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Empirical Characteristic Function Approach to Goodness-of-Fit Tests for the Cauchy Distribution with Parameters Estimated by MLE or EISE

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Abstract

We consider goodness-of-fit tests of Cauchy distribution based on weighted integrals of the squared distance of the difference between the empirical characteristic function of the standardized data and the characteristic function of the standard Cauchy distribution. For standardization of data Gürtler and Henze (2000) used the median and the interquartile range. In this paper we use maximum likelihood estimator (MLE) and an equivariant integrated squared error estimator (EISE), which minimizes the weighted integral. We derive an explicit form of the asymptotic covariance function of the characteristic function process with parameters estimated by MLE or EISE. The eigenvalues of the covariance function are numerically evaluated and the asymptotic distribution of the test statistics are obtained by the residue theorem. Simulation study shows that the proposed tests compare well to tests proposed by Gürtler and Henze (2000) and more traditional tests based on the empirical distribution function.

1 Introduction.

Let $C(\alpha, \beta)$ denote the Cauchy distribution with the location parameter α and the scale parameter β, with the density

$$f(x;\theta) = f(x;\alpha,\beta) = \frac{\beta}{\pi(\beta^2 + (x-\alpha)^2)}, \quad \theta = (\alpha, \beta).$$

Given a random sample $x_1, \ldots, x_n$ from an unknown distribution F, we want to test the null hypothesis H_0 that F belongs to the family of Cauchy distributions. Since Cauchy distributions form a location scale family, we consider affine invariant tests. The proposed tests are based on the empirical characteristic function

$$\Phi_n(t) = \Phi_n(t;\hat\alpha,\hat\beta) = \frac{1}{n}\sum_{j=1}^{n} \exp(ity_j), \qquad y_j = \frac{x_j - \hat\alpha}{\hat\beta}, \tag{1.1}$$

of the standardized data y_j. Here $\hat\alpha = \hat\alpha_n = \hat\alpha_n(x_1, \ldots, x_n)$ and $\hat\beta = \hat\beta_n = \hat\beta_n(x_1,\ldots,x_n)$ are affine equivariant estimators of α and β satisfying

$$\begin{aligned} \hat\alpha_n(a+bx_1, \ldots, a+bx_n) &= a + b\hat\alpha_n(x_1,\ldots,x_n), \\ \hat\beta_n(a+bx_1,\ldots,a+bx_n) &= b\hat\beta_n(x_1,\ldots,x_n). \end{aligned}$$

For $\hat{\alpha}_n$ and $\hat{\beta}_n$, we use maximum likelihood estimator (MLE) and an equivariant integrated squared error estimator (EISE) defined in (2.6) below. The reason for considering MLE is its asymptotic efficiency. Although optimality as estimators does not imply optimality for the goodness-of-fit tests, it seems natural to consider MLE. The reason for considering EISE is a possible extension to stable distributions other than the Cauchy distribution studied in this paper. Although the median and the interquartile range used by Gürtler and Henze (2000) are attractive estimators because of their simplicity, it seems theoretically more natural to consider MLE and EISE.

Following Gürtler and Henze (2000) we consider test statistic

$$D_{n,\kappa} := n \int_{-\infty}^{\infty} \left|\Phi_n(t) - e^{-|t|}\right|^2 w(t)dt, \qquad w(t) = e^{-\kappa|t|},\ \kappa > 0, \tag{1.2}$$

which is the weighted L^2-distance between $\Phi_n(t)$ and the characteristic function $e^{-|t|}$ of $C(0,1)$ with respect to the weight function $w(t) = e^{-\kappa|t|}$, $\kappa > 0$. This weight function is chosen for convenience, so that we can explicitly evaluate the asymptotic covariance function of the empirical characteristic function process under H_0. Using the relation

$$\int_{-\infty}^{\infty} \cos(ct) e^{-\kappa|t|} dt = \frac{2\kappa}{\kappa^2 + c^2},$$

the integral in (1.2) can be explicitly evaluated and an alternative convenient expression of $D_{n,\kappa}$ is given by

$$D_{n,\kappa} = \frac{2}{n} \sum_{j,k=1}^{n} \frac{\kappa}{\kappa^2 + (y_j - y_k)^2} - 4 \sum_{j=1}^{n} \frac{1+\kappa}{(1+\kappa)^2 + {y_j}^2} + \frac{2n}{2+\kappa}. \tag{1.3}$$

Our test statistic $D_{n,\kappa}$ is a quadratic form of the empirical characteristic function process. Although we derive an explicit form of the asymptotic covariance function of the empirical characteristic function process, it is not trivial to derive the asymptotic distribution of $D_{n,\kappa}$ under H_0 from the covariance function, especially when the parameters are estimated (e.g. chapter 7 of Durbin (1973a) and Durbin (1973b)). Therefore finite sample critical values of goodness-of-fit tests are often evaluated by Monte Carlo simulation, as was done in Gürtler and Henze (2000). Note that if we evaluate the critical values by Monte Carlo simulation only, there is no need to derive the explicit form of the asymptotic covariance function. Furthermore it is impossible to perform usual Monte Carlo simulation for the asymptotic case $n = \infty$. Therefore numerical evaluation of the asymptotic distribution is important in order to check the convergence of the finite sample distributions, which are evaluated by Monte Carlo simulations.

In this paper, we make use of the explicit form of the asymptotic covariance function for numerically evaluating the asymptotic critical values of the test statistics. We introduce a homogeneous integral equation of the second kind and consider the associated Fredholm determinant, which can be approximated by evaluating eigenvalues of the asymptotic covariance function numerically. Then we apply the residue theorem in Lévy's inversion formula and evaluate the asymptotic distribution function of $D_{n,\kappa}$.

This paper is organized as follows. In Section 2.1 we first define and summarize properties of MLE and EISE. Then we state theoretical results on asymptotic distribution of $D_{n,\kappa}$ under H_0 in Theorem 2.1 and Theorem 2.2. Our method for numerically evaluating the asymptotic critical values of $D_{n,\kappa}$ is discussed in Section 2.2. In Section 3 we present computational studies of the proposed tests in comparison to other testing procedures. We also give percentage points of some classical goodness-of-fit statistics when the parameters of the Cauchy distribution are estimated by MLE. Appendix A gives proofs of the theoretical results of Section 2.1. We utilize theorems of Csörgő (1983) and Gürtler and Henze (2002). Appendix B gives proofs and some detailed technical arguments of results of Section 2.2.

2 Main results

2.1 Asymptotic theory of the proposed test statistics

We first review maximum likelihood estimator of Cauchy distribution and define an equivariant integrated squared error estimator. Except for differences in estimators, we follow the line of arguments in Gürtler and Henze (2000).

1. MLE. The log-likelihood function is given by

$$L = n \log \beta - \sum_{j=1}^{n} \log\{\beta^2 + (x_j - \alpha)^2\} - n \log \pi.$$

Differentiation of L with respect to (α, β) gives the likelihood equation

$$\frac{\partial L}{\partial \alpha} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{x_j - \alpha}{\beta^2 + (x_j - \alpha)^2} = 0, \tag{2.1}$$

$$\frac{\partial L}{\partial \beta} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{\beta^2}{\beta^2 + (x_j - \alpha)^2} = \frac{1}{2} n. \tag{2.2}$$

Equivariance of MLE is easily checked. According to Copas (1975), except for pathological cases such that more than half of the observations are the same, the likelihood function L is unimodal. Therefore with probability one, a local maximum of the likelihood function is actually the global maximum and it is relatively easy to obtain MLE by solving the likelihood equation.

2. EISE. Here we define an affine equivariant version of the ISE (integrated squared error) estimator proposed by Besbeas and Morgan (2001). The original ISE estimator of Besbeas and Morgan (2001) is not equivariant. EISE is based on standardized empirical characteristic function. Let

$$\Phi_n(t; \alpha, \beta) = \frac{1}{n} \sum_{j=1}^{n} \exp(it(x_j - \alpha)/\beta),$$

which is the same as (1.1) with $\hat{\alpha}_n$ and $\hat{\beta}_n$ replaced by α and β. Write

$$I(\alpha, \beta) = \int_{-\infty}^{\infty} \left|\Phi_n(t; \alpha, \beta) - e^{-|t|}\right|^2 w(t) dt, \tag{2.3}$$

where we use the following weight function

$$w(t) = \exp(-\nu|t|), \quad \nu > 0. \tag{2.4}$$

As in (1.3) the integral $I(\alpha, \beta)$ can be calculated as

$$I(\alpha, \beta) = \frac{2}{n^2} \sum_{j,k=1}^{n} \frac{\nu \beta^2}{\nu^2 \beta^2 + (x_j - x_k)^2} - \frac{4}{n} \sum_{j=1}^{n} \frac{(1+\nu)\beta^2}{(1+\nu)^2 \beta^2 + (x_j - \alpha)^2} + \frac{2}{2+\nu}. \tag{2.5}$$

EISE $(\hat{\alpha}_n, \hat{\beta}_n)$ is defined to be the minimizer of $I(\alpha, \beta)$:

$$I(\hat{\alpha}_n, \hat{\beta}_n) = \min_{\alpha, \beta} I(\alpha, \beta). \tag{2.6}$$

It is easy to see that EISE is affine equivariant by definition.

Note that the weighting constant κ in the test statistic (1.2) and the weighting constant ν in (2.4) for EISE may be different. In our theoretical results on EISE and $D_{n,\kappa}$ we treat ν and κ separately. However for performing goodness-of-fit test, it seems natural to set $\nu = \kappa$. In our simulation studies in Section 3 we set $\nu = \kappa$.

