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LIQUIDITY MOTIVES OF HOLDING MONEY UNDER INVESTMENT RISK: A DYNAMIC ANALYSIS*

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Abstract

Jones and Ostroy (1984) argue that money, as an asset of the least transaction cost, offers flexibility to its holder, which other assets cannot provide. We extend the idea of Jones and Ostroy into a truely dynamic framework of infinite horizon with a risk-neutral decision-maker. We then investigate the effect of an increase in investment risk on the demand for liquidity à la Jones and Ostroy. In particular, we prove that the opitmal strategy exists, that it has a reservation property, and that the reservation value increases when investment risk increases in the sense of a mean-preserving spread. While the effect of a mean-preserving spread on the reservation value is unambiguous, its effect on money demand is ambiguous. We then provide conditions on increasing investment risk under which money demand unambiguously increases.

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1. Introduction

Without doubt, money is the most liquid asset. To convert money to other assets is immediate and costless, whereas to convert non-money assets to other assets including money involves time and substantial transaction costs. Thus, money enables prompt moves among various investment, both financial and real. In a sense, money offers liquidity services. It is natural to assume that these liquidity services are one of the most important determinants of money demand. In fact, this is the heart of the speculative demand for money, as opposed to transaction and precautionary demand for money. Unfortunately, however, there are relatively few examinations of the liquidity or speculative motive of holding money, as compared with the transaction and precautionary motives.

Among existing literature of the liquidity motive of holding money, Jones and Ostroy (1984)'s formulation has attracted much attention. They argue that money, as an asset of the least transaction cost, offers flexibility to its holder, which other assets cannot provide. Under the presence of liquidation (transaction) costs on other assets, money is held to enable the option of waiting for tomorrow to resolve uncertainty rather than investing today under uncertainty. Thus, their formulation of liquidity services of money can be considered as an enabler of options.

Their argument suggests that if the degree of uncertainty about the future increases and the resolution of uncertainty is still gradual, the value of waiting increases and hence the demand for liquidity also increases in the spirit of Jones and Ostroy's "liquidity as flexibility," or money as an enabler of options. In the current paper, we extend the idea of Jones and Ostroy into a truly dynamic framework of infinite horizon with a risk-neutral decision-maker and we then investigate the effect of an increase in investment risk on the demand for money.

To achieve this objective, we start from Taub's (1988) model of pure-currency economy where money is required to purchase commodities (which itself specifies Lucas' (1980) model by assuming that a decision-maker is risk-neutral) and extend it into Markovian economic environment. Instead of assuming that money is required to buy commodities, we assume that money is required to make investment. This assumption of a liquidity constraint in investment, which might be called a cash-in-advance constraint in investment, is rationalized when there are sizable transaction costs and transaction delays in trading with other assets than money, and it substantiates Jones and Ostroy's idea of "liquidity as flexibility" in our framework.

Specifically, we consider a behavior of a risk-neutral fund-manager who contemplates in each period whether or not to make an irreversible investment which, if made, generates a stochastic future return. We impose the liquidity constraint by assuming that the amount of investment at one time is constrained by the fund-manager's cash-holdings at that time. In fact, this framework of dynamic setting can be reinterpreted as a variant of a real-option model, with one additional assumption that money is required to make investment, either financial or real.

We formally prove that the optimal strategy for the fund-manager exists and that it takes the form of a "trigger" strategy. It is called a trigger strategy because there exists a function of the return realized in the current period and the fund-manager puts all his available money into investment if and only if the return exceeds the value of this function. In general, the trigger is a function of the return. We show that when the stochastic kernel which describes the Markov process satisfies some regularity conditions, the trigger is constant. In such a case, we say that the optimal strategy has a reservation property and the fund-manager makes an investment if and only if the return exceeds the constant reservation value.

We then investigate effects of an increase in investment risk on the optimal strategy and the resulting demand for money. Suppose that the stochastic kernel satisfies the conditions which induce the reservation property of the optimal strategy and that the degree of uncertainty increases in the sense of a mean-preserving spread. Then, the results of this paper show that a new trigger level is always above the initial reservation value. Therefore, if making investment is not an optimal strategy before the risk increases, it cannot be so after the risk has increased. This result is totally consistent with a fact which is well-known in the theory of options: An increase in risk or volatility increases the value of a waiting option. Since money as liquidity is an enabler of this option, the value of money as liquidity increases when the risk increases.

Despite this fact, the demand for liquidity, which we define as a long-run time-average cash-holdings of the fund-manager, may or may not increase as a result of a mean-preserving spread. This ambiguity follows because while the trigger level increases by a mean-preserving spread, the probability that the return exceeds the trigger level might also increase at the same time. If the latter increase dominates the former one, the fund-manager becomes more likely to invest, and as a result, the demand for liquidity defined as an average cash-holdings *de*creases. This result suggests that the mean-preserving spread as a concept of an increase in risk is not strong enough for us to derive the unambiguous effect on money demand of an increase in investment risk. We then present conditions under which an increase in investment unambiguously increases the demand for money: When the return is distributed independently and identically, an increase in investment risk characterized by a single-crossing property as well as one additional regularity condition implies that the money demand increases unambiguously.

The organization of this paper is as follows. Section 2 gives a brief overview of the model and then illustrates some of the results of this paper. In particular, we specify the distribution of the state variable as the uniform distribution to construct an example in which both the reservation value and the average cash-holdings increase by a mean-preserving spread in risk. In Section 3, we formally develop a model of an investment-fund manager with uncertain investment opportunities in the future. Money, or cash in our formulation, is assumed to be needed to make investment in a form of liquidity constraint, which captures flexibility that liquidity provides. We show that the optimal strategy for the fund-manager exists, that it can be characterized by a trigger strategy and that it has a reservation property when the Markov process satisfies some additional assumptions. We also define there the demand for liquidity as a long-run time-average cash-holdings of the fund-manager. Section 4 conducts a sensitivity analysis in which an increase in risk is examined in the framework described in Section 3. In particular, we show that an increase in risk in the sense of a mean-preserving spread increases the trigger level. In the same time, it turns out that the mean-preserving spread does not always imply that the money demand also increases. We provide the conditions under which an increase in money demand unambiguously follows. All proofs as well as some lemmas are given in Section 5.

We heavily draw on techniques of dynamic programming in order to prove the claims made in the current paper. For such techniques, an approach based upon the contraction-mapping theorem is common in the literature. However, we need to allow the possibility that the money is accumulated without bound, and hence, a simple adaptation of the contraction-mapping approach does not work here because it requires that the value function should be bounded, which is not the case in our model. We, therefore, invoke the dynamic programming technique developed by Ozaki and Streufert (1996) for a wide class of objective functions which includes unbounded ones. We adapt it for the framework of this paper and provide a series of results along the line developed there. We do this in the Appendix. The proofs given in Section 5 rely on these results.

2. An Illustrative Example

This section gives a brief overview of the model and then illustrates some of the results of this paper by specifying the distribution which governs state-variable's evolution. The formal analysis of the model starts in the next section.

2.1. Optimal Investment and Average Cash-holdings

Consider a manager of an investment fund, such as venture capital invested in venture businesses, specialized in investing projects involving substantial fixed costs that are sunk after investment. This fund has a constant (positive) cash inflow of y in each period. Competition is intense to find good investment opportunities among such funds and the manager must be agile in capturing these opportunities. This means that, at the time the manager has a promising opportunity, he must have sufficient liquid assets ready to invest, rather than illiquid but higher-return ones. In other word, the amount of investment at one time in those opportunities is constrained by the fund's holdings of liquid assets at that time. Since cash is the most liquid asset, we hereafter consider cash as liquid assets, and to make analysis simple, we abstract away from other less liquid financial assets. We assume for simplicity that the general price level is constant over periods, so that there is neither inflation nor deflation.

Under these assumptions, the fund-manager has a choice between investing in particular opportunities in this period or wait until the next period setting cash inflow aside. Let m_t be the (non-negative) amount of cash that the fund-manager has in the beginning of period t and let x_t be the (non-negative) amount of investment which is made in period t and sunk afterward. Then, the fund-manager's budget constraint is given by

$$(\forall t \geq 0) \quad x_t + m_{t+1} \leq y + m_t$$

and his liquidity constraint in investment is given by

$$(\forall t \geq 0) \quad x_t \leq m_t\,. \tag{1}$$

The latter condition may also be considered as a cash-in-advance constraint in investment.

Let r be a net rate of interest and suppose that a unit investment opportunity in period t yields the same net return $\hat{z}_t$ in each period in indefinite future starting at period t. Then, the current value of all the future net returns on a unit investment made in period t is given by $z_t \equiv (1+r)\hat{z}_t/r$. We assume that investment is uncertain, in the sense that z_t is stochastic. For simplicity, we assume that z_t is distributed according to a distribution function F_0 independently and identically in each period. (We relax this assumption later.) The fund-manager maximizes the expected present value of the future net returns on whole investment,

$$E\left[\sum_{t=0}^{\infty} \beta^t x_t z_t\right], \tag{2}$$

evaluated with a constant discount factor $\beta \in (0,1)$, by appropriately choosing an investment strategy x_t, taking account of both the budget constraint and the liquidity constraint. In the above formula, $E[\cdot]$ denotes the expectation with respect to the infinite-dimensional product probability measure constructed from F_0. (The exact form of the objective function will be presented below by (16)).[1]

We will show in Section 3 that the optimal investment x_t^* has a reservation property and can be characterized by

$$(\forall t \geq 0) \quad x_t^* = \begin{cases} 0 & \text{if} \quad z_t \leq R \\ m_t & \text{if} \quad z_t > R\,, \end{cases} \tag{3}$$

where R is a unique solution to the following equation:

$$R = \frac{\beta}{1-\beta}\int_R^{+\infty} (1 - F_0(z'))\, dz' \tag{4}$$

(see Theorem 3 and the equation (25) below). The optimal investment strategy, (3), dictates the investment of all available cash when the current shock (return) is greater than the reservation

[1] This model is also applicable for a financial fund with cash (perfectly liquid assets) and consols (less liquid assets), so long as transaction of the less illiquid assets involves substantial fixed costs. There is no transaction cost converting cash into the illiquid assets, while conversion of these assets into cash involves considerable transaction costs. So long as these transaction costs are large, a portfolio manager does not sell the illiquid assets, and the model is essentially the same as that in this section. See, for example, Leland (1999) for a recent treatment of transaction costs in financial markets.

level, R, and no investment at all if otherwise. Furthermore, the money demand, m_0^*, defined as the average cash-holdings turns out to be given by

$$m_0^* = \frac{y}{1 - F_0(R)} \tag{5}$$

(see (27) below).

