}}\alpha_2(\omega_2,\omega_2')\}=0\,.$$

Since the above equalities hold for all $(\omega_1,\omega_2)\in\Omega_1\times\Omega_2$, it follows that

$$p_1^2(\omega_2\mid\omega_1)\mu_1(\omega_1)=p_2^1(\omega_1\mid\omega_2)\mu_2(\omega_2)\quad\text{for all}\quad(\omega_1,\omega_2)\in\Omega_1\times\Omega_2\,,$$

which implies that there exists $k\geq 0$ such that

$$\mu_1(\omega_1)=kp_1(\omega_1)\quad\text{for all}\quad \omega_1\in\Omega_1\,.$$

Hence, it follows from the equalities (8) that

$$\lambda(\omega)=kp(\omega)\quad\text{for all}\quad \omega\in\Omega\,.$$

Fix $i\in N/\{1,2\}$ and $\omega_i\in\Omega_i$ arbitrarily. Let

$$z_i(\omega_i)=kp_i(\omega_i)-\sum_{\omega_i'\in\Omega_i/\{\omega_i\}}\alpha_i(\omega_i,\omega_i')-\mu_i(\omega_i)\,,$$

and

$$z_i(\omega_i')=\alpha_i(\omega_i',\omega_i)\quad\text{for all}\quad \omega_i'\in\Omega_i/\{\omega_i\}\,.$$

Note that the equalities (6) hold. Condition 2 implies that the equalities (7) hold, and therefore, for every $i\in N/\{1,2\}$, and every $\omega_i\in\Omega_i$,

$$\alpha_i(\omega_i',\omega_i)=0\quad\text{for all}\quad \omega_i'\in\Omega_i/\{\omega_i\}\,,$$

and

$$\mu_i(\omega_i)=kp_i(\omega_i)\,.$$

From the above arguments and the inequality (1), we have proved that

$$\sum_{i\in N,\omega\in\Omega}\sum_{\omega_i'\in\Omega_i}\{u_i(f(\omega),\omega)-u_i(f(\omega/\omega_i'),\omega)\}p_i(\omega_{-i}\mid\omega_i)\alpha_i(\omega_i,\omega_i')$$

$$+\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_{-i}\in\Omega_{-i}}u_i(f(\omega),\omega)p_i(\omega_{-i}\mid\omega_i)-U_i^*(\omega_i)\}\mu_i(\omega_i)$$
$$=k\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_{-i}\in\Omega_{-i}}u_i(f(\omega),\omega)p_i(\omega_{-i}\mid\omega_i)-U_i^*(\omega_i)\}p_i(\omega_i)\geq 0,$$

which implies the inequality (9).

Q.E.D.

Myerson and Satterthwaite (1983) investigated a model of bilateral trading and showed the non-existence of a budget-balancing side payment function satisfying IC and IIR. Their negative result depends on the assumptions that there exist only two players, i.e., $n=2$, and that players' private signals are independent, i.e., for every $i\in N$, $p_i(\cdot\mid\omega_i)$ is independent of $\omega_i\in\Omega_i$. In contrast, Theorem 2 of this paper assumes that $n\geq 3$, and players' private signals are *correlated*. The following proposition states that when $n=2$, there may exist no budget-balancing side payment function x satisfying IC and IIR, irrespective of whether players' private signals are correlated.

Proposition 3: *Suppose* $n=2$*, that the inequality (1) holds with equality, i.e.,*

(10) $$\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_{-i}\in\Omega_{-i}}u_i(f(\omega),\omega)p_i(\omega_{-i}\mid\omega_i)-U_i^*(\omega_i)\}p_i(\omega_i)=0,$$

and that there exists $\hat{\omega}\in\Omega$ *such that*

(11) $$\sum_{\omega\in\Omega}\{u_1(f(\omega),\hat{\omega}_1,\omega_2)+u_2(f(\omega),\omega_1,\hat{\omega}_2)\}p_1(\omega_2\mid\hat{\omega}_1)p_2(\omega_1\mid\hat{\omega}_2)$$
$$>U_1^*(\hat{\omega}_1)+U_2^*(\hat{\omega}_2).$$

Then, there exists no budget-balancing side payment function x *that satisfies IIR and IC.*

Proof: From IIR and the equality (10), it must hold that for every $i\in N$, and every $\omega_i\in\Omega_i$,

$$\sum_{\omega_j\in\Omega_j}\{u_i(f(\omega),\omega)+x_i(\omega)\}p_i(\omega_j\mid\omega_i)=U_i^*(\omega_i).$$

We denote $\eta_i:\Omega_i\to R$ and $\eta=(\eta_i)_{i\in N}$. From the theorem in Fan (1956), it follows that there exists a budget-balancing side payment function x that satisfies IIR and IC, if and only if for every (α,η,λ), whenever

(12) $$\lambda(\omega)=p_i(\omega_j\mid\omega_i)\sum_{\omega_i'\in\Omega_i}\alpha_i(\omega_i,\omega_i')-\sum_{\omega_i'\in\Omega_i}p_i(\omega_j\mid\omega_i')\alpha_i(\omega_i',\omega_i)$$
$$+p_i(\omega_j\mid\omega_i)\eta_i(\omega_i)\text{ for all } i\in N \text{ and all } \omega\in\Omega,$$

then

(13) $$\sum_{i\in N,\omega\in\Omega}\sum_{\omega_i'\in\Omega_i}\{u_i(f(\omega),\omega)-u_i(f(\omega_i',\omega_j),\omega)\}p_i(\omega_{-i}\mid\omega_i)\alpha_i(\omega_i,\omega_i')$$
$$+\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_j\in\Omega_j}u_i(f(\omega),\omega)p_i(\omega_j\mid\omega_i)-U_i^*(\omega_i)\}\eta_i(\omega_i)\geq 0.$$

We specify

$$\alpha_i(\hat{\omega}_i,\omega_i)=p_j(\omega_i\mid\hat{\omega}_j)\text{ for all } \omega_i\in\Omega_i,$$
$$\alpha_i(\omega_i,\omega_i')=0\text{ for all } \omega_i\in\Omega_i/\{\hat{\omega}_i\}\text{ and } \omega_i'\in\Omega_i,$$
$$\eta_i(\hat{\omega}_i)=-1,$$

and

$$\eta_i(\omega_i) = 0 \quad \text{for all} \quad \omega_i \in \Omega_i / \{\hat{\omega}_i\}.$$

Note from $n = 2$ that for every $i \in N$, and every $\omega \in \Omega$,

$$p_i(\omega_j \mid \omega_i) \sum_{\omega_i' \in \Omega_i} \alpha_i(\omega_i, \omega_i') - \sum_{\omega_i' \in \Omega_i} p_i(\omega_j \mid \omega_i') \alpha_i(\omega_i', \omega_i) + p_i(\omega_j \mid \omega_i) \eta_i(\omega_i)$$

$$= -p_1(\omega_2 \mid \hat{\omega}_1) p_2(\omega_1 \mid \hat{\omega}_2).$$

Let

$$\lambda(\omega) = -p_1(\omega_2 \mid \hat{\omega}_1) p_2(\omega_1 \mid \hat{\omega}_2) \quad \text{for all} \quad \omega \in \Omega.$$

Note that (α, η, λ) satisfies the equalities (12). From the inequality (11), it follows that

$$\sum_{i \in N, \omega \in \Omega} \sum_{\omega_i' \in \Omega_i} \{u_i(f(\omega), \omega) - u_i(f(\omega_i', \omega_j), \omega)\} p_i(\omega_{-i} \mid \omega_i) \alpha_i(\omega_i, \omega_i')$$

$$+ \sum_{i \in N, \omega_i \in \Omega_i} \{ \sum_{\omega_j \in \Omega_j} u_i(f(\omega), \omega) p_i(\omega_j \mid \omega_i) - U_i^*(\omega_i)\} \eta_i(\omega_i)$$

$$= -\sum_{\omega \in \Omega} \{u_1(f(\omega), \hat{\omega}_1, \omega_2) + u_2(f(\omega), \omega_1, \hat{\omega}_2)\} p_1(\omega_2 \mid \hat{\omega}_{i1}) p_1(\omega_2 \mid \hat{\omega}_1)$$

$$+ U_1^*(\hat{\omega}_1) + U_2^*(\hat{\omega}_2) < 0,$$

which contradicts the inequality (13).

