CIRJE-F-237

Asymptotic Expansions of the Distributions of Semi-Parametric Estimators in a Linear Simultaneous Equations System

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August 2003

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July 2003

Abstract

Asymptotic expansions are made of the distributions of a class of semi-parametric estimators including the Maximum Empirical Likelihood (MEL) method and the Generalized Method of Moments (GMM) for the coefficients of a single structural equation in the linear simultaneous equations system. The expansions in terms of the sample size, when the non-centrality parameters increase proportionally, are carried out to the order of $O(n^{-2})$. Comparisons of the distribuitions of the MEL and GMM estimators are also made.

Key Words

Asymptotic Expansions, Maximum Empirical Likelihood (MEL), Generalized Method of Moments (GMM), Linear Simultaneous Equations System

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*This is a preliminary draft and it has been prepared as a supplementary note for Kunitomo (2002), and Kunitomo and Matsushita (2003).

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1. Introduction

The study of estimating a single structural equation in econometric models has led to develop several estimation methods as the alternatives to the least squares estimation method. The classical examples in the econometric literatures are the limited information maximum likelihood (LIML) method and the instrumental variables (IV) method including the two-stage least squares (TSLS) method. See Anderson, Kunitomo, and Sawa (1982) and Anderson, Kunitomo, and Morimune (1986) for their finite sample properties, for instance. In addition to these classical methods the maximum empirical likelihood (MEL) method has been proposed and has gotten some attention recently in the statistical and econometric literatures. It is probably because the MEL method gives asymptotically efficient estimator in the semi-parametric sense and also improves the serious bias problem known in the generalized method of moments (GMM) method when the number of instruments is large in econometric models. See Owen (2001), Qin and Lawless (1994), and Kitamura, Tripathi, and Ahn (2001) on the details of the MEL method.

In the econometric literatures the generalized method of moments (GMM) estimation method has been quite popular in the past two decades. The GMM method was originally proposed by Hansen (1982) in the econometric literature and it is essentially the same as the estimating equation (EE) method proposed by Godambe (1960) which has been used in statistical applications. This approach has an attractive feature that it has rather broad applicability and it is easily implemented in econometric analyses. It has been known that both the MEL estimator and the GMM estimator are asymptotically normally distributed and efficient when the sample size is large. Because we have two semi-parametric estimation methods for econometric models and they are asymptotically equivalent, it is interesting to make comparison of the finite sample properties of alternatives estimation methods.

The main purpose of this study is to give the asymptotic expansions of the distributions of a class of semi-parametric estimators for the coefficients of a single structural equation in the linear simultaneous equations system. The estimation methods under the present study include both the MEL estimator and the GMM estimator as special cases. Since it is quite difficult to investigate the exact distributions of these estimators in the general case, their asymptotic expansions give useful information on their finite sample properties. In this paper the asymptotic expansions shall be carried out in terms of the sample size which is proportional to the non-centrality parameters. Comparisons of the distributions of the MEL and GMM methods can be made based on the results reported in this paper and they have been reported partially in Kunitomo (2002), and Kunitomo and Matsushita (2003). Our formulation of this paper is intentionally parallel to the classical studies on the single equation estimation methods in the linear simultaneous equations by Fujikoshi et. al. (1982) and Anderson et. al. (1986). It is mainly because the interpretation can be drawn in the light of past studies on the finite sample properties of estimators in the classical parametric framework.

In Section 2 we define the single equation econometric models and their estimation methods. Then in Section 3 we give the stochastic expansions of a class of estimators. In Section 4, we give the results of the asymptotic expansions of the distribution functions of estimators under a set of assumptions on the disturbances. Then we shall briefly mention to the asymptotic bias and mean squared errors of estimators in the more general case and some discussions on the use of our results in Section 5. The proofs of Lemmas and Theorems and some useful formulas for our results will be given in Appendices.

2. Estimating a Single Structural Equation by the Maximum Empirical Likelihood Method

Let a single structural equation in the econometric model be given by

$$y_{1i} = h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \theta) + u_i \quad (i = 1, \cdots, n) , \tag{2.1}$$

where $h(\cdot,\cdot,\cdot)$ is a function, y_{1i} and $\mathbf{y}_{2i}$ are 1×1 and $G_1 \times 1$ (vector of) endogenous variables, $\mathbf{z}_{1i}$ is a $K_1 \times 1$ vector of exogenous variables, θ is an $r \times 1$ vector of unknown parameters, and $\{u_i\}$ are mutually independent disturbance terms with $E(u_i) = 0$ $(i = 1, \cdots, n)$.

We assume that (2.1) is the first equation in a system of $(G_1 + 1)$ structural equations relating the vector of $G_1 + 1$ endogenous variables $\mathbf{y}_i^{'} = (y_{1i}, \mathbf{y}_{2i}^{'})$ and the vector of K $(= K_1 + K_2)$ exogenous variables $\{\mathbf{z}_i\}$ which includes $\{\mathbf{z}_{1i}\}$. The set of exogenous variables $\{\mathbf{z}_i\}$ are often called the instrumental variables and we have the orthogonal condition

$$E(u_i\, \mathbf{z}_i) = \mathbf{0} \quad (i = 1, \cdots, n) . \tag{2.2}$$

Because we do not specify the equations except (2.1) and we only have the limited information on the set of instrumental variables (or instruments), we only consider the limited information estimation methods.

When the function $h(\cdot,\cdot,\cdot)$ is of the linear form, (2.1) can be written as

$$y_{1i} = (\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})\begin{pmatrix} \beta \\ \gamma \end{pmatrix} + u_i \quad (i = 1, \cdots, n), \tag{2.3}$$

where $\theta^{'} = (\beta^{'}, \gamma^{'})$ is a $1 \times p$ $(p = K_1 + G_1)$ vector of unknown coefficients. Furthermore, when all structural equations in the econometric model are linear, the reduced form equations for $\mathbf{y}_i^{'} = (y_{1i}, \mathbf{y}_{2i}^{'})$ can be represented as

$$\mathbf{y}_i = \mathbf{\Pi}^{'}\mathbf{z}_i + \mathbf{v}_i \quad (i = 1, \cdots, n) , \tag{2.4}$$

where $\mathbf{v}_i^{'} = (v_{1i}, \mathbf{v}_{2i}^{'})$ is a $1 \times (1 + G_1)$ disturbance terms with $E[\mathbf{v}_i] = \mathbf{0}$ and

$$\mathbf{\Pi}^{'} = (\pi_1, \mathbf{\Pi}_2)^{'} \tag{2.5}$$

is a $(1 + G_1) \times K$ partitioned matrix of the linear reduced form coefficients. By multiplying $(1, -\beta^{'})$ to (2.4) from the left-hand side, we have the linear restriction

$$(1, -\beta^{'})\mathbf{\Pi}^{'} = (\gamma^{'}, \mathbf{0}^{'}) \tag{2.6}$$

and $u_i = v_{1i} - \beta^{'}\mathbf{v}_{2i}$ $(i = 1, \cdots, n)$.