Setting $\partial I/\partial\alpha = \partial I/\partial\beta = 0$ in (2.5), we obtain the following estimating equations for EISE.

$$\frac{\partial I}{\partial \alpha} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{x_j - \alpha}{((\nu+1)^2\beta^2 + (x_j - \alpha)^2)^2} = 0, \tag{2.7}$$

$$\frac{\partial I}{\partial \beta} = 0 \quad \Leftrightarrow \quad \frac{1}{n}\sum_{j,k=1}^{n} \frac{\nu(x_j - x_k)^2}{(\nu^2\beta^2 + (x_j - x_k)^2)^2} - \sum_{j=1}^{n} \frac{2(1+\nu)(x_j-\alpha)^2}{((1+\nu)^2\beta^2 + (x_j - \alpha))^2} = 0. \tag{2.8}$$

Although these estimating equations are somewhat more complicated than the likelihood equations in (2.1) and (2.2), we can employ standard theory of U-statistics to study the asymptotic behavior of the estimating equations. We could not establish unimodality of $I(\alpha, \beta)$, but in our experiences the estimating equations can be solved numerically if an appropriate initial value is chosen and apparently produced a unique solution.

The test statistics $D_{n,\kappa}$ has yet another alternative representation, which is useful for obtaining its asymptotic distribution.

$$D_{n,\kappa} = \int_{-\infty}^{\infty} \hat{Z}_n(t)^2 \hat{\beta}_n e^{-\hat{\beta}_n \kappa |t|} dt,$$

where

$$\hat{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Big\{ \cos(tx_j) + \sin(tx_j) - e^{-\hat{\beta}_n|t|}\big(\cos(t\hat{\alpha}_n) + \sin(t\hat{\alpha}_n)\big) \Big\}. \tag{2.9}$$

$\hat{Z}_n(t)$ corresponds to the empirical characteristic process. We use the Fréchet space $C(\mathbf{R})$ of continuous functions on the real line $\mathbf{R}$ for considering the random processes. The metric of $C(\mathbf{R})$ is given by

$$\rho(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(x,y)}{1 + \rho_j(x,y)},$$

where $\rho_j(x,y) = \max_{|t|\le j} |x(t) - y(t)|$.

In the rest of this paper we use the following notations. $\xrightarrow{D}$ means weak convergence of random variables or stochastic processes, $\xrightarrow{P}$ means convergence in probability and i.i.d. means "independently and identically distributed" as usual.

Now we state results on weak convergence of $\hat{Z}_n(t)$ and weak convergence of test statistics $D_{n,\kappa}$ in the following two theorems. Note that because our tests are affine invariant, we can assume without loss of generality that $X_1, \dots, X_n$ is a random sample from $C(0,1)$.

Theorem 2.1 *Let $X_1, \dots, X_n$ be* i.i.d. *$C(0,1)$ random variables and let $\hat{Z}_n$ be defined in* (2.9). *Then $\hat{Z}_n \xrightarrow{D} Z$ in $C(\mathbf{R})$, where Z is a zero mean Gaussian process with covariance functions given below for*

MLE *and* EISE, *respectively.*

$$(2.10)\quad \text{MLE}: \quad \Gamma(s,t) = e^{-|t-s|} - \{1 + 2(st + |st|)\}e^{-|s|-|t|},$$

$$(2.11)\quad \text{EISE}: \quad \Gamma(s,t) = e^{-|t-s|} - e^{-|s|-|t|} + M_1(st + |st|)e^{-|s|-|t|} - M_2\left\{(t \cdot \operatorname{sgn} s + |t|)(1 - e^{-\nu|s|}) + (s \cdot \operatorname{sgn} t + |s|)(1 - e^{-\nu|t|})\right\}e^{-|s|-|t|} + M_3(e^{-\nu|s|} + e^{-\nu|t|})(st + |st|)e^{-|s|-|t|},$$

where

$$M_1 = \frac{(\nu+2)^2(5\nu^2 + 14\nu + 10)}{16(\nu+1)^3}, \quad M_2 = \frac{(\nu+1)(\nu+2)}{\nu^2}, \quad M_3 = \frac{(\nu+2)^2}{2\nu},$$

and ν *is the weighting constant in* (2.4).

These asymptotic covariance functions do not involve definite integrals as was the case of the median and the interquartile range in Gürtler and Henze (2000). In particular the case of MLE is very simple.

Note that for both cases $\Gamma(s,t)$ is symmetric with respect to the origin and $\Gamma(s,t) = 0$ for s,t such that $st < 0$. This implies that $\{Z(t) \mid t > 0\}$ and $\{Z(-t) \mid t > 0\}$ are independently and identically distributed for both cases. Covariance functions $\Gamma(s,t)$ for MLE and for EISE with $\nu = 1.0$ are plotted in Figure 1 and Figure 2.

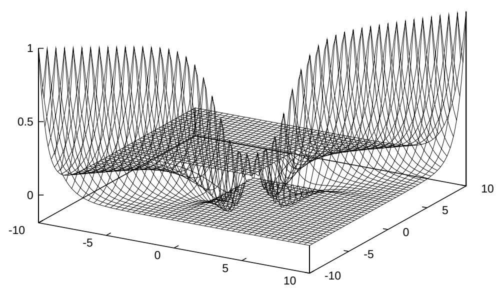

Figure 1: MLE

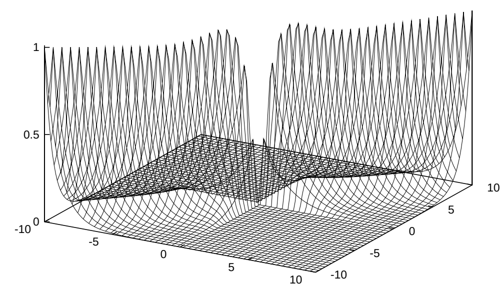

Figure 2: EISE ($\nu = 1.0$)

Theorem 2.2 *Under the conditions of Theorem 2.1*

$$D_{n,\kappa} = \int_{-\infty}^{\infty} \hat{Z}_n(t)^2 \hat{\beta}_n e^{-\hat{\beta}_n \kappa |t|} dt \xrightarrow{D} D_\kappa := \int_{-\infty}^{\infty} Z(t)^2 e^{-\kappa|t|} dt.$$

By Fubini, the exact expectation of D_κ can be evaluated as

$$\mathrm{E}(D_\kappa) = \int_{-\infty}^{\infty} \mathrm{E}(Z(t)^2) e^{-\kappa|t|} dt = \int_{-\infty}^{\infty} \Gamma(t,t) e^{-\kappa|t|} dt.$$

Substituting (2.10) and (2.11) we obtain $\mathrm{E}(D_\kappa)$ for the case of MLE and EISE as

$$(2.12)\quad \text{MLE}: \quad \mathrm{E}(D_\kappa) = \frac{4}{\kappa(\kappa+2)} - \frac{16}{(\kappa+2)^3},$$

$$(2.13)\quad \text{EISE}: \quad \mathrm{E}(D_\kappa) = \frac{4}{\kappa(\kappa+2)} + \frac{8M_1}{(\kappa+2)^3} - \frac{8M_2}{(\kappa+2)^2} + \frac{8M_2}{(\kappa+\nu+2)^2} + \frac{16M_3}{(\kappa+\nu+2)^3}.$$

These exact expectations of D_κ for both cases will be used as numerical checks in approximating the eigenvalues of the covariance function $\Gamma(s,t)e^{-\kappa(|s|+|t|)/2}$ in Section 3.4.

2.2 Approximation of the asymptotic critical values of the proposed test statistics

In this section we investigate the distribution of D_κ. For convenience, we briefly review some basic material on a homogeneous integral equation of the second kind and the associated Fredholm determinant. The Fredholm determinant gives the characteristic function of D_κ. Detailed treatments of this approach in statistical applications are given in Tanaka (1996) or Anderson and Darling (1952). We refer to Hochstadt (1973) for standard material on integral equation. Later in this section we transform our kernels $\Gamma(s,t)$ on $\mathbf{R}^2$ to kernels $K(s,t)$ on $[-1,1]^2$.

Theorem 2.3 (Mercer's Theorem, Chapter 5 of Hochstadt (1973)) *Let $K(s,t)$ be the kernel of a positive self-adjoint operator on $L^2[-1,1]$ and suppose that $K(s,t)$ is continuous in both variables. Then*

$$K(s,t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(s) f_j(t), \qquad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty, \tag{2.14}$$

where λ_j is an eigenvalue and $f_j(t)$ is the corresponding orthonormal eigenfunction of the integral equation

$$\lambda \int_{-1}^{1} K(s,t) f(t) dt \;\; = \;\; f(s). \tag{2.15}$$

The series (2.14) *converges uniformly and absolutely to $K(s,t)$.*

Usually Mercer's theorem is stated for functions in $L^2[0,1]$, but for convenience in our proofs we stated it in terms of $L^2[-1,1]$. For our problem we need to deal with kernels which are not continuous at $(-1,-1)$ and $(1,1)$. As in Anderson and Darling (1952) the following version of Mercer's theorem by Hammerstein (1927) is useful.

Theorem 2.4 *Suppose that the covariance function $K(s,t)$ of a Gaussian process is continuous except at $(-1,-1)$ and $(1,1)$ with $\partial K(s,t)/\partial s$ continuous for $|s|,|t| < 1, s \neq t$, and bounded in $|s| \leq 1-\epsilon$ for every $t \in [-1,1]$ and every $\epsilon > 0$. Then the right hand side of* (2.14) *converges uniformly in every domain in the interior of* $[-1,1]^2$.