Q.E.D.

We must note that the inequality (11) holds if $U_1^*(\hat{\omega}_1)$ and $U_2^*(\hat{\omega}_2)$ are very small. Hence, Proposition 3 implies that there may exists no budget balancing side payment function satisfying IC and IIR in the two-player case, if for every player there exists a private signal for this player such that the associated interim outside value is very small. Further implications will be shown in the next two sections.

4. Full Surplus Extraction by a Risk-Averse Principal

Consider the following situation in which a *risk-averse* principal hires n risk-neutral agents. Each agent $i \in N$ receives her private signal $\omega_i \in \Omega_i$, whereas the principal receives no private signal. Each agent $i \in N$ announces her message $m_i \in M_i = \Omega_i$ in a direct mechanism (f, x) where the side payment function x may not be budget balancing.[3] Based on their message profile $m \in M$, the agents collectively choose the alternative $f(m) \in A$. Each agent i receives her profit $u_i(f(m), \omega)$ and pays the monetary amount $-x_i(m) \in R$ to the principal. The principal's utility is given by $v(-\sum_{i \in N} x_i(m))$ where $v : R \to R$ is an increasing and concave function. Each agent $i's$ utility is given by $u_i(f(m), \omega) + x_i(m)$. We assume that the interim outside value for each player is set equal to zero, i.e.,

$$U_i^*(\omega_i) = 0 \text{ for all } i \in N \text{ and all } \omega_i \in \Omega_i .$$

In the same way as in the proof of Proposition 1, we can check that there exists a side payment function x such that

$$(14) \qquad -\sum_{i \in N} x_i(\omega) = \sum_{i \in N, \omega' \in \Omega} u_i(f(\omega'), \omega') p(\omega') \text{ for all } \omega \in \Omega ,$$

and for every $i \in N$, player $i's$ interim expected utility always equals zero, i.e.,

$$\sum_{\omega_{-i} \in \Omega_{-i}} \{\pi_i(f(\omega), \omega) - \delta_i(\omega)\} p_i(\omega_{-i} \mid \omega_i) = 0 \text{ for all } \omega_i \in \Omega_i .$$

Here, when all agents announce honestly according to ϕ^*, the principal receives a constant monetary amount $\sum_{i \in N, \omega' \in \Omega} u_i(f(\omega'), \omega') p(\omega')$ irrespective of what is the true private signal profile, and therefore, the principal's expected utility is given by

$$\sum_{\omega \in \Omega} v(-\sum_{i \in N} x_i(\omega)) p(\omega) = v(\sum_{i \in N, \omega' \in \Omega} u_i(f(\omega'), \omega') p(\omega')) .$$

Since v is concave, this values is equivalent to the maximal expected utility for the principal with the constraints of IIR for all agents. We will say that *the principal can extract the full surplus* if there exists x satisfying IC, IIR, and the equalities (14). In the same way as in the proofs of Theorem 2 and Proposition 3, we can prove the following proposition.

Proposition 4: *If $n = 2$ and there exists $\hat{\omega} \in \Omega$ such that*

$$(15) \qquad \sum_{\omega \in \Omega} \{u_1(f(\omega), \hat{\omega}_1, \omega_2) + u_2(f(\omega), \omega_1, \hat{\omega}_2)\} p_1(\omega_2 \mid \hat{\omega}_1) p_2(\omega_1 \mid \hat{\omega}_2)$$
$$> \sum_{\omega \in \Omega} \{u_1(f(\omega), \omega) + u_2(f(\omega), \omega)\} p(\omega),$$

then the principal cannot extract the full surplus. If $n \geq 3$ and p satisfies Conditions 1 and 2, then the principal can extract the full surplus.

Note that the inequality (15) holds in the two-player case if each player $i \in \{1,2\}$ has high productivity when she receives the private signal $\hat{\omega}_i$, irrespective of the other player's private signal and the collective decision, in the sense that for every $\omega \in \Omega$,

[3] The side payments between the principal and the agents are always budget balancing.

$$u_1(f(\omega),\hat{\omega}_1,\omega_2) > \sum_{\omega'\in\Omega} u_1(f(\omega'),\omega')p(\omega'),$$

and

$$u_2(f(\omega),\omega_1,\hat{\omega}_2) > \sum_{\omega'\in\Omega} u_2(f(\omega'),\omega')p(\omega').$$

Hence, the first part of Proposition 4 implies that the principal cannot extract the full surplus in the two-agent case if for each player there exists a private signal for this player with which her productivity is very high. On the other hand, the latter part of Proposition 4 implies that in the case of three or more agents, the principal can extract the full surplus for generic prior distributions, if $|\Omega_{-1-2}| \geq |\Omega_1| + |\Omega_2| - 1$, and $|\Omega_{-i}| \geq |\Omega_i|$ for all $i \in N/\{1,2\}$.

As an example, we consider *auctions* with a single risk-averse seller and multiple risk-neutral buyers. The seller has one unit of commodity for sale. Each buyer $i's$ private signal ω_i implies her own valuation, where Ω_i is given by a finite set of positive real numbers. The set of alternatives is given by $A = N$, where the alternative $i \in A$ implies that the commodity is transferred to buyer i. Hence, for every buyer $i \in N$,

$$u_i(a,\omega) = \omega_i \text{ if } a = i,$$

and

$$u_i(a,\omega) = 0 \text{ if } a \neq i.$$

The social choice function f is efficient, i.e., for every $\omega \in \Omega$, and every $i \in N$,

$$\omega_i \geq \omega_j \text{ for all } j \in N \text{ whenever } f(\omega) = i.$$ [4]

The expected full surplus is given by

$$\sum_{i\in N,\omega\in\Omega} u_i(f(\omega),\omega)p(\omega) = \sum_{\omega\in\Omega} (\max_{i\in N} \omega_i)p(\omega).$$

For every $i \in N$, we denote by $\overline{\omega}_i \in \Omega_i$ buyer $i's$ highest possible valuation, where

$$\overline{\omega}_i > \omega_i \text{ for all } \omega_i \in \Omega_i/\{\overline{\omega}_i\}.$$

We assume that each buyer's highest possible valuation is greater than the expected full surplus, i.e.,

$$\overline{\omega}_i > \sum_{\omega\in\Omega} (\max_{j\in N} \omega_j)p(\omega) \text{ for all } i \in N.$$

This assumption is trivial when each buyer has the same highest possible valuation. We must note that when $n = 2$, this assumption implies the inequality (15). Hence, it follows from the first part of Proposition 4 that when there exist only two buyers, the risk-averse seller cannot extract the full surplus. On the other hand, the latter part of Proposition 4 implies that with three or more buyers, if $|\Omega_{-1-2}| \geq |\Omega_1| + |\Omega_2| - 1$ and $|\Omega_{-i}| \geq |\Omega_i|$ for all $i \in N/\{1,2\}$, then the full surplus extraction by the seller is generically possible.

Proposition 1 implies that if incentive compatibility is not required, the seller can always extract the full surplus even in the two-buyer case. Moreover, we can check that if, instead of requiring interim individual rationality, we require only ex ante individual rationality that

$$\sum_{\omega\in\Omega} \{u_i(f(\omega),\omega) + x_i(\omega)\}p(\omega) = 0 \text{ for all } i \in N,$$

then the seller can extract the full surplus in a wide class of environments with two buyers. Suppose that (p, f) satisfies the 'sorting' condition that for every $i \in \{1,2\}$, every

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[4] Without any substantial change of our arguments, we can allow $f(\omega)$ to be a lottery over multiple agents whose evaluations are the highest.

