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On the Structure, Asymptotic Theory and Applications of STAR-GARCH Models

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Abstract

Non-linear time series models, especially regime-switching models, have become increasingly popular in the economics, finance and financial econometrics literature. However, much of the research has concentrated on the empirical applications of various models, with little theoretical or statistical analysis associated with the structure of the models or asymptotic theory. Some structural and statistical properties have recently been established for the Smooth Transition Autoregressive (STAR) - Generalised Autoregressive Conditional Heteroscedasticity (GARCH), or STAR-GARCH, model, including the necessary and sufficient conditions for the existence of moments, and the sufficient condition for consistency and asymptotic normality of the (Quasi)-Maximum Likelihood Estimator ((Q)MLE). While these moment conditions are straightforward to verify in practice, they may not be satisfied for the GARCH model if the underlying long run persistence is close to unity. A less restrictive condition for consistency and asymptotic normality may alleviate this problem. The paper establishes a weak sufficient, or log-moment, condition for consistency and asymptotic normality of (Q)MLE for STAR-GARCH. This condition can easily be extended to any non-linear conditional mean model with GARCH errors, subject to reasonable regularity conditions. Although the log-moment condition cannot be verified as easily as the second and fourth moment conditions, it allows the long run persistence of the GARCH process to exceed one. Monte Carlo experiments show that the log-moment condition is more reliable in practice than the second and fourth moment conditions when the underlying long run persistence is close to unity. These experiments also show that the correct specification of the conditional mean is crucial in obtaining unbiased estimates for the GARCH component. The sufficient conditions for consistency and asymptotic normality are verified empirically using S&P 500 returns, 3-month US Treasury Bill returns, and exchange rates between Australia and the USA. The effects of outliers and extreme observations on the empirical moment conditions are also analysed in detail.

1 Introduction

Engle's (1982) Autoregressive Conditional Heteroscedasticity (ARCH) model and Bollerslev's (1986) Generalised ARCH (GARCH) model are the most popular models for capturing time-varying symmetric volatility in financial and economic time series data. Despite their popularity, the structural and statistical properties of these models were not fully established until recently.

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However, most of the theoretical results on GARCH models have assumed a constant or linear conditional mean, and it has not yet been established whether those results would also hold if the conditional mean were non-linear.

Ling and McAleer (2003) proposed a multivariate ARMA - GARCH model, and established its structural and statistical properties. Jeantheau (1998) established consistency results of estimators for multivariate GARCH models. His proofs of consistency did not assume a particular functional form for the conditional mean, but assumed a log-moment condition and some regularity conditions for purposes of identification. Chan and McAleer (2002) established the structural and statistical properties for the GARCH components in the Smooth Transition Autoregressive - GARCH (STAR-GARCH) model. They showed that the results in Ling (1999) and Ling and McAleer (2002a, b, 2003) also applied in the case of STAR-GARCH, including the necessary and sufficient conditions for the existence of moments, and a sufficient condition for consistency and asymptotic normality of the (Quasi-) Maximum Likelihood Estimator ((Q)MLE).

This paper extends the results of Elie and Jeantheau (1995), Jeantheau (1998), Boussama (2000) and Chan and McAleer (2002), and shows that a weaker log-moment condition derived by Bougerol and Picard (1992) is sufficient to ensure consistency and asymptotic normality of the (Q)MLE for the GARCH component in a STAR-GARCH model. Moreover, the results of this paper can easily be extended to a wide class of non-linear time series models with GARCH errors, subject to appropriate regularity conditions.

Since the existence of the second and fourth moments of the unconditional shocks implies consistency and asymptotic normality of (Q)MLE for the GARCH model (see Ling (1999), Ling and McAleer (2002a, b, 2003), verifying the moment conditions is a diagnostic check regarding the adequacy of the estimator. It is important to note that the moment conditions are functions of the true parameters, and these must be estimated in practice. The (Q)MLE of the parameters are often used to verify these moment conditions by direct substitution (see McAleer et al. (2003) and Hoti et al. (2002) for univariate and multivariate models, respectively), but the reliability of this approach has not yet been investigated.

This paper also conducts two Monte Carlo experiments. Experiment 1 shows that the empirical version of the fourth moment condition established in Ling and McAleer (2002b) (see also Chan and McAleer (2002)) for asymptotic normality can easily be violated if the true long run persistence in the GARCH component is close to, but less than, unity. The advantage of the log-moment condition is that it allows the long run persistence to exceed one, and thus provides a more reliable means of checking consistency and asymptotic normality for the QMLE. Experiment 2 investigates the effects on the empirical moment conditions of the conditional mean being misspecified, and shows that the correct specification of the conditional mean is important when verifying the moment conditions using QMLE.

Finally, the Logistic STAR-GARCH (LSTAR - GARCH) and Exponential STAR-GARCH (ESTAR - GARCH) models are estimated using S&P 500 Composite Returns, 3-month US Treasury

Bill returns, and the exchange rate between the USA and Australia. The rolling empirical log-moment and second and fourth moment conditions, and their sensitivity to outliers and extreme observations, are also examined in detail.

The plan of the paper is as follows: Section 2 provides a brief review of the GARCH and STAR-GARCH models, with a particular emphasis on their theoretical developments. A new theoretical result regarding the statistical properties of the QMLE for STAR-GARCH is also established. This is followed by two Monte Carlo experiments in Section 3. The empirical results are presented in Section 4, and Section 5 gives some concluding remarks.

2 The Models

This section discusses some of the most recent theoretical results on the GARCH, STAR and STAR-GARCH models. Definitions, regularity conditions and sufficient conditions for the existence of moments, stationarity and ergodicity, and sufficient conditions for consistency and asymptotic normality of the QMLE for these models, will be discussed in detail. A new and weaker sufficient condition for consistency and asymptotic normality for the QMLE of the STAR-GARCH model will also be presented.

Let (Ω, A, P) be a probability space, $\{y_t, t \in \mathbb{Z}\}$ an $\mathbb{R}$-valued process, and $\theta = (\phi, \omega, \alpha, \beta)'$ a parameter in $\Theta \in \mathbb{R}^{k+1}$, so that $\phi = (\phi_1, \phi_2, ..., \phi_r)'$, $\alpha = (\alpha_1, ..., \alpha_p)'$, $\beta = (\beta_1, ..., \beta_q)'$, $r+p+q = k$, and θ_0 denote the true parameter vector. Define y_t as a discrete-time stochastic process with generalised conditional heteroscedastic errors if, $\forall\ t \in \mathbb{Z}$,

$$y_t = f(x_t; \phi) + \varepsilon_t \tag{2.1}$$

$$\varepsilon_t = \eta_t \sqrt{h_t}, \qquad \eta_t \sim \text{iid}(0, 1) \tag{2.2}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}, \tag{2.3}$$

where $x_t = (y_{t-1}, y_{t-2}, ..., \varepsilon_{t-1}, \varepsilon_{t-2}, ..., z_t)'$ and z_t is a $1 \times g$ vector of exogenous. Moreover, it is assumed that $\alpha_i > 0$ for all $i = 1, ..., p$ and $\beta_i > 0$ for all $i = 1, ..., q$ to ensure the positivity of h_t. When $q = 0$, equation (2.3) reduces to Engle's (1982) ARCH(p) process.

Define the likelihood function to be

$$l(\theta) = -\frac{1}{2T} \sum_{t=1}^{T} \big(\log h_t + \frac{\varepsilon_t^2}{h_t} \big). \tag{2.4}$$

The maximum likelihood estimator (MLE) for the model defined in equations (2.1) - (2.3) is the solution to the following maximisation problem:

$$\hat{\theta} = \text{argmax}_{\theta \in \Theta} l(\theta), \tag{2.5}$$

if η_t is normally distributed. Otherwise, $\hat{\theta}$ is defined as the Quasi-MLE (QMLE).

For $q = 0$ and $f(x_t;\phi) = \phi' x_t$, where c is a constant, and by assuming that the process ε_t started infinitely far in the past with finite $2m$th moment, Engle (1982) showed that ε_t is second-order stationary if and only if all the roots of the characteristic polynomial $(1 - \sum_{i=1}^{p} \alpha_i z^i) = 0$ lie outside the unit circle.

Milhøj (1985) avoided Engle's assumption and showed that

$$\sum_{i=1}^{p} \alpha_i < 1$$

is necessary and sufficient for ε_t to be second-order stationary. Furthermore, Milhøj (1985) also derived the regularity condition for the existence of moments without the restrictive assumption. The result is identicial to that of Engle in the case of ARCH(1) with normal η_t, but cannot be given an explicit form in the case of ARCH(p) and $m > 2$.

Weiss (1986) and Pantula (1989) showed that $E(\varepsilon_t^4) < \infty$ is sufficient for consistency and asymptotic normality for QMLE in the case of $q = 0$ and $f(x_t;\phi) = c$. This result was further improved by Ling and McAleer (2003), who showed that the QMLE is consistent and asymptotically normal if $E(\varepsilon_t^2) < \infty$.

For $q > 0$ and $f(x_t;\phi) = \phi' x_t$, Bollerslev (1986) showed that the necessary and sufficient condition for (2.2) to be second-order stationary is

$$\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{q} \beta_i < 1.$$

Under the assumption that $\beta_1 > 0$, Nelson (1990) derived the necessary and sufficient condition for stationarity and ergodicity of GARCH(1,1) to be

$$E(\log(\alpha_1 \eta_t^2 + \beta_1)) < 0, \qquad \beta_1 > 0. \tag{2.6}$$

This condition is not easy to apply in practice as it involves the expectation of an unknown random variable and unknown parameters. However, it is attractive because the condition allows the long run persistence, namely $\alpha_1 + \beta_1$, to be greater than one. Therefore, it is a stronger result based on a weaker condition than Bollerslev's.

The log-moment condition in (2.6) was extended to GARCH(p,q) with $f(x_t;\phi) = 0$ by Bougerol and Picard (1992). They showed that $E(\log(\alpha_1 \eta_t^2 + \beta_1))$ is, in fact, the Liapunov exponent for GARCH(1,1), and that the negativity of the associated Liapunov exponent is necessary and sufficient for strict stationarity and ergodicity of GARCH(p,q).