We apply the above theorems to a continuous covariance function $K(s,t)$ of a zero mean continuous Gaussian process $Z(t)$, $-1 < t < 1$, with a finite trace $\int_{-1}^{1} K(t,t)dt < \infty$. Let $X_1, X_2, \ldots,$ be i.i.d. standard normal random variables. Then the series

$$Y(t) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} f_j(t) X_j$$

converges in the mean and with probability one for each $t \in (-1,1)$. Then $Y(t)$ is a Gaussian process with $\mathrm{E}Y(t) = 0$ and $\mathrm{E}[Y(t)Y(s)] = K(s,t)$. Thus $Y(t)$ defines the same stochastic process as $Z(t)$. Let

$$W^2 = \int_{-1}^{1} Y^2(t) dt = \int_{-1}^{1} \left\{ \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} f_j(t) X_j \right\}^2 dt = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} X_j^2. \tag{2.16}$$

The characteristic function of W^2 is given as

$$\mathrm{E}(e^{iuW^2}) = \mathrm{E}[\exp(iu\sum_{j=1}^{\infty} X_j^2/\lambda_j)] = \prod_{j=1}^{\infty} \mathrm{E}[\exp(iuX_j^2/\lambda_j)] = \prod_{j=1}^{\infty}(1-2iu/\lambda_j)^{-\frac{1}{2}}.$$

The characteristic function has an alternative expression $1/\sqrt{D(2it)}$ where $D(\lambda)$ is the associated Fredholm determinant

$$D(\lambda) = \prod_{j=1}^{\infty}\left(1-\frac{\lambda}{\lambda_j}\right).$$

There are two problems in treating the characteristic function in the form of Fredholm determinant. One is in the approximation of $D(\lambda)$ itself and the other is in the Lev́y's inversion formula. In many previous works, the exact Fredholm determinant was available and the exact characteristic function was inverted analytically or numerically. For example, Anderson and Darling (1952) inverted the exact characteristic function of their statistics and expressed the distribution as series, although each term in the series needs numerical computation. Many examples of characteristic functions given in Tanaka (1996) are numerically inverted.

In our problem Fredholm determinant can not be explicitly evaluated and we need to first approximate $D(\lambda)$. We discretize the homogeneous integral equation and use the eigenvalues of resulting finite system of linear equations. Since the eigenfunctions $\{f_j(x)\}$ are continuous, we can approximate f_j by a step function

$$\tilde{f}_{j,N}(x) = \sum_{k=1}^{N} f_j(\xi_k)\varphi_k(x),$$

where

$$\varphi_k(x) = \begin{cases} 1, & \text{if} \quad -1+\frac{2(k-1)}{N} < x \le -1+\frac{2k}{N}, \\ 0, & \text{otherwise,} \end{cases}$$

is the indicator function of the interval $(-1+2(k-1)/N, -1+2k/N]$ and $\xi_k \in (-1+2(k-1)/N, -1+2k/N)$, $k=1,\dots,N$. As shown in Section 3.4 our approximation works well for the present problem. In the literature on numerical treatment of integral equations (e.g. Baker (1977)), many other approximations of the eigenfunctions are considered.

By the above discretization the integral equation (2.15) is approximated by the following finite system of linear equations

$$\tilde{f} = \frac{\lambda}{N}\tilde{K}\tilde{f},$$

where

$$\tilde{K} = \begin{pmatrix} K(\xi_1,\xi_1) & \dots & K(\xi_1,\xi_N) \\ \vdots & & \vdots \\ K(\xi_N,\xi_1) & \dots & K(\xi_N,\xi_N) \end{pmatrix}, \qquad \tilde{f} = \begin{pmatrix} f(\xi_1) \\ \vdots \\ f(\xi_N) \end{pmatrix}.$$

Then Fredholm determinant is approximated as

$$\tilde{D}_N(\lambda) = \left|I - \frac{\lambda}{N}\tilde{K}\right| = \prod_{j=1}^{N}\left(1-\frac{\lambda}{\tilde{\lambda}_j}\right), \qquad 0 < \tilde{\lambda}_1 \le \dots \le \tilde{\lambda}_N,$$

where $1/\tilde{\lambda}_j = 1/\tilde{\lambda}_j(N)$ are the eigenvalues of $\tilde{K}/N$. This method is called a quadrature method and we state a version of Theorem 3.4 of Baker (1977) concerning the convergence of eigenvalues.

Theorem 2.5 *Let the eigenvalues $\tilde{\lambda}_j(N)$ be obtained by the quadrature method. If $K(s,t)$ is positive definite and continuous in $s,t \in [-1,1]$,*

$$\lim_{N\to\infty} \tilde{\lambda}_j(N) = \lambda_j,$$

for each j and

$$\lim_{N\to\infty} \tilde{D}_N(\lambda) = D(\lambda)$$

for each λ.

Remark 2.1 *The covariance functions in* (2.19) *and* (2.20) *below do not satisfy the conditions of this theorem if $\kappa \le 1$. However this theorem gives only a sufficient condition for the convergence. In our problem the values of $\tilde{D}_N(\lambda)$ seems to converge as we increase N even for the case $\kappa \le 1$ and the resulting value is consistent with our Monte Carlo simulations. Therefore in Section 3.4 we use the approximation of this theorem even for the case $\kappa \le 1$. It remains to theoretically prove that the approximation is valid for the case $\kappa \le 1$.*

Inverting the characteristic functions gives the distribution functions and the probability density functions of the proposed statistics. As we have seen, the characteristic function is usually expressed as the square root of a complex valued function, which sometimes causes computational difficulty (Section 6.1 of Tanaka (1996)). But in our problem the eigenvalues appear in pairs and we do not need the square root in the characteristic function. This follows from the independence and the identical distribution of $\{Z(t) \mid t < 0\}$ and $\{Z(-t) \mid t > 0\}$. We state this as a theorem.

Theorem 2.6 *Suppose that $K(s,t)$, $-1 \le s,t \le 1$ satisfies the conditions of Theorem 2.3 or 2.4. Furthermore suppose that $K(s,t)$ is symmetric with respect to the origin, $K(s,t) = 0$ if $st < 0$ and has finite trace $\int_{-1}^{1} K(t,t)dt < \infty$. Then the characteristic function of the quadratic form W^2 in* (2.16) *is given by*

$$\phi(t) = E(e^{itW^2}) = \frac{1}{D(2it)},$$

where $D(\lambda)$ is the Fredholm determinant of the kernel $K(s,t)$ restricted to $[0,1]^2$.

A proof of this theorem is given in Appendix B.

Using the residue theorem, we can now invert characteristic function analytically. Assuming that the kernel $K(s,t)$ restricted to $[0,1]^2$ has only single eigenvalues, the corresponding density function and distribution function are calculated as

$$f_{W^2}(y) = \sum_{j=1}^{\infty} \frac{\lambda_j}{2} \frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{\infty}(1-\frac{\lambda_j}{\lambda_k})}, \tag{2.17}$$

$$F_{W^2}(y) = 1 - \sum_{j=1}^{\infty} \frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{\infty}(1-\frac{\lambda_j}{\lambda_k})}. \tag{2.18}$$

Note that the series on the right hand side is alternating and we can bound the $f_{W^2}(y)$ and $F_{W^2}(y)$ relatively easily.

Remark 2.2 *In the case of multiple eigenvalues, the residue calculation becomes somewhat more complicated and we omit the results for the case of multiple eigenvalues. As shown in Table 12, our numerical study indicates that our kernels restricted to* $[0,1]^2$ *have single eigenvalues only.*

Note that our kernel $\Gamma(s,t)$ on $\mathbf{R}^2$ does not satisfy finite interval condition of Mercer's theorem. In such a case, integral equation (2.15) is called a singular integral equation and is difficult to treat. Therefore we will make a transformation of variable and map $\mathbf{R}$ into $[-1,1]$. First, in view of Theorem 2.6 we only consider $s,t \geq 0$ in $\Gamma(s,t)$. Furthermore for deriving the distribution of D_κ, we have to incorporate the weight function $e^{-\kappa t}$ into the kernel, i.e., we consider the following kernels

$$\begin{aligned}
\text{MLE}: \quad & \Gamma^M(s,t) = \{e^{-|t-s|} - (1+4st)e^{-(s+t)}\}e^{-\frac{\kappa}{2}(s+t)}, \\
\text{EISE}: \quad & \Gamma^I(s,t) = [e^{-|t-s|} - \{1 - 2M_1 st + 2M_2(s+t)\}e^{-(s+t)} \\
& \qquad +2\{(M_2 t + M_3 st)e^{-\kappa s} + (M_2 s + M_3 st)e^{-\kappa t}\}e^{-(s+t)}]e^{-\frac{\kappa}{2}(s+t)}.
\end{aligned}$$

Now we make the transformation $s \mapsto u$ defined by

$$u = \int_0^s e^{-x} dx = 1 - e^{-s}, \qquad 0 \leq u \leq 1.$$

Then

$$s = -\log(1-u), \quad ds = \frac{1}{1-u} du = e^s du.$$

The kernel and the eigenfunctions are transformed as

$$\begin{aligned}
\Gamma(s,t) \quad &\mapsto \quad K(u,v) = \frac{\Gamma(-\log(1-u), -\log(1-v))}{\sqrt{(1-u)(1-v)}}, \\
f_j(s) \quad &\mapsto \quad \frac{f_j(-\log(1-u))}{\sqrt{1-u}}.
\end{aligned}$$

Eigenvalues of (2.15) do not change by this transformation and so is Fredholm determinant.