This result has a natural interpretation. Consider one unit of cash in the fund-manager's hand at period t. Recall that if he invests this cash in the current investment opportunity, this decision yields $\hat{z}_t$ from this period onward. If he postpones his decision, he foregoes $\hat{z}_t$ this period, but he may get better opportunity yielding net return $z \geq \hat{z}_t$ in the future. Suppose that the fund-manager has only *one* unit of cash to invest and he has to determine the optimal timing of investment. This is a special case of classical optimal stopping problems about when to exercise an option (to invest a unit of cash, in this case). The optimal strategy for this problem is well-known to be (under some assumptions) a reservation strategy such that: "stop and invest" if $\hat{z}_t > \hat{R}$ and "wait and continue to the next period" if otherwise, where $\hat{R}$ is such that

$$\hat{R} = \frac{\beta}{1 - \beta} \int_{\hat{R}}^{+\infty} (1 - \hat{F}(z'))\, dz' \tag{6}$$

and $\hat{z}_t$ is assumed to be independently and identically distributed according to the distribution function, $\hat{F}$ (see, for example, Lippman and McCall, 1976; Sargent, 1987). Notice (6)'s resemblance to (4).[2]

In the optimal stopping problem of the previous paragraph, the fund-manager has an option to invest only one unit of cash (that is, he has only two alternatives: to invest or not) and he makes a once-and-for-all decision about when to exercise this option. In the model of this paper, the fund-manager is allowed to invest any amount of money as long as the liquidity constraint is met and he decides whether or not to invest not only once but also in every period. Despite these differences, the similarity between (4) and (6) indicates that the two models have

[2] In fact, if we define R by $(1 + r)\hat{R}/r$ and F_0 by $(\forall x)\ F_0(x) = \hat{F}(rx/1 + r)$, then (6) is turned into

$$R = \frac{\beta}{1 - \beta} \int_{R}^{+\infty} (1 - F_0(z'))\, dz' \,,$$

which is identical to (4).

basically the very similar structure. This observation supports our view that cash is endowed with a function of an "enabler" of the call option by the liquidity services it offers.[3]

We now turn back to the equation (4). To make our analysis as simple as possible, we further specify F_0 to be a uniform distribution over $[a, b]$, where $0 < a < b$. The uniform-distribution assumption greatly reduces complexity and allows us to obtain an explicit solution R to (4). We assume that the parameters of the model satisfy the following condition:

$$\beta > \frac{2a}{a+b} \tag{7}$$

in order to assure that $R \in (a, b)$ holds. If $R \in (a, b)$ holds, then by (4), we have

$$R = \frac{\beta}{1-\beta} \int_R^b \frac{b - z'}{b-a}\, dz' = \frac{\beta (b-R)^2}{2(1-\beta)(b-a)}\,. \tag{8}$$

By solving this quadratic equation, we get

$$\begin{aligned} R \ &= \ \frac{1}{\beta}\left(b - (1-\beta)a - D^{1/2}\right) \\ \text{where} \ \ D &\equiv (1-\beta)(b-a)\left[b - a + \beta(a+b)\right]\,. \end{aligned} \tag{9}$$

It can be verified that $R \in (a, b)$ certainly holds under the assumption (7), which justifies (8).

2.2. An Increase in Risk

We now consider effects on the reservation level and average cash-holdings caused by an increase in risk. To be more specific, suppose that the uniform distribution F_0 is slightly more dispersed by γ, over $[a - \gamma, b + \gamma]$. This is a mean-preserving spread, a way of characterizing *increased risk* (see Rothschild and Stiglitz, 1970).

If this mean-preserving spread takes place, (8) is modified to

$$R = \frac{\beta (b + \gamma - R)^2}{2(1-\beta)(b - a + 2\gamma)}\,. \tag{10}$$

[3] The liquidity constraint in investment, (1), certainly makes it easier for us to interpret money as a provider of liquidity services. It is *not*, however, indispensable in order to derive the reservation property of the optimal investment strategy (see Subsection 3.2.3). It is the assumption that investment is irreversible that is essential to derive it. The existence of money makes it possible to postpone such irreversible investment and this *is* the liquidity service money provides.

Denote the solution R to this equation by $R(\gamma)$ as a function of γ. Then, the implicit function theorem shows that

$$\left.\frac{dR(\gamma)}{d\gamma}\right|_{\gamma=0} = \frac{\beta(b-R)(R-a)}{(b-a)[(1-\beta)(b-a)+\beta(b-R)]}, \tag{11}$$

where R $(= R(0))$ in the right-hand side is given by (9). Since $a < R < b$ by (7), we conclude that

$$\left.\frac{dR(\gamma)}{d\gamma}\right|_{\gamma=0} > 0\,.$$

This result shows that increased risk in the form of a mean-preserving spread increases the reservation level.

In Section 4, we formally show that basically the same result holds not only locally (that is, at $\gamma = 0$) but also globally (that is, $R(\gamma) > R(0)$ for any positive γ), and not only for the uniform distribution but also for general distributions. Furthermore, it is shown that the result can be extended to a Markovian setting under appropriate assumptions.

We now turn to the effect of a mean-preserving spread on the average cash-holdings. Some algebra shows that

$$\begin{aligned}\left.\frac{dF_0(R(\gamma))}{d\gamma}\right|_{\gamma=0} &= \left.\frac{d}{d\gamma}\left(\frac{R(\gamma)-a+\gamma}{b-a+2\gamma}\right)\right|_{\gamma=0} \\ &= \frac{R'(0)(b-a)+a+b-2R}{(b-a)^2} \\ &= \frac{(1-\beta)(a+b)}{(1-\beta)(b-a)^2+\beta(b-a)(b-R)}\end{aligned}$$

where $R(\gamma)$ is the solution to (10), $R'(0)$ is given by (11) and R is given by (9). Since $R < b$ by (7), we conclude that

$$\left.\frac{dF_0(R(\gamma))}{d\gamma}\right|_{\gamma=0} > 0\,.$$

This and (5) together show that increased risk in the form of a mean-preserving spread increases the money demand defined as the average cash-holdings.

It might seem that this result holds regardless of a specification of the distribution function in the light of the fact that a mean-preserving spread always increases the reservation level. However, we provide an example (Example 1) in Section 4, which exhibits a *de*crease in money demand when a mean-preserving spread takes place. Thus, the effect of a mean-preserving spread on money demand is ambiguous in general. In the same section, we consider another concept of being riskier which is stronger than the mean-preserving spread and show that the money demand increases whenever a risk increases according to this concept under an appropriate condition (see Proposition 2 below).

3. The Formal Model

3.1. Stochastic Kernel

The stochastic environment of the model is governed by a stochastic kernel, which we discuss in this subsection. Let Z be a subset of $\mathbb{R}_+$ and let $\mathcal{B}_Z$ be the Borel σ-algebra on Z. We assume that Z is compact and convex, and hence, we may write as $Z = [\underline{z}, \bar{z}]$ where $\underline{z}$ and $\bar{z}$ are defined by $\underline{z} = \min Z$ and $\bar{z} = \max Z$, respectively. We assume that $\underline{z} < \bar{z}$. The net rate of returns on investment made in period t, z_t, is a random variable on $(Z, \mathcal{B}_Z)$. In our model, z_t also serves as a state variable. We assume that z_t is distributed according to a Markov process and we let $P_0(\cdot|\cdot) : \mathcal{B}_Z \times Z \to [0,1]$ be a stochastic kernel which governs the transition of z_t. That is, P_0 is such that $(\forall z)$ $P_0(\cdot|z)$ is a probability measure on $\mathcal{B}_Z$ and $(\forall E)$ $P_0(E|\cdot)$ is a $\mathcal{B}_Z$-measurable function. We denote by F_0 the (cumulative) distribution function derived from P_0, that is, $(\forall z', z)$ $F_0(z'|z) = P_0(\,[\underline{z}, z']\,|\,z\,)$.

Throughout the paper, we assume that P_0 is *weakly continuous* in the sense that for any sequence $\langle z_n \rangle_{n=1}^{\infty}$ in Z which converges to z_0, $P_0(\cdot|z_n)$ converges to $P_0(\cdot|z_0)$ in the weak topology.[4]

Subsection 3.2.1 and Section 4 assume that P_0 is stochastically increasing and stochastically convex (Topkis, 1998). A stochastic kernel P_0 is *stochastically increasing* if for each

[4]The convergence in the weak topology requires by definition that for any bounded continuous function $h : Z \to \mathbb{R}$,

$$\int_Z h(z')\, P_0(dz'|z_n) \to \int_Z h(z')\, P_0(dz'|z_0)\,.$$

The weak continuity of P_0 is sometimes referred to as the *Feller property* (see, for example, Stokey and Lucas, 1989).

nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined on Z by

$$z \mapsto \int_Z h(z')\, P_0(dz'|z) \tag{12}$$

is nondecreasing. A stochastic kernel P_0 is *stochastically convex* if for each nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined on Z by (12) is convex.

Section 3.3 assumes that P_0 has a limiting probability measure. A stochastic kernel P_0 *has a limiting probability measure* if there exists a probability measure π_0 on $(Z, \mathcal{B}_Z)$ such that for any probability measure μ on $(Z, \mathcal{B}_Z)$, the sequence of probability measures defined by

$$\left\langle \int_Z \int_Z \cdots \int_Z P_0(\cdot|z_t) P_0(dz_t|z_{t-1}) \cdots P_0(dz_1|z_0)\mu(dz_0) \right\rangle_{t=1}^{\infty}$$

converges to π_0 in the weak topology as t goes to $+\infty$. Such a π_0 is called a *limiting probability measure* of P_0 and is unique when it exists. Furthermore, under the maintained assumption that P_0 is weakly continuous, π_0 satisfies[5]

$$(\forall E \in \mathcal{B}_Z) \;\; \pi_0(E) = \int_Z P_0(E|z)\, d\pi_0(z)\,. \tag{13}$$

We may think of π_0, when it exists, as describing the long-run behavior of P_0.

We say that z_t is *independently and identically distributed* (*i.i.d.*) if $P_0(\cdot|z)$ is independent of z. In the *i.i.d.* case, it is clear that the properties of P_0 introduced so far are all satisfied. The next proposition provides another example of a stochastic kernel which satisfies all of them.

Proposition 1. *Let $Z = [0,1]$ and let P_0 be a stochastic kernel defined by*

$$(\forall z, z' \in Z) \quad F_0(z'|z) = P_0(\,[0, z']\,|\,z\,) = \int_0^{z'} (2 - z)\, dz\,.$$

That is, $P_0(\cdot|z)$ is the uniform distribution on $[0, 1/(2-z)]$. Then, P_0 is weakly continuous, stochastically increasing, stochastically convex and has a limiting probability measure.

3.2. Dynamic Programming Problem and Optimal Investment Strategy

Construct the t-fold self-product measurable space from $(Z, \mathcal{B}_Z)$ and denote it by $(Z^t, \mathcal{B}_{Z^t})$, that is, $(Z^t, \mathcal{B}_{Z^t}) = (Z \times \cdots \times Z, \mathcal{B}_Z \otimes \cdots \otimes \mathcal{B}_Z)$, where the products are t-fold. A generic element

[5] An argument similar to that of Stokey and Lucas (1989, p.376, Theorem 12.10) may be applied to prove this.

of $(Z^t, \mathcal{B}_{Z^t})$, which is denoted by $(z_1, \ldots, z_t)$ or ${}_1\boldsymbol{z}_t$, is a history of states' realized up to period t. An *investment strategy* is any $\mathbb{R}_+$-valued, $\langle \mathcal{B}_{Z^t} \rangle$-adapted stochastic process and denoted by ${}_0\boldsymbol{x}$ or $\langle x_t \rangle_{t=0}^{\infty}$. Here, the $\langle \mathcal{B}_{Z^t} \rangle$-adaptedness requires that $x_0 \in \mathbb{R}_+$ and $(\forall t \geq 1)$ $x_t : Z^t \to \mathbb{R}_+$ should be $\mathcal{B}_{Z^t}$-measurable. Similarly, a *money-holding strategy*, denoted ${}_1\boldsymbol{m}$ or $\langle m_t \rangle_{t=1}^{\infty}$, is any $\mathbb{R}_+$-valued, $\langle \mathcal{B}_{Z^{t-1}} \rangle$-adapted stochastic process. That is, $m_1 \in \mathbb{R}_+$ and $(\forall t \geq 2)$ m_t is $\mathcal{B}_{Z^{t-1}}$-measurable. Given $m_0 \geq 0$, an investment strategy ${}_0\boldsymbol{x}$ is *feasible from* m_0 if there exists a money-holding strategy ${}_1\boldsymbol{m}$ such that the *budget constraint*:

$$(\forall t \geq 0) \quad x_t + m_{t+1} \leq y + m_t \,, \tag{14}$$

and the *liquidity constraint in investment* (or, the *cash-in-advance constraint in investment*):

$$(\forall t \geq 0) \quad x_t \leq m_t \tag{15}$$

are both met.