The maximum empirical likelihood (MEL) estimator for the vector of unknown parameters θ in (2.1) is defined by maximizing the Lagrange form

$$L_n^*(\lambda, \theta) = \sum_{i=1}^{n} \log p_i - \mu(\sum_{i=1}^{n} p_i - 1) - n\lambda^{'} \sum_{i=1}^{n} p_i\, \mathbf{z}_i[y_{1i} - h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \theta)] , \tag{2.7}$$

where μ and λ are a scalar and a $K \times 1$ vector of Lagrange multipliers, and p_i ($i = 1, \cdots, n$) are the weighted probability functions to be chosen. It has been known (see Qin and Lawles (1994) or Owen (2001)) that the above maximization problem is the same as to maximize

$$L_n(\lambda, \theta) = -\sum_{i=1}^{n} \log\{1 + \lambda^{'}\mathbf{z}_i\ [y_{1i} - h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \theta)]\}\ , \tag{2.8}$$

where we used the conditions $\hat{\mu} = n$, and

$$[n\hat{p}_i]^{-1} = 1 + \lambda^{'}\mathbf{z}_i[y_{1i} - h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \theta)]\ . \tag{2.9}$$

By differentiating (2.8) with respect to λ and combining the resulting equation with (2.9), we have the relation

$$\sum_{i=1}^{n} \hat{p}_i\mathbf{z}_i\ [y_{1i} - h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \theta)] = \mathbf{0} \tag{2.10}$$

and

$$\hat{\lambda} = [\sum_{i=1}^{n} \hat{p}_i u_i^2(\hat{\theta})\mathbf{z}_i\mathbf{z}_i^{'}]^{-1}[\frac{1}{n}\sum_{i=1}^{n} u_i(\hat{\theta})\mathbf{z}_i]\ , \tag{2.11}$$

where $u_i(\hat{\theta}) = y_{1i} - h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \hat{\theta})$ and $\hat{\theta}$ is the maximum empirical likelihood (MEL) estimator for the vector of unknown parameters θ . From (2.8) the MEL estimator of θ is the solution of the set of p equations

$$\hat{\lambda}^{'}\sum_{i=1}^{n} \hat{p}_i\mathbf{z}_i[-\frac{\partial h(\mathbf{y}_{2i}, \mathbf{z}_{1i}, \hat{\theta})}{\partial \theta_j}] = 0\ (j = 1, \cdots, p)\ . \tag{2.12}$$

When the structural equation of (2.1) is linear, the MEL estimator of the vector of coefficient parameters $\theta^{'} = (\beta^{'}, \gamma^{'})$ can be simplified as the solution of

$$\begin{aligned} & [\sum_{i=1}^{n} \hat{p}_i \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\sum_{i=1}^{n} \hat{p}_i u_i(\hat{\theta})^2\mathbf{z}_i\mathbf{z}_i^{'}]^{-1}[\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\ y_{1i}] \\ = \ & [\sum_{i=1}^{n} \hat{p}_i \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\sum_{i=1}^{n} \hat{p}_i u_i(\hat{\theta})^2\mathbf{z}_i\mathbf{z}_i^{'}]^{-1}[\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})] \begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} . \end{aligned} \tag{2.13}$$

If we substitute $1/n$ for $\hat{p}_i$ ($i = 1, \cdots, n$) in (2.13), then we have the generalized method of moments (GMM) estimator for the vector of coefficient parameters $\theta^{'} = (\beta^{'}, \gamma^{'})$, which can be written as the solution of

$$\begin{aligned} & [\frac{1}{n}\sum_{i=1}^{n} \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\frac{1}{n}\sum_{i=1}^{n} u_i(\hat{\theta})^2\mathbf{z}_i\mathbf{z}_i^{'}]^{-1}[\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i\ y_{1i}] \\ = \ & [\frac{1}{n}\sum_{i=1}^{n} \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\frac{1}{n}\sum_{i=1}^{n} u_i(\hat{\theta})^2\mathbf{z}_i\mathbf{z}_i^{'}]^{-1}[\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})] \begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} , \end{aligned} \tag{2.14}$$

where $\hat{\theta}$ is an initial (consistent) estimator of θ . (See Hayashi (2000) on the details of the GMM method in econometrics, for instance.) By generalizing the weight probabilities p_i ($i = 1, \cdots, n$) in (2.13), we can introduce a class of estimators. Let

$$\hat{p}_i^* = \frac{1}{n[1 + \delta\ \lambda^{'}\mathbf{z}_i u_i(\hat{\theta})]}\ , \tag{2.15}$$

where δ is a positive constant ($0 \leq \delta \leq 1$) and $\hat{\theta}$ is the MEL estimator of θ . Then we define the modification of the MEL estimator (the MMEL estimator) by substituting $\hat{p}_i$ ($i = 1, \cdots, n$) into (2.9)-(2.11). We shall denote the resulting Lagrange multiplier and the modified estimator as $\hat{\lambda}$ and $\hat{\theta}$ whenever we can avoid any confusion.

In the rest of the present paper we shall consider the standardized error of estimators as

$$\hat{\mathbf{e}} = \sqrt{n}\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{pmatrix} , \tag{2.16}$$

where $\hat{\theta}^{'} = (\hat{\beta}^{'}, \hat{\gamma}^{'})$. We sometimes denote $\hat{\mathbf{e}}$ for the MEL estimator and its modification as $\hat{\mathbf{e}}_{EL}$ and $\hat{\mathbf{e}}^{*}$, respectively, in order to avoid some confusion.
Under a set of regularity conditions, the asymptotic variance-covariance matrix of the asymptotically efficient estimators in the semi-parametric framework is given by

$$\mathbf{Q}^{-1} = \mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D} , \tag{2.17}$$

where

$$\mathbf{C} = \text{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}u_i^2 \tag{2.18}$$

and

$$\mathbf{D} = [\mathbf{\Pi}_2 , \begin{pmatrix} \mathbf{I}_{K_1} \\ \mathbf{O} \end{pmatrix}] , \tag{2.19}$$

$$\mathbf{M} = \text{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'} . \tag{2.20}$$

When the disturbance terms are homoscedastic random variables such as an i.i.d. sequence, then we have $\mathbf{C} = \sigma^2\mathbf{M}$ and $E(u_i^2) = \sigma^2$. We assume that the (constant) matrix $\mathbf{M}$ is positive definite, and the rank condition

$$\mathbf{rank}(\mathbf{D}) = p \ (= G_1 + K_1) . \tag{2.21}$$

These conditions assure that the limiting variance-covariance matrix $\mathbf{Q}$ is non-degenerate. The above rank condition implies the order condition

$$L = K - p \geq 0 , \tag{2.22}$$

which is called the degree of over-identification in the econometric literatures.