Ling and McAleer (2002b) established the sufficient condition of the stationary solution of a family of GARCH(1,1) models investigated by He and Terašvirta (1999a) with $f(x_t;\phi) = 0$. Ling and McAleer (2002b) showed that the moment condition in He and Teräsvirta (1999a) is necessary but not sufficient, and provided the sufficient condition. He and Teräsvirta (1999b) also investigated the fourth moment structure of the GARCH(p,q) process with $f(x_t;\phi) = 0$. In the case of GARCH(1,1), the fourth moment condition under normality of η_t is

$$(\alpha_1 + \beta_1)^2 + 2\alpha_1^2 < 1. \tag{2.7}$$

Ling (1999) obtained a sufficient condition for the existence of the $2m$th moment for a double threshold ARMA conditional heteroscedasticity model (DTARMACH). This model includes the ARCH, GARCH, TAR, TAR-ARCH, TAR-GARCH and TARMA models as special cases. Although Ling (1999) only applied his results to GARCH(p, q) with $f(x_t; \phi) = 0$ and threshold ARMA models with independently and identically distributed innovations, it is clear that the same result holds for TARMA(r, s)-GARCH(p, q) (see Ling and McAleer (2003) and Chan and McAleer (2002)), that is, a GARCH(p, q) process with

$$f(x_t; \phi) = \phi_0^{(0)} + \sum_{i=1}^{r_j} \phi_{1i}^{(j)} y_{t-i} + \sum_{i=1}^{s_j} \phi_{2i}^{(j)} \varepsilon_{t-i} + \varepsilon_t, \qquad a_{j-1} < y_{t-b} \leq a_j,$$

where $j = 1, ..., v_1$, b is the delay parameter, the threshold parameters satisfy $-\infty = a_0 < a_1 < ... < a_{v1} = \infty$, and the coefficients $\phi_{1i}^{(j)}$ and $\phi_{2i}^{(j)}$ are constant.

The sufficient condition derived in Ling (1999) is also necessary for the existence of the $2m$th moment for GARCH(p, q) when $f(x_t; \phi)$ follows a general Autoregressive Moving Average (ARMA(r, s)) process (see Ling and McAleer (2002a, 2003)). Furthermore, Ling and McAleer (2002a) also derived the necessary and sufficient moment conditions of the asymmetric power GARCH(p, q) model of Ding et al. (1993).

Estimation of the parameters of the ARMA(r, s)-GARCH(p, q) model is typically by MLE, or by QMLE when η_t is not normal. Ling and Li (1997) showed that the local QMLE for GARCH(p, q) is consistent and asymptotically normal if $E(\varepsilon_t^4) < \infty$. For the global QMLE, Ling and McAleer (2002a) showed that $E(\varepsilon_t^2) < \infty$ is sufficient for consistency, and $E(\varepsilon_t^6) < \infty$ is sufficient for asymptotic normality. Elie and Jeantheau (1995) and Jeantheau (1998) showed that the log-moment condition is sufficient for consistency of QMLE for GARCH(p, q), while Boussama (2000) proved that the same condition is also sufficient for asymptotic normality.

McAleer et al. (2003) showed that the moment conditions in Ling and McAleer (2002a, 2003), and the conditions for consistency and asymptotic normality in Ling and McAleer (2003), Elie and Jeantheau (1995), Jeantheau (1998) and Boussama (2000), also hold for the Glosten et al. (1992) (GJR) asymmetric GARCH model. Hoti et al. (2002) established a multivariate GJR model, and showed that the moment conditions and consistency results in Ling and McAleer (2003), the consistency results in Elie and Jeantheau (1995) and Jeantheau (1998), and the asymptotic normality results in Ling and McAleer (2003), also hold for their model.

Chan and Tong (1986) and Terašvirta (1994) extended the Threshold Autoregressive (TAR) model of Tong (1978) and Tong and Lim (1980) to allow for smooth transition behaviour, that is,

$$y_t = \sum_{i=1}^{m} \phi_i' x_t (G_{i-1}(s_t; \gamma_{i-1}, c_{i-1}) - G_i(s_t; \gamma_i, c_i)) + \varepsilon_t, \qquad \varepsilon_t \sim iid(0, \sigma^2), \tag{2.8}$$

where $\phi_i = (\phi_{i1}, ..., \phi_{ir})'$. $G_i(s_t; \gamma, c)$ are often called transition functions, which are required to be at least twice differentiable and range from zero to one, s_t is the threshold variables, γ_i is the transition rate, which reflects the speed of switching from one regime to another, and c_i is the threshold value, with $c_{i-1} < c_i$ for all $i = 1, ..., m$. A comprehensive survey of recent developments of this model can be found in van Dijk et al. (2002).

The most popular choice of transition functions, $G(s_t; \gamma, c)$, are the logistic function given by

$$G(s_t; \gamma, c) = \frac{1}{1 + \exp(-\gamma(s_t - c))}, \tag{2.9}$$

and the exponential function given by

$$G(s_t; \gamma, c) = 1 - \exp(-\gamma(s_t - c)^2). \tag{2.10}$$

A two-regime ($m = 2$) STAR model with a logistic (exponential) transition function is called an LSTAR (ESTAR) model. STAR models, especially LSTAR models, have been successfully applied in a number of areas. Teräsvirta and Anderson (1992) and Teräsvirta et al. (1994) characterised the different dynamics of industrial production indexes for various OECD countries during expansions and recessions using LSTAR models. Moreover, Lundbergh and Teräsvirta (2000) examined the forecast performance of the LSTAR model for unemployment rates in Denmark and Australia, arguing that many unemployment rates exhibit asymmetries in that the rate of increase is often higher than the rate of decrease. Their results showed that the STAR model is superior to its AR counterpart.

A STAR-GARCH model allows ε_t in equation (2.8) to follow a GARCH process, as defined in (2.2)-(2.3) or, equivalently, by setting $f(x_t; \phi)$ to follow a STAR process, as defined in (2.8). Lundbergh and Teräsvirta (1999) give a comprehensive exposition of this model, but do not provide any regularity conditions for stationarity or the existence of moments, or any statistical properties. Recently, Chan and McAleer (2002) showed that the results in Ling (1999) and Ling and McAleer (2002a, b, 2003) also hold for STAR-GARCH. They showed that $E(\varepsilon_t^2) < \infty$ is sufficient for consistency and $E(\varepsilon_t^4) < \infty$ is sufficient for asymptotic normality for the QMLE of STAR-GARCH. Chan and McAleer (2003) investigated the effects of outliers and extreme observations on the QMLE of the STAR-GARCH model.

A less restrictive condition, namely the log-moment condition, is given below for the consistency and asymptotic normality of QMLE for the STAR-GARCH model with $p = q = 1$.

Proposition 1: *Denote $\hat{\theta}$ as the solution to the maximisation problem as defined in (2.5), with $p = q = 1$ in (2.3). Under strict stationarity and ergodicity (see Proposition 1 in Chan and McAleer (2002)), and $E(\log(\alpha_1\eta_t^2 + \beta_1)) < 0$, it follows that $\hat{\theta}$ is consistent for θ_0 and asymptotically normal.*

Proof: The proof of consistency is similar to that of Elie and Jeantheau (1995) and Jeantheau

(1998), with the conditional mean replaced by a stationary univariate STAR process. The proof of asymptotic normality is similar to that of Boussama (2000), with the conditional mean replaced by a stationary univariate STAR process. ■

Corollary 1: *If $E(\varepsilon_t^2) < \infty$, it follows that $\hat{\theta}$ is consistent for θ_0 and asymptotically normal.*

Proof: The second moment condition implies the log-moment condition, and hence the result in Proposition 1. This completes the proof. ■

The theoretical results presented above will be analysed in two Monte Carlo experiments in Section 3

3 Monte Carlo Experiments

This section reports the results of two Monte Carlo experiments. Unless stated otherwise, all the STAR-GARCH models considered have the following specification:

$$\begin{aligned} y_t &= (\phi_{10} + \phi_{11} y_{t-1})(1 - G(y_{t-1};\gamma,c)) + (\phi_{20} + \phi_{21} y_{t-1})G(y_{t-1};\gamma,c) + \varepsilon_t, \\ \varepsilon_t &= \eta_t \sqrt{h_t}, \qquad \eta_t \sim NID(0,1), \\ h_t &= \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} \end{aligned} \tag{3.1}$$

where the transition function, $G(y_{t-1};\gamma,c)$, is either the logistic function, as defined in equation (2.9), or the exponential function, as defined in equation (2.10). The number of replications is 1000, with 3000 observations used throughout.

The following table contains the two sets of parameters used in the data generating process (DGP):

	ϕ_{10}	ϕ_{11}	ϕ_{20}	ϕ_{21}	γ	c	ω	α	β
Set I	0.1	0.9	-0.3	-0.9	1	0	0.01	0.2	0.75
Set II	0.1	0.9	-0.3	-0.9	1	0	0.0001	0.09	0.9

Table 1: Parameter Values

The NID random error, η_t, is generated by a normal random generator written in Ox version 3, with zero mean and unit variance. Furthermore, all estimation routines are written in Ox, with initial values set to the true values.

3.1 Experiment 1: Empirical Moment Condition

The aim of the first experiment is to show that the fourth moment condition can easily be violated if the true long run persistence in the GARCH model is close to, but less than, unity. In these cases, the log-moment and second moment conditions provide more reliable information than the fourth moment regarding the statistical properties of the QMLE.

Recall the necessary and sufficient condition for $E(\varepsilon_t^2) < \infty$ for GARCH(1,1) is

$$\alpha + \beta < 1. \tag{3.2}$$

As α and β are unknown in practice, McAleer et al. (2003) and Hoti et al. (2002) suggest examining

$$\hat{\alpha} + \hat{\beta} < 1, \tag{3.3}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the QMLE of α and β, respectively. Equation (3.3) is the empirical second moment condition. Similarly, the empirical fourth moment condition under normality is

$$(\hat{\alpha} + \hat{\beta})^2 + 2\hat{\alpha}^2 < 1, \tag{3.4}$$

and the empirical log-moment condition is

$$T^{-1} \sum_{t=1}^{T} \log(\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}) < 0, \tag{3.5}$$

where $\hat{\eta}_t$ is the estimated standardised residual.

The parameter space that would satisfy the fourth moment condition is clearly smaller than for the second moment. Moreover, when the long run persistence, namely $\alpha + \beta$, is close to unity, the empirical fourth moment condition may not be satisfied, even though the fourth moment condition based on the true parameters is satisfied. This is particularly important as many financial time series exhibit long run persistence that is close to unity, and obtaining statistical properties of the estimators is important for purposes of statistical inference. Thus, examining the performance of the empirical moment conditions should be a useful practical exercise.

The steps for Experiment 1 are as follows:

Step 1. Generate data using a STAR-GARCH model, as defined in equation (3.1).

Step 2. The MLE is obtained by estimating the true model.

Step 3. The estimates are then substituted into the second and fourth moment conditions to obtain their respective empirical counterparts.

Step 4. The empirical log-moment condition is obtained by using the MLE and the empirical standardised residuals.

The above steps are repeated for both LSTAR-GARCH and ESTAR-GARCH for the two parameter sets, as listed in Table 1, and the summary statistics of these moment conditions for each model are given in Tables 2 - 5.