After this transformation, writing s,t instead of u,v again, we have the following kernels on $[0,1]^2$:

$$\begin{aligned}
\text{(2.19)}\ \text{MLE}: \quad & K^M(s,t) = ((1-s)(1-t))^{\frac{\kappa-1}{2}} \left[\min\left\{\frac{1-t}{1-s}, \frac{1-s}{1-t}\right\} \right. \\
& \qquad \left. -\{1 + 4\log(1-s)\log(1-t)\}(1-s)(1-t) \right].
\end{aligned}$$

$$\begin{aligned}
\text{(2.20)}\ \text{EISE}: \quad & K^I(s,t) = ((1-s)(1-t))^{\frac{\kappa-1}{2}} \left[\min\left\{\frac{1-t}{1-s}, \frac{1-s}{1-t}\right\} \right. \\
& \qquad -\{1 - 2M_1\log(1-s)\log(1-t) - 2M_2\log((1-s)(1-t))\}(1-s)(1-t) \\
& \qquad -2\Big\{\big(M_2\log(1-t) - M_3\log(1-s)\log(1-t)\big)(1-s)^\kappa \\
& \qquad \left. +\big(M_2\log(1-s) - M_3\log(1-s)\log(1-t)\big)(1-t)^\kappa\Big\}(1-s)(1-t) \right].
\end{aligned}$$

If $\kappa > 1$, the conditions of Mercer's theorem are satisfied. In the case of $0 < \kappa \leq 1$ these kernels are discontinuous at $(1,1)$, but Theorem 2.4 is applicable. Note that by (2.12) and (2.13) traces of these kernels are finite. Figure 3 and Figure 4 show the graphs of ISE and MLE kernels $K(s,t)$ respectively with $\kappa = 1.0$. Figure 5 and Figure 6 are for the case $\kappa = 3.0$. In the case $\kappa = 1.0$, we can see discontinuity of $K(s,t)$ at $(1,1)$.

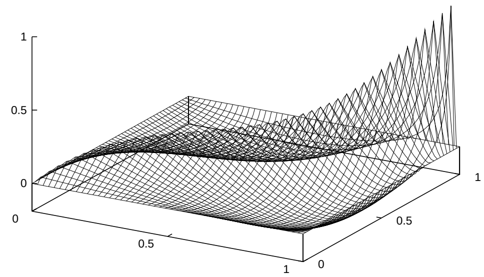

Figure 3: MLE ($\kappa = 1.0$)

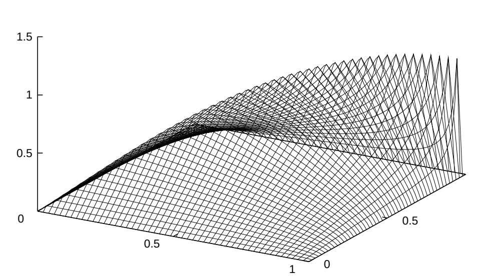

Figure 4: EISE ($\kappa = 1.0$)

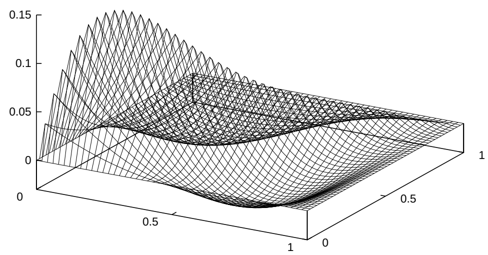

Figure 5: MLE ($\kappa = 3.0$)

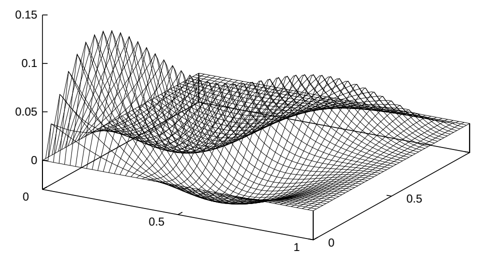

Figure 6: EISE ($\kappa = 3.0$)

3 Computational studies

In this section, some simulation results are given. Since the exact finite sample distributions are difficult to obtain, first we approximate the percentage points of $D_{n,\kappa}$ by Monte Carlo simulation. Then the power of both tests for the finite sample is evaluated. In the end of this section percentage points of D_κ is computed by the residue theorem.

For MLE, the estimates are easily found by Newton method. Convenient initial values are suggested by Copas (1975). But for EISE simple Newton method does not work well and we often need grid search of initial values. In the case of EISE we use the same initial value as MLE for Newton method. If it fails to converge we do grid search of initial values and we obtain the parameter value which minimizes (2.3) among, say, 20000 points, and use it as the initial value of Newton method. When the values of the estimators have converged, we can compute $D_{n,\kappa}$ by (1.3). Based on 100,000 Monte Carlo replications, the upper 10 and 5 percentage points of the statistics $D_{n,\kappa}, \kappa \in \{0.1, 0.5, 1.0, 2.5, 5.0, 10.0\}$ are tabulated in Table 1, Table 2 for MLE and Table 3, Table 4 for EISE.

3.1 Tests based on empirical distribution function

Here we give critical values for some classical procedures based on the empirical distribution function of Cauchy distribution $C(0,1)$, which were not tabulated in Section 4.14 of D'Agostino and Stephens (1986). They stated that MLE was computationally difficult and only gave critical values based on estimators proposed in Chernoff *et al.* (1967). Let $F_0(x) = 1/2+\pi^{-1}\arctan x$ be the distribution function of $C(0,1)$, $Y_{(j)} = (X_{(j)}-\hat{\alpha})/\hat{\beta}$, $j = 1,\ldots,n$, be the order statistic of the standardized data, and let $Z_{(j)} = F_0(Y_{(j)})$. We consider the Anderson-Darling statistic AD, Cramér-von Mises statistic CM, Kolmogorov-Smirnov statistic KS, Watson statistic W, defined by

$$\begin{aligned}
\mathrm{KS} &= \max\left\{\max_{1\le j\le n}(\frac{j}{n} - Z_{(j)}), \max_{1\le j\le n}(Z_{(j)} - \frac{j-1}{n})\right\}, \\
\mathrm{CM} &= \sum_{j=1}^{n}\left(Z_{(j)} - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}, \\
\mathrm{AD} &= -n - \frac{1}{n}\sum_{j=1}^{n}\left\{(2j-1)\log Z_{(j)} + (2n+1-2j)\log(1-Z(j))\right\}, \\
\mathrm{W} &= \mathrm{CM} - n\left(\frac{1}{n}\sum_{j=1}^{n} Z_{(j)} - \frac{1}{2}\right)^2.
\end{aligned}$$

We tabulate critical values of above tests in Table 5 and Table 6 for the case that the parameters are estimated by MLE. It is interesting to compare these classical tests when estimators other than MLE are used. However our main concern here is the comparison of the tests based on the empirical characteristic function to those based on empirical distribution function when the parameters are estimated.

3.2 Alternative hypotheses

For studying the power functions of various tests considered, we use the following family of distributions containing the Cauchy distribution as a special case.

1. $t(j)$. Student's t distribution with j degrees of freedom for $j = 1,2,3,4,5,10,\infty$. Note $t(1) = C(1,0)$ and $t(\infty) = N(0,1)$.

2. $st(a,b)$. Stable distributions with the characteristic function

$$\Phi(t) = \begin{cases} \exp\bigl(-|t|^a[1 - ib\,\mathrm{sgn}\,t\tan(a\pi/2)]\bigr), & \text{if} \quad a \neq 1, \\ \exp\bigl(-|t|[1 + ib(2/\pi)\,\mathrm{sgn}\,t\log t]\bigr), & \text{if} \quad a = 1. \end{cases}$$

   Here we only consider symmetric stable distributions ($b = 0$). Characteristic exponent $a \in (0,2]$ concerns the tail behavior of the distribution. Note that $st(1,0) = C(0,1)$ and $st(2,0) = N(0,1)$.

3.3 Analysis of finite sample power

For the significance levels $\xi = 0.1,\ 0.05$, finite sample power of the tests are tabulated in Table 7, Table 9, Table 11 for MLE and Table 8, Table 10 for EISE based on 10,000 Monte Carlo replications. In these tables '*' stands for 100, i.e. the power of 100 percent. We omit the case of EISE with the sample size

$n = 200$, because the simulation study becomes computationally very heavy due to frequent need of grid search of initial values under various alternative hypotheses.

In comparison to the test proposed by Gürtler and Henze (2000) for the nominal level of 10 percent, our test has about 10 percent less power than their test for the case of $n = 50$ and the alternatives $t(2), t(3), t(4)$. Our tests are slightly more powerful than their test when the alternative distribution is far away from the null (e.g. $N(0,1)$) or the weight κ is small and the alternative is close to the null (e.g. $st(1.2, 0)$).

We also see that our tests compare well to more traditional tests based on the empirical distribution function tabulated in Gürtler and Henze (2000).

For future research, it is worth investigating the problem of choosing weights κ and ν depending on alternatives in order to maximize power.

3.4 Approximation of D_κ

First we calculate the eigenvalues of kernels (2.19) and (2.20) numerically. 500 eigenvalues are easily approximated by the above simple algorithm except for the case $\kappa = 0.1$. For the case of $\kappa = 0.1$, we had some numerical difficulty and the approximated sum of $2\sum_{j=1}^{\infty} 1/\lambda_j$ did not converge to $\mathrm{E}[D_\kappa]$ quickly. Therefore we omit the case $\kappa = 0.1$ and present results for the cases $\kappa \in \{0.5, 1.0, 2.5, 5.0, 10.0\}$. Note that the powers of both tests are the lowest for the case of $\kappa = 0.1$. Table 12 gives the largest 10 values of $1/\lambda_j$ for both MLE and EISE with $\kappa \in \{0.5, 1.0, 2.5, 5.0, 10.0\}$.