In order to compare alternative efficient estimation methods in the finite sample sense, we need to derive the asymptotic expansions of the density functions of the standardized estimators (2.16) in the form of

$$f(\xi) = \phi_{\mathbf{Q}}(\xi)[\ 1 + \frac{1}{\sqrt{n}}H_1(\xi) + \frac{1}{n}H_2(\xi)\ ] + o(\frac{1}{n}) , \tag{2.23}$$

where $\xi = (\xi_1, \cdots, \xi_p)^{'}$, $\phi_{\mathbf{Q}}(\xi)$ is the multivariate normal density function with mean $\mathbf{0}$ and the variance-covariance matrix $\mathbf{Q}$, and $H_i(\xi)$ ($i = 1, 2$) are some polynomial functions of elements of ξ. In order to derive the asymptotic expansions of the distributions of estimators in a simple manner, however, we need a set of regularity conditions.

Assumption I :
(i) The sequence $\{u_i\}$ are i.i.d. random variables which have the positive density with respect to Lebesgue measure with $E[|u_i|^6] < \infty$ and $E[\|\mathbf{v}_{2i}\|^2 u_i^4] < \infty$.
(ii) The matrix $\mathbf{M}$ is positive definite and the rank condition in (2.21), where

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'} = \mathbf{M} + o_p(\frac{1}{n}) \ . \tag{2.24}$$

(iii) The sequence of $\mathbf{z}_i = (z_i^{(j)})$ $(i = 1, \cdots, n; j = 1, \cdots, K)$ are independent of u_i $(i = 1, \cdots, n)$ and there exist finite $M_3(j_1, j_2, j_3)$ and $M_4(j_1, j_2, j_3, j_4)$ such that

$$\frac{1}{n}\sum_{i=1}^{n} z_i^{(j_1)} z_i^{(j_2)} z_i^{(j_3)} = M_3(j_1, j_2, j_3) + o_p(\frac{1}{\sqrt{n}}) \ , \tag{2.25}$$

$$\frac{1}{n}\sum_{i=1}^{n} z_i^{(j_1)} z_i^{(j_2)} z_i^{(j_3)} z_i^{(j_4)} = M_4(j_1, j_2, j_3, j_4) + O_p(\frac{1}{\sqrt{n}}) \ . \tag{2.26}$$

The conditions in (ii) and (iii) of **Assumption I** are rather strong and it is possible to weaken them. Then the resulting formulas and their derivations become more complicated than those reported in this paper while the essential method of derivations will not to be changed. We can treat both cases when $\{\mathbf{z}_i\}$ are deterministic and stochastic and also it is possible to replace the independence assumption with $\{u_i\}$ by using a martingale assumption on the random vector sequence of $\sum_{i=1}^{n} \mathbf{z}_i u_i$. In order to use the inessential arguments, however, we mostly treat $\{\mathbf{z}_i\}$ as if they were deterministic variables.

We shall use the mean operator $AM_n(\hat{\mathbf{e}})$, which is defined as the mean of $\hat{\mathbf{e}}$ with respect to the asymptotic expansion of its density function of the standardized estimator up to $O(n^{-1})$ in the form of (2.23). We write the asymptotic bias and the asymptotic MSE of the standardized estimator by

$$ABIAS_n(\hat{\mathbf{e}}) = AM_n(\hat{\mathbf{e}}) \ , \tag{2.27}$$

and

$$AMSE_n(\hat{\mathbf{e}}) = AM_n(\hat{\mathbf{e}}\,\hat{\mathbf{e}}^{'}) \ . \tag{2.28}$$

These quantities are useful because the asymptotic expansion of the distribution of estimators are quite complicated in the general case. However, it should be noted that they are not necessarily the same as the asymptotic expansions of the exact moments and some care should be taken in this respect.

3. Stochastic Expansions of Estimators

3.1 Stochastic Expansions

First we apply the similar arguments used in Owen (1990) and Qin and Lawless (1994) on the probability limits and the consistency of the MEL estimator. Then we have $n\hat{p}_i \xrightarrow{p} 1$, $\hat{\theta}_{EL} \xrightarrow{p} \theta_0$, ($\theta_0$ is the true value of θ) and $\sqrt{n}\hat{\lambda}$ converges to a random vector as $n \to \infty$.

In the linear case we substitute (2.16) into (2.13) and we have the corresponding representation of the standardized estimator $\hat{\mathbf{e}}$ as

(3.1)
$$[\sum_{i=1}^{n} \hat{p}_i \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\sum_{i=1}^{n} \hat{p}_i u_i(\hat{\theta})^2 \mathbf{z}_i \mathbf{z}_i^{'}]^{-1}[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{z}_i u_i]$$
$$= [\sum_{i=1}^{n} \hat{p}_i \begin{pmatrix} \mathbf{y}_{2i} \\ \mathbf{z}_{1i} \end{pmatrix} \mathbf{z}_i^{'}][\sum_{i=1}^{n} \hat{p}_i u_i(\hat{\theta})^2 \mathbf{z}_i \mathbf{z}_i^{'}]^{-1}[\frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i (\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})]\hat{\mathbf{e}} ,$$

where we use the notation $\hat{\theta}$ for $\hat{\theta}_{EL}$ without any subscript whenever we do not have any confusion. As $n \to \infty$, we write the first order term of $\hat{\mathbf{e}}$ as $\mathbf{e}_0$, which is given by

(3.2)
$$\tilde{\mathbf{e}}_0 = [\mathbf{D}^{'}(\text{plim}\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'})(\text{plim}\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'} u_i^2(\hat{\theta}))^{-1}(\text{plim}\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'})\mathbf{D}]^{-1}$$
$$\times[\mathbf{D}^{'}(\text{plim}\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'})(\text{plim}\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'} u_i^2(\hat{\theta}))^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i) .$$

The probability limits and the random variable on the right hand side of (3.1) have been defined properly because the matrices $\mathbf{M}$ and $\mathbf{C}$ are non-singular and $\mathbf{D}$ is of full rank by our assumptions. By using the central limit theorem (CLT) to the last term, we have the weak convergence

(3.3)
$$\tilde{\mathbf{e}}_0 \xrightarrow{d} N_p(\mathbf{0}, \mathbf{Q}) ,$$

where a $p \times p$ matrix $\mathbf{Q}$ has been defined by (2.17) and $\xrightarrow{d}$ means the convergence of distribution as $n \to \infty$. Also as $n \to \infty$ we notice that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i(\hat{\theta}) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i + \frac{1}{n}[-\sum_{i=1}^{n} \mathbf{z}_i (\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})\hat{\mathbf{e}}]$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i - (\frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'})\mathbf{D}\hat{\mathbf{e}} + O_p(\frac{1}{\sqrt{n}}) .$$