Statistics	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.018	0.286	0.813	-0.051	0.987	1.137
Minimum	0.006	0.139	0.648	-0.163	0.886	0.864
Mean	0.010	0.201	0.746	-0.087	0.947	0.979
Variance	3.25E-06	3.94E-04	5.27E-04	2.53E-04	1.84E-04	9.95E-04
Skewness	0.650	0.041	-0.214	-0.680	-0.553	-0.041
Kurtosis	3.761	2.980	3.152	3.918	3.709	3.584

Table 2: Summary Statistics for LSTAR-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set I

Statistics	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.018	0.285	0.814	-0.052	0.987	1.136
Minimum	0.006	0.138	0.649	-0.163	0.886	0.865
Mean	0.010	0.202	0.746	-0.087	0.947	0.979
Variance	3.29E-06	3.96E-04	0.001	2.55E-04	1.86E-04	0.001
Skewness	0.638	0.030	-0.202	-0.666	-0.544	-0.049
Kurtosis	3.721	2.944	3.138	3.845	3.668	3.548

Table 3: Summary Statistics for ESTAR-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set I

Statistics	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	2.19E-04	0.133	0.922	-0.007	1.001	1.028
Minimum	5.04E-05	0.062	0.854	-0.043	0.963	0.939
Mean	1.12E-04	0.094	0.894	-0.019	0.988	0.995
Variance	8.38E-10	1.39E-04	1.47E-04	3.16E-05	2.88E-05	1.47E-04
Skewness	0.649	0.066	-0.050	-0.892	-0.939	-0.549
Kurtosis	3.397	2.768	2.660	4.317	4.602	3.901

Table 4: Summary Statistics for LSTAR-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set II

Statistics	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	2.58E-04	0.148	0.930	-0.007	1.006	1.056
Minimum	4.87E-05	0.060	0.850	-0.044	0.962	0.939
Mean	1.12E-04	0.095	0.894	-0.019	0.988	0.995
Variance	8.75E-10	1.31E-04	1.39E-04	3.13E-05	2.89E-05	1.48E-04
Skewness	0.725	0.132	-0.093	-0.807	-0.746	-0.302
Kurtosis	3.928	3.287	3.104	4.010	4.102	4.050

Table 5: Summary Statistics for ESTAR-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set II

As shown in Table 2, the log-moment and second and fourth moment conditions perform well, on average, for LSTAR-GARCH. The means of the second and fourth moment conditions are 0.947 and 0.979, respectively, which are very close to their respective true values, namely 0.95 and 0.9825.

A similar conclusion can be drawn for ESTAR-GARCH, as shown in Table 3. Interestingly, the mean empirical log-moment and second and fourth moment conditions from both models are equal to 3 decimal places. It is important to note that the maximum values of the fourth moment conditions in LSTAR-GARCH and ESTAR-GARCH are 1.137 and 1.136, respectively. In fact, there are 244 and 247 replications which failed to satisfy the empirical fourth moment for LSTAR-GARCH and ESTAR-GARCH, respectively.

However, the empirical log-moment and second moment conditions were satisfied in both cases. This seems to suggest that the empirical fourth moment condition may be too restrictive to satisfy in practice. Such a result reflects the importance of Proposition 1 and Corollary 1, namely, the existence of the second moment, and hence the existence of the log-moment condition, are sufficient to ensure consistency and asymptotic normality for the purpose of valid inference.

Although the second moment condition is more restrictive than the log-moment condition, it has the advantage of computational simplicity. It is more computational intensive to obtain the empirical log-moment condition, whereas the empirical second moment condition for GARCH(1,1) is merely the sum of the ARCH and GARCH coefficients.

As the long run persistence approaches unity, the empirical second moment condition will also suffer from the same problem as the empirical fourth moment condition. In order to investigate the seriousness of this issue for the empirical second moment, Experiment 1 is repeated using Parameter Set II in which the true long run persistence is 0.99.

Tables 4-5 present the summary statistics of the empirical moment conditions for all replications. It is worth noting that the true fourth moment condition for Parameter Sets I and II are 0.9825 and 0.9963, respectively. Interestingly, the maximum empirical fourth moment condition from both LSTAR-GARCH and ESTAR-GARCH using Parameter Set I is larger than those using Set II. However, the number of replications failing to satisfy the fourth moment condition has increased for both models, as expected. There are 356 and 352 replications which fail to meet the fourth moment condition for LSTAR-GARCH and ESTAR-GARCH, respectively. Thus, the number of replications failing to satisfy the empirical fourth condition increases as the long run persistence approaches unity, regardless of the specification of the conditional mean.

Moreover, there are 2 and 3 replications which fail to satisfy the empirical second moment condition for LSTAR-GARCH and ESTAR-GARCH, respectively. As the true long run persistence is very close to unity, the performance of the empirical second moment condition seems to be reasonable. The more computationally intensive log-moment condition is still performing well for Parameter Set II. As the maximum empirical log-moment condition is -0.007 for both LSTAR-GARCH and ESTAR-GARCH, all replications satisfy the log-moment condition.

3.2 Experiment 2: Misspecification Analysis

The previous experiment made the explicit assumption that the specification of the conditional mean is known and is correctly specified. However, this is seldom the case in practice, so it is important to investigate the performance of these moment conditions when the conditional mean is misspecified. Recently, Chan and McAleer (2002) investigated the effects of misspecifying the conditional mean on the QMLE of the conditional variance. They showed that the likelihood function to be maximised when the conditional mean is misspecified can be written as

$$L(\theta) = -\frac{1}{2}\sum_t \big(\log h_t + \frac{\varepsilon_t^2}{h_t}\big) - \frac{1}{2}\sum_t \frac{\delta^2(x_t;\theta)}{h_t}, \tag{3.6}$$

where $\delta(x_t;\theta)$ is the difference between the correctly specified and misspecifed conditional means. Essentially, the more accurate does the misspecified conditional mean approximate the correct mean (that is, as $\delta(x_t;\theta) \to 0$), the lower is the bias in the QMLE of the parameters in the conditional variance. However, the effects of misspecifying the conditional mean on the empirical moment conditions are still unknown.

The aim of Experiment 2 is to investigate the performance of the empirical moment condition when the conditional mean is misspecified. This experiment replicates the results of Chan and McAleer (2002) regarding the bias of the QMLE in the conditional variance when the conditional mean is misspecified. The steps for Experiment 2 are as follows:

Step 1. Generate data using a STAR-GARCH model, as defined in equation (3.1).

Step 2. Estimate the true model.

Step 3. Estimate an AR(1)-GARCH(1,1) model, namely

$$\begin{aligned} y_t &= \phi_0 + \phi_1 y_{t-1} + \varepsilon_t, \\ \varepsilon_t &= \eta_t\sqrt{h_t}, \qquad \eta_t \sim NID(0,1) \\ h_t &= \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1} \end{aligned}$$

Step 4. Estimate a Constant-GARCH(1,1) model, namely

$$\begin{aligned} y_t =& \phi_0 + \varepsilon_t, \\ \varepsilon_t =& \eta_t \sqrt{h_t}, \qquad \eta_t \sim NID(0,1) \\ h_t =& \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} \end{aligned}$$

The above steps are repeated for both LSTAR-GARCH and ESTAR-GARCH for the two parameters sets, as defined in Table 1. This leads to four sets of results, as shown in Tables 6 - 13.

The choice of the AR(1)-GARCH(1,1) and the Constant-GARCH(1,1) models is due primarily to their popularity. The Constant-GARCH(1,1) model is often chosen when the autocorrelations of the data are close to zero, but the autocorrelation does not reveal any non-linear structure in the conditional mean. Other empirical studies have preferred the AR(1)-GARCH(1,1) specification to accommodate the possible presence of serial correlation, and using the autoregressive process to approximate the unknown conditional mean. This experiment will reveal the effects of these decisions on the empirical moment conditions.

Panels A and B in Figure 1 contain the plots of some simulated data against their lagged values from LSTAR-GARCH and ESTAR-GARCH, respectively, using Parameter Set I. Similar plots are also available for data generated using Parameter Set II for both processes, as shown in Panels C and D in Figure 1. The number of observations in each case is 3000.

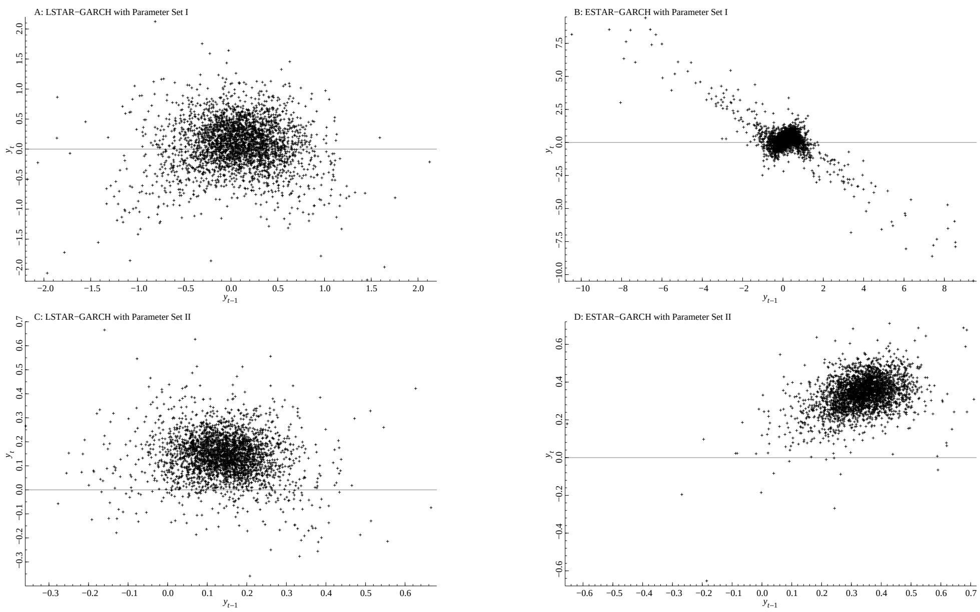

Figure 1: Simulated Data Using Parameter Sets I and II

As shown in Panel A of Figure 1, there is a clear LSTAR pattern in the conditional mean. For lagged values (y_{t-1}) less (greater) than the threshold value ($c = 0$), there is a clear positive (negative) trend in the data. This shows clearly the regime switching behaviour displayed in Parameter Set I. In this case, neither a constant mean nor a first-order autoregressive model can capture the non-linear pattern, so that a substantial bias in the QMLE of the parameters in the GARCH component of the AR-GARCH and Constant-GARCH models would be expected.

A similar conclusion can be drawn for data generated by ESTAR-GARCH using Parameter Set I. As shown in Panel B of Figure 1, the data show a clear ESTAR pattern. There is a positive relationship between y_t and y_{t-1} for $y_{t-1} \in (-1, 1)$, but slowly becomes negative for $|y_{t-1}| > 1$. Again, neither a constant nor a simple first-order autoregressive model can capture the non-linear pattern in the data, so that it will bias the QMLE of the parameters in the GARCH component of the two misspecified models. Therefore, the empirical moment conditions will also be affected.