The infinite sum and the infinite products in (2.18) have to be approximated by a finite sum and finite products. Let l and m $(l < m)$ denote the number of terms in the sum and the products, respectively. Then $F_{W^2}(y)$ in (2.18) is approximated as

$$F_{W^2}(y) \approx 1 - \sum_{j=1}^{l} \frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{m}(1 - \frac{\lambda_j}{\lambda_k})}.$$

For the remaining part of the product $1/\prod_{m+1}^{\infty}(1 - \frac{\lambda_j}{\lambda_k})$, we can give bounds by using the equation

$$\frac{1}{1-x} = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right).$$

A lower bound of the j-th term of the series is given by

$$\frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{m}(1 - \frac{\lambda_j}{\lambda_k})} \exp\left(\lambda_j \sum_{k=m+1}^{\infty} \frac{1}{\lambda_k}\right),$$

and an upper bound is given by

$$\frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{m}(1 - \frac{\lambda_j}{\lambda_k})} \exp\left(\lambda_j \left\{1 + \frac{\lambda_j}{2(\lambda_{m+1} - \lambda_j)}\right\} \sum_{k=m+1}^{\infty} \frac{1}{\lambda_k}\right).$$

For evaluating $\sum_{k=m+1}^{\infty} 1/\lambda_k$, we can utilize the expected value $\mathrm{E}(D_\kappa) = 2\sum_{k=1}^{\infty} 1/\lambda_k$. The sum $\sum_{k=1}^{m} 1/\lambda_k$ is evaluated numerically by approximating the first m eigenvalues as $\sum_{k=1}^{m} 1/\lambda_k \approx \sum_{k=1}^{m} 1/\tilde{\lambda}_k$ and then $\mathrm{E}(D_\kappa)/2 - \sum_{k=1}^{m} 1/\tilde{\lambda}_k$ approximates $\sum_{k=m+1}^{\infty} 1/\lambda_k$.

Note that the series is alternating. Therefore the range of the critical values is obtained by substituting the above bounds for positive terms and negative terms separately, i.e., by substituting the upper bound for positive terms and the lower bound for negative terms, or vice versa. We present Tables 13 and 14 of approximate percentage points of D_κ.

More extensive numerical results on the ranges are tabulated in Tables 15–18. In these tables four values are presented for each combination of l, m and κ. The upper left value gives a lower bound (a) for percentage points of D_κ and the upper right values gives the corresponding upper bound (b). The lower left value is difference of the upper bound and the lower bound. Furthermore the lower right value gives $\sum_{k=m+1}^{\infty} 1/\tilde{\lambda}_k$. These are summarized in the following table.

	κ	
l/m	a	b
	$b-a$	$\mathrm{E}(D_\kappa)/2 - \sum_{k=1}^{m} 1/\tilde{\lambda}_k$

where

a : a lower bound for percentage points of D_κ ,

b : an upper bound for percentage points of D_κ ,

$b-a$: the range for percentage points of D_κ.

Note that more accurate approximations may be obtained if we evaluate higher order moments of D_κ.

A Proofs of the results in Section 2.1.

Proofs of the theorems are essentially the same as those of Gürtler and Henze (2000). Although we could state only the differences in our case from Gürtler and Henze (2000), for convenience we reproduce here the outline of the whole proof.

Before considering Fréchet space $C(\mathbf{R})$, we first assume the restricted space $C(S)$ of continuous functions on a compact subset S with the supremum norm $\|f\| = \sup_{t\in s} |f(t)|$. Defining $k(x,t) = \cos(tx) + \sin(tx)$, alternative representation of $\hat{Z}_n(t)$ is given by

$$\begin{aligned}\hat{Z}_n(t) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Big\{ \cos(tX_j) + \sin(tX_j) - e^{-\hat{\beta}_n|t|}(\cos(t\hat{\alpha}_n) + \sin(t\hat{\alpha}_n)) \Big\} \\ &= \int k(x,t) d \Big\{ \sqrt{n}\big(F_n(x) - F(x, \hat{\theta}_n)\big) \Big\} .\end{aligned}$$

We only have to check the conditions (iv) and (v) of Csörgő (1983). The first step is to obtain the "Bahadur representations"

$$\sqrt{n}\hat{\alpha}_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_1(X_j) + r_{1n}, \quad \sqrt{n}(\hat{\beta}_n - 1) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_2(X_j) + r_{2n}, \qquad r_{1n}, r_{2n} \xrightarrow{P} 0$$

of the estimators for the standard Cauchy case $C(0, 1)$.

Lemma A.1 *The Bahadur representations $l(x) = (l_1(x), l_2(x))$ and their covariance matrices for MLE and for EISE are given by*

1. MLE

$$l_1(x) = \frac{4x}{1+x^2}, \qquad l_2(x) = \frac{2(x^2-1)}{1+x^2},$$

$$E[l(X)l(X)'] = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

2. EISE

$$\begin{aligned} l_1(x) &= (\nu+1)(\nu+2)^3 \frac{x}{((\nu+1)^2+x^2)^2}, \\ l_2(x) &= \frac{1}{2}(\nu+2) - \frac{1}{2}(\nu+2)^3 \frac{(\nu+1)^2 - x^2}{((\nu+1)^2+x^2)^2}, \end{aligned}$$

$$E[l(X)l(X)'] = \frac{(\nu+2)^2(5\nu^2+14\nu+10)}{16(\nu+1)^3} \times I_2,$$

where ν is the weighting constant in (2.4) and I_2 is the 2×2 identity matrix.

Proof. In the case of MLE, l_1 and l_2 are easily obtained from the score functions and the Fisher information matrix.

For EISE we apply the delta method to the estimating equations (2.7), (2.8) for the case of $C(\alpha, \beta)$ and obtain

$$\sqrt{n}(\hat{\alpha}_n - \alpha) = \frac{1}{\sqrt{n}} \frac{\sum_{j=1}^n g_1(X_j)}{\sum_{j=1}^n g_2(X_j)/n} + r_{1n}, \qquad r_{1n} \xrightarrow{P} 0,$$

$$\sqrt{n}(\hat{\beta}_n - \beta) = \frac{\sqrt{n}}{4\beta} \frac{\frac{1}{n(n-1)} \sum_{j,k=1}^n h_1(X_j, X_k) - \frac{1}{n-1} \sum_{j=1}^n 2h_2(X_j)}{\frac{1}{n(n-1)} \sum_{j,k=1}^n h_3(X_j, X_k) - \frac{1}{n-1} \sum_{j=1}^n 2h_4(X_j)} + r_{2n}, \quad r_{2n} \xrightarrow{P} 0, \tag{A.1}$$

where

$$g_1(x) = -\frac{(x-\alpha)}{((x-\alpha)^2 + (\nu+1)^2\beta^2)^2}, \qquad g_2(x) = \frac{3(x-\alpha)^2 - (\nu+1)^2\beta^2}{((x-\alpha)^2 + (\nu+1)^2\beta^2)^3},$$

$$h_1(x_1, x_2) = \frac{\nu(x_1-x_2)^2}{((x_1-x_2)^2 + \nu^2\beta^2)^2}, \qquad h_3(x_1, x_2) = \frac{\nu^3(x_1-x_2)^2}{((x_1-x_2)^2 + \nu^2\beta^2)^3},$$

$$h_2(x) = \frac{(\nu+1)(x-\alpha)^2}{((x-\alpha)^2 + (\nu+1)^2\beta^2)^2}, \qquad h_4(x) = \frac{(\nu+1)^3(x-\alpha)^2}{((x-\alpha)^2 + (\nu+1)^2\beta^2)^3}.$$

Returning to the standard case $(\alpha, \beta) = (0, 1)$, the numerator of $\sqrt{n}(\hat{\beta}_n - 1)$ in (A.1) can be expressed in the form of a U-statistic

$$\sqrt{n}\{U_n - \frac{1}{n(n-1)} \sum_{j=1}^n 2h_2(X_j)\} = \sqrt{n}U_n + r_{3n}, \qquad r_{3n} \xrightarrow{P} 0,$$

where

$$U_n = \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1} \sum_{1 \le j < k \le n}^n h(X_j, X_k) = \frac{2}{n(n-1)} \sum_{1 \le j < k \le n}^n \{h_1(X_j, X_k) - h_2(X_j) - h_2(X_k)\}.$$

By standard argument on U-statistic (Chapter 3 of Maesono (2001), Chapter 5 of Serfling (1980)) we only need to evaluate

$$a(x_1) = \mathrm{E}[h(X_1, X_2) \mid X_1 = x_1],$$

since

$$\sqrt{n}U_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} 2a(X_j) + r_{4n}, \qquad r_{4n} \xrightarrow{P} 0.$$

It can be shown that $a(x_1)$ is written as

$$a(x_1) = \frac{1}{2}\frac{(\nu+1)^2 - x_1^2}{(x_1^2 + (\nu+1)^2)^2} - \frac{1}{2(\nu+2)^2}.$$

The denominators of $\sqrt{n}\hat{\alpha}_n$ and $\sqrt{n}(\hat{\beta}_n - 1)$ converge in probability to their expectations

$$\mathrm{E}\left[\sum_{j=1}^{n} g_2(X_j)/n\right] = -\frac{1}{(\nu+1)(\nu+2)^3},$$

$$\mathrm{E}\left[\frac{4}{n(n-1)}\sum_{j,k=1}^{n} h_3(X_j, X_k) - \frac{8}{n-1}\sum_{j=1}^{n} h_4(X_j)\right] = -\frac{2}{(\nu+2)^3}.$$

Thus the Bahadur representation of EISE is given by

$$\begin{aligned}
\sqrt{n}\hat{\alpha}_n &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\nu+1)(\nu+2)^3\frac{X_j}{(X_j^2 + (\nu+1)^2)^2} + r_{1n},\\
\sqrt{n}(\hat{\beta}_n - 1) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left\{\frac{(\nu+2)^3}{2}\frac{X_j{}^2 - (\nu+1)^2}{\left(X_j{}^2 + (\nu+1)^2\right)^2} + \frac{(\nu+2)}{2}\right\} + r_{2n}.
\end{aligned}$$

□

Note that the covariance matrix $\mathrm{E}[l(X)l(X)']$ for both MLE and EISE is finite and positive definite. Therefore condition (iv) of Csörgő (1983) is satisfied. Since l_1 and l_2 are bounded and differentiable, condition (v) of Csörgő (1983) is satisfied. Therefore the weak convergence of $\hat{Z}_n(t)$ to a zero mean Gaussian process Z is proved in the space $(C(S), \|\cdot\|_\infty)$. Since the compact set S is arbitrary, the space $(C(s), \|\cdot\|_\infty)$ can be extended to Fréchet space $C(\mathbf{R})$ easily.