Then by utilizing the representation of (2.11) for $\hat{\lambda}$, we have

(3.4)
$$\sqrt{n}\lambda - \lambda_0 \xrightarrow{p} 0 ,$$

where

$$\lambda_0 = \mathbf{C}_n^{-1/2}[\mathbf{I}_K - \mathbf{C}_n^{-1/2}\mathbf{M}_n\mathbf{D}(\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{M}_n\mathbf{D})^{-1}\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1/2}][\mathbf{C}_n^{-1/2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i] ,$$

and we denote $K \times K$ matrices $\mathbf{M}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i \mathbf{z}_i^{'}$ and

(3.5)
$$\mathbf{C}_n = \frac{1}{n}\sum_{i=1}^{n} u_i^2 \mathbf{z}_i \mathbf{z}_i^{'} .$$

The random vector

(3.6)
$$\mathbf{X}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{z}_i u_i$$

converges to $N_K(\mathbf{0},\mathbf{C})$ by CLT, and $\mathbf{M}_n \xrightarrow{p} \mathbf{M}$, $\mathbf{C}_n \xrightarrow{p} \mathbf{C}$ as $n \longrightarrow +\infty$, where $\mathbf{C}$ is given by $\mathbf{C} = \mathbf{lim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} E[u_i^2\mathbf{z}_i\mathbf{z}_i^{'}]$, which is the same as (2.18). We note that the limiting distribution of $\mathbf{B}_n = \mathbf{C}^{-1/2}\mathbf{X}_n$ is given by $N_K(\mathbf{0},\mathbf{I}_K)$. Hence we have the convergence

$$\mathbf{C}_n^{1/2}\sqrt{n}\hat{\lambda} \xrightarrow{d} N_K(\mathbf{0},\bar{\mathbf{P}}_E) \ , \tag{3.7}$$

and the projection operator is defined by

$$\bar{\mathbf{P}}_E = \mathbf{I}_K - \mathbf{E}(\mathbf{E}^{'}\mathbf{E})^{-1}\mathbf{E}^{'} \ ,$$

which is constructed by a $K \times p$ matrix $\mathbf{E} = \mathbf{C}^{-1/2}\mathbf{MD}$. Its sample analogue $\mathbf{E}_n = \mathbf{C}_n^{-1/2}\mathbf{M}_n\mathbf{D} \xrightarrow{p} \mathbf{E}$ as $n \longrightarrow +\infty$.

The method we shall use to derive the asymptotic expansion of the density function of the standardized estimator $\hat{\mathbf{e}}$ is similar to the one used in Fujikoshi et. al. (1982) and Anderson et. al. (1986). We shall expand $\hat{\mathbf{e}}$ by the perturbation method in terms of the random variables $\mathbf{X}_n = (X_n^{(j)})$, $\mathbf{Y}_n = (Y_n^{(jk)})$ in (3.21) and

$$\mathbf{U}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} \ , \tag{3.8}$$

where we define $p \times 1$ random vectors

$$\mathbf{w}_i = (\mathbf{v}_{2i}^{'},\mathbf{0}^{'})^{'} - \mathbf{q}u_i \tag{3.9}$$

and

$$\mathbf{q} = \frac{1}{\sigma^2}E[\begin{pmatrix}\mathbf{v}_{2i}\\ \mathbf{0}\end{pmatrix}u_i] \ .$$

Then if $E[|u_i|^s] < \infty$ for $s \geq 3$, we can take a positive (bounded) constant $c_n(1,n)$ depending on n which satisfies

$$\mathbf{P}(\|\mathbf{X}_n\| > [(s-1)\Lambda_n(1)\log n]^{1/2}) \leq c_n(1,s)\frac{(1/\sqrt{n})^{s-2}}{(\log n)^{s/2}} \ , \tag{3.10}$$

where $\Lambda_n(1)$ as the maximum of the characteristic roots of $\mathbf{M}_n$. Also for the random variables $\mathbf{Y}_n$ and $\mathbf{U}_n$ we can take positive constants $c_n(i,s)$ $(i = 2,3)$ and similar inequalities for $s \geq 3$ under **Assumption I**. These arguments on the validity of the asymptotic expansions of random variables have been given by Bhattacharya and Ghosh (1978) for the i.i.d. random variables sequence. They can be easily extended to the present situation while their derivations and resulting explanations become quite lengthy, however, we omit their details. The validity of the asymptotic expansions based on the inversion of the characteristic functions, which will be utilized in Section 4 of this paper, was also briefly discussed by Fujikoshi et. al. (1982). In the econometric literatures the asymptotic expansion method has been previously discussed by Sargan and Mikhail (1971), and Phillips (1983), for instance.

By expanding (2.13) and (2.16) with respect to $\mathbf{e}_0$, formally we can write

$$\hat{\mathbf{e}} = \tilde{\mathbf{e}}_0 + [\mathbf{e}_0 - \tilde{\mathbf{e}}_0] + \frac{1}{\sqrt{n}}\mathbf{e}_1 + \frac{1}{n}\mathbf{e}_2 + o_p(\frac{1}{n}) \ , \tag{3.11}$$

and

$$\sqrt{n}\hat{\lambda} = \lambda_0 + \frac{1}{\sqrt{n}}\lambda_1 + \frac{1}{n}\lambda_2 + o_p(\frac{1}{n}) \ , \tag{3.12}$$

where we denote

$$\mathbf{e}_0 = [\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{M}_n\mathbf{D}]^{-1}[\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i u_i] \ . \tag{3.13}$$

By substituting these expansions into (2.9), we can also expand the estimated probability function as

$$n\ \hat{p}_i = 1 + \frac{1}{\sqrt{n}}p_i^{(1)} + \frac{1}{n}p_i^{(2)} + o_p(\frac{1}{n}) \ , \tag{3.14}$$

where $p_i^{(1)} = -\lambda_0^{'}\mathbf{z}_i u_i$ and

$$\begin{aligned} p_i^{(2)} &= -\lambda_1^{'}\mathbf{z}_i[u_i - \frac{1}{\sqrt{n}}(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})\mathbf{e}_0] \\ &+ \lambda_0^{'}\mathbf{z}_i[\frac{1}{\sqrt{n}}(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})\mathbf{e}_1 + \mathbf{z}_i^{'}\mathbf{D}\mathbf{e}_0 + (\mathbf{v}_{2i}^{'}, \mathbf{0}^{'})\mathbf{e}_0] + (\lambda_0^{'}\mathbf{z}_i)^2[u_i - \frac{1}{\sqrt{n}}(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'})\mathbf{e}_0]^2 \ . \end{aligned}$$