On the contrary, a simple autoregressive process, or even a constant mean model, may be able to fit reasonably well the data generated by LSTAR-GARCH and ESTAR-GARCH using Parameter Set II. Panels C and D in Figure 1 show a simple autoregressive pattern, and there are no sign of non-linearity. In particular, Panel C in Figure 1 shows a very weak negative relationship between the data and the lagged values, in which case a simple constant mean model may also fit the data reasonably well. This is due primarily to the different parameters in the conditional variance. In particular, the smaller unconditional variance in Parameter Set II has restricted the variability of the data, and subsequently hidden the non-linear nature of the data from these plots.

Statistics	ϕ_0	ϕ_1	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.133	0.046	0.023	0.344	0.793	-0.061	0.990	1.217
Minimum	0.083	-0.095	0.007	0.172	0.585	-0.191	0.877	0.866
Mean	0.108	-0.021	0.014	0.250	0.687	-0.115	0.937	1.006
Variance	5.75E-05	0.001	6.23E-06	0.001	0.001	4.43E-04	2.78E-04	0.002
Skewness	-0.033	0.081	0.586	0.070	-0.127	-0.653	-0.551	0.147
Kurtosis	3.037	2.768	3.549	2.652	3.001	3.617	3.579	3.278

Table 6: Summary Statistics for AR-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set I

Statistics	ϕ_0	ϕ_1	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.187	0.363	0.044	0.472	0.682	-0.116	0.981	1.358
Minimum	0.117	0.116	0.013	0.209	0.413	-0.365	0.811	0.873
Mean	0.149	0.262	0.025	0.347	0.563	-0.193	0.910	1.074
Variance	1.44E-04	1.19E-03	2.47E-05	0.002	0.002	0.002	0.001	0.006
Skewness	0.149	-0.135	0.554	0.010	-0.162	-0.677	-0.541	0.198
Kurtosis	3.139	3.345	3.263	2.846	2.939	3.529	3.334	2.909

Table 7: Summary Statistics for AR-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set I

Statistics	ϕ_0	ϕ_1	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.163	-0.045	2.09E-04	0.131	0.934	-0.009	1.000	1.028
Minimum	0.147	-0.151	5.86E-05	0.054	0.857	-0.038	0.967	0.947
Mean	0.156	-0.097	1.11E-04	0.091	0.897	-0.019	0.988	0.993
Variance	8.25E-06	3.32E-04	6.45E-10	1.32E-04	1.34E-04	2.49E-05	2.41E-05	1.27E-04
Skewness	-0.088	0.032	0.660	0.070	-0.034	-0.786	-0.743	-0.360
Kurtosis	2.673	2.656	3.598	2.980	3.020	3.847	3.911	3.467

Table 8: Summary Statistics for AR-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set II

Statistics	ϕ_0	ϕ_1	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.216	0.505	4.88E-04	0.178	0.898	-0.019	1.001	1.060
Minimum	0.166	0.375	1.41E-04	0.078	0.770	-0.123	0.899	0.843
Mean	0.190	0.444	2.65E-04	0.120	0.850	-0.044	0.970	0.970
Variance	7.48E-05	5.19E-04	3.12E-09	1.83E-04	2.85E-04	1.41E-04	1.22E-04	5.18E-04
Skewness	0.105	-0.120	0.786	0.298	-0.419	-1.052	-0.980	-0.638
Kurtosis	2.830	2.830	3.790	3.473	3.450	5.695	5.421	4.666

Table 9: Summary Statistics for AR-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set II

Statistics	ϕ_0	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.128	0.023	0.343	0.794	-0.060	0.990	1.216
Minimum	0.085	0.007	0.171	0.585	-0.190	0.877	0.865
Mean	0.105	0.014	0.249	0.688	-0.115	0.938	1.005
Variance	4.06E-05	6.22E-06	0.001	0.001	4.37E-04	2.76E-04	0.002
Skewness	-0.075	0.587	0.083	-0.123	-0.656	-0.560	0.159
Kurtosis	3.025	3.558	2.657	2.987	3.642	3.619	3.290

Table 10: Summary Statistics for Constant-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set I

Statistics	ϕ_0	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.249	0.045	0.418	0.755	-0.087	0.974	1.299
Minimum	0.157	0.013	0.172	0.456	-0.332	0.812	0.828
Mean	0.206	0.025	0.299	0.610	-0.171	0.908	1.008
Variance	1.71E-04	2.42E-05	0.002	0.002	0.001	0.001	0.005
Skewness	-0.022	0.514	0.080	-0.122	-0.631	-0.553	0.212
Kurtosis	3.135	3.272	2.747	2.935	3.512	3.323	2.983

Table 11: Summary Statistics for Constant-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set I

Statistics	ϕ_0	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.146	2.23E-04	0.132	0.935	-0.009	1.000	1.026
Minimum	0.138	5.89E-05	0.053	0.856	-0.044	0.963	0.941
Mean	0.142	1.14E-04	0.092	0.896	-0.020	0.988	0.993
Variance	1.44E-06	7.25E-10	1.35E-04	1.42E-04	2.70E-05	2.53E-05	1.30E-04
Skewness	0.067	0.790	0.017	-0.083	-0.958	-0.852	-0.389
Kurtosis	2.998	3.884	2.976	3.050	4.499	4.369	3.542

Table 12: Summary Statistics for Constant-GARCH and Empirical Moment Conditions from Data Generated by LSTAR-GARCH with Parameter Set II

Statistics	ϕ_0	ω	α	β	Log-moment	2^{nd} moment	4^{th} moment
Maximum	0.353	0.002	0.359	0.756	-0.065	1.004	1.198
Minimum	0.334	0.001	0.168	0.362	-0.589	0.626	0.530
Mean	0.345	0.001	0.244	0.660	-0.158	0.904	0.940
Variance	7.62E-06	6.23E-08	0.001	0.003	0.003	0.002	0.006
Skewness	-0.302	1.467	0.420	-1.231	-1.873	-1.392	-0.767
Kurtosis	3.266	6.268	3.555	5.808	9.751	6.635	4.565

Table 13: Summary Statistics for Constant-GARCH and Empirical Moment Conditions from Data Generated by ESTAR-GARCH with Parameter Set II

Tables 2 - 13 contain the summary statistics for the QMLE of the GARCH component and the respective empirical moment conditions for the three models in all four cases. As shown in Tables 6 and 10, misspecifying the LSTAR-GARCH process as an AR-GARCH or a Constant-GARCH models can lead to substantial bias in the QMLE of the conditional variance. In either case, the $\hat{\alpha}$ ($\hat{\beta}$) estimate is biased upward (downward). Interestingly, this is broadly similar to the effects of extreme observations and outliers (see Verhoeven and McAleer (2002)). The empirical log-moment and second moment conditions have decreased for both models due to the downward bias in $\hat{\beta}$. However, the empirical fourth moment condition has now increased due to the upward bias in $\hat{\alpha}$, with means greater than one for both models.

Interestingly, misspecifying the conditional mean does not seem to yield a serious bias in the QMLE of the conditional variance for data generated by LSTAR-GARCH using Parameter Set II. In both cases, $\hat{\alpha}$ and $\hat{\beta}$ are very close to the true values, and subsequently the empirical log-moment and second and fourth moment conditions are very close to those generated from estimating the correct model. This result seems to suggest that this particular LSTAR-GARCH process can be approximated well by AR-GARCH, or even a Constant-GARCH model, as observed in previous discussions regarding Figure 1.

Similar conclusions can be drawn for ESTAR-GARCH. As shown in Tables 9 and 13, substantial bias is observed in $\hat{\alpha}$ and $\hat{\beta}$ for both AR-GARCH and the Constant-GARCH models. As in the case of LSTAR-GARCH for Parameter Set I, the $\hat{\alpha}$ ($\hat{\beta}$) was biased upward (downward), indicating that neither an AR process nor a Constant-GARCH model provids a good approximation to the ESTAR-GARCH process for Parameter Set I. The bias in the GARCH estimates has affected the empirical moment conditions slightly. It is interesting to note that both the empirical log-moment and second moment conditions have decreased due to the downward bias in $\hat{\beta}$. However, the empirical fourth moment condition had increased in both models due to the upward bias in $\hat{\alpha}$.

Data generated by the ESTAR-GARCH process for Parameter Set II provide some interesting conclusions. The $\hat{\alpha}$ and $\hat{\beta}$ estimates in the AR-GARCH model show a smaller degree of bias than the $\hat{\alpha}$ and $\hat{\beta}$ estimates in the Constant-GARCH model. This seems to suggest that the AR-GARCH model provides a better approximation to the conditional mean of the ESTAR-GARCH process than does the Constant-GARCH model, which corresponds to the previous discussion regarding Panel D in Figure 1.

4 Empirical Results

This section examines the empirical moment conditions of STAR-GARCH models for three sets of empirical data, namely Standard and Poor's 500 Composite Index (S&P), US Treasury Bill 3-month Middle Rate returns (USTB), and the US/Australia Exchange Rate (US/AUD). Daily data for S&P are obtained from DataStream Service, with the sample period 1/1/1986 to 12/4/2000, giving 3726 observations in total. Weekly data for USTB are obtained from DataStream Service, with sample period 1/1/1986 to 30/12/1998, giving 689 observations. Daily data for US/AUD are obtained from dX EconData, with sample period 1/1/1986 to 12/4/2000, giving 3726 observations.

Of primary interest are the returns for these series, which are calculated as follows:

$$r_t = \log y_t - \log y_{t-1}. \tag{4.1}$$

The plots of the data and their respective returns for the three series can be found in Figures 2 to 7.