Let $\beta = 1/(1+r)$, where r is the net rate of interest. The fund-manager maximizes the expected present value of all the future net returns on investment, which is given by

$$\begin{aligned} I_{z_0}({}_0\boldsymbol{x}) \equiv \lim_{T \to +\infty} x_0 z_0 + \beta \int_Z \cdots \\ \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z x_T z_T \, P_0(dz_T | z_{T-1}) \right) P_0(dz_{T-1} | z_{T-2}) \cdots P_0(dz_1 | z_0) \,, \end{aligned} \tag{16}$$

when the initial state is z_0 and the investment strategy ${}_0\boldsymbol{x}$ is chosen.[6] Since each component of the sequence is well-defined and the sequence is non-decreasing, the limit exists (allowing $+\infty$). Note that the monotone convergence theorem shows that this objective function satisfies *Koopmans' equation*:

$$(\forall z_0)(\forall_0 \boldsymbol{x}) \quad I_{z_0}({}_0\boldsymbol{x}) = x_0 z_0 + \beta \int_Z I_{z_1}({}_1\boldsymbol{x}) \, P_0(dz_1 | z_0) \,,$$

[6]Let $(Z^\infty, \mathcal{B}_{Z^\infty})$ be the infinite-dimensional self-product measurable space constructed from $(Z, \mathcal{B}_Z)$ and let its generic element be denoted by ${}_1\boldsymbol{z} = (z_1, z_2, \ldots)$. If we construct the probability measure $P_0^\infty(\cdot | z_0)$ on $(Z^\infty, \mathcal{B}_{Z^\infty})$ from P_0 and $z_0 \in Z$ (for such a construction, see Stokey and Lucas, 1989), the objective (16) turns out to be equal to

$$\int_{Z^\infty} \sum_{t=0}^{\infty} \beta^t x_t z_t \, P_0^\infty(d\,{}_1\boldsymbol{z} | z_0) \,,$$

which may be simply denoted by $E\left[\sum_{t=0}^{\infty} \beta^t x_t z_t \middle| \, z_0 \right]$ (see (2)).

where ${}_1\boldsymbol{x}$ is a continuation of ${}_0\boldsymbol{x}$ after the realization of z_1.

In order to describe the optimal investment strategy, we need some definitions. A function $v^* : \mathbb{R}_+ \times Z \to \mathbb{R}$ is the *value function* for the fund-manager's problem if it satisfies

$$(\forall m, z) \quad v^*(m, z) = \max \{ \, I_z({}_0\boldsymbol{x}) \mid {}_0\boldsymbol{x} \text{ is feasible from } m \, \} \, .$$

An investment strategy ${}_0\boldsymbol{x}$ is *optimal from* $(m, z) \in \mathbb{R}_+ \times Z$ if it is feasible from m and satisfies $I_z({}_0\boldsymbol{x}) = v^*(m, z)$. Define the *feasibility correspondence* $\Gamma : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}^2_+$ by

$$(\forall m) \quad \Gamma(m) = \left\{ \, (x, m') \in \mathbb{R}^2_+ \, \middle| \, x + m' \leq y + m \text{ and } x \leq m \, \right\} \, .$$

When v^* exists, we define the *policy correspondence* $g : \mathbb{R}_+ \times Z \to \mathbb{R}^2_+$ by

$$(\forall m, z) \quad g(m, z) = \arg\max \left\{ \, xz + \beta \int_Z v^*(m', z') \, P_0(dz'|z) \, \middle| \, (x, m') \in \Gamma(m) \, \right\} \, . \tag{17}$$

For the policy correspondence to be well-defined, v^* must be such that $(\forall m')$ $v^*(m', \cdot)$ is $\mathcal{B}_Z$-measurable and the right-hand side of (17) is nonempty, which turns out to be the case for the current model (see Theorem 1). An investment strategy ${}_0\boldsymbol{x}$ is *recursively optimal* from $(m, z) \in \mathbb{R}_+ \times Z$ if there exists a money-holding strategy ${}_1\boldsymbol{m}$ such that[7]

$$(x_0, m_1) \in g(m, z) \quad \text{and} \quad (\forall t \geq 1) \;\; (x_t, m_{t+1}) \in g(m_t, z_t) \, .$$

Now, let us define the "trigger" level of z. (The reason this is called a trigger shall be apparent later in Theorem 1.) Suppose that $R : Z \to \mathbb{R}$ is a $\mathcal{B}_Z$-measurable function and define the operator T which maps such a function R to another $\mathcal{B}_Z$-measurable function TR by

$$(\forall R)(\forall z) \quad TR(z) = \beta \int_Z \max \left\{ z', R(z') \right\} P_0(dz'|z) \, . \tag{18}$$

Lemma 1 (Section 5) proves that there exists a solution R to the functional equation defined by $R = TR$ (that is, a fixed point of T) which satisfies $(\forall z)$ $0 \leq R(z) \leq \bar{z}$. Lemma 1 also shows that such a function R is unique and we denote it by R^*.

Given $t \geq 1$, $z \in Z$ and a $\mathcal{B}_Z$-measurable function $h : Z \to \mathbb{R}$, we denote by $E^t_0[h|z]$ the t-fold iterated expectation of h with respect to P_0:

$$E^t_0[h|z] = \int_Z \cdots \int_Z \int_Z h(z_t) \, P_0(dz_t|z_{t-1}) \, P_0(dz_{t-1}|z_{t-2}) \cdots P_0(dz_1|z) \, .$$

[7] Note that among the requirements of recursive optimality is the existence of a measurable selection of g.

Also, we define E_0^0 by $(\forall h, z)$ $E_0^0[h|z] = h(z)$ and we often write E_0^1 as E_0. We then define a function $A : Z \to \mathbb{R}_+$ by

$$(\forall z) \quad A(z) = y \sum_{s=0}^{+\infty} \beta^s E_0^s[R^*|z]\,. \tag{19}$$

Note that A is well-defined and finite-valued since R^* is a $\mathcal{B}_Z$-measurable bounded function and $\beta \in (0, 1)$. Then, the following theorem characterizes the value function and the policy correspondence for the fund-manager's problem.

Theorem 1. *The value function v^* exists and is given by*

$$(\forall m, z) \quad v^*(m, z) = \begin{cases} R^*(z)m + A(z) & \textit{if} \quad z \le R^*(z) \\ zm + A(z) & \textit{if} \quad z > R^*(z) \end{cases} \tag{20}$$

and the policy correspondence g exists and is given by

$$(\forall m, z) \quad g(m, z) = \begin{cases} \{\,(0, m + y)\,\} & \textit{if} \quad z < R^*(z) \\ \{\,(x, m') \in \Gamma(m) \,|\, x + m' = y + m\,\} & \textit{if} \quad z = R^*(z) \\ \{\,(m, y)\,\} & \textit{if} \quad z > R^*(z)\,. \end{cases} \tag{21}$$

Furthermore, recursive optimality implies optimality.

We now construct an investment strategy ${}_0\boldsymbol{x}^*$ (and its associated money-holding strategy ${}_1\boldsymbol{m}$) which is recursively optimal from (m, z) as follows:

$$(\forall t \ge 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \text{if} \quad z_t \le R^*(z_t) \\ (m_t, y) & \text{if} \quad z_t > R^*(z_t) \end{cases} \tag{22}$$

where $m_0 \equiv m$ and $z_0 \equiv z$. The stochastic process ${}_0\boldsymbol{x}^*$ thus defined is $\langle \mathcal{B}_{Z^t} \rangle$-adapted since R^* is $\mathcal{B}_Z$-measurable. Therefore, it is certainly an investment strategy and recursively optimal from (m, z) by (21) and the definition of recursive optimality. By the last statement of Theorem 1, we know that ${}_0\boldsymbol{x}^*$ is an optimal investment strategy from (m, z). We define a function $g^* : \mathbb{R}_+ \times Z \to \mathbb{R}_+$ by $(\forall m_t, z_t)$ $x_t^* = g^*(m_t, z_t)$, where ${}_0\boldsymbol{x}^* = \langle x_t^* \rangle_{t=0}^{\infty}$ is constructed by (22).

The optimal investment strategy has a simple form described by (22). It is now clear why R^* is called a trigger. When the realized value of z is greater than $R^*(z)$, the fund-manager invests all cash available into current investment. On the contrary, if the value of z is no greater than $R^*(z)$, he does not invest and carries over the whole cash to the next period.

The existence of cash reserve provides the fund-manager with an option not to invest in the current period but to wait until next period. This shows that the fund-manager has a call option when he has cash in hand. Here, cash is an "enabler" of this call option, or flexibility in terms of Jones and Ostroy (1984). Cash is endowed with this function by the liquidity services it provides, and ultimately by transaction costs implicit in the irreversibility of investment. This observation will be further illustrated in the two special cases and in the case where the liquidity constraint is absent.

3.2.1. Stochastically Increasing and Stochastically Convex Kernels

In general, the trigger level R^* is a function, and hence, the shape of the "continuation region," $\{ z \in Z \,|\, z \leq R^*(z) \}$, depends on the shape of R^*. Depending on R^*, the shape of the continuation region may be complicated. It may not be even a connected set. However, when the Markov process under consideration meets some requirements, the continuation region is largely simplified.

Specifically, assume that the kernel P_0 is stochastically increasing and stochastically convex (Section 3.1). In such a case, the optimal investment strategy, as well as the continuation region, can be characterized in a simple manner. The optimal investment strategy ${}_0\boldsymbol{x}^*$ *has a reservation property* if there exists a *constant* $z^* \geq 0$ such that

$$(\forall t \geq 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \text{if} \quad z_t \leq z^* \\ (m_t, y) & \text{if} \quad z_t > z^* \,. \end{cases}$$

Here, the constant z^*, which we call a *reservation value*, serves as a trigger for the investment: if the value of z_t is greater than z^*, the whole available cash is put into the current investment.

Theorem 2. *Suppose that the kernel P_0 is stochastically increasing and stochastically convex. Then, the optimal investment strategy has a reservation property. Furthermore, if $\beta E_0[z'|\underline{z}] \geq \underline{z}$, then there exists a unique $z^* \in Z$ such that $z^* = R^*(z^*)$ and the reservation level equals z^*.*

The theorem shows that when the stochastic kernel satisfies the given conditions, the reservation value z^* exists in $[0, \bar{z}]$. If $z^* < \underline{z}$, then the fund-manager always invests and the continuation region is given by the empty set. If an additional assumption of $\beta E_0[z'|\underline{z}] \geq \underline{z}$ is satisfied, such a case is ruled out and it holds that $z^* \geq \underline{z}$, and hence, the continuation region is given by $[\underline{z}, z^*]$.

3.2.2. I.i.d. Kernels

If the stochastic kernel is *i.i.d.*, it is clearly stochastically increasing and stochastically convex. Therefore, Theorem 2 shows that the trigger level R^* is constant and that it equals the reservation value z^*. In such an *i.i.d.* case, we can further characterize the constant trigger level R^* (or equivalently, the reservation value z^*) as a unique solution to some simple equation.

To be more precise, suppose that z_t is *i.i.d.* according to P_0 and that F_0 is its associated distribution function. The expectation of z_t with respect to P_0 (or F_0) is denoted by $E_0[z]$. In this *i.i.d.* case, it turns out from (18) that the trigger level R^* is a constant which satisfies both $0 \leq R \leq \bar{z}$ and

$$R = \beta \int_Z \max\left\{z', R\right\} \ P_0(dz') \, . \tag{23}$$

By Lemma 1 (Section 5), such a constant exists and is unique.

There are two possible cases. First, assume that $\beta E_0[z] < \underline{z}$. Then, it is easy to see that $R = \beta E_0[z]$ solves (23). Therefore, by Lemma 1 (Section 5) and the fact that $0 \leq \beta E_0[z] \leq \bar{z}$, $R^* = \beta E_0[z]$. In this case, it always holds that $(\forall t)$ $z_t > R^*$, and hence, (22) implies that the fund-manager always invests all the money available to him regardless of the realization of z_t. Note that this case happens when the fund-manager discounts the future a lot (that is, when the interest rate is quite high) and/or when the distribution of z_t is largely skewed toward its lower tail. Clearly, waiting is not a good strategy in such cases.