By using the representation

$$(\mathbf{y}_{2i}^{'}, \mathbf{z}_{1i}^{'}) = \mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'} + \mathbf{q}^{'}u_i \ , \tag{3.15}$$

we shall expand

$$\begin{aligned} \hat{\mathbf{C}}_n &= \sum_{i=1}^{n}\hat{p}_i u_i^2(\hat{\theta})\mathbf{z}_i\mathbf{z}_i^{'} \\ &= \mathbf{C}_n^{(0)} + \frac{1}{\sqrt{n}}\hat{\mathbf{C}}_n^{(1)} + \frac{1}{n}\hat{\mathbf{C}}_n^{(2)} + o_p(\frac{1}{n}) \end{aligned} \tag{3.16}$$

and

$$\sum_{i=1}^{n}\hat{p}_i\begin{pmatrix}\mathbf{y}_{2i} \\ \mathbf{z}_{1i}\end{pmatrix}\mathbf{z}_i^{'} = \mathbf{E}_n^{(0)} + \frac{1}{\sqrt{n}}\mathbf{E}_n^{(1)} + \frac{1}{n}\mathbf{E}_n^{(2)} + o_p(\frac{1}{n}) \ , \tag{3.17}$$

where we define the matrices as $\mathbf{E}_n^{(0)} = \mathbf{D}^{'}\mathbf{M}_n$, $\mathbf{C}_n^{(0)} = \mathbf{C}_n$ and

$$\begin{aligned} \hat{\mathbf{C}}_n^{(j)} &= \frac{1}{n}\sum_{i=1}^{n}p_i^{(j)}\mathbf{z}_i\mathbf{z}_i^{'}u_i^2(\hat{\theta}) \ \ (j = 1, 2) \ , \\ \mathbf{E}_n^{(1)} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}\mathbf{v}_{2i} \\ \mathbf{0}\end{pmatrix}\mathbf{z}_i^{'} + \mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^{n}p_i^{(1)}\mathbf{z}_i\mathbf{z}_i^{'} + \frac{1}{n}\sum_{i=1}^{n}p_i^{(1)}\begin{pmatrix}\mathbf{v}_{2i} \\ \mathbf{0}\end{pmatrix}\mathbf{z}_i^{'} \ , \\ \mathbf{E}_n^{(2)} &= \mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\mathbf{z}_i\mathbf{z}_i^{'} + \frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\begin{pmatrix}\mathbf{v}_{2i} \\ \mathbf{0}\end{pmatrix}\mathbf{z}_i^{'} \ . \end{aligned}$$

By using (3.12)-(3.14), it is convenient to rewrite (3.16) as

$$\hat{\mathbf{C}}_n = \mathbf{C}_n + \frac{1}{\sqrt{n}}\mathbf{C}_n^{(1)} + \frac{1}{n}\mathbf{C}_n^{(2)} + o_p(\frac{1}{n}) \ , \tag{3.18}$$

where we have

$$\mathbf{C}_n^{(1)} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}[p_i^{(1)}u_i^2 - 2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0] \ ,$$

$$\mathbf{C}_n^{(2)} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}[\{(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0\}^2 - 2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_1 - 2u_ip_i^{(1)}(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0 + u_i^2p_i^{(2)}].$$

By substituting the above expressions into (3.1) for $\hat{\mathbf{e}}$, $\hat{\lambda}$, and $\hat{p}_i$ $(i = 1, \cdots, n)$, we can determine each terms of the stochastic expansions of $\hat{\mathbf{e}}$ in a recursive way. The leading two terms are given by

(3.19)
$$\mathbf{e}_1 = -\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0})]\mathbf{e}_0 + \mathbf{Q}_n[\mathbf{A}_1][\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_iu_i - \mathbf{M}\mathbf{D}^{'}\mathbf{e}_0] \ ,$$

(3.20)
$$\mathbf{e}_2 = \mathbf{Q}_n[\mathbf{A}_2][\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_iu_i - \mathbf{M}_n\mathbf{D}^{'}\mathbf{e}_0] - \mathbf{Q}_n[\mathbf{A}_1][\mathbf{M}_n\mathbf{D}\mathbf{e}_1 + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0}^{'})\mathbf{e}_0] - \mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0}^{'})\mathbf{e}_1 \ ,$$

where we use the corresponding notations

$$\mathbf{Q}_n^{-1} = \mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{M}_n\mathbf{D} \ ,$$
$$\mathbf{A}_1 = -\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1} + \mathbf{E}_n^{(1)}\mathbf{C}_n^{-1} \ ,$$
$$\mathbf{A}_2 = \mathbf{D}^{'}\mathbf{M}_n[-\mathbf{C}_n^{-1}\mathbf{C}_n^{(2)}\mathbf{C}_n^{-1} + \mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1}] - \mathbf{E}_n^{(1)}\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1} + \mathbf{E}_n^{(2)}\mathbf{C}_n^{-1}.$$

3.2 Effects of $\mathbf{C}_n$

We investigate the effects of estimating the variance-covariance matrix $\mathbf{C}$ by $\hat{\mathbf{C}}_n$. For this purpose we define a $K \times K$ random matrix

(3.21)
$$\mathbf{Y}_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(u_i^2 - \sigma^2) \ .$$

Under a set of regularity conditions, each components of $\mathbf{Y}_n$ have the asymptotic normality as $n \to +\infty$. We notice that the covariance of the (j,k)-th elements of $\mathbf{Y}_n$ and the l-th element of $\mathbf{X}_n$ is given by

(3.22)
$$Cov(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i^{(j)}z_i^{(k)}(u_i^2 - \sigma^2),\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}z_i^{(l)}u_i) = \mu_3\frac{1}{n}\sum_{i=1}^{n}z_i^{(j)}z_i^{(k)}z_i^{(l)} \ ,$$

where $\mu_3 = E[u_i^3]$.
From this relation we notice that $\mathbf{X}_n$ and $\mathbf{Y}_n$ are asymptotically independent when $\mu_3 = 0$ and then our analyses can be simplified considerably. It is the case when the disturbances are normally distributed, for instance.

We shall use the assumption of $\mathbf{M}_n = \mathbf{M} + o_p(n^{-1})$ ((2.24) in **Assumption I**) for the simplification. It is straightforward to relax this condition, but the resulting expressions become some complications. In our situation we expand the inverse of the variance-covariance matrix of estimators as

$$\begin{aligned}
\mathbf{Q}_n^{-1} &= \mathbf{D}^{'}\mathbf{M}_n[\mathbf{C}^{-1} + \mathbf{C}_n^{-1}(\mathbf{C} - \mathbf{C}_n)\mathbf{C}]\mathbf{M}_n\mathbf{D} \\
&= \mathbf{D}^{'}\mathbf{M}_n\{\mathbf{C}^{-1} - [\mathbf{C}^{-1} + \mathbf{C}_n^{-1}(\mathbf{C} - \mathbf{C}_n)\mathbf{C}^{-1}](\mathbf{C}_n - \mathbf{C})\mathbf{C}^{-1}\}\mathbf{M}_n\mathbf{D} \\
&= \mathbf{Q}^{-1} - \frac{1}{\sqrt{n}}[\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}] \\
&\quad + \frac{1}{n}[\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}] + O_p(n^{-3/2}) \ ,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1} &= \mathbf{D}^{'}\mathbf{M}_n[\mathbf{C}^{-1} + \mathbf{C}_n^{-1}(\mathbf{C} - \mathbf{C}_n)\mathbf{C}^{-1}] \\
&= \mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} - \frac{1}{\sqrt{n}}[\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}] \\
&\quad + \frac{1}{n}[\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}] + O_p(n^{-3/2}),
\end{aligned}$$