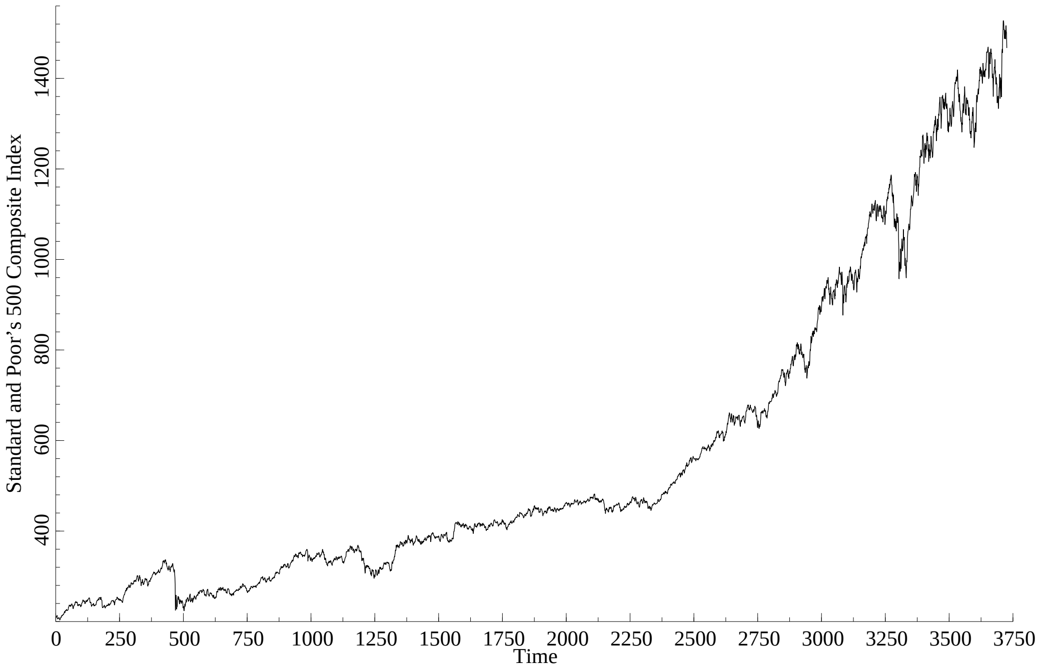

Figure 2: Standard and Poor's Composite Index

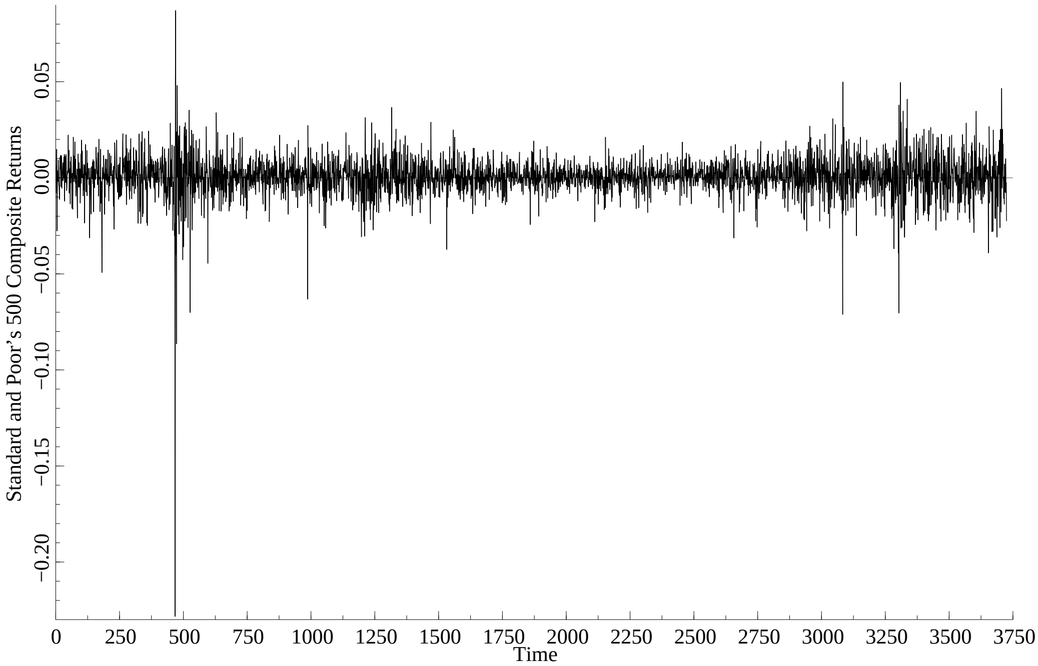

Figure 3: Standard and Poor's Composite Returns

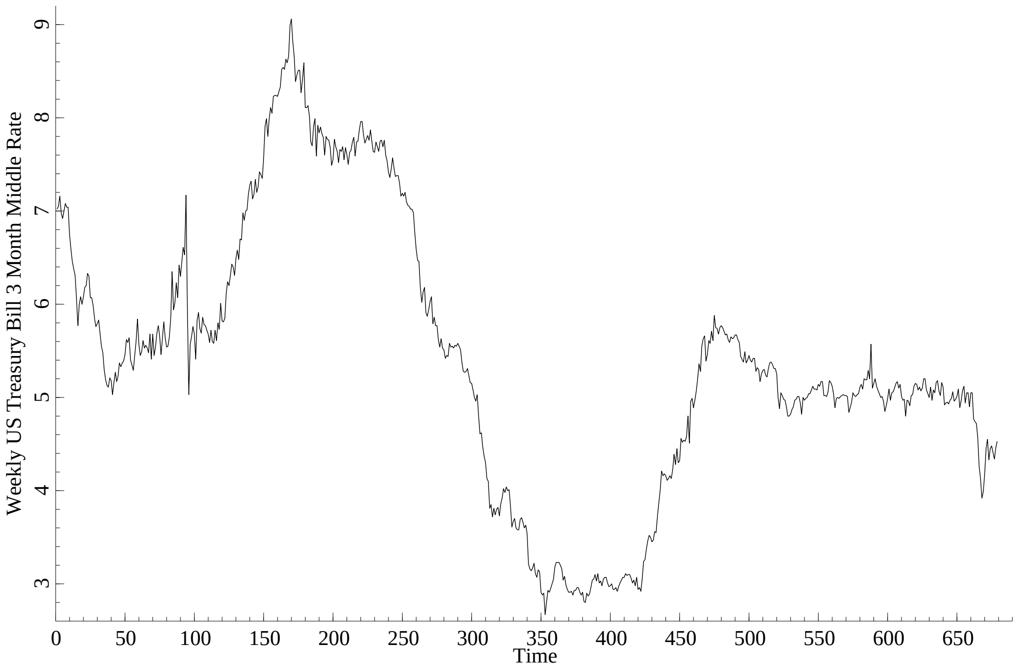

Figure 4: 3-month US Treasury Bill Rate

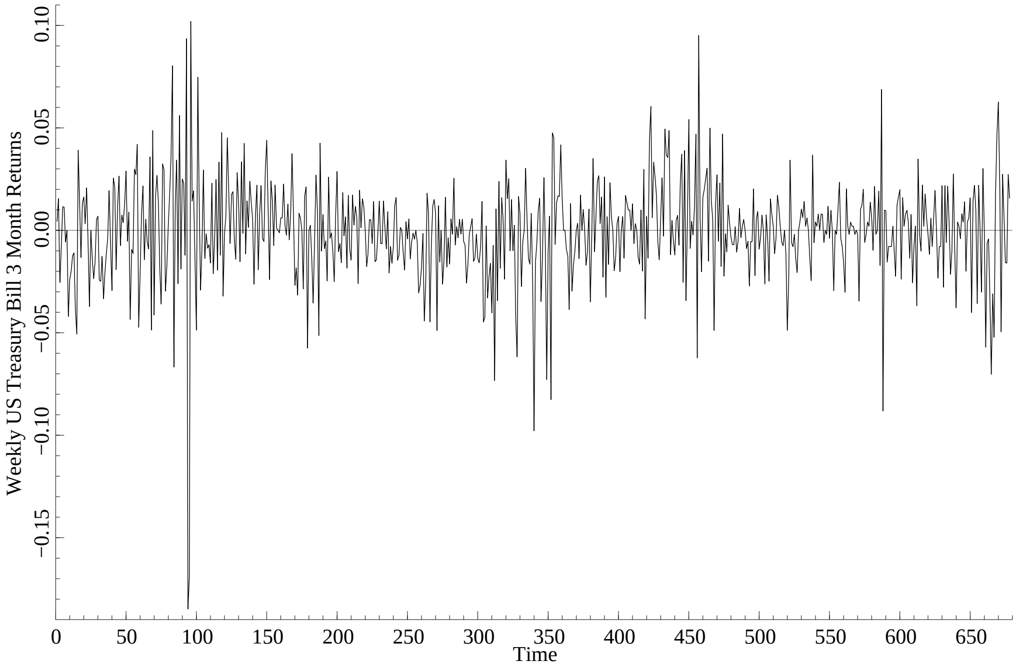

Figure 5: 3-month US Treasury Bill Returns

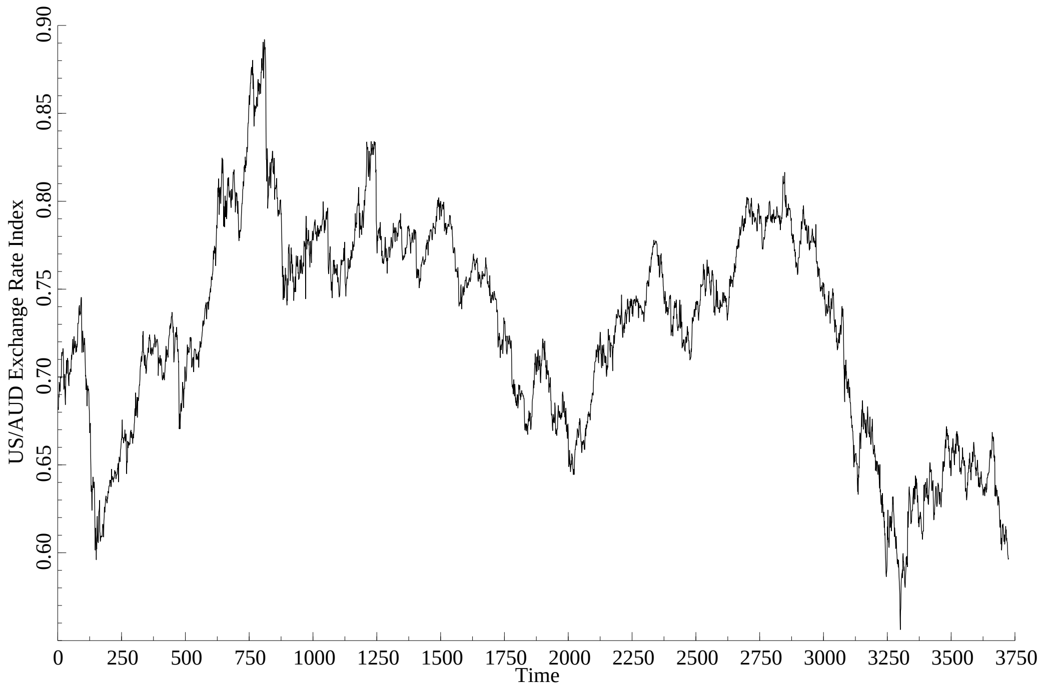

Figure 6: US/AUD Exchange Rate

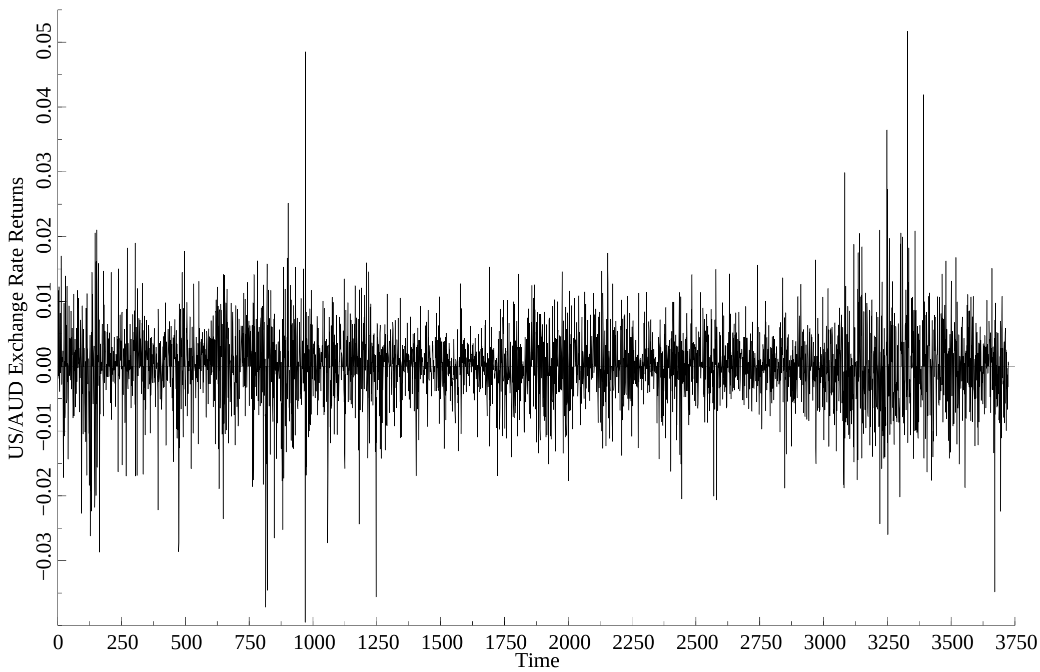

Figure 7: US/AUD Exchange Rate Returns

As shown in Figure 2, one of the most significant features of the S&P returns is the negative outlier in observation 466, which corresponds to the stock market crash in October 1987. Furthermore, the presence of extreme observations seems to be common throughout the series. Similar features are also found in USTB and US/AUD, as shown in Figures 4 and 6. US/AUD would appear to be a more volatile series than S&P and USTB, and also contains a larger number of extreme observations and outliers.

In order to examine the effects of outliers and extreme observations on the estimates, as well as on the empirical moment conditions of STAR-GARCH models, rolling estimates of LSTAR-GARCH and ESTAR-GARCH, as defined in equation (3.1), are obtained using each of the three series. In order to strike a balance between efficiency in estimation and a sensible number of rolling windows, the rolling window sizes are selected to be 3000 for both the S&P and US/AUD returns, and 500 for the USTB returns.

Recall that the empirical moment conditions are functions of the estimates, and that aberrant observations are known to affect the QMLE substantially with a predicable patterns (see Chan and McAleer (2003), and Verhoeven and McAleer (2002)). Therefore, the derivatives of the empirical moments with respect to the estimates should provide important information about the sensitivity of the empirical moment conditions to extreme observations and outliers.

Recall that the empirical second moment is

$$C_2 = \hat{\alpha} + \hat{\beta}, \tag{4.2}$$

and hence

$$\begin{aligned} dC_2 &= \frac{\partial C_2}{\partial \hat{\alpha}} d\hat{\alpha} + \frac{\partial C_2}{\partial \hat{\beta}} d\hat{\beta} \\ &= d\hat{\alpha} + d\hat{\beta}. \end{aligned} \tag{4.3}$$

Similarly, the empirical fourth moment is

$$C_4 = (\hat{\alpha} + \hat{\beta})^2 + 2\hat{\alpha}^2, \tag{4.4}$$

so that

$$\begin{aligned} dC_4 &= \frac{\partial C_4}{\partial \hat{\alpha}} d\hat{\alpha} + \frac{\partial C_4}{\partial \hat{\beta}} d\hat{\beta} \\ &= (2(\hat{\alpha} + \hat{\beta}) + 4\hat{\alpha}) d\hat{\alpha} + 2(\hat{\alpha} + \hat{\beta}) d\hat{\beta}. \end{aligned} \tag{4.5}$$

The empirical log-moment is

$$\begin{aligned} C_L &= E(\log(\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta})) \\ &= \int_{-\infty}^{\infty} \log(\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}) P(\hat{\eta}_t) d\hat{\eta}_t, \end{aligned} \tag{4.6}$$

where $P(\hat{\eta}_t)$ denotes the probability density of $\hat{\eta}_t$, so that

$$
\begin{aligned}
dC_L &= \frac{\partial C_L}{\partial \hat{\alpha}} d\hat{\alpha} + \frac{\partial C_L}{\partial \hat{\beta}} d\hat{\beta} \\
&= E(\frac{\hat{\eta}_t^2}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) d\hat{\alpha} + E(\frac{1}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) d\hat{\beta}.
\end{aligned} \tag{4.7}
$$