Second, assume that $\beta E_0[z] \geq \underline{z}$. Then, any solution R to (23) satisfies that $\underline{z} \leq R < \bar{z}$.[8] Therefore, (23) is further simplified to

$$R \;\; = \;\; \beta \int_Z \max\left\{z', R\right\} \ P_0(dz')$$

[8]This is shown as follows. First, suppose that $R < \underline{z}$. Then, (23) and the assumption that $\beta E_0[z] \geq \underline{z}$ imply that $R = \beta E_0[z] \geq \underline{z}$, which is a contradiction. Therefore, it holds that $\underline{z} \leq R$. Second, suppose that $R \geq \bar{z}$. Then, (23) implies that $R = \beta R$, which in turn implies that $R = 0$ since $\beta < 1$. This contradicts that $R \geq \bar{z} > \underline{z} \geq 0$, where the strict inequality is assumed throughout the paper. Therefore, it holds that $R < \bar{z}$.

$$
\begin{aligned}
&= \beta \int_{[\underline{z},R]} \max\{z', R\}\ P_0(dz') + \beta \int_{(R,\bar{z}]} \max\{z', R\}\ P_0(dz') \\
&= \beta \int_{[\underline{z},R]} R\, P_0(dz') + \beta \int_{(R,\bar{z}]} z'\, P_0(dz') \\
&= \beta R F_0(R) + \beta \int_{(R,\bar{z}]} z'\, dF_0(z') \\
&= \beta R F_0(R) + \beta \left(\bar{z} F_0(\bar{z}) - R F_0(R) - \int_R^{\bar{z}} F_0(z')\, dz' \right) \\
&= \beta \bar{z} - \beta \int_R^{\bar{z}} F_0(z')\, dz' \\
&= \beta R + \beta \int_R^{\bar{z}} (1 - F_0(z'))\, dz'\ ,
\end{aligned} \tag{24}
$$

where the fifth equality holds by the integration-by-parts formula (see, for example, Folland, 1984, p.100, Theorem 3.30). Finally, R^* is characterized as a unique solution R to the following equation:

$$
R = \frac{\beta}{1-\beta} \int_R^{\bar{z}} (1 - F_0(z'))\, dz'\ . \tag{25}
$$

In a summary, we have the following theorem.

Theorem 3. *Suppose that the kernel P_0 is i.i.d. and assume that $\beta E_0[z] \geq \underline{z}$ holds. Then, the value function v^* for the fund-manager is given by*

$$
(\forall m, z) \quad v^*(m,z) = \begin{cases} R^* m + \dfrac{R^* y}{1-\beta} & \text{if} \quad z \leq R^* \\ zm + \dfrac{R^* y}{1-\beta} & \text{if} \quad z > R^* \end{cases}
$$

and an optimal investment strategy ${}_0\boldsymbol{x}^*$ is given by

$$
(\forall t \geq 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \text{if} \quad z_t \leq R^* \\ (m_t, y) & \text{if} \quad z_t > R^*\ , \end{cases}
$$

where a constant R^* is a unique solution to the equation (25).

Theorem 3 derives (3) and (4) in Subsection 2.1 and it substantiates our discussion there. Note that the condition (7) in the uniform-distribution example given in the same subsection implies that $\beta E_0[z] \geq \underline{z}$ holds as assumed in Theorem 3.

3.2.3. Absence of Liquidity Constraint in Investment

Consider the fund-manager's investment problem which is the same as the one in Subsection 3.2.2 except that the liquidity constraint in investment (or, the cash-in-advance constraint in investment), (15), is absent. That is, suppose that the fund-manager may invest any amount of money as long as only the budget constraint (14) is satisfied. In such a case, an optimal investment strategy is characterized by the following theorem.

Theorem 4. *Suppose that the liquidity constraint in investment, (15), is absent. Also assume that the kernel P_0 is i.i.d. and that $\beta E_0[z] \geq \underline{z}$ holds. Then, the value function v^* for the fund-manager is given by*

$$(\forall m, z) \quad v^*(m, z) = \begin{cases} R^* m + R^* y + \dfrac{R^* y}{1-\beta} & \textit{if} \quad z \leq R^* \\ zm + zy + \dfrac{R^* y}{1-\beta} & \textit{if} \quad z > R^* \end{cases}$$

and an optimal investment strategy ${}_0\boldsymbol{x}^*$ is given by

$$(\forall t \geq 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \textit{if} \quad z_t \leq R^* \\ (m_t + y, 0) & \textit{if} \quad z_t > R^*, \end{cases}$$

where a constant R^* is a unique solution to the equation (25).

Theorem 4 shows that the liquidity constraint in investment, (15), is not essential to derive the reservation property of the optimal investment strategy. Furthermore, the reservation values R^* in Theorems 3 and 4 are identical. This shows that while the presence of the liquidity constraint affects the value function and the optimal investment level, it does not affect the reservation value at all. These facts indicate that the irreversibility of investment is the key feature that determines the basic structure of the model. The liquidity constraint in investment endows money with the role of an enabler of an option to postpone such irreversible investment.

3.3. Long-run Time-average Cash-holdings

We now turn to the issue of liquidity, or cash, demand. Throughout this subsection, we assume that P_0 has a limiting probability measure π_0. By this, z_t may be regarded as being *i.i.d.* according to π_0 in the long-run.

We define the long-run time-average cash-holdings by the expectation of the optimal cash-holdings with respect to its limiting probability measure. In view of (13), we consider the limiting probability measure μ of the optimal cash-holdings to be characterized by

$$(\forall B \in \mathcal{B}_{\mathbb{R}_+}) \quad \mu(B) = \int_Z \int_{\mathbb{R}_+} \chi_{\{\, g^*(m,z) \in B\,\}}(m,z)\, d\mu(m) d\pi_0(z)\,,$$

where χ is the indicator function.[9] It turns out[10] that such a μ exists, its support is given by $M \equiv \{\, iy \,|\, i = 1, 2, \ldots\}$ and it can be explicitly calculated as

$$\mu(\{y\}) = 1 - \pi_0^* \quad \text{and} \quad (\forall i \geq 2) \;\; \mu(\{iy\}) = (1 - \pi_0^*)(\pi_0^*)^{i-1}$$

as far as $\pi_0^* \in (0,1)$, where we abbreviate $\pi_0(\{\, z \,|\, z \leq R^*(z)\,\})$ to π_0^*.

Alternatively, we can derive μ more constructively as the limiting probability measure of the stochastic kernel which governs the transition of the optimal cash-holdings as follows. First, we derive the stochastic kernel Q over $\mathbb{R}_+ \times Z$ when the fund-manager follows the optimal strategy (22) by

$$(\forall B \in \mathcal{B}_{\mathbb{R}_+})(\forall E \in \mathcal{B}_Z)(\forall m, z) \quad Q(B \times E \,|\, m, z) = \chi_{\{\, g^*(m,z) \in B\,\}}(m,z) P_0(E|z)\,.$$

It turns out that Q thus defined only over the measurable rectangles can be extended to the unique stochastic kernel $Q : (\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_Z) \times (\mathbb{R}_+ \times Z) \to [0,1]$.[11] Second, we define the long-run stochastic kernel for the optimal money-holdings, $Q^M : \mathcal{B}_{\mathbb{R}_+} \times \mathbb{R}_+ \to [0,1]$, to be the "marginal probability" of Q with respect to π_0, that is,

$$\begin{aligned} (\forall B)(\forall m) \quad Q^M(B|m) &= \int_Z Q(B \times Z \,|\, m, z)\, d\pi_0(z) \\ &= \int_Z \chi_{\{\, g^*(m,z) \in B\,\}}(m,z)\, d\pi_0(z)\,, \end{aligned}$$

an equivalent and somewhat simpler expression of which is

$$(\forall m \neq 0) \quad Q^M(\{m'\}|m) = \begin{cases} \pi_0^* & \text{if } \; m' = m + y \\ 1 - \pi_0^* & \text{if } \; m' = y\,. \end{cases}$$

[9]That is, $\chi_A : \mathbb{R}_+ \times Z \to \{0,1\}$ is a function defined by

$$(\forall m, z) \quad \chi_A(m,z) = \begin{cases} 1 & \text{if} \quad (m,z) \in A \\ 0 & \text{if} \quad (m,z) \notin A\,. \end{cases}$$

[10]See Taub (1988).

[11]See, for example, Stokey and Lucas (1989, p.284), Theorem 9.13.

Third, when $\pi_0^* \in (0,1)$, the limiting probability measure of Q^M (in the sense defined in Subsection 3.1) uniquely exists and equals μ.[12]

Suppose that $\pi_0^* \in (0,1)$. Then, the *long-run time-average cash-holdings*, m_0^*, is given by

$$m_0^* = \sum_{m=y,2y,\ldots} m\,\mu(\{m\}) = \frac{y}{1-\pi_0^*} = \frac{y}{1-\pi_0\left(\{\, z \,|\, z \leq R^*(z)\,\}\right)}\,. \tag{26}$$

If the probability that the investment is the optimal strategy measured by the limiting probability measure π_0 is higher, then the long-run average cash-holdings is smaller, as we would expect.

When z_t is *i.i.d.*, π_0 always exists and equals P_0 and R^* is a constant which is characterized by (25). Therefore, the long-run average cash-holdings, m_0^*, is given by

$$m_0^* = \frac{y}{1-P_0\left(\{\, z \,|\, z \leq R^*\,\}\right)}\,, \tag{27}$$

which justifies (5) in Subsection 2.1.

4. An Increase in Risk

4.1. Effects on Trigger Level

This section investigates an increase in risk and its effects on the model's outcomes. We first consider the effect on the trigger level caused by an increase in risk in the sense of a mean-preserving spread (Rothschild and Stiglitz, 1970). To be precise, let P_0 and P_1 be two probability measures. We denote by E_0 and E_1 the expectations with respect to P_0 and P_1, respectively and we denote by F_0 and F_1 the distribution functions associated with P_0 and P_1, respectively. By definition, P_1 *is obtained from* P_0 *by a mean-preserving spread* if it holds that

$$E_1[z] = E_0[z] \quad \text{and} \quad (\forall x \in \mathbb{R}_+) \quad \int_{-\infty}^{x} F_1(z)\,dz \geq \int_{-\infty}^{x} F_0(z)\,dz\,. \tag{28}$$

We say that a stochastic kernel P_1 *is obtained from* P_0 *by a mean-preserving spread* if $(\forall z \in Z)$ $P_1(\cdot|z)$ is obtained from $P_0(\cdot|z)$ by a mean-preserving spread.

[12] See, for example, Hoel, Port and Stone (1987, p.73), Theorem 7. Also note that μ satisfies

$$(\forall m' \in M) \quad \mu(\{m'\}) = \sum_{m\in M} Q^M(\{m'\}|m)\mu(\{m\})\,.$$

The following theorem states that the increase in risk in the sense of a mean-preserving spread raises (or at least unchanges) the trigger level, R^*.

Theorem 5. *Let P_0 be a stochastic kernel which is stochastically increasing and stochastically convex and let P_1 be a stochastic kernel which is obtained from P_0 by the mean-preserving spread. Then, $(\forall z \in Z)$ $R_1^*(z) \geq R_0^*(z)$, where R_i^* is the trigger level corresponding to P_i for each $i = 0, 1$.*

Note that both P_0 and P_1 are assumed to be weakly continuous, which is a maintained assumption throughout the paper. Also note that while the assumption that P_0 is stochatically increasing and stochastically convex is essential (see the proof of Theorem 5 in Section 5), P_1 need to be *neither* stochastically increasing *nor* stochastically convex.