where we define $\mathbf{Q} = \mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}$. Then by using the inversion of the matrix $\mathbf{Q}_n$, we can represent

$$\begin{aligned}
\mathbf{Q}_n &= \mathbf{Q} + \mathbf{Q}_n(\mathbf{Q}^{-1} - \mathbf{Q}_n^{-1})\mathbf{Q} \\
&= \mathbf{Q} - \mathbf{Q}(\mathbf{Q}_n^{-1} - \mathbf{Q}^{-1})\mathbf{Q} + \mathbf{Q}(\mathbf{Q}_n^{-1} - \mathbf{Q}^{-1})\mathbf{Q}(\mathbf{Q}_n^{-1} - \mathbf{Q}^{-1})\mathbf{Q} \\
&= \mathbf{Q} + \frac{1}{\sqrt{n}}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}] \\
&\quad + \frac{1}{n}[-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q} \\
&\quad + \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}] + O_p(n^{-3/2}),
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1} \\
= & \ \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} + \frac{1}{\sqrt{n}}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}] \\
& + \frac{1}{n}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1} - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1} \\
& \quad - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} \\
& \quad + \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}] \\
& + O_p(n^{-3/2}) \\
= & \ \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} + \frac{1}{\sqrt{n}}[-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}] + \frac{1}{n}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{Y}_n\mathbf{A}] + O_p(n^{-3/2}) \ ,
\end{aligned}$$

where we define a $K \times K$ matrix

$$\mathbf{A} = \mathbf{C}^{-1/2}\bar{\mathbf{P}}_E\mathbf{C}^{-1/2} = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} \ . \tag{3.23}$$

By using the above relations, the first term of the stochastic expansion of $\hat{\mathbf{e}}$ can be represented as

$$
\begin{aligned}
(3.24) \qquad \mathbf{e}_0 &= \tilde{\mathbf{e}}_0 + \frac{1}{\sqrt{n}}[-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] \\
&\quad + \frac{1}{n}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] + O_p(n^{-3/2}) \\
&= \tilde{\mathbf{e}}_0 + \frac{1}{\sqrt{n}}\mathbf{e}_0^{(1)} + \frac{1}{n}\mathbf{e}_0^{(2)} + O_p(n^{-3/2}) ,
\end{aligned}
$$

where $\tilde{\mathbf{e}}_0 = \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{X}_n$ and we define $\mathbf{e}_0^{(1)}$ and $\mathbf{e}_0^{(2)}$ by the right-hand side of the equation.

Similarly, we can derive a representation of λ_0, which is defined by

$$
(3.25) \qquad \lambda_0 = [\mathbf{C}_n^{-1} - \mathbf{C}_n^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}\mathbf{C}_n^{-1}]\mathbf{X}_n .
$$

By using the stochastic expansions of $\mathbf{C}_n$ and $\mathbf{Q}_n$ we have derived in this subsection, it is straightforward to obtain an important expression

$$
(3.26) \qquad \lambda_0 = \mathbf{A}\mathbf{X}_n + \frac{1}{\sqrt{n}}[-\mathbf{A}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] + O_p(n^{-1}) .
$$

This formula helps simplifying our derivations considerably.

3.3 Terms involving $\mathbf{e}_1$

We shall investigate each terms involving $\mathbf{e}_1$. For this purpose we decompose $\mathbf{e}_1$ as $\mathbf{e}_1 = \mathbf{e}_{1.1} + \mathbf{e}_{1.2} + \mathbf{e}_{1.3}$, where

$$
\begin{aligned}
(3.27) \qquad \mathbf{e}_{1.1} &= \mathbf{Q}_n[\mathbf{A}_1][\mathbf{X}_n - \mathbf{M}\mathbf{D}^{'}\mathbf{e}_0] , \\
(3.28) \qquad \mathbf{e}_{1.2} &= -\mathbf{e}_0(\mathbf{q}^{'}\mathbf{e}_0) , \\
(3.29) \qquad \mathbf{e}_{1.3} &= -\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}\mathbf{C}_n^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0 .
\end{aligned}
$$

The last two terms can be investigated rather easily and we treat these terms first. By using the representations in Section 3.2, we can rewrite

$$
\begin{aligned}
&\mathbf{e}_{1.2} \\
= \; & -[\tilde{\mathbf{e}}_0 - \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n]\mathbf{q}^{'}[\tilde{\mathbf{e}}_0 - \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] \\
= \; & -\tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + \frac{1}{\sqrt{n}}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + \tilde{\mathbf{e}}_0\mathbf{q}^{'}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] \\
& + O_p(n^{-1}) \\
= \; & \mathbf{e}_{1.2}^{(0)} + \frac{1}{\sqrt{n}}\mathbf{e}_{1.2}^{(1)} + O_p(n^{-1}) ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{e}_{1.3} \\
= \; & -[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} - \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}](\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})
\end{aligned}
$$

$$\times[\tilde{\mathbf{e}}_0 - \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] + O_p(n^{-1})$$

$$= -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0$$

$$+\frac{1}{\sqrt{n}}[\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n$$

$$+\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0] + O_p(n^{-1})$$

$$= \mathbf{e}_{1.3}^{(0)} + \frac{1}{\sqrt{n}}\mathbf{e}_{1.3}^{(1)} + O_p(n^{-1}) \ .$$

Here we have defined $\mathbf{e}_{1.2}^{(1)}$ and $\mathbf{e}_{1.3}^{(1)}$ implicitly. The analysis of $\mathbf{e}_{1.1}$ becomes substantially more complicated because there are many terms involved in the random matrices $\mathbf{C}_n^{(1)}$ and $\mathbf{E}_n^{(1)}$. We rewrite