It is clear that a unit change in $\hat{\alpha}$ has the same impact on the empirical second moment as a unit change in $\hat{\beta}$. However, assuming that the sufficient condition for $h_t > 0$ is satisfied, that is, $\hat{\alpha} > 0$ and $\hat{\beta} > 0$, then a unit change in $\hat{\alpha}$ has a larger impact than a unit change in $\hat{\beta}$. Moreover, it is straightforward to show that, if $dC_4 = 0$, then

$$
\frac{d\hat{\beta}}{d\hat{\alpha}} = -(1 + \frac{2\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}).
$$

Thus, the effect of a unit change in $\hat{\alpha}$ on the empirical fourth moment is $1 + \frac{2\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$ times larger than the effect of a unit change in $\hat{\beta}$.

The following inequality is useful for examining the effect of a unit change in $\hat{\alpha}$ and $\hat{\beta}$ on the empirical log-moment condition.

Inequality 1:

$$
E(\frac{\hat{\eta}_t^2}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) - E(\frac{1}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) \leq 0.
$$

Proof: Let

$$
f(x) = \frac{x}{\alpha x + \beta}, \quad \text{and} \quad g(x) = \frac{1}{\alpha x + \beta}.
$$

Taking the second derivatives of $-f(x)$ and $g(x)$, and assuming that $\alpha > 0$ and $\beta > 0$, yields

$$
\begin{aligned}
\frac{d^2 h}{dx^2} &= \frac{2\alpha\beta}{(\alpha x + \beta)^3} > 0, \qquad \forall x \geq 0, \\
\frac{d^2 g}{dx^2} &= \frac{2\alpha}{(\alpha x + \beta)^3} > 0, \qquad \forall x \geq 0,
\end{aligned}
$$

where $h(x) = -f(x)$. These imply $f(x)$ is concave and $g(x)$ is convex for $x \in \mathbb{R}^+$. Therefore,

$$
E(f(x)) \leq f(E(x)) \quad \text{and} \quad E(g(x)) \geq g(E(x)), \qquad \forall x \in \mathbb{R}^+
$$

by Jensen's Inequality. Letting $x = \hat{\eta}_t^2$, the above expressions can be rewritten as

$$
\begin{aligned}
E(\frac{\hat{\eta}_t^2}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) &= E(\frac{x}{\hat{\alpha}x + \hat{\beta}}) \\
&\leq \frac{E(x)}{\hat{\alpha}E(x) + \hat{\beta}}.
\end{aligned}
$$

As $E(x) = E(\hat{\eta}_t^2) = 1$, by assumption, it follows that

$$E(\frac{\hat{\eta}_t^2}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) \leq \frac{1}{\hat{\alpha} + \hat{\beta}}. \tag{4.8}$$

It is also straightforward to show that

$$E(\frac{1}{\hat{\alpha}\hat{\eta}_t^2 + \hat{\beta}}) \geq \frac{1}{\hat{\alpha} + \hat{\beta}}. \tag{4.9}$$

Since both $\hat{\alpha}$ and $\hat{\beta}$ are positive, by assumption, subtracting (4.8) from (4.9) yields the result. This completes the proof. ■

Using Inequality 1, it is clear that a unit change in $\hat{\beta}$ has a greater impact on the empirical log-moment than a unit change in $\hat{\alpha}$.

Verhoeven and McAleer (2002) and Chan and McAleer (2002) provided empirical evidence to suggest that aberrant observations generally have positive (negative) effects on $\hat{\alpha}$ ($\hat{\beta}$). Thus, the relative effects on the estimates from these observations will determine their effects on the empirical moment conditions. Let $\Delta\hat{\alpha}$ and $\Delta\hat{\beta}$ denote the changes in $\hat{\alpha}$ and $\hat{\beta}$, respectively, due to aberrant observations. If either $\Delta\hat{\alpha} > 0$ and $\Delta\hat{\beta} < 0$, or $\Delta\hat{\alpha} < 0$ and $\Delta\hat{\beta} > 0$, which seem to occur in practice in the presence of aberrant observations, and if

$$\frac{\Delta\hat{\beta}}{\Delta\hat{\alpha}} < -1,$$

then the empirical second moment will decrease. Furthermore, if

$$\frac{\Delta\hat{\beta}}{\Delta\hat{\alpha}} < -(1 + \frac{2\hat{\alpha}}{\hat{\alpha} + \hat{\beta}}),$$

then the empirical fourth moment will also decrease.