$\omega_i \in \Omega_i / \{\overline{\omega}_i\}$, and every $\tilde{\omega}_i \in \Omega_i / \{\omega_i\}$,

$$\sum_{\omega_j \in \Omega_j} \{u_i(f(\omega_i', \omega_j), \omega) - u_i(f(\omega), \omega)\} p_i(\omega_j \mid \omega_i)$$
$$\geq \sum_{\omega_j \in \Omega_j} \{u_i(f(\omega_i', \omega_j), \tilde{\omega}_i, \omega_j) - u_i(f(\omega), \tilde{\omega}_i, \omega_j)\} p_i(\omega_j \mid \tilde{\omega}_i) \text{ if } \tilde{\omega}_i < \omega_i,$$

and

$$\sum_{\omega_j \in \Omega_j} \{u_i(f(\omega_i', \omega_j), \omega) - u_i(f(\omega), \omega)\} p_i(\omega_j \mid \omega_i)$$
$$\leq \sum_{\omega_j \in \Omega_j} \{u_i(f(\omega_i', \omega_j), \tilde{\omega}_i, \omega_j) - u_i(f(\omega), \tilde{\omega}_i, \omega_j)\} p_i(\omega_j \mid \tilde{\omega}_i) \text{ if } \tilde{\omega}_i > \omega_i,$$

where $\omega_i' \in \Omega_i / \{\omega_i\}$ is the smallest element of Ω_i that is larger than ω_i. Then, according to the standard analysis, it follows that for every $i \in \{1,2\}$, there exists $s_i : \Omega_i \to R$ such that

$$\sum_{\omega_j \in \Omega_j} \{u_i(f(\omega), \omega) + s_i(\omega_i)\} p_i(\omega_j \mid \omega_i)$$
$$\geq \sum_{\omega_j \in \Omega_j} \{u_i(f(\tilde{\omega}_i, \omega_j), \omega) - s_i(\tilde{\omega}_i)\} p_i(\omega_j \mid \omega_i) \text{ for all } \tilde{\omega}_i \in \Omega_i.$$

For every $i \in \{1,2\}$, let

$$x_i(\omega) = s_i(\omega_i) - s_j(\omega_j) - \sum_{\tilde{\omega} \in \Omega} \{u_i(f(\tilde{\omega}), \tilde{\omega}) + s_i(\tilde{\omega}_i) - s_j(\tilde{\omega}_j)\} p(\tilde{\omega}),$$

which satisfies incentive compatibility and ex ante individual rationality with equality. Hence, without interim individual rationality, the seller can extract the full surplus even in the two-buyer case.

5. Full Surplus Extraction by a Risk-Neutral Player

Fix any player $i^* \in N$ arbitrarily. We will say that *player* i^* *can extract the full surplus* if there exists a budget-balancing side payment function x satisfying IC and IIR where the properties of IIR hold with equality for every player except player i^*, i.e., for every $i \in N/\{i^*\}$, and every $\omega_i \in \Omega_i$,

(16) $$\sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega)), \omega) + x_i(\omega)\} p_i(\omega_{-i} \mid \omega_i) = U_i^*(\omega_i).$$

Proposition 5: *Suppose that* p *satisfies Conditions 1 and 2. Then, player* i^* *can extract the full surplus.*

Proof: The inequality (1) implies that we can choose $U_{i^*}^{**}(\omega_{i^*})$ for each $\omega_{i^*} \in \Omega_{i^*}$ such that

$$U_{i^*}^{**}(\omega_{i^*}) \geq U_{i^*}^{*}(\omega_{i^*}) \text{ for all } \omega_{i^*} \in \Omega_{i^*},$$

and

(17) $$\{ \sum_{\omega_{-i^*} \in \Omega_{-i^*}} u_{i^*}(f(\omega), \omega) p_{i^*}(\omega_{-i^*} \mid \omega_{*_i}) - U_{i^*}^{**}(\omega_{*_i})\} p_{i^*}(\omega_{*_i})$$
$$+ \sum_{i \in N/\{i^*\}, \omega_i \in \Omega_i} \{ \sum_{\omega_{-i} \in \Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} \mid \omega_i) - U_i^*(\omega_i)\} p_i(\omega_i) = 0.$$

From Theorem 2 and the equality (17), it follows that if Conditions 1 and 2 holds, then there exists a budget balancing side payment function x satisfying IC such that the equality (16) holds for all $i \in N/\{i^*\}$ and all $\omega_i \in \Omega_i$, and

$$\sum_{\omega_{-i^*} \in \Omega_{-i^*}} u_{i^*}(f(\omega), \omega) p_{i^*}(\omega_{-i^*} \mid \omega_{*_i}) = U_{i^*}^{**} \text{ for all } \omega_{i^*} \in \Omega_{i^*}.$$

This implies that player i^* can extract the full surplus.

Q.E.D.

Crèmer and McLean (1985, 1988) investigated auctions with a single seller and three or more bidders, and showed that the seller can extract the full surplus if the bidders' private signals are correlated. Crèmer and McLean assumed that the seller receives no private signals. In contrast, the present paper permits the seller to receive her private signal and takes into account the constraints of IC and IIR for this seller as well as the bidders. The following proposition states that if there exists a player whose private signal is independent, player i^* may not be able to extract the full surplus.

Proposition 6: *Fix* $i \in N/\{i^*\}$ *arbitrarily. Suppose that there exist* $\hat{\omega}_i \in \Omega_i$ *and* $\tilde{\omega}_i \in \Omega_i/\{\hat{\omega}_i\}$ *such that*

(18) $$\sum_{\tilde{\omega}_{-i} \in \Omega_{-i}} \{u_i(f(\tilde{\omega}), \tilde{\omega}) p_i(\tilde{\omega}_{-i} \mid \tilde{\omega}_i) - u_i(f(\tilde{\omega}), \tilde{\omega}_{-i}, \hat{\omega}_i) p_i(\tilde{\omega}_{-i} \mid \hat{\omega}_i)\}$$
$$< U_i^*(\tilde{\omega}_i) - U_i^*(\hat{\omega}_i).$$

Suppose that $p_i(\cdot \mid \omega_i)$ *is independent of* $\omega_i \in \Omega_i$. *Then, there exists no side payment* *function* x_i *for player* i *such that for every* $\omega_i \in \Omega_i$,

(19) $$\sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega)), \omega) + x_i(\omega)\} p_i(\omega_{-i} \mid \omega_i)$$

$$\geq \sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega_{-i}, \omega_i')), \omega) + x_i(\omega_{-i}, \omega_i')\} p_i(\omega_{-i} \mid \omega_i) \text{ for all } \omega_i' \in \Omega_i,$$

and

(20) $$\sum_{\omega_{-i} \in \Omega_{-i}} \{u_i(f(\omega)), \omega) + x_i(\omega)\} p_i(\omega_{-i} \mid \omega_i) = U_i^*(\omega_i).$$