This theorem shows that the mean-preserving spread in the distribution of return shocks increases the "trigger return level" that induces the fund-manager to invest. Theorem 2 implies that under the assumption of Theorem 5, there exists the reservation value z_0^*. Suppose that $z_t \leq z_0^*$. Then, it follows that $z_t \leq R_1^*(z_t)$ because $z_t \leq R_0^*(z_t) \leq R_1^*(z_t)$ where the first inequality holds since $z_t \leq z_0^*$ if and only if $z_t \leq R_0^*(z_t)$ by Theorem 2 and the second inequality holds by Theorem 5. This shows that if making investment is not an optimal strategy before the risk increases, it cannot be so after the risk has increased. Therefore, an increase in risk tends to increase cash balances to be carried over to the next period in order to exploit potentially more favorable future opportunities. Money cash balances work as a provider of this option, which is more favorable under more risk.

Let us assume that z_t is *i.i.d.* according to P_0 and let P_1 be an *i.i.d.* stochastic kernel which is obtained from P_0 by a mean-preserving spread. Since any *i.i.d.* kernel is stochastically increasing and stochastically convex, Theorem 5 implies that $R_1^* \geq R_0^*$, where R_i^* is the reservation levels corresponding to P_i for each $i = 1, 2$.

Alternatively, we may verify this fact directly without invoking Theorem 5 as follows. We need to consider two cases separately. First, assume that $\beta E_0[z] < \underline{z}$. In this case, we have $R_0^* = \beta E_0[z]$ (see Subsection 3.2.2). By the definition of the mean-preserving spread, it holds that $E_0[z] = E_1[z]$ and hence that $R_1^* = R_0^*$. Therefore, an increase in risk does not alter the trigger level in this case. Second, assume that $\beta E_0[z] \geq \underline{z}$. In this case, for each $i = 0, 1$, R_i^* is characterized as a unique fixed point of a function $H_i : Z \to \mathbb{R}$, which is defined by[13]

$$(\forall x) \quad H_i(x) = \beta E_i[z] + \beta \int_{\underline{z}}^{x} F_i(z)\, dz\,. \tag{29}$$

It is easy to see that for each $i = 0, 1$, H_i is continuous and nondecreasing. Furthermore, the slope of H_i is less than unity everywhere.[14] Therefore, H_i crosses the 45-degree line once from above because $H_i(\underline{z}) = \beta E_i[z] \geq \underline{z}$ by the assumption. The unique intersection of H_i and the 45-degree line, which must be no less than $\underline{z}$, is the trigger level R_i^*. Finally, note that $(\forall x)\ H_1(x) \geq H_0(x)$ by (28), (29) and the fact that P_1 is obtained from P_0 by the mean-preserving spread. We thus conclude that $R_1^* \geq R_0^*$.

The argument of the previous paragraph verifies that the comparative-static analysis made in Subsection 2.2 for the uniform-distribution example remains to be true also for any specification of the distribution. Furthermore, Theorem 5 extends this to a Markovian setting under the stated assumption on the original distribution.

4.2. Effects on Average Cash-holdings

[13]This is because

$$\begin{aligned}
R_i^* &= \beta R_i^* F_i(R_i^*) + \beta \int_{(R_i^*, \bar{z}]} z\, dF_i(z) \\
&= \beta R_i^* F_i(R_i^*) + \beta E_i[z] - \beta \int_{[\underline{z}, R_i^*]} z\, dF_i(z) \\
&= \beta R_i^* F_i(R_i^*) + \beta E_i[z] - \beta \left(R_i^* F_i(R_i^*) - \underline{z} \lim_{z \to \underline{z}^-} F_i(z) - \int_{\underline{z}}^{R_i^*} F_i(z)\, dz \right) \\
&= \beta E_i[z] + \beta \int_{\underline{z}}^{R_i^*} F_i(z)\, dz\,,
\end{aligned}$$

where the first equality holds by the fourth equality in (24), the third equality holds by a version of the integration-by-parts formula (Folland, 1984, p.103, 34(b)) and the fourth equality holds by the fact that $\lim_{z \to \underline{z}^-} F_i(z) = 0$.

[14]This is because for any x_1 and x_2 such that $x_2 \geq x_1$,

$$\begin{aligned}
&\left(\beta \int_{\underline{z}}^{x_2} F_i(z)\, dz - \beta \int_{\underline{z}}^{x_1} F_i(z)\, dz \right) \Big/ (x_2 - x_1) \\
= \ & \beta \int_{x_1}^{x_2} F_i(z)\, dz \Big/ (x_2 - x_1) \\
= \ & \beta \left(F_i(x_2) x_2 - F_i(x_1) x_1 - \int_{(x_1, x_2]} z\, dF_i(z) \right) \Big/ (x_2 - x_1) \\
\leq \ & \beta \left(F_i(x_2) x_2 - F_i(x_1) x_1 - (F_i(x_2) - F_i(x_1))\, x_1 \right) / (x_2 - x_1) \\
= \ & \beta F_i(x_2) \ < \ 1\,,
\end{aligned}$$

where we invoked the integration-by-parts formula (see (24)) to show the inequality.

We now turn to a discussion of the effect of an increase in risk on the average cash-holdings. To do this, suppose that the stochastic kernel P_0 has a limiting probability measure so that the long-run time-average cash-holdings m_0^* may be well-defined. Theorem 5 in the previous subsection proved that a mean-preserving spread in the distribution increases the trigger level under the appropriate assumption. Furthermore, the comparative-static analysis made in Subsection 2.2 for the uniform-distribution example showed that it increases the average cash-holdings as well as the trigger level. Nevertheless, the effect of a mean-preserving spread on the average cash-holdings is not clear in general for the following reasons.

First, we do not know whether or not a stochastic kernel P_1, which is obtained from another stochastic kernel P_0 by a mean-preserving spread, has a limiting probability measure even if P_0 is assumed to have one, say π_0. If P_1 does not have it, the average cash-holdings under P_1 is not well-defined. Second, even if a limiting probability measure of P_1, say π_1, exists, we do not know whether or not π_1 can be obtained from π_0 by a mean-preserving spread. Third and most importantly, even if π_1 is obtained from π_0 by a mean-preserving spread, it might be the case that the *probability* that the return exceeds the trigger level increases while the trigger level also increases. Then, the average cash-holdings would *de*crease, rather than increase, in response to the mean-preserving spread (see (26) above).

The third point made in the previous paragraph is essential while the others are not. In fact, if we assume that P_0 and P_1 are *i.i.d.* and that P_1 is obtained from P_0 by a mean-preserving spread, then the first two points become vacuous because in such a case, it holds trivially that $\pi_0 = P_0$, $\pi_1 = P_1$ and π_1 is obtained from π_0 by a mean-preserving spread, but third point still remains.

To see this point more closely, let P_0 and P_1 be *i.i.d.* stochastic kernels such that P_1 is obtained from P_0 by a mean-preserving spread and let R_0 and R_1 be the reservation levels corresponding to P_0 and P_1, respectively. Recall that Theorem 5 shows that $R_1^* \geq R_0^*$. In view of (27), if it were the case that $P_1(\{\, z \mid z \leq R_1^* \,\}) \geq P_0(\{\, z \mid z \leq R_0^* \,\})$, then we would conclude that an increase in risk in the form of mean-preserving spread increases (or at least, unchanges) the average cash-holdings. Unfortunately, it is not always the case as the following example illustrates.

Example 1. Let $Z = [0,3]$, let $\beta > 2/3$, let P_0 be such that $P_0(\{1\}) = P_0(\{2\}) = 1/2$, and let P_1 be such that $P_1(\{0\}) = 1/4$ and $P_1(\{2\}) = 3/4$. It can be shown that $R_0^* = 2\beta/(2-\beta)$ and $R_1^* = 6\beta/(4-\beta)$.[15] It holds that $R_1^* > R_0^*$ as we expect. (This holds true regardless of the value of β as far as $\beta \in (0,1)$.) However, $R_0^* \in (1,2)$ and $R_1^* < 2$ when $\beta > 2/3$, and hence, $F_0(R_0^*) = 1/2$ and $F_1(R_1^*) = 1/4$. We thus have $P_1(\{\, z \,|\, z \le R_1^* \,\}) = 1/4 < 1/2 = P_0(\{\, z \,|\, z \le R_0^* \,\})$.

As this example suggests, in order to determine the effect of an increase in risk on the average cash-holdings, we need a stronger concept of being riskier than the mean-preserving spread. We say that P_0 and P_1 satisfy a *single-crossing property* (Sargent, 1987, p.64) if there exists $\hat{z} \in Z$ such that $F_1(z) \ge F_0(z)$ when $z < \hat{z}$ and $F_1(z) \le F_0(z)$ when $z \ge \hat{z}$. It can be seen immediately that if $E_0[z] = E_1[z]$ and P_0 and P_1 satisfy a single-crossing property, then P_1 is obtained from P_0 by a mean-preserving spread. Therefore, we may regard the single-crossing property as a stronger concept of being riskier than the mean-preserving spread. The single-crossing property, however, is not yet strong enough to guarantee that the average cash-holdings increase since P_0 and P_1 in Example 1 have the same expectation and satisfy the single-crossing property. To guarantee it, we need a further assumption on P_0, which is provided in the following proposition.

Proposition 2. *Suppose that two i.i.d. stochastic kernels P_0 and P_1 have the same expectation and satisfy the single-crossing property with $\hat{z}$. Assume further that*

$$\beta E_0[z] + \beta \int_{\underline{z}}^{\hat{z}} F_0(z)\, dz < \hat{z}\,. \tag{30}$$

Then, we have $m_1^* \ge m_0^*$, where m_i^* is the average cash-holdings under P_i for each $i = 1, 2$.

In Example 1, P_0 and P_1 have the same expectation which equals $3/2$ and they also satisfy the single-crossing property with $\hat{z} = 1$. The left-hand side of (30) is $(3/2)\beta$, and hence, (30) is violated when $\beta > 2/3$ as assumed in Example 1. Now suppose that $\beta < 2/3$ in Example 1. Then, (30) holds true and both R_0^* and R_1^* would be less than 1. Therefore, $F_1^*(R_1^*) = 1/4 > 0 = F_0^*(R_0^*)$ as the proposition predicts.

[15] Conjecture that $R_0^* \in (1,2)$. Then, it follows from (25) that $R_0^* = 2\beta/(2-\beta)$, which certainly satisfies the conjecture when $\beta > 2/3$. Since we know that (25) has a unique solution, we conclude that $R_0^* = 2\beta/(2-\beta)$. A similar argument applies for R_1^*.

Note that the condition (30) is sufficient but not necessary. In the uniform-distribution example given in Subsection 2.2, (30) is not always satisfied.[16] Nevertheless, it always holds there that $m_1^* \geq m_0^*$ at least for a small change in risk which satisfies a single-crossing property.