In order to derive the covariance function of Z, we set up some notations. Define

$$\nabla_\theta F(x,\theta) := \left(\frac{\partial}{\partial\alpha}F(x,\theta), \frac{\partial}{\partial\beta}F(x,\theta)\right)' = -\frac{(\beta, x-\alpha)'}{\pi\left(\beta^2 + (x-\alpha)^2\right)},$$

where $F(x,\theta) = 1/2 + \pi^{-1}\arctan((x-\alpha)/\beta)$ and define its kernel transform

$$H(t,\theta) = \left(H_1(t,\theta), H_2(t,\theta)\right)' := \int k(x,t)d\nabla_\theta F(x,\theta).$$

In the Cauchy case $H(t,\theta)$ is written as

$$\begin{aligned}
H_1(t,\theta) &= \frac{2}{\pi\beta}\int\left[\cos\left(t(\alpha+\beta y)\right) + \sin\left(t(\alpha+\beta y)\right)\right]\frac{y}{(1+y^2)^2}dy,\\
H_2(t,\theta) &= \frac{1}{\pi\beta}\int\left[\cos\left(t(\alpha+\beta y)\right) + \sin\left(t(\alpha+\beta y)\right)\right]\frac{1-y^2}{(1+y^2)^2}dy.
\end{aligned}$$

Putting $(\alpha, \beta) = \theta_0 = (0, 1)$ we obtain

$$H(t, \theta_0) = \left(te^{-|t|}, -|t|e^{-|t|}\right)'.$$

Write $F_0(x) = F(x, \theta_0)$ for simplicity. The kernel transform of Bahadur representations are given as follows.

1. MLE

$$\text{(A.2)} \qquad \begin{aligned} \int k(x,s) l_1(x) dF_0(x) &= 2se^{-|s|}. \\ \int k(x,s) l_2(x) dF_0(x) &= -2|s|e^{-|s|}. \end{aligned}$$

2. EISE

$$\text{(A.3)} \qquad \begin{aligned} \int k(x,s) l_1(x) dF_0(x) &= \frac{(\nu+1)(\nu+2)}{\nu^2}\left(e^{-|s|}\operatorname{sgn} s - e^{-(\nu+1)|t|}\right) - \frac{(\nu+2)^2}{2\nu} se^{-(\nu+1)|t|}. \\ \int k(x,s) l_2(x) dF_0(x) &= \frac{(\nu+1)(\nu+2)}{\nu^2}\left(e^{-(\nu+1)|t|} - e^{-|t|}\right) + \frac{(\nu+2)^2}{2\nu} |t|e^{-(\nu+1)|t|}. \end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbf{R}^2$. Write

$$\begin{aligned} F_n(x) - F(x, \hat{\theta}_n) &= F_n(x) - F_0(x) - (F(x, \hat{\theta}_n) - F_0(x)) \\ &= F_n(x) - F_0(x) - \langle \hat{\theta}_n - \theta_0, \nabla_\theta F(x, \theta_n^*) \rangle, \end{aligned}$$

where θ_n^* is some value between θ_0 and θ_n. Note that $\theta_n^* \xrightarrow{P} \theta_0$. Now replace $\sqrt{n}(\hat{\theta}_n - \theta_0)$ by its Bahadur representation. Then $\hat{Z}_n(t)$ is written as

$$\begin{aligned} \hat{Z}_n(t) &= \int k(x,t) d\left\{\sqrt{n}\big(F_n(x) - F_0(x)\big)\right\} - \left\langle \sqrt{n}(\hat{\theta}_n - \theta_0), H(t, \theta_n^*) \right\rangle \\ &= Z_n^*(t) + \Delta_n^{(2)}(t) + \Delta_n^{(3)}(t), \end{aligned}$$

where

$$\begin{aligned} Z_n^*(t) &:= \int k(x,t) d\left\{\sqrt{n}\big(F_n(x) - F_0(x)\big)\right\} - \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^n l(X_j), H(t, \theta_n^*) \right\rangle \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \left[\cos(tX_j) + \sin(tX_j) - e^{-|t|} - te^{-|t|} l_1(X_j) + |t|e^{-|t|} l_2(X_j)\right]. \end{aligned}$$

Z_n^* also converges to Z. The remainder terms $\Delta_n^{(2)}$ and $\Delta_n^{(3)}$ are defined by

$$\begin{aligned} \Delta_n^{(2)} &:= \left\langle \sqrt{n}(\hat{\theta}_n - \theta_0), H(t, \theta_0) - H(t, \theta_n^*) \right\rangle, \\ \Delta_n^{(3)} &:= -\langle \epsilon_n, H(t, \theta_0) \rangle, \qquad \epsilon_n = (r_{n1}, r_{n2})'. \end{aligned}$$

These remainder terms satisfy $\sup_{t \in S} |\Delta_n^{(2)}| \xrightarrow{P} 0$, and $\sup_{t \in S} |\Delta_n^{(3)}| \xrightarrow{P} 0$ by conditions (iv) and (vi) of Csörgő (1983). The asymptotic process Z has an alternative expression

$$Z(t) = \int k(x,t) dB_{F_0}(x) - \left\langle \int l(x) dB_{F_0}(x), H(t, \theta_0) \right\rangle,$$

where $B_{F_0}(x)$ is the Brownian bridge corresponding to the distribution function F_0, having covariance function $\mathrm{E}[B_{F_0}(s)B_{F_0}(t)] = F_0(s \wedge t) - F_0(s)F_0(t)$. Z^* and Z have the same covariance function

$$
\begin{aligned}
\text{(A.4)} \qquad \Gamma(s,t) &= \tilde{K}_0(s,t) - K_0(s)K_0(t) + H(s,\theta_0)'\mathrm{E}[l(X_1)l(X_1)']H(s,\theta_0) \\
&\quad - \left\langle H(t,\theta_0), \int k(x,s)l(x)dF_0(x) \right\rangle - \left\langle H(s,\theta_0), \int k(x,t)l(x)dF_0(x) \right\rangle,
\end{aligned}
$$

where

$$K_0(t) = \int k(x,t)dF_0(x) = e^{-|t|}, \qquad \tilde{K}_0(s,t) = \int k(x,s)k(x,t)dF_0(x) = e^{-|t-s|}.$$

Evaluating (A.4) for the case of MLE and EISE using (A.2) and (A.3) proves Theorem 2.1. □

We here remark a relation between the Bahadur representation l^M of MLE and the Bahadur representation l^I of EISE. From the asymptotic efficiency of MLE it follows that l^M and $l^I - l^M$ are orthogonal, i.e.,

$$\mathrm{E}[l^M(l^I - l^M)] = 0.$$

B Proofs of the results in Section 2.2.

We first give a proof of Theorem 2.6. Because $K(s,t)$ is symmetric with respect to the origin and $K(s,t) = 0$ for $st < 0$

$$K(s,t) = \begin{cases} \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(s)f_j(t), & \text{if} \quad 0 \le s,t < 1, \\ \sum_{j=1}^{\infty} \frac{1}{\lambda_j} g_j(s)g_j(t), & \text{if} \quad -1 < s,t \le 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $f_i(t) = g_i(-t)$ satisfies the integral equation (2.15) and $f_j(t) = 0$ for $t < 0$. Then

$$
\begin{aligned}
W^2 &= \int_{-1}^{1} Y^2(t)dt = \int_{-1}^{0} Y^2(t)dt + \int_{0}^{1} Y^2(t)dt \\
&= \int_0^1 \left[\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} X_{2j-1} f_j(t)\right]^2 dt + \int_{-1}^{0} \left[\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} X_{2j} g_j(t)\right]^2 dt \\
&= \sum_{j=1}^{\infty} \frac{1}{\lambda_j}(X_{2j-1}^2 + X_{2j}^2).
\end{aligned}
$$

Therefore

$$E(e^{itW^2}) = [E(\exp(it\int_0^1 Y^2(t)dt))]^2 = \frac{1}{D(2it)}.$$

□

We now derive the density function and the distribution function in (2.17) and (2.18). By the inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}}{\prod_{j=1}^{\infty}(1 - \frac{2it}{\lambda_j})}dt.$$