Proof: Suppose that x_i satisfies the inequalities (19) and the equalities (20). Since $p_i(\cdot \mid \omega_i)$ is independent of $\omega_i \in \Omega_i$, it must hold that $\sum_{\omega_{-i} \in \Omega_{-i}} x_i(\omega) p_i(\omega_{-i} \mid \omega_i)$ is independent of $\omega_i \in \Omega_i$. Hence, it follows from the equalities (20) that for every $\omega_i \in \Omega_i$,

$$\sum_{\omega_{-i} \in \Omega_{-i}} x_i(\omega) p_i(\omega_{-i} \mid \omega_i) = U_i^*(\omega_i) - \sum_{\omega_{-i} \in \Omega_{-i}} u_i(f(\omega), \omega) p_i(\omega_{-i} \mid \omega_i),$$

which, together with the inequality (18), implies that

$$\sum_{\tilde{\omega}_{-i} \in \Omega_{-i}} \{u_i(f(\tilde{\omega})), \tilde{\omega}_{-i}, \hat{\omega}_i) + x_i(\tilde{\omega})\} p_i(\tilde{\omega}_{-i} \mid \hat{\omega}_i)$$

$$- \sum_{\tilde{\omega}_{-i} \in \Omega_{-i}} \{u_i(f(\tilde{\omega}_{-i}, \hat{\omega}_i)), \tilde{\omega}_{-i}, \hat{\omega}_i) + x_i(\tilde{\omega}_{-i}, \hat{\omega}_i)\} p_i(\tilde{\omega}_{-i} \mid \hat{\omega}_i)$$

$$= \sum_{\tilde{\omega}_{-i} \in \Omega_{-i}} \{u_i(f(\tilde{\omega}), \tilde{\omega}_{-i}, \hat{\omega}_i) p_i(\tilde{\omega}_{-i} \mid \hat{\omega}_i) - u_i(f(\tilde{\omega}), \tilde{\omega}) p_i(\tilde{\omega}_{-i} \mid \tilde{\omega}_i)\}$$

$$- U_i^*(\hat{\omega}_i) + U_i^*(\tilde{\omega}_i) > 0.$$

This contradicts the inequalities (19).

Q.E.D.

Proposition 6 implies that if there exists a player $i \in N / \{i^*\}$ such that $p_i(\cdot \mid \omega_i)$ is independent of $\omega_i \in \Omega_i$, and the inequality (18) holds for some $\hat{\omega}_i \in \Omega_i$ and $\tilde{\omega}_i \in \Omega_i / \{\hat{\omega}_i\}$, then player i^* cannot extracts the full surplus. Note that the inequality (18) holds when the difference in interim outside values between the private signals $\hat{\omega}_i$ and $\tilde{\omega}_i$ is large. Note also that even if the interim outside values $U_i^*(\hat{\omega}_i)$ and $U_i^*(\tilde{\omega}_i)$ are equivalent, the inequality (18) holds when player $i's$ payoff in the case of $\hat{\omega}_i$ is better than that in the case of $\tilde{\omega}_i$, i.e.,

$$u_i(f(\tilde{\omega}), \tilde{\omega}_{-i}, \hat{\omega}_i) > u_i(f(\tilde{\omega}), \tilde{\omega}) \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}.$$

6. Uniqueness and Interim Individual Rationality

We introduce the following two conditions on p.

Condition 3: For every $i \in N/\{2\}$, the collection of probability functions on Ω_{-2-i} given by

$$\{p_{2i}(\cdot \mid \omega_i) \mid \omega_i \in \Omega_i\}$$

is linearly independent in that for every $(w(\omega_i))_{\omega_i \in \Omega_i} \in R^{|\Omega_i|}$, whenever

$$\sum_{\omega_i \in \Omega_i} w(\omega_i) p_{2i}(\omega_{-2-i} \mid \omega_i) = 0 \text{ for all } \omega_{-2-i} \in \Omega_{-2-i},$$

then

$$w(\omega_i) = 0 \text{ for all } \omega_i \in \Omega_i.$$

Condition 4: The collection of probability functions on Ω_{-1-2} given by

$$\{p_{12}(\cdot \mid \omega_1, \omega_2) \mid (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2\}$$

is linearly independent in that for every $(w(\omega_1, \omega_2))_{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2} \in R^{|\Omega_1| \times |\Omega_2|}$, whenever

$$\sum_{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2} w(\omega_1, \omega_2) p_{12}(\omega_{-1-2} \mid \omega_1, \omega_2) = 0 \text{ for all } \omega_{-1-2} \in \Omega_{-1-2},$$

then

$$w(\omega_1, \omega_2) = 0 \text{ for all } (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2.$$

We must note that if $|\Omega_{-2-i}| \geq |\Omega_i|$ for all $i \in N/\{2\}$, then Condition 3 holds for generic prior distributions. We must note that if $|\Omega_{-1-2}| \geq |\Omega_1| \times |\Omega_2|$, then Condition 4 holds for generic prior distributions.[5] The following proposition will be useful to construct a side budget balancing payment function satisfying IIR without harming other properties such as incentive compatibility and uniqueness.

Proposition 7: *Consider an arbitrary collection of functions (v_i) where $v_i : \Omega_i \to R$ for all $i \in N$, and*

$$\sum_{i \in N, \omega_i \in \Omega_i} v_i(\omega_i) p_i(\omega_i) = 0.$$

Suppose that p satisfies Conditions 3 and 4. Then there exists a collection of functions (y_i) such that $y_i : \Omega_{-i} \to R$ for all $i \in N$,

$$\sum_{i \in N} y_i(\omega_{-i}) = 0 \text{ for all } \omega \in \Omega,$$

and

$$\sum_{\omega_{-i} \in \Omega_{-i}} y_i(\omega_{-i}) p_i(\omega_{-i} \mid \omega_i) = v_i(\omega_i) \text{ for all } i \in N \text{ and all } \omega_i \in \Omega_i.$$

[5] We must note that if $|\Omega_{-2-i}| \geq |\Omega_i|$ for all $i \in N/\{1\}$, and $|\Omega_{-1-2}| \geq |\Omega_1| \times |\Omega_2|$, then it must hold that $n \geq 4$.

Proof: Condition 3 implies that for every $i \in N/\{1,2\}$, there exists t_i such that

$$\sum_{\omega_{-2-i}\in\Omega_{-2-i}} t_i(\omega_{-2-i}) p_{i2}(\omega_{-2-i} \mid \omega_i) = v_i(\omega_i) \quad \text{for all } \omega_i \in \Omega_i .$$

Let

$$w_2(\omega_2) = v_2(\omega_2) + \sum_{\omega_{-2}\in\Omega_{-2}} \{ \sum_{i\in N/\{1,2\}} t_i(\omega_{-2-i})\} p_2(\omega_{-2} \mid \omega_2) .$$

Note that

(21) $$\sum_{\omega_1\in\Omega_1} v_1(\omega_1) p_1(\omega_1) + \sum_{\omega_2\in\Omega_2} w_2(\omega_2) p_2(\omega_2) = 0 .$$

From the theorem in Fan (1956), there exists $z : \Omega_1 \times \Omega_2 \to R$ such that

$$\sum_{\omega_2\in\Omega_2} z(\omega_1,\omega_2) p_1(\omega_2 \mid \omega_1) = v_1(\omega_1), \text{ and}$$

$$\sum_{\omega_1\in\Omega_1} z(\omega_1,\omega_2) p_2(\omega_1 \mid \omega_2) = -w_2(\omega_2),$$

if and only if for every (η_1, η_2), whenever

(22) $$\eta_1(\omega_1) p_1(\omega_2 \mid \omega_1) = \eta_2(\omega_2) p_2(\omega_1 \mid \omega_2) \quad \text{for all } (\omega_1,\omega_2) \in \Omega_1 \times \Omega_2 ,$$

then

(23) $$\sum_{\omega_1\in\Omega_1} \eta_1(\omega_1) v_1(\omega_1) + \sum_{\omega_2\in\Omega_2} \eta_2(\omega_2) w_2(\omega_2) = 0 .$$

In the same way as in the proof of Theorem 2, it follows that whenever (η_1, η_2) satisfies the equalities (22), then there exists $k \geq 0$ such that

$$\eta_i(\omega_i) = k p_i(\omega_i) \quad \text{for all } i \in \{1,2\} \text{ and all } \omega_i \in \Omega_i .$$

Hence, it follows from the equality (21) that the equality (23) holds, and therefore, we have shown that such a z exists. Condition 4 implies that there exists $t_1 : \Omega_{-1-2} \to R$ such that

$$\sum_{\omega_{-1-2}\in\Omega_{-1-2}} t_1(\omega_{-1-2}) p_{12}(\omega_{-1-2} \mid \omega_1,\omega_2) = z(\omega_1,\omega_2) .$$