5. Lemmas and Proofs

Proof of Proposition 1. (a) Weak continuity. Scheffé's theorem (Billingsley, 1986, p.218, Theorem 16.11) shows that $\| P_0(\cdot|z_n) - P_0(\cdot|z_0) \| \to 0$ as $n \to +\infty$ since the density function of $P_0(\cdot|z_n)$ exists and converges to that of $P_0(\cdot|z_0)$ except at 0 and $1/(2 - z_0)$, where $\| \cdot \|$ is the total variation norm, which implies the weak continuity of P_0. (b) Stochastic increase. Note that

$$(\forall z, z') \quad 1 - F_0(z'|z) = \max\left\{0, 1 - (2 - z)z'\right\} . \tag{31}$$

Since $1 - F_0(z'|z)$ is nondecreasing in z for each z', the stochastic increase follows from Topkis (1998, p.161, Lemma 3.9.1(b)). (c) Stochastic convexity. From (31), we see that $1 - F_0(z'|z)$ is convex in z for each z'. Hence, the stochastic convexity follows from Topkis (1998, p.161, Lemma 3.9.1(d)). (d) Existence of a limiting probability measure. Note that Assumption 12.1 of Stokey and Lucas (1989, p.381) is now satisfied (say, let $a = 0$, $b = 1$, $c = 1/4$, $\varepsilon = 1/4$ and $N = 1$). Also, note that their "monotonicity" is the equivalent of weak increase here and their "Feller property" is the equivalent of weak continuity here. Therefore, by their Theorem 12.12 (Stokey and Lucas, 1989, p.381), P_0 has a limiting probability measure. ■

Lemma 1. *There exists a unique fixed point R^* to the operator T defined by (18) which satisfies $(\forall z)$ $0 \leq R^*(z) \leq \bar{z}$. Furthermore, R^* is upper semi-continuous (u.s.c.) and given by $R^* = \lim_{n\to\infty} T^n \bar{z} = \lim_{n\to\infty} T^n 0$.*

Proof. First, define R^+ by $(\forall z)$ $R^+(z) = \bar{z}$. Then, it follows that

$$(\forall z) \quad TR^+(z) = \beta \int_Z \max\left\{z', \bar{z}\right\} P_0(dz'|z) \leq \beta \int_Z \bar{z}\, P_0(dz'|z) = \beta\bar{z} \leq \bar{z} = R^+(z)\,.$$

[16] In the uniform-distribution example in Subsection 2.2, $\hat{z} = (a+b)/2$ and the left-hand side of (30) is $\beta((3/8)a + (5/8)b)$, which is greater than $\hat{z}$ when β is close to 1.

Since T is monotonic in the sense that $(\forall R, R')\ R \geq R' \Rightarrow TR \geq TR'$, $\langle T^n R^+ \rangle_{n=1}^{\infty}$, where T^n denotes the t-fold self-composition of T, is a nonincreasing sequence of functions. Hence, its limit exists and $\mathcal{B}_Z$-measurable. We denote it by R^∞. We now see that R^∞ is a fixed point of T because

$$
\begin{aligned}
(\forall z) \quad TR^\infty(z) &= \beta \int_Z \max\left\{z', R^\infty(z')\right\} P_0(dz'|z) \\
&= \beta \int_Z \max\left\{z', \lim_{n\to\infty} T^n R^+(z')\right\} P_0(dz'|z) \\
&= \beta \int_Z \lim_{n\to\infty} \max\left\{z', T^n R^+(z')\right\} P_0(dz'|z) \\
&= \lim_{n\to\infty} \beta \int_Z \max\left\{z', T^n R^+(z')\right\} P_0(dz'|z) \\
&= \lim_{n\to\infty} T^{n+1} R^+(z) \\
&= R^\infty(z)\,,
\end{aligned}
$$

where the fourth inequality holds by the monotone convergence theorem.

Second, define R^- by $(\forall z)\ R^-(z) = 0$. Then, it follows that

$$
(\forall z) \quad TR^-(z) = \beta \int_Z \max\left\{z', 0\right\} P_0(dz'|z) \geq 0 = R^-(z)\,.
$$

Since $(\forall R, R')\ R \geq R' \Rightarrow TR \geq TR'$, $\langle T^n R^+ \rangle_{n=1}^{\infty}$ is a nondecreasing sequence of functions. Hence, its limit exists and $\mathcal{B}_Z$-measurable. We denote it by R_∞. We now see that R_∞ is a fixed point of T because

$$
\begin{aligned}
(\forall z) \quad TR_\infty(z) &= \beta \int_Z \max\left\{z', R_\infty(z')\right\} P_0(dz'|z) \\
&= \beta \int_Z \max\left\{z', \lim_{n\to\infty} T^n R^-(z')\right\} P_0(dz'|z) \\
&= \beta \int_Z \lim_{n\to\infty} \max\left\{z', T^n R^-(z')\right\} P_0(dz'|z) \\
&= \lim_{n\to\infty} \beta \int_Z \max\left\{z', T^n R^-(z')\right\} P_0(dz'|z) \\
&= \lim_{n\to\infty} T^{n+1} R^-(z) \\
&= R_\infty(z)\,,
\end{aligned}
$$

where the fourth inequality holds by the monotone convergence theorem.

This paragraph shows that $R^\infty = R_\infty$. To this end, let $z \in Z$ and let $n \geq 1$. Then, we have

$$
\begin{aligned}
0 &\leq T^n R^+(z) - T^n R^-(z) \\
&= \beta \int_Z \left(T^{n-1}R^+(z_1) - T^{n-1}R^-(z_1)\right)\, P_0(dz_1|z) \\
&= \beta^2 \int_Z \int_Z \left(T^{n-2}R^+(z_2) - T^{n-2}R^-(z_2)\right)\, P_0(dz_2|z_1)P_0(dz_1|z) \\
&= \cdots \\
&= \beta^n \int_Z \cdots \int_Z \int_Z \left(R^+(z_n) - R^-(z_n)\right)\, P_0(dz_n|z_{n-1})P_0(dz_{n-1}|z_{n-2}) \cdots P_0(dz_1|z) \\
&= \beta^n \int_Z \cdots \int_Z \int_Z \bar{z}\, P_0(dz_n|z_{n-1})P_0(dz_{n-1}|z_{n-2}) \cdots P_0(dz_1|z) \\
&= \beta^n \bar{z}\,.
\end{aligned}
$$

Since the whole inequality holds for any n, taking the limit proves the claim.

Let R be any fixed point of T such that $R^- = 0 \leq R \leq \bar{z} = R^+$. Then, it holds that $TR^- \leq TR = R \leq TR^+$ by the monotonicity of T and the assumption that R is a fixed point of T. By iterating this procedure, we have $(\forall n)\ T^n R^- \leq R \leq T^n R^+$. Therefore, it follows that $R_\infty = \lim_{n\to\infty} T^n R^- \leq R \leq \lim_{n\to\infty} T^n R^+ = R^\infty$. This and the fact proven in the previous paragraph show that $R = R_\infty = R^\infty$, and hence, $R^* \equiv R^\infty$ is the unique fixed point of T satisfying $0 \leq R^* \leq \bar{z}$.

Finally, we show that R^* is u.s.c. The weak increase of P_0 and Gihmann and Skorohod (1979, Lemma 1.5) imply that $(\forall n)\ T^n R^+$ is u.s.c. in z. Therefore, R^* is u.s.c. since it is the infimum of u.s.c. functions by $R^* = \lim_{n\to\infty} T^n R^+ = \inf_{n\geq 1}\langle T^n R^+\rangle$. ■

Lemma 2. *The function A defined by (19) is u.s.c. and satisfies*

$$
(\forall z) \quad A(z) = R^*(z)y + \beta E_0[A|z] = R^*(z)y + \beta \int_Z A(z')\, P_0(dz'|z)\,. \tag{32}
$$

Proof. (U.s.c.) Since R^* is u.s.c. (Lemma 1) and bounded from above (by $\bar{z}$), $(\forall s \geq 0)\ E_0^s[R^*|z]$ is u.s.c. in z by Gihmann and Skorohod (1979, Lemma 1.5). Let $z_0 \in Z$ and let $\varepsilon > 0$. Since $E_0^s[R^*|z]$ is uniformly bounded from above in s and z and since $\beta < 1$, there exits $S \geq 1$ such that $y \sum_{s=S+1}^{+\infty} \beta^s E_0^s[R^*|z] < \varepsilon/2$. Furthermore, since $\sum_{s=0}^{S} \beta^s E_0^s[R^*|z]$ is u.s.c. in z (because it is a *finite* sum of u.s.c. functions), there exists a neighborhood N of z_0 such that $(\forall z \in$

$N)$ $y\sum_{s=0}^{S}\beta^s E_0^s[R^*|z] < A(z_0)+\varepsilon/2$. Finally, we have $(\forall z \in N)$ $A(z) < A(z_0)+\varepsilon$, which completes the proof.

(Equation (32)) The equation holds because

$$\begin{aligned}
A(z) &= y\sum_{s=0}^{+\infty}\beta^s E_0^s[R^*|z] \\
&= R^*(z)y + \beta y\sum_{s=0}^{+\infty}\beta^s E_0^{s+1}[R^*|z] \\
&= R^*(z)y + \beta y\sum_{s=0}^{+\infty}\beta^s E_0\left[E_0^s[R^*|z_1]|z\right] \\
&= R^*(z)y + \beta E_0\left[y\sum_{s=0}^{+\infty}\beta^s E_0^s[R^*|z_1]\,\middle|\, z\right] \\
&= R^*(z)y + \beta E_0[A|z]\,,
\end{aligned}$$

where the third equality holds by the law of iterated expectations and the fourth equality holds by the monotone convergence theorem. ■

Proof of Theorem 1. First, we show that the function $\hat{v}:\mathbb{R}_+\times Z\to\mathbb{R}$ defined by

$$(\forall m,z) \quad \hat{v}(m,z) = \max\big\{z, R^*(z)\big\}m + A(z)$$

is the solution to Bellman's equation:

$$(\forall m,z) \quad v(m,z) = \max\left\{ xz + \beta\int_Z v(m',z')\,P_0(dz'|z)\,\middle|\,(x,m')\in\Gamma(m)\right\}. \tag{33}$$

We have

$$\begin{aligned}
&\max\left\{ xz + \beta\int_Z \hat{v}(m',z')\,P_0(dz'|z)\,\middle|\,(x,m')\in\Gamma(m)\right\} \\
=\;&\max\left\{ xz + m'\beta\int_Z \max\big\{z',R^*(z')\big\}\,P_0(dz'|z) + \beta\int_Z A(z')\,P_0(dz'|z)\,\middle|\,(x,m')\in\Gamma(m)\right\} \\
=\;&\max\big\{ xz + m'TR^*(z) + \beta E_0[A|z]\,\big|\,(x,m')\in\Gamma(m)\big\} \\
=\;&\max\big\{ xz + m'R^*(z) + \beta E_0[A|z]\,\big|\,(x,m')\in\Gamma(m)\big\} \\
=\;&\begin{cases} R^*(z)(m+y) + \beta E_0[A|z] & \text{if} \quad z\le R^*(z) \\ mz + R^*(z)y + \beta E_0[A|z] & \text{if} \quad z > R^*(z) \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} R^*(z)m + R^*(z)y + \beta E_0[A|z] & \text{if} \quad z \le R^*(z) \\ zm + R^*(z)y + \beta E_0[A|z] & \text{if} \quad z > R^*(z) \end{cases} \\
&= \begin{cases} R^*(z)m + A(z) & \text{if} \quad z \le R^*(z) \\ zm + A(z) & \text{if} \quad z > R^*(z) \end{cases} \\
&= \hat{v}(m,z)\,,
\end{aligned}
$$

where the third equality holds by the fact that R^* is the fixed point of T and sixth equality holds by Equation (32).

Second, we show that $\hat{v}$ is admissible (the Appendix), that is, $\hat{v}$ is u.s.c. and satisfies

$$(\forall m,z) \quad 0 \le \hat{v}(m,z) \le \frac{m\bar{z}}{1-\beta} + \frac{\beta y \bar{z}}{(1-\beta)^2}\,.$$

That $\hat{v}$ is u.s.c. follows since R^* is u.s.c. (by Lemma 1) and A is u.s.c. (by Lemma 2). To show the inequalities, note that $R^* \le TR^+ \le \beta\bar{z}$ by (18). Therefore,

$$
\begin{aligned}
0 \le \hat{v}(m,z) &= \max\big\{z, R^*(z)\big\}m + y\sum_{s=0}^{+\infty}\beta^s E_0^s[R^*|z] \\
&\le \bar{z}m + y\sum_{s=0}^{+\infty}\beta^s\beta\bar{z} \\
&= \bar{z}m + y\frac{\beta\bar{z}}{1-\beta} \\
&\le \frac{m\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2}\,.
\end{aligned}
$$

Finally, since $\hat{v}$ is an admissible solution to Bellman's equation as shown in the preceding paragraphs, we conclude that v^* defined by (20), which equals $\hat{v}$, is the value function by Theorem A1. Furthermore, the first paragraph of this proof shows that g defined by (21) is the policy correspondence. Finally, that recursive optimality implies optimality is among the conclusions of Theorem A1. ■

Lemma 3. *Suppose that the kernel P_0 is stochastically increasing and stochastically convex. Then, R^* is nondecreasing and convex in z.*

Proof. First, we show that for each $n \ge 1$, $T^n R^+$ is nondecreasing and convex in z. We prove this by induction. The statement holds true when $n = 0$ since $T^0 R^+ = R^+ = \bar{v}$ is constant and hence both nondecreasing and convex in z. Suppose that $T^{n-1}R^+$ is nondecreasing and convex in z. Then, $\max\{z', T^{n-1}R^+(z')\}$ is also nondecreasing and convex in z'. Therefore, $T^n R^+$ is nondecreasing and convex by Topkis (1998, p.161, Corollary 3.9.1(a)(c)) since P_0 is stochastically increasing and stochastically convex.