(3.30) $\mathbf{C}_n^{(1)}$

$$= \frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\{-2u_i(\mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'} + \mathbf{q}^{\prime}u_i)[\tilde{\mathbf{e}}_0 - \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] + u_i^3(-\mathbf{z}_i^{'}\lambda_0)\}$$

$$= \Big\{-2(\mathbf{q}^{\prime}\tilde{\mathbf{e}}_0)\mathbf{C} - \mu_3\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})\Big\}$$

$$+\frac{1}{\sqrt{n}}\Big\{-2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{Y}_n - 2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{\prime}u_i(\mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'})\tilde{\mathbf{e}}_0$$

$$+2\mathbf{C}_n\mathbf{q}^{'}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})(u_i^3 - \mu_3)\Big\}$$

$$+O_p(n^{-1}) \ ,$$

where $\mu_3 = E[u_i^3]$. Then we have

$$-\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1}$$

$$= \Big\{2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} + \mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)\mathbf{C}^{-1}\Big\}$$

$$+\frac{1}{\sqrt{n}}\Big\{-2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}$$

$$-\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})\mathbf{C}\mathbf{Y}_n\mathbf{C}^{-1} - \mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})\mathbf{C}^{-1}$$

$$-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[-\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)(u_i^3 - \mu_3) - 2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{\prime}u_i(\mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'})\tilde{\mathbf{e}}_0$$

$$+2\mathbf{C}\mathbf{q}^{'}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n]\mathbf{C}^{-1}\Big\}$$

$$+O_p(n^{-1}) \ .$$

On the other hand, we represent

(3.31) $\mathbf{E}_n^{(1)}$

$$= \frac{1}{\sqrt{n}}(\mathbf{w}_i + \mathbf{q}u_i)\mathbf{z}_i^{'} + \mathbf{D'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'} - (\mathbf{z}_i^{'}\lambda_0)u_i$$

$$+\frac{1}{n}\sum_{i=1}^{n}(\mathbf{w}_i + \mathbf{q}u_i)\mathbf{z}_i^{'} - (\mathbf{z}_i^{'}\lambda_0)u_i$$

$$= \Big\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)\Big\}$$

$$+\frac{1}{\sqrt{n}}\Big\{-\mathbf{D'}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)u_i - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)u_i\Big\}$$

$$+O_p(n^{-1}) \,.$$

Then we have

$$\mathbf{Q}_n\mathbf{E}_n^{(1)}\mathbf{C}_n^{-1}$$

$$= [\mathbf{Q} + \frac{1}{\sqrt{n}}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}]\mathbf{E}_n^{(1)}[\mathbf{C}^{-1} - \frac{1}{\sqrt{n}}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}] + O_p(n^{-1})$$

$$= \Big\{\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{C}^{-1} + \mathbf{Q}\mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)\mathbf{C}^{-1}\Big\}$$

$$+ \frac{1}{\sqrt{n}}\Big\{\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1}$$

$$-\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}$$

$$+\mathbf{Q}[-\mathbf{D'}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)u_i - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)u_i]\mathbf{C}^{-1}\Big\}$$

$$+ O_p(n^{-1}) \,.$$

By using the relation $\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} = \mathbf{C}^{-1} - \mathbf{A}$ and

$$\mathbf{X}_n - \mathbf{M}_n\mathbf{D}\mathbf{e}_0 = \mathbf{C}\mathbf{A}\mathbf{X}_n + \frac{1}{\sqrt{n}}[\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n] + O_p(n^{-1}), \tag{3.32}$$

we rewrite

$$\mathbf{e}_{1.1}$$

$$= [-\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1} + \mathbf{Q}_n\mathbf{E}_n^{(1)}\mathbf{C}_n^{-1}][\mathbf{X}_n - \mathbf{M}_n\mathbf{D}\mathbf{e}_0]$$

$$= \Big\{[2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1} + \mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)\mathbf{C}^{-1}$$

$$+\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{C}^{-1} + \mathbf{Q}\mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)\mathbf{C}^{-1}][\mathbf{X}_n - \mathbf{M}_n\mathbf{D}\mathbf{e}_0]\Big\}$$

$$+ \frac{1}{\sqrt{n}}\Big\{\Big(-2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}$$

$$-\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_\mathbf{0})\mathbf{C}_n\mathbf{Y}_n\mathbf{C}^{-1} - \mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_\mathbf{0})\mathbf{C}^{-1}$$

$$-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[-2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{\prime}u_i(\mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{\prime}(\mathbf{z}_i^{'}\lambda_0)(u_i^3 - \mu_3)$$

$$
\begin{aligned}
&+2\mathbf{C}\mathbf{q}^{'}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n]\mathbf{C}^{-1} \\
&+\mathbf{Q}[-\mathbf{D}'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)u_i - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}(\lambda_0^{'}\mathbf{z}_i)u_i]\mathbf{C}^{-1} \\
&+\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1} \\
&-\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\Big)[\mathbf{X}_n - \mathbf{M}_n\mathbf{D}\tilde{\mathbf{e}}_0]\Big\} \ + \ O_p(n^{-1}).
\end{aligned}
$$

For the class of the modified estimators, we take an arbitrary δ $(0 \le \delta \le 1)$ and substitute $\delta\lambda_0$ (and $\delta\lambda_1$) into λ_0 (or λ_1). Then we can further derive the expression as

$$
\begin{aligned}
&\mathbf{e}_{1.1} \\
= \ &\Big\{\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}(\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n) + \delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2\Big\} \\
+ \ &\frac{1}{\sqrt{n}}\Big\{\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)(\mathbf{C}^{-1} - \mathbf{A})\mathbf{Y}_n\mathbf{A}\mathbf{X}_n \\
&\quad +\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})(\mathbf{C}^{-1} - \mathbf{A})\mathbf{Y}_n\mathbf{A}\mathbf{X}_n + \mathbf{Q}\mathbf{q}\tilde{\mathbf{e}}_0^{'}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n \\
+ \ &\Big(-\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})\mathbf{C}_n\mathbf{Y}_n\mathbf{A}\mathbf{X}_n - \delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_{\mathbf{0}})\mathbf{A}\mathbf{X}_n \\
&\quad -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[-2\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n u_i(\mathbf{z}_i^{'}\mathbf{D} + \mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 - \delta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2(u_i^3 - \mu_3)] \\
&\quad +\mathbf{Q}[-\delta\mathbf{D}'\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2 u_i - \delta\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2 u_i] \\
&\quad -\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \delta\lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{A}\mathbf{X}_n \\
&\quad +\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'} + \mathbf{q}(\mathbf{X}_n^{'} - \delta\lambda_0^{'}\mathbf{C}_n)]\mathbf{A}\mathbf{X}_n\Big)\Big\} \\
+ \ &O_p(n^{-1}) \\
= \ &\mathbf{e}_{1.1}^{(0)} + \frac{1}{\sqrt{n}}\mathbf{e}_{1.1}^{(1)} \ + \ O_p(n^{-1}) \ ,
\end{aligned}
$$

where we define $\mathbf{e}_{1.1}^{(0)}$ and $\mathbf{e}_{1.1}^{(0)}$ by the last equation. By collecting each terms of $\mathbf{e}_1$ and we summarize them as

$$
\mathbf{e}_1 = \mathbf{e}_1^{(0)} + \frac{1}{\sqrt{n}}\mathbf{e}_1^{(1)} + O_p(n^{-1}), \tag{3.33}
$$

where $\mathbf{e}_1^{(0)} = \mathbf{e}_{1.1}^{(0)} + \mathbf{e}_{1.2}^{(0)} + \mathbf{e}_{1.3}^{(0)}$ and $\mathbf{e}_1^{(1)} = \mathbf{e}_{1.1}^{(1)} + \mathbf{e}_{1.2}^{(1)} + \mathbf{e}_{1.3}^{(1)}$. Then the leading terms of $\mathbf{e}_1$ can be represented as