This implies that

$$\sum_{\omega_{-1-2}\in\Omega_{-1-2}} t_1(\omega_{-1-2}) p_{12}(\omega_{-1-2} \mid \omega_1) = v_1(\omega_1) \quad \text{for all } \omega_1 \in \Omega_1 ,$$

and

$$-\sum_{\omega_{-1-2}\in\Omega_{-1-2}} t_1(\omega_{-1-2}) p_{12}(\omega_{-1-2} \mid \omega_2) = w_2(\omega_2) \quad \text{for all } \omega_2 \in \Omega_2 .$$

We specify (y_i) as follows. For every $\omega \in \Omega$,

$$y_i(\omega_{-i}) = t_i(\omega_{-2-i}) \quad \text{for all } i \in N/\{2\},$$

and

$$y_2(\omega_{-2}) = -\sum_{i\in N/\{2\}} t_i(\omega_{-2-i}) .$$

From the above arguments it follows that

$$\sum_{i\in N} y_i(\omega_{-i}) = 0 \quad \text{for all } \omega \in \Omega ,$$

and

$$\sum_{\omega_{-i}\in\Omega_{-i}} y_i(\omega_{-i}) p_i(\omega_{-i} \mid \omega_i) = v_i(\omega_i) \quad \text{for all } i \in N \text{ and all } \omega_i \in \Omega_i .$$

Q.E.D.

The following theorem states that under Conditions 3 and 4, the existence of a mechanism with budget balancing that implements the social choice function implies the existence of a mechanism with interim individual rationality as well as budget balancing that implements the social choice function, and therefore, interim individual rationality is a trivial requirement.

Theorem 8: *Consider an indirect mechanism* (g,x) *with budget balancing. Suppose that* ϕ *is the unique k times iteratively undominated message rule profile in* (p,g,x). *Suppose that* p *satisfies Conditions 5 and 6, and that*

(24) $$\sum_{i\in N,\omega_i\in\Omega_i}\{\sum_{\omega_{-i}\in\Omega_{-i}}u_i(g(\phi(\omega)),\omega)p_i(\omega_{-i}\mid\omega_i)-U_i^*(\omega_i)\}p_i(\omega_i)\geq 0.$$

Then, there exists a budget balancing side payment function $\hat{x}$ *such that* ϕ *is the unique k times iteratively undominated message rule profile in* $(p,g,\hat{x})$, *and it satisfies interim individual rationality in that for every* $i\in N$, *and every* $\omega_i\in\Omega_i$,

$$\sum_{\omega_{-i}\in\Omega_{-i}}\{u_i(g(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p_i(\omega_{-i}\mid\omega_i)\geq U_i^*(\omega_i).$$

Proof: From the inequality (24), it follows that there exists (v_i) such that for every $i\in N$,

$$\sum_{i\in N,\omega_i\in\Omega_i}v_i(\omega_i)p_i(\omega_i)=0,$$

and for every $\omega_i\in\Omega_i$,

$$v_i(\omega_i)\geq U_i^*(\omega_i)-\sum_{\omega_{-i}\in\Omega_{-i}}\{u_i(g(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p_i(\omega_{-i}\mid\omega_i).$$

From Proposition 7, it follows that there exists (y_i) that satisfies the properties in Proposition 7. We specify $\hat{x}$ by

$$\hat{x}_i(m)=x_i(m)+y_i(m_{-i}^2)\ \text{ for all } i\in N \text{ and all } m\in\Omega.$$

Note that $(p,f,\hat{x})$ satisfies interim individual rationality in the above sense. Since y_i is independent of m_i, it follows that ϕ is the unique k-times iteratively undominated message rule profile in $(p,g,\hat{x})$.

Q.E.D.

Hence, all we have to do from the next section is to show the existence of a budget balancing mechanism uniquely implementing the social choice function without explicitly requiring interim individual rationality.

7. Twice Iterative Dominance

We introduce the following condition on p.

Condition 5: For every $i \in N/\{1\}$, every $\omega_i \in \Omega_i$, and every $\omega_i' \in \Omega_i$,

$$p_i^1(\cdot \mid \omega_i) \neq p_i^1(\cdot \mid \omega_i').$$

We must note that if $|\Omega_1| \geq 2$, then Condition 3 holds for generic prior distributions. We introduce the following conditions on f.

Condition 6: There exists $d : \Omega \to R$ such that for every $\omega \in \Omega$, and every $\omega' \in \Omega$ satisfying $\omega_1 \neq \omega_1'$,

$$u_1(f(\omega_{-1}', \omega_1), \omega) + d(\omega_{-1}', \omega_1) > u_1(f(\omega'), \omega) + d(\omega').$$

Suppose that player 1's utility $u_1(a, \omega)$ is independent of ω_{-1}, and that for every $i \in N/\{1\}$, player $i's$ utility $u_i(a, \omega)$ is independent of ω_1. These suppositions are regarded as a weaker version of the private value assumption that every player's utility is independent of the other players' private signals. Suppose that a social choice function f is *strictly efficient* in that for every $\omega \in \Omega$,

$$\sum_{i \in N} u_i(f(\omega), \omega) > \sum_{i \in N} u_i(f(a), \omega) \quad \text{for all } a \in A/\{f(\omega)\}.$$

We specify

$$d(\omega) = \sum_{i \in N/\{1\}} u_i(f(\omega), \omega).$$

We must note that Condition 6 holds in this case.[6] The following proposition states that under Condition 6, if $|\Omega_1| \geq 2$, then for generic prior distributions there exists a direct mechanism with budget balancing in which truth telling is the unique twice iteratively undominated message rule profile.

Proposition 9: *Suppose that Conditions 5 and 6 hold. Then, there exists a budget-balancing side payment function x such that ϕ^* is the unique twice iteratively undominated message rule profile in (p, f, x).*

Proof: For every $i \in N/\{1\}$, we define a function $s_i : \Omega_1 \times \Omega_i \to R$ in ways that for every $(\omega_1, \omega_i) \in \Omega_1 \times \Omega_i$,

$$s_i(\omega_1, \omega_i) = -\{1 - p_i^1(\omega_1 \mid \omega_i)\}^2 - \sum_{\omega_1' \in \Omega_1/\{\omega_1\}} p_i^1(\omega_1' \mid \omega_i)^2.$$

Condition 5 implies that for every $i \in N/\{1\}$, every $\omega_i \in \Omega_i$, and every $\omega_i' \in \Omega_i/\{\omega_i\}$,

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[6] This is originated in Groves (1973), D'Aspremont and Gerard-Varet (1979), and others. See also Matsushima (1990a) in terms of dominant strategies.