4.1 Standard and Poor's Composite Index

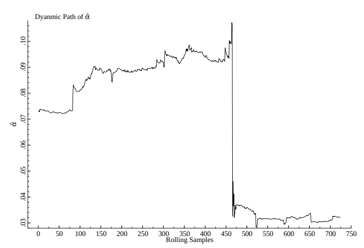

Figure 8: Dynamic Path of $\hat{\alpha}$ for LSTAR-GARCH

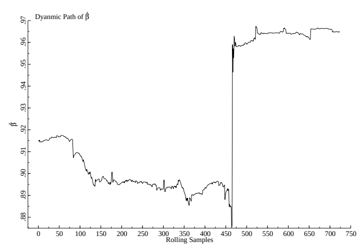

Figure 9: Dynamic Path of $\hat{\beta}$ for LSTAR-GARCH

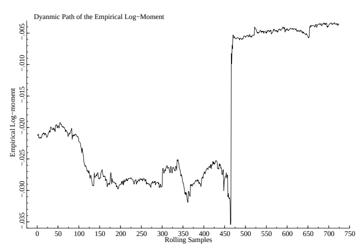

Figure 10: Dynamic Path of Empirical Log-moment for LSTAR-GARCH

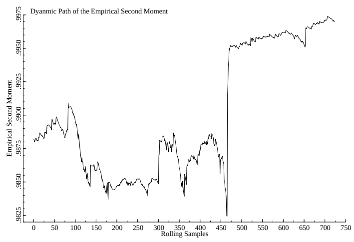

Figure 11: Dynamic Path of Empirical Second Moment for LSTAR-GARCH

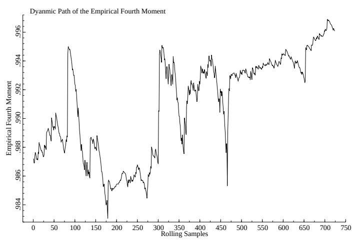

Figure 12: Dynamic Path of Empirical Fourth Moment for LSTAR-GARCH

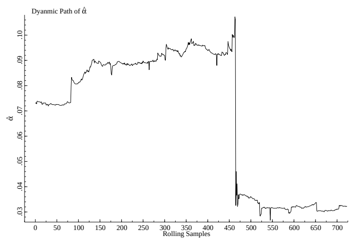

Figure 13: Dynamic Path of $\hat{\alpha}$ for ESTAR-GARCH

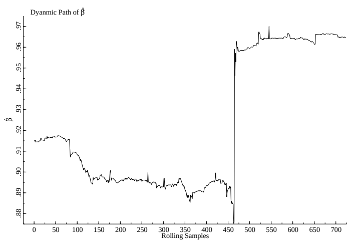

Figure 14: Dynamic Path of $\hat{\beta}$ for ESTAR-GARCH

Figure 15: Dynamic Path of Empirical Log-moment for ESTAR-GARCH

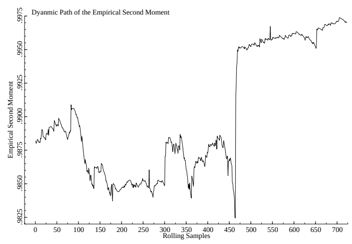

Figure 16: Dynamic Path of Empirical Second Moment for ESTAR-GARCH

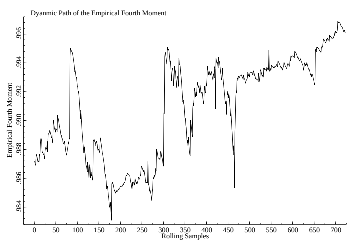

Figure 17: Dynamic Path of Empirical Fourth Moment for ESTAR-GARCH

Figures 8 - 12 contain the dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, as well as the empirical log-moment and second and fourth moment conditions of LSTAR-GARCH. The $\hat{\alpha}$ and $\hat{\beta}$ estimates seem to be affected greatly by the presence of outliers. When the outlier in observation 466 was removed from the rolling window, $\hat{\alpha}$ ($\hat{\beta}$) decreased (increased) from 0.106 (0.876) to 0.033 (0.959). This suggests that the outlier has a positive (negative) impact on $\hat{\alpha}$ ($\hat{\beta}$), which conforms with the empirical findings of Chan and McAleer (2003) and Verhoeven and McAleer (2002).

As the estimates are sensitive to the presence of outliers, the empirical moment conditions are subsequently affected. As shown in Figures 10 - 11, the movements of the empirical log-moment and second moments are similar to the movements in $\hat{\beta}$. When the outlier is removed from the rolling sample, the log-moment increased from -0.035 to -0.08, and the second moment increased from 0.982 to 0.992. This is primarily due to the fact that the outlier seemed to have a larger impact on $\hat{\beta}$ than on $\hat{\alpha}$. The mean empirical log-moment and second moment are -0.019 and 0.990, respectively.

Movements in the empirical fourth moment do not seem to be as dramatic as the log-moment and second moment. Although Figure 12 shows substantial fluctuations in the fourth moment, the range of variability is narrower than for the log-moment and second moment. Despite the upward trend in the empirical fourth moment, all rolling samples satisfy the fourth moment condition, with a mean of 0.991.

Rolling estimates for ESTAR-GARCH reveal a similar story for S&P with LSTAR-GARCH, as shown in Figure 13 - 17. Movements in $\hat{\alpha}$ and $\hat{\beta}$ are almost identical to the movements in $\hat{\alpha}$ and $\hat{\beta}$ for LSTAR-GARCH: $\hat{\alpha}$ decreased from 0.107 to 0.033 when the outlier was removed from the rolling sample, while $\hat{\beta}$ increased from 0.876 to 0.959.

Not surprisingly, the movements in the empirical log-moment and second and fourth moments are also very similar to those for LSTAR-GARCH. Again, the empirical log-moment increased from -0.035 to -0.008 when the outlier was removed from the rolling sample, while the second moment increased from 0.983 to 0.992. Furthermore, movements in the empirical fourth moment are also similar to the movements in the fourth moment for LSTAR-GARCH. As in the case of LSTAR-GARCH, all rolling samples satisfy the fourth moment condition for ESTAR-GARCH. The mean log-moment and second and fourth moments are -0.019, 0.990 and 0.991, respectively.

4.2 3-month US Treasury Bill Middle Rate

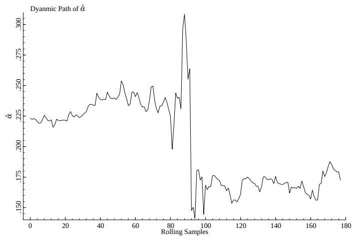

Figure 18: Dynamic Path of $\hat{\alpha}$ for LSTAR-GARCH

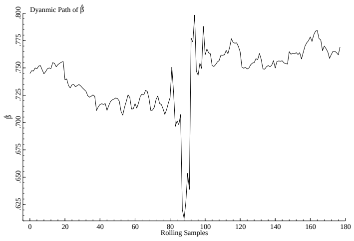

Figure 19: Dynamic Path of $\hat{\beta}$ for LSTAR-GARCH

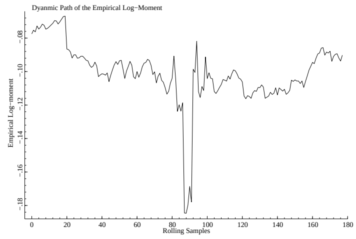

Figure 20: Dynamic Path of Empirical Log-moment for LSTAR-GARCH

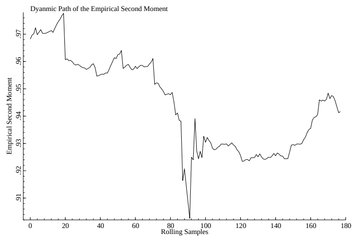

Figure 21: Dynamic Path of Empirical Second Moment for LSTAR-GARCH

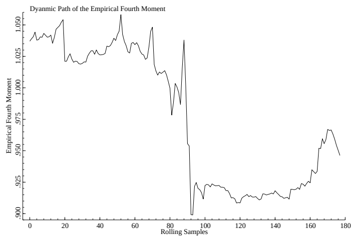

Figure 22: Dynamic Path of Empirical Fourth Moment for LSTAR-GARCH

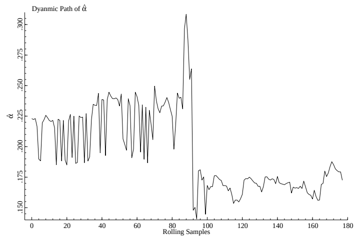

Figure 23: Dynamic Path of $\hat{\alpha}$ for ESTAR-GARCH

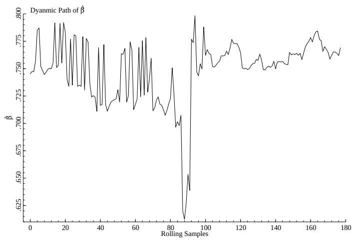

Figure 24: Dynamic Path of $\hat{\beta}$ for ESTAR-GARCH

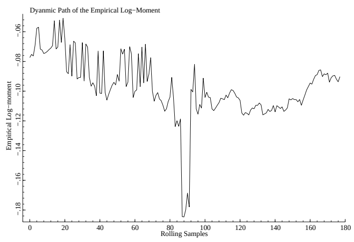

Figure 25: Dynamic Path of Empirical Log-moment for ESTAR-GARCH

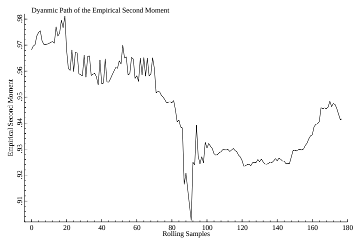

Figure 26: Dynamic Path of Empirical Second Moment for ESTAR-GARCH

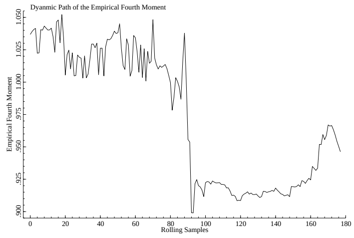

Figure 27: Dynamic Path of Empirical Fourth Moment for ESTAR-GARCH

Figures 18 - 22 contain the dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, as well as the empirical log-moment and second and fourth moment conditions for LSTAR-GARCH. Both $\hat{\alpha}$ and $\hat{\beta}$ moved steadily in the early rolling samples around means of 0.227 and 0.725, respectively. The inclusion of the two extreme observations in the rolling samples, namely observations 588 and 589, increased $\hat{\alpha}$ from 0.231 to 0.296, while $\hat{\beta}$ decreased from 0.707 to 0.620. However, when the outlier in observation 95 was removed from the rolling sample, $\hat{\alpha}$ decreased from 0.264 to 0.148, while $\hat{\beta}$ increased from 0.639 to 0.777. This suggests that QMLE is sensitive to extreme observations and outliers, and that the relative size of these aberrant observations would also seem to be a critical factor in determining their effects on the estimates.

All rolling samples satisfy the empirical log-moment and second moment conditions, as shown in Figures 20 and 21. The effects of aberrant observations on the empirical log-moment and second moment conditions are illustrated in rolling sample 87, when the log-moment decreased from -0.119 to -0.185, while the second moment decreased from 0.938 to 0.916. These changes are due primarily to the inclusion of the two extreme observations, and their effects on $\hat{\alpha}$ and $\hat{\beta}$. Similarly, the removal of the outlier in observation 95 has increased both the empirical log-moment and second moment due to the effects on $\hat{\beta}$.

The first 85 rolling samples fail to satisfy the fourth moment condition for LSTAR-GARCH. However, the fourth moment begins to decline as some of the extreme observations prior to observation 95 are removed from the rolling samples, and subsequently decreased $\hat{\alpha}$ dramatically. Since the empirical fourth moment is more sensitive to changes in $\hat{\alpha}$, the decline in the empirical fourth moment was to be expected.