Since R^* is a pointwise limit of a sequence of nondecreasaing and convex functions by Lemma 1 and the fact proven in the previous paragraph, R^* is nondecreasing and convex. ■

Proof of Theorem 2. By Lemma 3, R^* is convex and hence continuous on $(\underline{z}, \bar{z})$. Since R^* is u.s.c. by Lemma 1 and nondecreasing by Lemma 3, it is continuous on $[\underline{z}, \bar{z})$ with only possible discontinuity occurring at $z = \bar{z}$. Furthermore, note that $R^*(\bar{z}) \leq TR^+ = \beta\bar{z} < \bar{z}$. Therefore, Lemma 3 implies that the graph of R^* crosses the 45-degree line from above if and only if $R^*(\underline{z}) \geq \underline{z}$. First, suppose that $R^*(\underline{z}) < \underline{z}$. Then, any z^* such that $0 \leq z^* < \underline{z}$ serves as a reservation level and the optimal strategy clearly has a reservation property. Second, suppose that $R^*(\underline{z}) \geq \underline{z}$. Then, there exists a unique $z^* \in Z$ such that $z^* = R^*(z^*)$ and the optimal strategy has a reservation property since $\{\, z \in Z \,|\, z \leq R^*(z) \,\} = \{\, z \in Z \,|\, z \leq z^* \,\}$.

We complete the proof by showing that when $\beta E_0[z'|\underline{z}] \geq \underline{z}$, it holds that $R^*(\underline{z}) \geq \underline{z}$. To see this, suppose that it does not. Then, since R^* solves $R^* = TR^*$, it follows from (18) that $R^*(\underline{z}) = \beta E_0[z'|\underline{z}] \geq \underline{z} > R^*(\underline{z})$, which is a contradiction. ■

Proof of Theorem 3. This follows immediately from Theorem 1 since $A(z) = R^*y/(1-\beta)$ by (19). ■

Proof of Theorem 4. Define a correspondence $\hat{\Gamma} : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}^2_+$ by

$$(\forall m) \quad \hat{\Gamma}(m) = \left\{\, (x, m') \in \mathbb{R}^2_+ \,\middle|\, x + m' \leq y + m \,\right\}$$

and let R^* be a unique solution to (25). First, we show that the function $\hat{v} : \mathbb{R}_+ \times Z \to \mathbb{R}$ defined by

$$(\forall m, z) \quad \hat{v}(m, z) = \max\left\{z, R^*\right\}(m + y) + \frac{R^*y}{1-\beta}$$

is the solution to Bellman's equation:

$$(\forall m, z) \quad v(m,z) = \max\left\{ xz + \beta \int_Z v(m',z')\, P_0(dz') \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\}.$$

We have

$$\begin{aligned}
& \max\left\{ xz + \beta \int_Z \hat{v}(m',z')\, P_0(dz') \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
= \;& \max\left\{ xz + (m'+y)\beta \int_Z \max\left\{z', R^*\right\} P_0(dz') + \frac{\beta R^* y}{1-\beta} \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
= \;& \max\left\{ xz + (m'+y)R^* + \frac{\beta R^* y}{1-\beta} \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
= \;& \begin{cases} R^*(y+m+y) + \dfrac{\beta R^* y}{1-\beta} & \text{if} \quad z \le R^* \\ z(y+m) + R^* y + \dfrac{\beta R^* y}{1-\beta} & \text{if} \quad z > R^* \end{cases} \\
= \;& \max\left\{z, R^*\right\}(y+m) + R^* y + \frac{\beta R^* y}{1-\beta} \\
= \;& \max\left\{z, R^*\right\}(m+y) + \frac{R^* y}{1-\beta} \;=\; \hat{v}(m,z),
\end{aligned}$$

where the second equality holds by the fact that R^* is a solution to (23).

Second, we observe that all the results (in particular, Theorem A1) in the Appendix still holds if we replace Γ there by $\hat{\Gamma}$ here and v^+ there by $\hat{v}^+$, which is defined by

$$(\forall m) \quad \hat{v}^+(m) = \frac{m\bar{z}}{1-\beta} + \frac{y\bar{z}}{(1-\beta)^2}.$$

To do this, we only need to verify that $B\hat{v}^+ \le \hat{v}^+$ (Lemma A2), which holds true because

$$\begin{aligned}
(\forall m, z) \quad B\hat{v}^+(m,z) \;&=\; \max\left\{ xz + \beta \int_Z \hat{v}^+(m',z')\, P_0(dz') \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
&=\; \max\left\{ xz + \beta\hat{v}^+(m') \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
&=\; \max\left\{ xz + \frac{\beta m'\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2} \,\middle|\, (x,m') \in \hat{\Gamma}(m) \right\} \\
&\le\; (m+y)z + \frac{\beta(m+y)\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2} \\
&\le\; (m+y)\bar{z} + \frac{\beta(m+y)\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2} \\
&=\; \frac{m\bar{z}}{1-\beta} + \frac{y\bar{z}}{(1-\beta)^2} \;=\; \hat{v}^+(m,z).
\end{aligned}$$

Third, we show that $\hat{v}$ is admissible (the Appendix), that is, $\hat{v}$ is u.s.c. and satisfies $\hat{v} \leq \hat{v}^+$. The former is immediate and the latter holds because

$$
\begin{aligned}
(\forall m, z) \quad \hat{v}(m,z) &= \max\left\{z, R^*(z)\right\}(m+y) + \frac{R^* y}{1-\beta} \\
&\leq \bar{z}(m+y) + \frac{\beta \bar{z} y}{1-\beta} \\
&= \bar{z} m + \frac{\bar{z} y}{1-\beta} \\
&\leq \frac{m\bar{z}}{1-\beta} + \frac{y\bar{z}}{(1-\beta)^2} \;=\; \hat{v}^+(m,z)\,,
\end{aligned}
$$

where the first inequality holds since $R^* \leq \beta \bar{z}$ by (23).

Finally, note that $\hat{v}$ is an admissible solution to Bellman's equation as proven in the first and third paragraphs of this proof. Therefore, Theorem A1 (the Appendix) completes the proof.

■

Proof of Theorem 5. For each $i = 0, 1$, let T_i be the operator defined from P_i by (18). We show that for each $n \geq 1$, $T_1^n R^+ \geq T_0^n R^+$, which completes the proof since $(\forall i)$ $R_i^* = \lim_{n\to\infty} T_i^n R^+$. We prove the claim by induction. The statement clearly holds true when $n = 0$ since $T_1^0 R^+ = R^+ = T_0^0 R^+$. Suppose that $T_1^{n-1} R^+ \geq T_0^{n-1} R^+$. Then,

$$
\begin{aligned}
T_1^n R^+ &= T_1 \circ T_1^{n-1} R^+ \\
&\geq T_1 \circ T_0^{n-1} R^+ \\
&= \beta \int_Z \max\left\{z', T_0^{n-1} R^+(z')\right\} P_1(dz'|z) \\
&\geq \beta \int_Z \max\left\{z', T_0^{n-1} R^+(z')\right\} P_0(dz'|z) \\
&= T_0 \circ T_0^{n-1} R^+ \\
&= T_0^n R^+\,,
\end{aligned}
$$

where the first inequality holds by the induction hypothesis. To see that the second inequality holds, note that $\max\left\{z', T_0^{n-1} R^+(z')\right\}$ is convex in z' by Lemma 3 and the fact that the maximum of two convex functions is convex. Therefore, the inequality holds true by Rothschild and Stiglitz (1970) since $P_1(\cdot|z)$ is obtained from $P_0(\cdot|z)$ by the mean-preserving spread for each z.

■

Proof of Proposition 2. Note that the left-hand side of (30) can be written as $H_0(\hat{z})$, where a function $H_0 : Z \to \mathbb{R}$ is defined by (29). We showed there that H_0 crosses the 45-degree line once from above at $z = R_0^*$. Therefore, (30) implies that $R_0^* < \hat{z}$, and hence, that $F_1(R_0^*) \geq F_0(R_0^*)$ by the single-crossing property. Since $R_1^* \geq R_0^*$ by Theorem 5, we have $F_1(R_1^*) \geq F_0(R_0^*)$. Finally, (27) completes the proof. ■

APPENDIX

This appendix provides a method of dynamic programming which can be applied to the model in the current paper. The proofs given in Section 5 rely on the results in this appendix. We start with some definitions. Define the function $v^+ : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\begin{aligned}(\forall m) \quad v^+(m) &= \lim_{T\to+\infty} m\bar{z} + \cdots + \beta^{T-1}\left((T-1)y + m\right)\bar{z} + \beta^T(Ty+m)\bar{z} \\ &= \sum_{t=0}^{\infty} \beta^t m\bar{z} + \sum_{t=0}^{\infty} t\beta^t y\bar{z} \\ &= \frac{m\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2} \equiv B^+ m + A^+ .\end{aligned}$$

The function v^+ may be called the *overly-optimistic value function* since for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from m, it holds that $(\forall z)\ I_z({}_0\boldsymbol{x}) \leq v^+(m)$. A function $v : \mathbb{R}_+ \times Z \to \mathbb{R}$ is *admissible* if it is upper semi-continuous (u.s.c.) and satisfies $(\forall m, z)\ 0 \leq v(m,z) \leq v^+(m)$. Obviously, v^+ is admissible. Define the *Bellman operator*, which maps an admissible function v to another function Bv, by

$$(\forall v)(\forall m, z) \quad Bv(m,z) = \max\left\{ xz + \beta \int_Z v(m', z')\, P_0(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\}, \tag{34}$$

whose well-definition is proved below. We denote by B^n the n-fold self-composition of B, $B \circ \cdots \circ B$. Finally an admissible function v *solves Bellman's equation* if $v = Bv$.

Lemma A1. *The Bellman operator is well-defined.*

Proof. First, we show that for any admissible function v, the mapping defined by

$$(m', z) \mapsto \int_Z v(m', z')\, P_0(dz'|z)$$

is u.s.c. To do this, let v be an admissible function and let $\langle (m'_n, z_n) \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}_+ \times Z$ which converges to (m'_0, z_0). Then, by the admissibility of v and the u.s.c. of v^+, there exists $N \geq 1$ such that

$$(\forall n \geq N)(\forall z \in Z) \quad v(m'_n, z_n) \leq v^+(m'_n) < v^+(m'_0) + 1\,.$$

Therefore, the weak continuity of P_0, the u.s.c. of v and Lemma 1.5 of Gihman and Skorohod (1979) show that

$$\limsup_{n \to \infty} \int_Z v(m'_n, z')\, P_0(dz'|z_n) \leq \int_Z v(m'_0, z')\, P_0(dz'|z_0)\,,$$

which proves the claim.