(25) $$\sum_{\omega_1\in\Omega_1} s_i(\omega_1,\omega_i)p_i^1(\omega_1\mid\omega_i) > \sum_{\omega_1\in\Omega_1} s_i(\omega_1,\omega_i')p_i^1(\omega_1\mid\omega_i).$$ [7]

Fix a positive real number $k>0$ arbitrarily, and we specify x as follows. For every $m\in M$,

$$x_1(m)=d(m),$$

$$x_2(m)=-d(m)+k\{s_2(m_1,m_2)-\frac{1}{n-2}\sum_{j\in N/\{1,2\}} s_j(m_1,m_j)\},$$

and for every $i\in N/\{1,2\}$,

$$x_i(m)=k\{s_i(m_1,m_i)-\frac{1}{n-2}\sum_{j\in N/\{1,i\}} s_j(m_1,m_j)\}.$$

Note that x is budget balancing. From the inequalities (25), it follows that for every sufficiently large k, every $\phi\in\Phi$, and every $i\in N/\{1\}$, if $\phi_1=\phi_1^*$ and $\phi_i\neq\phi_i^*$, then

$$\sum_{\omega\in\Omega}\{u_i(f(\phi(\omega)/\phi_i^*(\omega_i)),\omega)+x_i(\phi(\omega)/\phi_i^*(\omega_i))\}p(\omega)$$
$$>\sum_{\omega\in\Omega}\{u_i(f(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p(\omega).$$

This implies that whenever player 1 announces honestly then all other players have strict incentive to announce honestly. Condition 6 implies that for every $\phi\in\Phi$, if $\phi_1\neq\phi_1^*$, then

$$\sum_{\omega\in\Omega}\{u_1(f(\phi(\omega)),\omega)+x_1(\phi(\omega))\}p(\omega)$$
$$=\sum_{\omega\in\Omega}\{u_1(f(\phi(\omega)),\omega_1)+d(\phi(\omega))\}p(\omega)$$
$$<\sum_{\omega\in\Omega}\{u_1(f(\omega_1,\phi_{-1}(\omega_{-1})),\omega_1)+d(\omega_1,\phi_{-1}(\omega_{-1}))\}p(\omega)$$
$$=\sum_{\omega\in\Omega}\{u_1(f(\phi_1^*(\omega_1),\phi_{-1}(\omega_{-1})),\omega)+x_1(\phi_1^*(\omega_1),\phi_{-1}(\omega_{-1}))\}p(\omega).$$

This implies that player 1 has strict incentive to announce honestly irrespective of what are the other players' message rules played. Hence, we have proved that ϕ^* is the unique twice iteratively undominated message rule profile in (p,f,x).

Q.E.D.

Matsushima (1990a) showed a sufficient condition on the common prior distribution under which with private values, there exists a budget balancing side payment function such that truth telling is the unique twice iteratively undominated message rule profile when the social choice function is strictly efficient. This sufficient condition requires the linear independence of the conditional distributions, which is more restrictive than Condition 5. Hence, Proposition 9 is regarded as the generalization of Matsushima (1990a).[8]

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[7] This is based on the idea of proper scoring rules. For the application of proper scoring rules to mechanism design, see Johnson, Pratt, and Zeckhauser (1990), Matsushima (1990b, 1993), Aoyagi (1998), and others.

[8] Arya, Glover, and Young (1995) also showed the possibility of uniquely and virtually implementing social choice functions in terms of twice iterative dominance on the private value assumption.

8. Virtual Implementation

Note that Condition 6 excludes a wide class of environments with interdependent values. The purpose of this section is to show that by using a modified version of direct mechanism, a social choice function is uniquely and, not exactly but virtually, implementable in terms of twice iterative dominance even in a wide class of environments with interdependent values.[9] We consider indirect mechanisms (g,x) where the set of messages for each player $i \in N$ is specified by

$$M_i = \Omega_i^2 .$$

Each player makes two announcements about her private signals at one time. We denote $m_i \equiv (m_i^1, m_i^2) \in M_i$ and $\phi_i \equiv (\phi_i^1, \phi_i^2) \in \Phi$, where

$$\phi_i^h : \Omega_i \to \Omega_i \quad \text{for each } h \in \{1,2\} .$$

The honest message profile is denoted by $\phi^{**} \in \Phi$, where

$$\phi_i^{**h}(\omega_i) = \omega_i \quad \text{for each } h \in \{1,2\} .$$

For every $\varepsilon > 0$, a mechanism (g,x) is said to satisfy the $\varepsilon -$*closeness to* f if for every $\omega \in \Omega$,

$$g(\phi^{**}(\omega))(f(\omega)) \geq 1-\varepsilon .$$

We introduce the following conditions on (p,f).

Condition 7: For every $i \in N$, every $\omega_i \in \Omega_i$, and every $\omega_i' \in \Omega_i / \{\omega_i\}$, there exists no $\beta \in R$ such that for every $a \in A$,

$$\sum_{\omega_{-i}\in\Omega_{-i}} u_i(a,\omega) p_i(\omega_{-i} \mid \omega_i) = \sum_{\omega_{-i}\in\Omega_{-i}} u_i(a,\omega_i',\omega_{-i}) p_i(\omega_{-i} \mid \omega_i') + \beta .$$

Condition 7 implies that each player has different preferences over pure alternatives across her private signals. Since Condition 7 is weaker than Condition 4, we can say that it holds in a wide class of environments with interdependent values. We introduce the following condition on p.

Condition 8: For every $i \in N$, there exists $\iota(i) \in N / \{i\}$ such that for every $\omega_i \in \Omega_i$, and every $\omega_i' \in \Omega_i / \{\omega_i\}$,

$$p_{i\iota(i)}(\cdot \mid \omega_i) \neq p_{i\iota(i)}(\cdot \mid \omega_i') .$$

Note that whenever $|\Omega_i| \geq 2$ for at least two players $i \in N$, then Condition 8 holds for generic prior distributions. The following proposition states that under Condition 7, for generic prior distributions, there exist mechanisms with budget-balancing that uniquely and virtually implement the social choice function in terms of twice iterative dominance.

Proposition 10: *Suppose that Conditions 7 and 8 hold. Then, for every* $\varepsilon > 0$*, there exists*

[9] There exist many papers on unique and virtual implementation with incomplete information such as Abreu and Matsushima (1992), Matsushima (1993), Duggan (1997), Serrano and Vohra (2000, 2001), and others.

an mechanism with budget-balancing (g,x) *satisfying the* $\varepsilon-$*closeness to* f *such that* ϕ^{**} *is the unique twice iteratively undominated message rule profile in* (p,g,x).

Proof: From Condition 7, it follows that for every $i\in N$, there exist $l_i:\Omega_i\to\Delta$ and $e_i:\Omega_i\to R$ such that for every $\omega_i\in\Omega_i$,

(26) $$\sum_{\omega_{-i}\in\Omega_{-i}} u_i(l_i(\omega_i),\omega)p_i(\omega_{-i}\mid\omega_i)+e_i(\omega_i)>\sum_{\omega_{-i}\in\Omega_{-i}} u_i(l_i(\omega_i'),\omega)p_i(\omega_{-i}\mid\omega_i)+e_i(\omega_i')$$

for all $\omega_i'\in\Omega_i\{\omega_i\}$.

We specify g by

$$g(m)=(1-\varepsilon)f(m^2)+\frac{\varepsilon}{n}\sum_{i\in N}l_i(m_i^1).$$

For every $i\in N/\{1\}$, we define a function $s_i:\Omega_{-\iota(i)}\to R$ in ways that for every $\omega_{-\iota(i)}\in\Omega_{-\iota(i)}$,

$$s_i(\omega_{-\iota(i)})=-\{1-p_{-i-\iota(i)}(\omega_{-i-\iota(i)}\mid\omega_i)\}^2-\sum_{\omega_i'\in\Omega_1/\{\omega_1\}}p_{-i-\iota(i)}(\omega_{-i-\iota(i)}\mid\omega_i')^2.$$

Condition 8 implies that for every $i\in N/\{1\}$, and every $\omega_i\in\Omega_i$,

(27) $$\sum_{\omega_{-i-\iota(i)}\in\Omega_{-i-\iota(i)}} s_i(\omega_{-\iota(i)})p_{i\iota(i)}(\omega_{-i-\iota(i)}\mid\omega_i)>\sum_{\omega_{-i-\iota(i)}\in\Omega_{-i-\iota(i)}} s_i(\omega_i',\omega_{-i-\iota(i)})p_{i\iota(i)}(\omega_{-i-\iota(i)}\mid\omega_i)$$

for all $\omega_i'\in\Omega_i/\{\omega_i\}$.

Fix a positive real number $k>0$ arbitrarily. We specify x in ways that for every $i\in N$, and every $m\in M$,

$$x_i(m)=\varepsilon e_i(m_i^1)-\frac{\varepsilon}{n-1}\sum_{j\in N/\{i\}}e_j(m_j^1)+k\{s_i(m_{-i-\iota(i)}^1,m_i^2)-\sum_{j\in N/\{i\}:i=\iota(j)}s_i(m_{-i-j}^1,m_j^2)\}$$

.