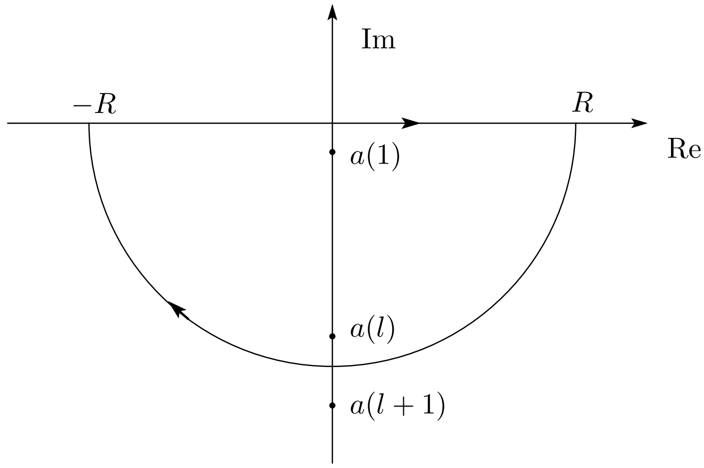

Figure 7: Complex integration

We consider complex integration

$$\frac{1}{2\pi}\oint_{-C}\frac{e^{-ixt}}{\prod_{j=1}^{\infty}(1-\frac{2it}{\lambda_j})}dt,$$

where a closed curve C consists of a semicircle C_R with radius R and a line segment $[-R, R]$ and $-C$ means the clockwise direction. See Figure 7. In the region $D = \{re^{-i\theta} \mid 0 < r < R,\ 0 < \theta < \pi\}$, the integrand has singular points at $a_j = -i\lambda_j/2$. Except for these points, the integrand is regular and continuous. The residual theorem tells us

$$\begin{aligned}\frac{1}{2\pi}\oint_{-C}\frac{e^{-ixt}}{\prod_{j=1}^{\infty}(1-\frac{2it}{\lambda_j})}dt &= -i\sum_{j=1}^{\infty}\text{Res}_{\ t=a_j}\left[\frac{e^{-ixt}}{\prod_{j=1}^{\infty}(1-\frac{2it}{\lambda_j})}\right]\\ &= \sum_{j=1}^{\infty}\frac{\lambda_j}{2}\frac{\exp(-\frac{\lambda_j}{2}y)}{\prod_{k\neq j}^{\infty}(1-\frac{\lambda_j}{\lambda_k})},\end{aligned}$$

In the integral on C_R we transform t by R and θ as $t = Re^{-i\theta}$. Then

$$\frac{1}{2\pi}\oint_{C_R}\frac{e^{-ixt}}{\prod_{j=1}^{\infty}(1-\frac{2it}{\lambda_j})}dt = \frac{1}{2\pi}\int_0^{\pi}\frac{iR\exp\{-i(xRe^{-i\theta}+\theta)\}}{\prod_{j=1}^{\infty}(1-\frac{2iRe^{-i\theta}}{\lambda_j})}d\theta \longrightarrow 0.$$

as $R \to \infty$. Here we can take R to be the midpoint $(a_j + a_{j+1})/2$ of neighboring a_j's, so that the denominator of the integrand never vanishes. Although the integrand is a function of all the eigenvalues λ_j, the convergence to zero of the integral over C_R is easily justified if $\prod_{j=1}^{\infty}(1-2iu/\lambda_j)^{-1/2}$ is of exponential order less than unity as $R \to \infty$ (see Slepian (1957)). Heuristic arguments can be easily given that $\prod_{j=1}^{\infty}(1-2iu/\lambda_j)^{-1/2}$ is of exponential order less than unity based on the fact $\sum_j 1/\lambda_j < \infty$. In general $1/\lambda_j = O(1/j^2)$ as discussed in Section 4 of Anderson and Darling (1952).

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Table 1: MLE: Upper 10 percentage points of $D_{n,\kappa}$

$n \setminus \kappa$	0.1	0.5	1.0	2.5	5.0	10.0
10	21.23	3.054	1.111	0.307	0.127	0.0498
20	21.60	3.093	1.103	0.292	0.118	0.0451
50	21.83	3.119	1.105	0.287	0.115	0.0432
100	21.90	3.131	1.108	0.285	0.115	0.0430
200	21.89	3.136	1.111	0.286	0.114	0.0426

Table 2: MLE: Upper 5 percentage points of $D_{n,\kappa}$

$n \setminus \kappa$	0.1	0.5	1.0	2.5	5.0	10.0
10	22.95	3.406	1.271	0.373	0.160	0.0675
20	23.31	3.481	1.263	0.349	0.147	0.0592
50	23.59	3.514	1.268	0.338	0.141	0.0543
100	23.66	3.546	1.271	0.337	0.138	0.0533
200	23.65	3.555	1.275	0.335	0.137	0.0524

Table 3: EISE: Upper 10 percentage points of $D_{n,\kappa}$

$n \setminus \kappa$	0.1	0.5	1.0	2.5	5.0	10.0
10	19.88	2.806	0.992	0.209	0.0541	0.0109
20	20.74	2.916	1.040	0.228	0.0645	0.0154
50	21.22	3.006	1.078	0.241	0.0709	0.0192
100	21.39	3.026	1.085	0.244	0.0727	0.0202
200	21.45	3.040	1.089	0.245	0.0736	0.0209

Table 4: EISE: Upper 5 percentage points of $D_{n,\kappa}$

$n \setminus \kappa$	0.1	0.5	1.0	2.5	5.0	10.0
10	20.95	3.103	1.118	0.234	0.0585	0.0113
20	22.20	3.277	1.182	0.264	0.0754	0.0180
50	22.85	3.394	1.231	0.281	0.0836	0.0226
100	23.07	3.416	1.241	0.285	0.0858	0.0240
200	23.14	3.438	1.251	0.286	0.0870	0.0247

Table 5: Upper 10 percentage points of KS CM AD W

$n \setminus \kappa$	KS	CM	AD	W
10	0.236	0.131	0.988	0.050
20	0.172	0.128	0.956	0.051
30	0.142	0.128	0.955	0.051
40	0.124	0.129	0.956	0.051
50	0.111	0.129	0.951	0.051
100	0.079	0.128	0.948	0.052
200	0.056	0.128	0.944	0.052

Table 6: Upper 5 percentage points of KS CM AD W

$n \setminus \kappa$	KS	CM	AD	W
10	0.259	0.168	1.272	0.057
20	0.188	0.166	1.228	0.058
30	0.156	0.168	1.230	0.060
40	0.136	0.169	1.228	0.060
50	0.122	0.169	1.233	0.060
100	0.087	0.168	1.221	0.060
200	0.062	0.169	1.223	0.061

Table 7: MLE: Power of $D_{n,\kappa}$ (Significance levels $\xi = 0.1,\ 0.05,\ n = 50$)

ξ	0.1						0.05					
κ	0.1	0.5	1.0	2.5	5.0	10.0	0.1	0.5	1.0	2.5	5.0	10.0
$C(0,1)$	10	10	10	11	10	10	5	5	5	6	5	5
$N(0,1)$	34	72	87	96	98	98	22	60	78	90	92	86
$t(2)$	15	22	25	26	27	24	8	13	16	15	14	9
$t(3)$	19	34	43	50	55	52	10	24	30	34	36	25
$t(4)$	22	43	54	64	71	70	13	30	42	49	52	40
$t(5)$	24	48	62	73	80	80	14	35	49	58	62	50
$t(10)$	29	60	75	87	92	92	17	47	63	77	80	71
$st(0.5,0)$	75	90	94	97	98	98	64	84	89	94	96	97
$st(0.8,0)$	14	21	27	35	40	42	7	13	17	24	29	31
$st(0.9,0)$	10	12	15	18	19	21	5	7	8	10	12	13
$st(1.1,0)$	6	8	9	10	11	10	2	3	4	5	6	6
$st(1.2,0)$	8	11	12	14	15	12	2	4	6	8	8	6
$st(1.5,0)$	20	34	40	42	44	34	11	23	28	28	26	15

Table 8: EISE: Power of $D_{n,\kappa}$ (Significance levels $\xi = 0.1,\ 0.05,\ n = 50$)

ξ	0.1						0.05					
κ	0.1	0.5	1.0	2.5	5.0	10.0	0.1	0.5	1.0	2.5	5.0	10.0
$C(0,1)$	10	10	10	10	10	10	5	5	5	5	5	5
$N(0,1)$	30	66	86	98	*	*	20	53	76	95	97	92
$t(2)$	13	19	24	30	29	19	8	11	15	19	16	8
$t(3)$	17	30	41	56	58	43	10	20	30	41	40	22
$t(4)$	19	37	52	71	74	62	11	25	40	57	58	36
$t(5)$	21	42	60	80	84	74	13	30	47	67	69	48
$t(10)$	24	53	73	91	95	91	15	40	62	84	88	73
$st(0.5,0)$	50	88	94	93	87	72	32	80	89	88	79	60
$st(0.8,0)$	12	21	27	31	28	25	6	12	17	20	18	15
$st(0.9,0)$	10	12	14	16	16	15	5	6	8	9	9	8
$st(1.1,0)$	7	8	9	10	10	10	3	3	4	5	5	6
$st(1.2,0)$	7	10	13	13	13	12	3	5	7	7	7	6
$st(1.5,0)$	18	29	39	47	41	24	10	19	28	33	26	11

Table 9: MLE: Power of $D_{n,\kappa}$ (Significance levels $\xi = 0.1,\ 0.05,\ n = 100$)