Figures 23 - 27 contain the dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, as well as the empirical log-moment and second and fourth moment conditions for ESTAR-GARCH. Although $\hat{\alpha}$ and $\hat{\beta}$ exhibit greater fluctuations in the early rolling samples, these estimates vary around similar means to their LSTAR-GARCH counterpart. Moreover, the effects of the aberrant observations on the estimates of ESTAR-GARCH are identical to those of LSTAR-GARCH. The inclusion of the two extreme observations, namely observations 588 and 589, increased $\hat{\alpha}$ from 0.231 to 0.296, while $\hat{\beta}$ decreased from 0.707 to 0.620. Furthermore, the removal of the outlier in observation 95 decreased $\hat{\alpha}$ from 0.264 to 0.148, while $\hat{\beta}$ increased from 0.639 to 0.777. Interestingly, $\hat{\alpha}$ and $\hat{\beta}$ in ESTAR-GARCH are equal to their LSTAR-GARCH counterparts up to 3 decimal places for the rolling samples described above.

The empirical log-moment for ESTAR-GARCH reveals a similar story as for LSTAR-GARCH. Again, the empirical log-moment decreased from -0.119 to -0.184 when the two extreme observations are included in the rolling samples, but increased from -0.178 to -0.099 when the outlier in observation 95 is removed from the rolling sample. The empirical second and fourth moments seem to be more volatile in the early rolling samples, due to the more volatile $\hat{\alpha}$ and $\hat{\beta}$ estimates of ESTAR-GARCH in the early periods. However, the effects of the aberrant observations are similar to those of LSTAR-GARCH, as shown in Figures 26 and 27. As in the case of LSTAR-GARCH, the first 85 rolling samples fail to satisfy the fourth moment condition due to the high $\hat{\alpha}$ estimates. However, the empirical fourth moment decreased to below one when the outlier was removed from the rolling sample, due to its positive effects on $\hat{\alpha}$.

4.3 US/AUD Exchange Rate

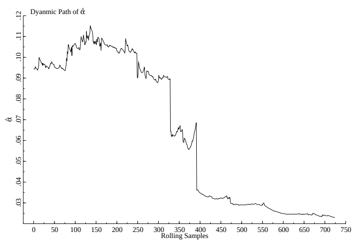

Figure 28: Dynamic Path of $\hat{\alpha}$ for LSTAR-GARCH

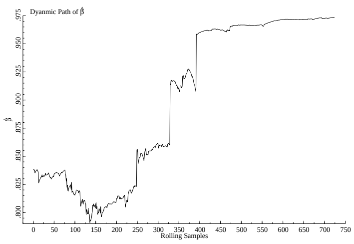

Figure 29: Dynamic Path of $\hat{\beta}$ for LSTAR-GARCH

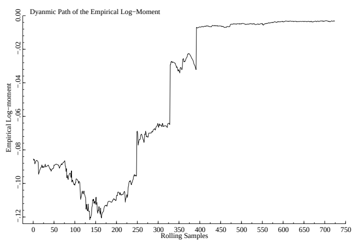

Figure 30: Dynamic Path of Empirical Log-moment for LSTAR-GARCH

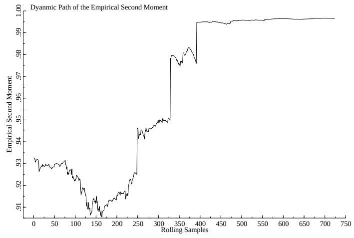

Figure 31: Dynamic Path of Empirical Second Moment for LSTAR-GARCH

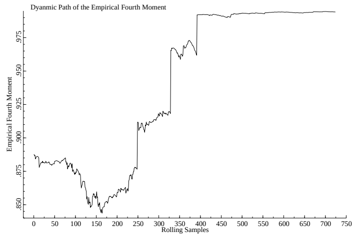

Figure 32: Dynamic Path of Empirical Fourth Moment for LSTAR-GARCH

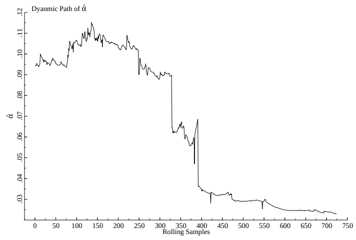

Figure 33: Dynamic Path of $\hat{\alpha}$ for ESTAR-GARCH

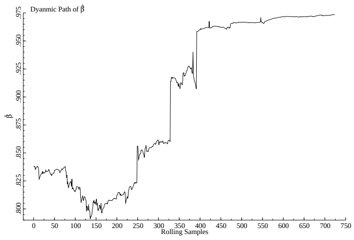

Figure 34: Dynamic Path of $\hat{\beta}$ for ESTAR-GARCH

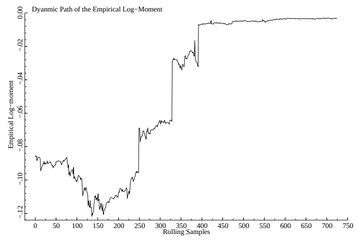

Figure 35: Dynamic Path of Empirical Log-moment for ESTAR-GARCH

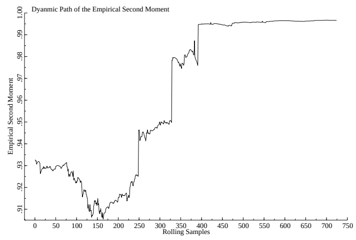

Figure 36: Dynamic Path of Empirical Second Moment for ESTAR-GARCH

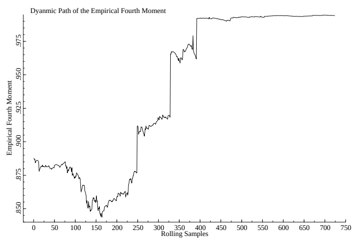

Figure 37: Dynamic Path of Empirical Fourth Moment for ESTAR-GARCH

Figures 28 to 32 give the dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, and the empirical log-moment and second and fourth moments, for LSTAR-GARCH. As shown in Figure 28, $\hat{\alpha}$ declines consistently throughout the rolling samples, with two dramatic drops, namely rolling samples 329 and 392: $\hat{\alpha}$

decreases from 0.09 to 0.064 in the first instance, and decreases further from 0.069 to 0.036 in the second.

Correspondently, $\hat{\beta}$ rises consistently throughout the rolling samples, with two dramatic increases in rolling samples 329 and 392, as shown in Figure 29. In fact, $\hat{\beta}$ increases from 0.860 to 0.914 in the first instance, and increases further from 0.907 to 0.959 in the second.

Interestingly, there is no obvious aberrant observation being removed or added in the two rolling samples. Thus, the dramatic movements do not seem to be caused by extreme observations or outliers in the data. This shows that not only aberrant observations can cause such changes in the estimates, and would be an interesting area of future research.

The following discussion provides two plausible explanations relating to intended future research in this area. One explanation is that the fitted values of the conditional mean from LSTAR-GARCH performed poorly in the rolling samples prior to observations 329 and 392, and hence created artificial outliers in the residuals. This could happen if the data fluctuate wildly and the conditional mean fails to capture the dynamics in the data. If this explanation were accurate, then it would suggest the following two additional points:

1. A different non-linear time series model would be required to capture the dynamics in the data. Since an autoregressive process and a constant mean are special cases of the STAR model, the failure of STAR to capture the dynamics in the data would imply that neither of the simpler models is appropriate.

2. STAR models are extremely sensitive to variations in the data, so that the QMLE of GARCH would be similarly affected.

The second explanation is that this is, in fact, a unique feature of the data. Notice that aberrant observations appear in clusters throughout the various samples, and the effects of consecutive outliers on the QMLE is still a relatively unresearched area.

Interestingly, the movements in the empirical log-moment and second and fourth moments seem to mimic the movements in $\hat{\beta}$. All the rolling samples satisfy the empirical log-moment and second and fourth moment conditions.

The dynamic paths of $\hat{\alpha}$ and $\hat{\beta}$, and the empirical log-moment and second and fourth moments, for ESTAR-GARCH reveal an identical story as for LSTAR-GARCH, as shown in Figures 33 to 37. Interestingly, $\hat{\alpha}$ and $\hat{\beta}$ for ESTAR-GARCH are often equal to their LSTAR-GARCH counterparts up to 4 decimal places. This suggests that both conditional means manage to capture the dynamics in the data. More importantly, all moment conditions are also satisfied for all the rolling windows for ESTAR-GARCH.

5 Concluding Remarks

This paper has provided a weak sufficient, or log-moment, condition for the consistency and asymptotic normality of QMLE for the STAR-GARCH(1,1) model. The condition can easily be extended to any non-linear time series model with GARCH(1,1) errors, subject to appropriate regularity conditions.

Monte Carlo experiments showed that the empirical log-moment condition is more suitable than the moment conditions to verify in practice when the true long run persistence underlying the GARCH process is close to unity. The empirical fourth moment condition is too restrictive in practice, and can often be violated. Although the performance of the empirical second moment is not as informative as the log-moment condition, it was superior to the empirical fourth moment, and has the advantage of computational simplicity over the log-moment condition.

The Monte Carlo experiments also showed that the correct specification of the conditional mean is crucial for obtaining consistent estimates of the parameters of the conditional variance. Moreover, the experiments also showed that the bias in the estimates of the conditional variance could be minimised if the conditional mean was approximated accurately. Such a result is useful in practice as the correct functional form of the conditional mean is typically unknown. This highlighted the importance of establishing the statistical properties of the estimators for purposes of valid statistical inference and the implementation of diagnostic tests.

Effects of aberrant observations on the empirical moments was discussed through the use of rolling estimates on three data sets, namely, Standard and Poor's Composite 500 Index (S&P), 3-month US Treasury Bill rate (USTB), and the exchange rate between the USA and Australia (US/AUS). The results showed that extreme observations and outliers affected the empirical moment conditions through their effects on the QMLE. Analytical expressions for the sensitivity of the empirical moments with respect to changes in the estimates were also derived. A unit change in $\hat{\alpha}$ was shown to have the same impact on the empirical second moment as a unit change in $\hat{\beta}$, a unit change in $\hat{\alpha}$ had a larger impact on the empirical fourth moment than a unit change in $\hat{\beta}$, and a unit change in $\hat{\beta}$ had a larger impact on the empirical log-moment then a unit change in $\hat{\alpha}$.

Although there have been some theoretical developments of STAR-GARCH models in recent years, the task of understanding the nature of non-linear models with conditionally heteroscedastic errors is far from complete. Lamoureux and Lastrapes (1990) examined the effects of structural shift in the conditional variance by including dummy variables in the GARCH equation. Lundbergh and Teräsvirta (1999) extended the concept of structural change in the conditional variance by incorporating the smooth transition mechanism in the GARCH equation, known as STAR-Smooth Transition GARCH (STAR-STGARCH). Although allowing smooth transition behaviour in both the conditional mean and the conditional variance would seem to be a useful extension of STAR-GARCH, the lack of structural and statistical properties for these models has prevented their widespread use in the literature. Future research in establishing the structural and statistical properties of these models is likely to provide invaluable insights into appropriate applications of these models.

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