Note that (g,x) is budget balancing and satisfies the $\varepsilon-$closeness to f. From the inequalities (26), it follows that for every $i\in N$, and every $\phi\in\Phi$, if $\phi_i^1\neq\phi_i^{**1}$, then

$$\begin{aligned}&\sum_{\omega\in\Omega}\{u_i(g(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p(\omega)\\&-\sum_{\omega\in\Omega}\{u_i(g(\phi_{-i}(\omega_{-i}),\phi_i'(\omega_i)),\omega)+x_i(\phi_{-i}(\omega_{-i}),\phi_i'(\omega_i)))\}p(\omega)\\&=\varepsilon\sum_{\omega\in\Omega}\{u_i(l_i(\phi_i^1(\omega_i)),\omega)+e_i(\phi_i^1(\omega_i))-u_i(l_i(\omega_i),\omega)-e_i(\omega_i)\}p(\omega)<0,\end{aligned}$$

where $\phi_i'=(\phi_i^{**1},\phi_i^2)$. This implies that each player has strict incentive to make the honest announcement of her first message irrespective of what are the other players' messages announced. From the inequalities (27), it follows that for every sufficiently large k, every $\phi\in\Phi$, and every $i\in N$, if $\phi_j^1=\phi_j^{**1}$ for all $j\in N$, and $\phi_i^2\neq\phi_i^{**2}$, then

$$\begin{aligned}&\sum_{\omega\in\Omega}\{u_i(g(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p(\omega)\\&-\sum_{\omega\in\Omega}\{u_i(g(\phi_{-i}(\omega_{-i}),\phi_i^{**}(\omega_i)),\omega)+x_i(\phi_{-i}(\omega_{-i}),\phi_i^{**}(\omega_i)))\}p(\omega)\\&=\sum_{\omega\in\Omega}\{(1-\varepsilon)u_i(f(\phi^2(\omega)),\omega)+ks_i(\omega_{-i-\iota(i)},\phi_i^2(\omega_i)))\\&-(1-\varepsilon)u_i(f(\phi_{-i}^2(\omega),\omega_i),\omega)-ks_i(\omega_{-\iota(i)})\}p(\omega)<0.\end{aligned}$$

This implies that whenever all players make the honest announcements of their first messages, then each player has strict incentive to make the honest announcement of her second message irrespective of what are the other players' second messages announced. Hence, we have proved that ϕ^{**} is the unique twice iteratively undominated message rule profile in (p,g,x).

Q.E.D.

9. Triple Iterative Dominance

Note that Condition 7 excludes an important class of environments in which each player's private signal may have only information about payoff-irrelevant factors such as interim outside values. As Matsushima (2002) has shown, no inconstant social choice function is uniquely implementable whenever players' private signals have no payoff-relevant information, and therefore, players' *ex post* preference profile is common knowledge in the sense that for every $i \in N$, every $\omega \in \Omega$, and every $\omega' \in \Omega / \{\omega\}$, there exists $\beta \in R$ such that

$$u_i(\cdot, \omega') = u_i(\cdot, \omega) + \beta .$$

Serrano and Vohra (2001) also showed this point by providing an example with interim individual rationality. The purpose of this section is to show that by using another modification of direct mechanism, any inconstant social choice function is virtually implementable in terms of triple iterative dominance whenever players' interim preferences are not common knowledge. We introduce the following conditions on (p, f).

Condition 9: There exist a nonempty proper subset $D_1 \subset \Omega_1$, two lotteries $\hat{\alpha} \in \Delta$, $\tilde{\alpha} \in \Delta$, and two real numbers $\hat{t} \in R$ and $\tilde{t} \in R$ that satisfy the following properties.

(i) For every $\omega_1 \in D_1$,

$$\sum_{\omega_{-1} \in \Omega_{-1}} u_1(\hat{\alpha}, \omega) p_1(\omega_{-1} \mid \omega_1) + \hat{t} > \sum_{\omega_{-1} \in \Omega_{-1}} u_1(\tilde{\alpha}, \omega) p_1(\omega_{-1} \mid \omega_1) + \tilde{t} ,$$

and for every $\omega_1 \in \Omega_1 / D_1$,

$$\sum_{\omega_{-1} \in \Omega_{-1}} u_1(\hat{\alpha}, \omega) p_1(\omega_{-1} \mid \omega_1) + \hat{t} < \sum_{\omega_{-1} \in \Omega_{-1}} u_1(\tilde{\alpha}, \omega) p_1(\omega_{-1} \mid \omega_1) + \tilde{t} .$$

(ii) For every $i \in N / \{1\}$, every $\omega_i \in \Omega_i$, and every $\omega_i' \in \Omega_i / \{\omega_i\}$,

$$p_i^1(D_1 \mid \omega_i) \neq p_i^1(D_1 \mid \omega_i') .$$

The property (i) of Condition 9 implies that each player's *interim* preference is not common knowledge. Note that with private values, players' *ex post* preferences are not common knowledge only if the property (i) holds. Note that the property (ii) holds for generic prior distributions. We introduce the following condition on p.

Condition 10: There exists $\iota(1) \in N / \{1\}$ such that for every $\omega_1 \in \Omega_1$, and every $\omega_1' \in \Omega_1 / \{\omega_1\}$,

$$p_{1\iota(1)}(\cdot \mid \omega_1) \neq p_{1\iota(1)}(\cdot \mid \omega_1') .$$

Note that Condition 10 holds for generic prior distributions. Serrano and Vohra (2000) showed a condition on the prior distribution under which every incentive compatible social choice function is virtually implementable in Bayesian Nash equilibrium. Conditions 9 and 10 are weaker than this condition.

We consider indirect mechanisms (g, x) where the set of messages for each player $i \in N / \{1\}$ is specified by

$$M_i = \Omega_i ,$$

and the set of messages for player 1 is specified by

$$M_1 = M_1^1 \times M_1^2 = \{0,1\} \times \Omega_1 .$$

Each player other than player 1 makes a single announcement about her private signal like in a direct mechanism. Player 1 not only makes a single announcement about her private signal but also announces either 0 or 1. We denote $m_1 = (m_1^1, m_1^2) \in M_1$ and $\phi_1 = (\phi_1^1, \phi_1^2) \in \Phi$, where

$$\phi_1^1 : \Omega_1 \to \{0,1\} \text{ and } \phi_1^2 : \Omega_1 \to \Omega_1 .$$

The honest message profile is denoted by $\hat{\phi} \in \Phi$, where $\hat{\phi}_i = \phi_i^*$ for all $i \in N/\{1\}$,

$$\hat{\phi}_1^2(\omega_1) = \omega_1 \text{ for all } \omega_1 \in \Omega_1 ,$$

$$\hat{\phi}_1^1(\omega_1) = 0 \text{ for all } \omega_1 \in D_1 ,$$

and

$$\hat{\phi}_1^1(\omega_1) = 1 \text{ for all } \omega_1 \in \Omega_1 / D_1 .$$

The following proposition states that under Conditions 9 and 10, for generic prior distributions there exist indirect mechanisms with budget-balancing that virtually implement the social choice function f in terms of triple iterative dominance.