Second, we completes the proof by showing that for any admissible function v, Bv is well-defined. However, this follows immediately because the maximand in (34) is u.s.c. by the fact proven in the previous paragraph and because Γ is compact-valued. ∎

Lemma A2. *$Bv^+ \leq v^+$ and for any admissible function v, Bv is admissible.*

Proof. The first half of the lemma follows because

$$\begin{aligned}(\forall m, z) \quad Bv^+(m,z) &= \max\left\{\, xz + \beta B^+ m' + \beta A^+ \,\middle|\, (x, m') \in \Gamma(m) \,\right\} \\ &\leq mz + \beta B^+(m+y) + \beta A^+ \\ &\leq m\bar{z} + \beta B^+(m+y) + \beta A^+ \\ &= \frac{m\bar{z}}{1-\beta} + \frac{\beta y \bar{z}}{(1-\beta)^2} \;=\; v^+(m,z)\,.\end{aligned}$$

To show the latter half of the lemma, let v be an admissible function. Then, the admissibility of v, the fact that B is monotonically non-decreasing in v and the inequality proven in the previous paragraph show that $0 \leq B0 \leq Bv \leq Bv^+ \leq v^+$. Furthermore, Bv is u.s.c. by the maximum theorem (Berge, 1963) because the maximand in (34) is u.s.c. by Lemma A1 and because Γ is continuous. ∎

Lemma A3. *For any $m \geq 0$, any investment strategy ${}_0\boldsymbol{x}$ which is feasible from m and any admissible function v, it holds that*

$$(\forall z) \quad I_z({}_0\boldsymbol{x}) = \lim_{T\to+\infty} x_0 z + \beta \int_Z \cdots$$
$$\beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z v(Ty+m, z_T)\, P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z)$$

Proof. Let $(m,z) \in \mathbb{R}_+ \times Z$, let ${}_0\boldsymbol{x}$ be an investment strategy which is feasible from m and let v be an admissible function. The iterated applications of Koopmans' equation to ${}_0\boldsymbol{x}$ shows that

$$(\forall T \geq 1) \quad I_z({}_0\boldsymbol{x}) = x_0 z + \beta \int_Z \cdots$$
$$\beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x})\, P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z)\,,$$

where ${}_T\boldsymbol{x}$ is a continuation of ${}_0\boldsymbol{x}$ after the realization of ${}_1z_T$. Therefore, for any $T \geq 1$, it follows that

$$
\begin{aligned}
& \Bigg| I_z({}_0\boldsymbol{x}) - \Bigg[x_0 z + \beta \int_Z \cdots \\
& \qquad \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z v(Ty+m, z_T) P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \Bigg] \Bigg| \\
= \; & \beta \Bigg| \int_Z \cdots \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x}) P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \\
& - \int_Z \cdots \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z v(Ty+m, z_T) P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \Bigg| \\
\leq \; & \beta \int_Z \Bigg| \left(\cdots \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x}) P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots \right) \\
& - \left(\cdots \beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z v(Ty+m, z_T)\, P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots \right) \Bigg| P_0(dz_1|z) \\
\leq \; & \cdots \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z |I_{z_T}({}_T\boldsymbol{x}) - v(Ty+m, z_T)|\, P_0(dz_T|z_{T-1}) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z \max\{I_{z_T}({}_T\boldsymbol{x}), v(Ty+m, z_T)\}\, P_0(dz_T|z_{T-1}) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z v^+(Ty+m) P_0(dz_T|z_{T-1}) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \\
= \; & \beta^T (B^+ Ty + B^+ m + A^+)\,,
\end{aligned}
$$

where the fifth inequality holds since $(\forall z_T)\ I_{z_T}({}_T\boldsymbol{x}) \leq v^+(Ty+m)$ by the fact that for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from m, $x_T \leq Ty + m$. Since the last term of the above inequalities goes to 0 as $T \to +\infty$, we have

$$\lim_{T\to\infty} \Bigg| I_z({}_0\boldsymbol{x}) - \Bigg[x_0 z + \beta \int_Z \cdots$$

$$\beta \int_Z \left(x_{T-1} z_{T-1} + \beta \int_Z v(Ty + m, z_T)\, P_0(dz_T|z_{T-1}) \right) P_0(dz_{T-1}|z_{T-2}) \cdots P_0(dz_1|z) \Bigg] \Bigg| = 0\,,$$

which completes the proof. ■

Lemma A4. *Any admissible solution to Bellman's equation is the value function.*

Proof. Let v be an admissible function which solves Bellman's equation and let $(m, z) \in \mathbb{R}_+ \times Z$. This paragraph shows that for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from m, it holds that $v(m, z) \geq I_z({}_0\boldsymbol{x})$. Let ${}_0\boldsymbol{x}$ be such an investment strategy and let ${}_1\boldsymbol{m}$ be its associated money-holding strategy. Then,

$$\begin{aligned} Bv(m,z) &\geq x_0 z + \beta \int_Z v(m_1, z_1)\, P_0(dz_1|z) \\ &\geq x_0 z + \beta \int_Z \left(x_1 z_1 + \beta \int_Z v(m_2, z_2)\, P_0(dz_2|z_1) \right) P_0(dz_1|z) \\ &\geq \cdots \\ &\geq x_0 z + \beta \int_Z \left(x_1 z_1 + \cdots \beta \int_Z v(m_T, z_T)\, P_0(dz_T|z_{T-1}) \cdots \right) P_0(dz_1|z) \end{aligned}$$

where the first inequality holds since v solves Bellman's equation and $(x_0, m_1) \in \Gamma(m)$ by the feasibility, the second inequality holds since v solves Bellman's equation and $(x_1, m_2) \in \Gamma(m_1)$ by the feasibility, and so on. Since the whole inequality holds for any $T \geq 1$, Lemma A3 proves the claim.

This paragraph completes the proof by showing that there exists an investment strategy ${}_0\boldsymbol{x}$ which is feasible from m and satisfies $v(m, z) = I_z({}_0\boldsymbol{x})$. Define the investment strategy ${}_0\boldsymbol{x}$ and the money-holding strategy ${}_1\boldsymbol{m}$ recursively by

$$\begin{aligned} (x_0, m_1) &\in \arg\max \left\{ xz + \beta \int_Z v(m', z')\, P_0(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \quad \text{and} \\ (\forall t \geq 1) \quad (x_t, m_{t+1}) &\in \arg\max \left\{ xz + \beta \int_Z v(m', z')\, P_0(dz'|z_t) \,\middle|\, (x, m') \in \Gamma(m_t) \right\}. \end{aligned}$$

Such strategies are well-defined by the measurable selection theorem (Wagner, 1977, p.880, Theorem 9.1(ii)). Then,

$$\begin{aligned} Bv(m,z) &= x_0 z + \beta \int_Z v(m_1, z_1)\, P_0(dz_1|z) \\ &= x_0 z + \beta \int_Z \left(x_1 z_1 + \beta \int_Z v(m_2, z_2)\, P_0(dz_2|z_1) \right) P_0(dz_1|z) \end{aligned}$$

$$= \cdots$$

$$= x_0 z + \beta \int_Z \left(x_1 z_1 + \cdots \beta \int_Z v(m_T, z_T)\, P_0(dz_T | z_{T-1}) \cdots \right) P_0(dz_1|z)$$

where the equalities hold by the definition of ${}_0\boldsymbol{x}$ and ${}_1\boldsymbol{m}$ and because v solves Bellman's equation. Since the whole inequality holds for any $T \geq 1$, Lemma A3 proves the claim. ■

Lemma A5. *A function v^∞ defined by $v^\infty \equiv \lim_{n\to\infty} B^n v^+$ is an admissible solution to Bellman's equation.*

Proof. Since by Lemma A2 and the fact that B is non-decreasing in v, $\langle B^n v^+ \rangle_{n=1}^\infty$ is a non-increasing sequence of u.s.c. functions which are bounded from below by 0, its limit exists and is u.s.c. Therefore, v^∞ is a well-defined admissible function. In the rest of this proof, we show that v^∞ solves Bellman's equation.

Note that $(\forall n \geq 1)$ $B^{n+1}v^+ = B \circ B^n v^+ \geq B \circ \lim_{n\to\infty} B^n v^+ = Bv^\infty$. Therefore, we have $v^\infty = \lim_{n\to\infty} B^{n+1} v^+ \geq Bv^\infty$.

To show the opposite inequality, let $(m, z) \in \mathbb{R}_+ \times Z$ and let $\langle (x_n, m'_n) \rangle_{n=1}^\infty$ be a sequence in $\mathbb{R}_+^2$ such that

$$(\forall n \geq 1) \quad (x_n, m'_n) \in \arg\max \left\{ xz + \beta \int_Z B^n v^+(m', z')\, P_0(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\}.$$

Such a sequence exists since the right-hand side is nonempty by Lemma A1 and the admissibility of $B^n v^+$. Since $\Gamma(m)$ is compact, there exists a subsequence $\langle (x_{n(i)}, m'_{n(i)}) \rangle_{i=1}^\infty$ which converges to $(x_0, m'_0) \in \Gamma(m)$. Then,

$$\begin{aligned}
Bv^\infty(m, z) &= \max \left\{ xz + \beta \int_Z v^\infty(m', z')\, P_0(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
&\geq x_0 z + \beta \int_Z v^\infty(m'_0, z')\, P_0(dz'|z) \\
&= x_0 z + \beta \int_Z \lim_{n\to\infty} B^n v^+(m'_0, z')\, P_0(dz'|z) \\
&= x_0 z + \beta \int_Z \lim_{i\to\infty} B^{n(i)} v^+(m'_0, z')\, P_0(dz'|z) \\
&\geq x_0 z + \beta \int_Z \lim_{i\to\infty} \limsup_{j\to\infty} B^{n(i)} v^+(m'_{n(j)}, z')\, P_0(dz'|z) \\
&\geq x_0 z + \beta \int_Z \lim_{i\to\infty} \limsup_{j\to\infty} B^{n(j)} v^+(m'_{n(j)}, z')\, P_0(dz'|z)
\end{aligned}$$

$$
\begin{aligned}
&= x_0 z + \beta \int_Z \limsup_{j\to\infty} B^{n(j)} v^+(m'_{n(j)}, z')\, P_0(dz'|z) \\
&\geq x_0 z + \beta \limsup_{j\to\infty} \int_Z B^{n(j)} v^+(m'_{n(j)}, z')\, P_0(dz'|z) \\
&= \limsup_{j\to\infty} \left(x_{n(j)} z + \beta \int_Z B^{n(j)} v^+(m'_{n(j)}, z')\, P_0(dz'|z) \right) \\
&= \limsup_{j\to\infty} B^{n(j)+1} v^+(m, z) \\
&= \lim_{j\to\infty} B^{n(j)+1} v^+(m, z) \\
&= v^\infty(m, z)\,,
\end{aligned}
$$

where the second inequality holds by the u.s.c. of $B^{n(i)}v^+$. To show the fourth inequality, let $J \geq 1$ be such that $(\forall j \geq J)\ m'_{n(j)} < m'_0 + 1$. Then, it follows that

$$
(\forall j \geq J)(\forall z' \in Z) \quad B^{n(j)} v^+(m'_{n(j)}, z') \leq v^+(m'_{n(j)}, z') = B^+ m'_{n(j)} + A^+ < B^+(m'_0 + 1) + A^+\,.
$$

Therefore, the desired inequality holds by Fatou's lemma. ■

Theorem A1. *The value function exists, it is the unique admissible solution to Bellman's equation, and recursive optimality implies optimality.*

Proof. Lemmas A4 and A5 show that v^∞ is a value function, and hence, the value function certainly exists. Suppose that v and v' are two admissible solutions to Bellman's equation. Then, it must be that $v = v'$ because both v and v' must be the value function by Lemma A4 and because the value function is unique by its definition. Therefore, the admissible solution to Bellman's equation is unique and equals v^∞ since v^∞ is admissible by Lemma A5. Finally, the second paragraph of the proof of Lemma A4 shows that recursive optimality implies optimality.

■

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