ξ	0.1						0.05					
κ	0.1	0.5	1.0	2.5	5.0	10.0	0.1	0.5	1.0	2.5	5.0	10.0
$C(0,1)$	10	10	10	10	10	11	5	5	5	6	5	5
$N(0,1)$	64	98	*	*	*	*	95	*	*	*	*	*
$t(2)$	21	38	49	60	67	71	12	26	36	44	52	53
$t(3)$	31	63	79	90	95	97	20	50	67	82	89	92
$t(4)$	39	75	90	97	99	*	25	63	82	94	97	98
$t(5)$	43	83	94	*	*	*	30	72	90	97	99	*
$t(10)$	54	93	99	*	*	*	40	86	97	*	*	*
$st(0.5,0)$	96	*	*	*	*	*	94	99	*	*	*	*
$st(0.8,0)$	20	33	42	51	56	59	12	22	29	39	45	48
$st(0.9,0)$	11	15	18	22	25	26	6	8	10	13	16	17
$st(1.1,0)$	11	13	12	11	11	10	6	7	6	5	5	4
$st(1.2,0)$	14	20	22	22	22	20	7	12	14	13	13	10
$st(1.5,0)$	31	61	74	82	85	84	20	48	62	71	74	70

Table 10: EISE: Power of $D_{n,\kappa}$ (Significance levels $\xi = 0.1,\ 0.05,\ n = 100$)

ξ	0.1						0.05					
κ	0.1	0.5	1.0	2.5	5.0	10.0	0.1	0.5	1.0	2.5	5.0	10.0
$C(0,1)$	10	10	10	10	10	10	5	5	5	5	5	5
$N(0,1)$	55	97	*	*	*	*	41	92	*	*	*	*
$t(2)$	17	32	47	63	66	57	10	22	35	50	51	40
$t(3)$	24	56	77	92	95	92	14	43	66	86	89	82
$t(4)$	30	70	89	98	99	98	19	57	81	96	98	96
$t(5)$	35	77	94	*	*	*	23	66	88	99	99	99
$t(10)$	43	90	99	*	*	*	31	81	97	*	*	*
$st(0.5,0)$	83	99	*	*	99	92	71	98	*	99	97	87
$st(0.8,0)$	15	31	42	46	40	32	8	20	30	34	29	21
$st(0.9,0)$	11	15	18	20	19	17	5	8	10	12	11	10
$st(1.1,0)$	9	11	12	12	12	11	4	5	6	6	6	6
$st(1.2,0)$	13	18	22	24	20	14	7	11	14	14	11	6
$st(1.5,0)$	16	63	80	84	72	55	25	46	68	75	61	42

Table 11: MLE: Power of $D_{n,\kappa}$ (Significance levels $\xi = 0.1,\ 0.05,\ n = 200$)

ξ	0.1						0.05					
κ	0.1	0.5	1.0	2.5	5.0	10.0	0.1	0.5	1.0	2.5	5.0	10.0
$C(0,1)$	10	10	10	11	10	10	5	5	5	5	5	5
$N(0,1)$	96	*	*	*	*	*	91	*	*	*	*	*
$t(2)$	34	67	82	93	96	98	22	53	73	86	93	95
$t(3)$	58	94	98	*	*	*	42	88	98	*	*	*
$t(4)$	69	99	*	*	*	*	55	96	*	*	*	*
$t(5)$	77	99	*	*	*	*	64	98	*	*	*	*
$t(10)$	54	93	99	*	*	*	40	86	97	*	*	*
$st(0.5,0)$	96	*	*	*	*	*	94	99	*	*	*	*
$st(0.8,0)$	32	54	63	72	78	81	21	41	51	62	69	72
$st(0.9,0)$	14	20	23	29	32	34	7	11	14	20	22	24
$st(1.1,0)$	12	15	16	17	17	16	7	8	9	9	9	8
$st(1.2,0)$	18	32	39	43	47	46	11	21	27	30	34	31
$st(1.5,0)$	57	92	97	99	*	99	85	95	98	98	99	99

Table 12: Approximate values of $1/\lambda_j$ $(j = 1, \ldots, 10)$ for D_κ

κ	j	1	2	3	4	5	6	7	8	9	10
0.5	MLE	0.2835	0.1672	0.1083	0.0798	0.0637	0.0505	0.0396	0.0315	0.0255	0.0210
	EISE	0.2713	0.1589	0.1054	0.0782	0.0627	0.0497	0.0391	0.0311	0.0253	0.0209
1.0	MLE	0.1131	0.0648	0.0366	0.0256	0.0180	0.0138	0.0107	0.00865	0.00706	0.00594
	EISE	0.1130	0.0603	0.0366	0.0249	0.0180	0.0136	0.0107	0.00858	0.00706	0.00591
2.5	MLE	0.0349	0.0171	0.0084	0.0056	0.0037	0.0028	0.0021	0.0017	0.0013	0.0011
	EISE	0.0298	0.0137	0.0076	0.0050	0.0034	0.0026	0.0020	0.0016	0.0013	0.0010
5.0	MLE	0.0165	0.0061	0.0029	0.0017	0.0011	0.0008	0.0006	0.0005	0.0004	0.0003
	EISE	0.0097	0.0040	0.0026	0.0014	0.0009	0.0007	0.0005	0.0004	0.0003	0.0003
10.0	MLE	0.0070	0.0019	0.0009	0.0005	0.0003	0.0002	0.0002	0.0001	9.9E-5	8.1E-5
	EISE	0.0029	0.0011	0.0006	0.0004	0.0002	0.0002	0.0001	0.0001	8.5E-5	7.0E-5

Table 13: MLE: Upper ξ percentage points of D_κ

$\xi \backslash \kappa$	0.5	1.0	2.5	5.0	10.0
0.1	3.153	1.111	0.286	0.114	0.0431
0.05	3.571	1.276	0.336	0.137	0.0527

Table 14: EISE: Upper ξ percentage points of D_κ

$\xi \backslash \kappa$	0.5	1.0	2.5	5.0	10.0
0.1	3.057	1.093	0.248	0.0750	0.0213
0.05	3.458	1.256	0.290	0.0886	0.0254

Table 15: MLE: Upper 10 percentage points of D_κ

$l/m \setminus \kappa$	0.5		1.0		2.5		5.0		10.0	
3/10	3.14298	3.16643	1.10959	1.11398	0.28608	0.28645	0.11443	0.11448	0.04307	0.04307
	0.02344	0.21733	0.00439	0.06614	0.00036	0.01135	4.8E-05	0.00298	7.4E-06	0.00077
25/50	3.15279	3.15303	1.11141	1.11147	0.28622	0.28623	0.11445	0.11445	0.04307	0.04307
	0.00024	0.05641	0.00076	0.01385	5.6E-05	0.0025	7E-06	0.00062	1E-06	0.00015
50/100	3.15286	3.1529	1.11143	1.11144	0.28623	0.28623	0.11445	0.11445	0.04307	0.04307
	3.6E-05	0.03801	5.3E-05	0.00608	4.1E-06	0.00123	4.9E-07	0.00029	7.2E-08	4.9E-05

Table 16: MLE: Upper 5 percentage points of D_κ

$l/m \setminus \kappa$	0.5		1.0		2.5		5.0		10.0	
3/10	3.56374	3.58307	1.27423	1.27802	0.33548	0.33581	0.1374	0.13745	0.05273	0.05273
	0.01933	0.21733	0.00379	0.06614	0.00033	0.01135	4.7E-05	0.00298	7.4E-06	0.00077
25/50	3.57127	3.57147	1.27572	1.27577	0.3356	0.3356	0.13742	0.13742	0.05273	0.05273
	0.0002	0.05641	4.5E-05	0.01385	3.8E-06	0.0025	4.8E-07	0.00062	7.2E-08	0.00015
50/100	3.57132	3.57135	1.27574	1.27575	0.3356	0.3356	0.13742	0.13742	0.05273	0.05273
	3E-05	0.03801	5.1E-06	0.00608	5E-07	0.00123	6.2E-08	0.00029	7.9E-09	4.9E-05

Table 17: EISE: Upper 10 percentage points of D_κ

$l/m \setminus \kappa$	0.5		1.0		2.5		5.0		10.0	
3/10	3.04655	3.07109	1.0915	1.09561	0.2478	0.24819	0.07497	0.07505	0.02133	0.02135
	0.02454	0.21675	0.00411	0.06607	0.00039	0.01113	7.4E-05	0.00283	1.6E-05	0.00072
25/50	3.05684	3.0571	1.09317	1.09322	0.24795	0.24795	0.075	0.075	0.02134	0.02134
	0.00026	0.0564	4.9E-05	0.01385	4.6E-06	0.00249	8.4E-07	0.00062	1.7E-07	0.00014
50/100	3.05692	3.05695	1.09319	1.0932	0.24795	0.24795	0.075	0.075	0.02134	0.02134
	3.8E-05	0.03801	5.5E-06	0.00608	6.1E-07	0.00123	1.1E-07	0.00029	1.9E-08	4.9E-05

Table 18: EISE: Upper 5 percentage points of D_κ

$l/m \setminus \kappa$	0.5		1.0		2.5		5.0		10.0	
3/10	3.44976	3.46987	1.25429	1.25788	0.28975	0.29011	0.08859	0.08866	0.02538	0.0254
	0.0201	0.21675	0.0036	0.06607	0.00036	0.01113	7E-05	0.00283	1.5E-05	0.00072
25/50	3.4576	3.45781	1.25568	1.25572	0.28988	0.28989	0.08861	0.08861	0.02539	0.02539
	0.00021	0.0564	4.3E-05	0.01385	4.3E-06	0.00249	8E-07	0.00062	1.7E-07	0.00014
50/100	3.45766	3.45769	1.25569	1.2557	0.28988	0.28988	0.08861	0.08861	0.02539	0.02539
	3.1E-05	0.03801	3E-06	0.00608	5.7E-07	0.00123	1.1E-07	0.00029	1.9E-08	4.9E-05