Proposition 11: *Suppose that Conditions 9 and 10 hold. Then, for every $\varepsilon > 0$, there exists an indirect mechanism with budget-balancing (g, x) satisfying the $\varepsilon-$closeness to f such that $\hat{\phi}$ is the unique triple iteratively undominated message rule profile in (p, g, x).*

Proof: Let

$$l(0) \equiv \hat{\alpha} \text{ and } l(1) \equiv \tilde{\alpha} .$$

We specify g by

$$g(m) = (1-\varepsilon) f(m_1^2, m_{-1}) + \varepsilon l(m_1^1) .$$

For every $i \in N/\{1\}$, we define a function $s_i : \{0,1\} \times \Omega_i \to R$ in ways that for every $\omega_i \in \Omega_i$,

$$s_i(0,\omega_i) = -\{1 - p_i^1(D_1 \mid \omega_i)\}^2 - \sum_{\omega_i' \in \Omega_i / \{\omega_i\}} p_i^1(D_1 \mid \omega_i')^2 ,$$

and

$$s_i(1,\omega_i) = -\{1 - p_i^1(\Omega_1 / D_1 \mid \omega_i)\}^2 - \sum_{\omega_i' \in \Omega_i / \{\omega_i\}} p_i^1(\Omega_1 / D_1 \mid \omega_i')^2 .$$

We define a function $s_1 : \Omega_{-\iota(1)} \to R$ in ways that for every $\omega_{-\iota(1)} \in \Omega_{-\iota(1)}$,

$$s_1(\omega_{-\iota(1)}) = -\{1 - p_{1\iota(1)}(\omega_{-1-\iota(1)} \mid \omega_1)\}^2 - \sum_{\omega_1' \in \Omega_1 / \{\omega_1\}} p_{1\iota(1)}(\omega_{-1-\iota(1)} \mid \omega_1')^2 .$$

The property (ii) of Condition (9) implies that for every $i \in N/\{1\}$, and every $\omega_i \in \Omega_i$,

(28) $$s_i(0,\omega_i) p_i^1(D_1 \mid \omega_i) + s_i(1,\omega_i) p_i^1(\Omega_1 / D_1 \mid \omega_i)$$
$$> s_i(0,\omega_i) p_i^1(D_1 \mid \omega_i') + s_i(1,\omega_i) p_i^1(\Omega_1 / D_1 \mid \omega_i') \text{ for all } \omega_i' \in \Omega_i / \{\omega_i\} .$$

Condition (10) implies that for every $\omega_1 \in \Omega_1$,

(29) $$\sum_{\omega_{-1-\iota(1)}\in\Omega_{-1-\iota(1)}} s_1(\omega_{-\iota(1)})p_{1\iota(1)}(\omega_{-1-\iota(1)}\mid\omega_1) > \sum_{\omega_{-1-\iota(1)}\in\Omega_{-1-\iota(1)}} s_1(\omega_1',\omega_{1-\iota(1)})p_{1\iota(1)}(\omega_{-1-\iota(1)}\mid\omega_1)$$

for all $\omega_1'\in\Omega_1/\{\omega_1\}$.

Let

$$e(0)\equiv\hat{t} \text{ and } e(1)\equiv\tilde{t}.$$

Fix a positive real number $k>0$ arbitrarily, and we specify x in ways that for every $i\in N/\{1,\iota(1)\}$, and every $m\in M$,

$$x_i(m) = -\frac{\varepsilon e(m_1^1)}{n-2} + k\{s_i(m_1^1,m_i) - \frac{\sum_{j\in N/\{i,1\}} s_j(m_1^1,m_j)}{n-2}\},$$

$$x_1(m) = \varepsilon e(m_1^1) + ks_1(m_1^2, m_{-1-\iota(1)}),$$

and

$$x_{\iota(1)}(m) = k\{s_{\iota(1)}(m_1^1,m_{\iota(1)}) - s_1(m_1^2,m_{-1-\iota(1)}) - \frac{\sum_{j\in N/\{1,\iota(1)\}} s_j(m_1^1,m_j)}{n-2}\}.$$

Note that (g,x) is budget balancing and satisfies the $\varepsilon-$closeness to f. From the property (i) of Condition 9, it follows that for every $\phi\in\Phi$, if $\phi_1^1\neq\hat{\phi}_1^1$, then

$$\begin{aligned}&\sum_{\omega\in\Omega}\{u_1(g(\phi(\omega)),\omega)+x_1(\phi(\omega))\}p(\omega)\\&-\sum_{\omega\in\Omega}\{u_1(g(\phi_{-1}(\omega_{-1}),\phi_1'(\omega_1)),\omega)+x_1(\phi_{-1}(\omega_{-1}),\phi_1'(\omega_1)))\}p(\omega)\\&=\varepsilon\sum_{\omega\in\Omega}\{u_1(l(\phi_1^1(\omega_i)),\omega)+e(\phi_1^1(\omega_1))-u_1(l_1(\omega_1),\omega)-e(\omega_1)\}p(\omega)<0,\end{aligned}$$

where $\phi_1'=(\hat{\phi}_1^1,\phi_i^2)$. This implies that player 1 has strict incentive to make the honest announcement of her first message. From the inequalities (28), it follows that for every sufficiently large k, every $\phi\in\Phi$, and every $i\in N/\{1\}$, if $\phi_1^1=\hat{\phi}_1^1$, and $\phi_i\neq\hat{\phi}_i$, then

$$\begin{aligned}&\sum_{\omega\in\Omega}\{u_i(g(\phi(\omega)),\omega)+x_i(\phi(\omega))\}p(\omega)\\&-\sum_{\omega\in\Omega}\{u_i(g(\phi_{-i}(\omega_{-i}),\hat{\phi}_i(\omega_i)),\omega)+x_i(\phi_{-i}(\omega_{-i}),\hat{\phi}_i(\omega_i))\}p(\omega)\\&=\sum_{\omega\in\Omega}\{u_i(g(\phi(\omega)),\omega)+ks_i(\hat{\phi}_1^1(\omega_1),\phi_i(\omega_i))\\&-u_i(g(\phi_{-i}(\omega_{-i}),\hat{\phi}_i(\omega_i)),\omega)-ks_i(\hat{\phi}_1^1(\omega_1),\hat{\phi}_i(\omega_i))\}p(\omega)<0.\end{aligned}$$

This implies that whenever player 1 make the honest announcement of her first message, then each player except player 1 has strict incentive to make the honest announcement. From the inequalities (29), it follows that for every sufficiently large k, and every $\phi\in\Phi$, if $\phi_1^1=\hat{\phi}_1^1$, and $\phi_i=\hat{\phi}_i$ for all $i\in N/\{1\}$, then

$$\begin{aligned}&\sum_{\omega\in\Omega}\{u_1(g(\phi(\omega)),\omega)+x_1(\phi(\omega))\}p(\omega)\\&-\sum_{\omega\in\Omega}\{u_1(g(\phi_{-1}(\omega_{-1}),\phi_1'(\omega_1)),\omega)+x_1(\phi_{-1}(\omega_{-1}),\phi_1'(\omega_1)))\}p(\omega)\\&=\sum_{\omega\in\Omega}\{u_1(g(\phi(\omega)),\omega)+ks_1(\phi_1^2(\omega_1),\hat{\phi}_{-1-\iota(1)}(\omega_{-1-\iota(1)}))\\&-u_1(g(\hat{\phi}(\omega)),\omega)-ks_1(\hat{\phi}(\omega))\}p(\omega)<0.\end{aligned}$$

Hence, we have proved that $\hat{\phi}$ is the unique triple iteratively undominated message rule profile in (p, g, x).

Q.E.D.

10. Conclusion

This paper investigated mechanism design with incomplete information and quasi-linearity. We showed that with three or more players and with a restriction on the size of the set of private signals for each player, there exists a side payment function that satisfies budget balancing, incentive compatibility, and interim individual rationality. We showed that in agency problems with adverse selection, the risk-averse principal could extract the full surplus without harming the agents' incentive compatibility and interim individual rationality. These possibility results depended on the assumptions that there exist three or more players and their private signals are correlated. We showed that the full surplus extraction might be impossible either when there exists only two players or when players' private signals are independent. We also investigated the possibility of uniquely implementing social choice functions by practicing only a small number of iterative removals of undominated strategies. We showed that whenever players' interim preferences are not common knowledge, then for generic prior distributions, every social choice function is uniquely and virtually implementable in terms of triple iterative dominance via a simple modification of direct mechanism.

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