$$
\begin{aligned}
&\mathbf{e}_1^{(0)} \\
= \ &\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n + \delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2
\end{aligned}
$$

$$-\tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 \ .$$

The random vector $\tilde{\mathbf{e_0}}$ is asymptotically normal and it is asymptotically uncorrelated with the random vector $\mathbf{A}\mathbf{X}_n$ by using CLT. Then by using Lemma 4.3, given $\tilde{\mathbf{e}}_0 = \mathbf{x}$ the conditional expectation of $\mathbf{e}_1^{(0)}$ is given by [1]

$$E[\mathbf{e}_1^{(0)}|\tilde{\mathbf{e}}_0 = \mathbf{x}] = (1-\delta)L\mathbf{Q}\mathbf{q} - \mathbf{x}\mathbf{x}^{'}\mathbf{q} + \delta\kappa_3\mathbf{Q}\mathbf{D}^{'}\mathbf{m}_3 \ + \ O_p(n^{-1/2}) \ , \tag{3.34}$$

where we denote $\kappa_3 = E(u_i^3)/\sigma^2$ and

$$\mathbf{m}_3 = \text{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i) \ . \tag{3.35}$$

Also the conditional second order moments of $\mathbf{e}_1^{(0)}$ can be calculated as

$$\begin{aligned}
& E[\mathbf{e}_1^{(0)}\mathbf{e}_1^{(0)'}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
= \ & (\delta\kappa_3)^2 E\{[\mathbf{Q}\mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2\mathbf{D}\mathbf{Q}|\mathbf{x}\} \\
+ \ & (\delta\kappa_3)E\{[\mathbf{Q}\mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2] \\
\times \ & [\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n - \mathbf{e}_0(\mathbf{q}^{'}\mathbf{e}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0]^{'}|\mathbf{x}\} \\
+ \ & (\delta\kappa_3)E\{[\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n - \mathbf{e}_0(\mathbf{q}^{'}\mathbf{e}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0] \\
\times \ & [\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2\mathbf{D}\mathbf{Q}]|\mathbf{x}\} \\
+ \ & E\{[\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n - \mathbf{e}_0(\mathbf{q}^{'}\mathbf{e}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0] \\
\times \ & [\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n - \mathbf{e}_0(\mathbf{q}^{'}\mathbf{e}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0]|\mathbf{x}\} \\
= \ & (\delta\kappa_3)^2\Big\{\mathbf{Q}\mathbf{D}^{'}\mathbf{m}_3\mathbf{m}_3^{'}\mathbf{D}\mathbf{Q} + 2\mathbf{Q}\mathbf{D}^{'}(\frac{1}{n})^2\sum_{i,j}\mathbf{z}_i\mathbf{z}_j^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_j)^2\mathbf{D}\mathbf{Q}\Big\} \\
+ \ & (\delta\kappa_3)\{\mathbf{Q}\mathbf{D}^{'}\mathbf{m}_3[(1-\delta)(L+2)\mathbf{Q}\mathbf{q} - \mathbf{x}\mathbf{x}^{'}\mathbf{q}]^{'} + [(1-\delta)(L+2)\mathbf{Q}\mathbf{q} - \mathbf{x}\mathbf{x}^{'}\mathbf{q}]\mathbf{m}_3^{'}\mathbf{D}\mathbf{Q}\Big\} \\
+ \ & \Big\{(\mathbf{x}^{'}\mathbf{C}_1^{*}\mathbf{x}^{'}\mathbf{x}\mathbf{x}^{'} + \mathbf{Q}\mathbf{Q}^{*}\mathbf{Q}\mathbf{x}^{'}\mathbf{C}_2^{*}\mathbf{x} + \mathbf{Q}^{'}\mathbf{C}_2^{*}\mathbf{Q}\mathbf{tr}(\mathbf{A}\mathbf{M}) \\
& \qquad +(1-\delta)^2L(L+2)\mathbf{Q}\mathbf{C}_1^{*}\mathbf{Q} - (1-\delta)L[\mathbf{Q}\mathbf{C}_1^{*}\mathbf{x}\mathbf{x}^{'} + \mathbf{x}\mathbf{x}^{'}\mathbf{C}_1^{*}\mathbf{Q}]\Big\} \\
+ \ & O_p(n^{-1/2}),
\end{aligned}$$

where we use the notations $\mathbf{C}_1^{*} = \mathbf{q}\mathbf{q}^{'}$, $\mathbf{C}_2^{*} = E[\mathbf{w}_i\mathbf{w}_i^{'}]$ and $\mathbf{Q}^{*} = \mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}$. In the above calculations we have used some relations on moments by applying Lemma

[1] We need to evaluate the terms of $O_p(n^{-1})$ in order to derive the asymptotic expansions of distributions which shall be done in Section 4. Two terms in $\tilde{\mathbf{e}}_{1.1}^{(1)}$ have some important roles to the final results.

4.2 as follows.

$$E[(\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n)^2] = L(L+2)+O_p(n^{-1/2}),$$
$$E[(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2(\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n)] = (L+2)\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i+O_p(n^{-1/2}).$$

Because it is straightforward to derive these relations by using the fact that the limiting random vectors $\mathbf{A}\mathbf{X}_n$ are normal, we omit the details.

3.4 Terms involving $\mathbf{e}_2$

We shall investigate the terms associated with $\mathbf{e}_2$. For this purpose we decompose $\mathbf{e}_2 = \mathbf{e}_{2.1} + \mathbf{e}_{2.2} + \mathbf{e}_{2.3}$ and $\mathbf{e}_{2.i}$ $(i = 1, 2, 3)$ which correspond to each terms of (3.11). Because we can estimate $\mathbf{Q}$ and $\mathbf{C}$ consistently by using $\mathbf{Q}_n$ and $\mathbf{C}_n$, their estimates do not affect the terms much involving $\mathbf{e}_2$. We first consider $\mathbf{e}_{2.3}$ and by using the stochactic expansion of $\mathbf{e}_1$, we have

$$(3.36)\ \mathbf{e}_{2.3} = -\mathbf{Q}_n\mathbf{D}^{'}\mathbf{M}_n\mathbf{C}_n^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0})^{'})\mathbf{e}_1$$
$$= -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{z}_i\mathbf{w}_i^{'})+(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{z}_iu_i)\mathbf{q}^{'}]\mathbf{e}_1^{(0)}+O_p(n^{-1/2}).$$

Then the first term of $\mathbf{e}_{2.3}$ can be expressed as

$$\begin{aligned}&\mathbf{e}_{2.3.1}\\ =\ & -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{z}_i\mathbf{w}_i^{'})\\ &\times[\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n+(1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n+\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^n\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2\\ &\quad -\tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0]\ .\end{aligned}$$

Because $\tilde{\mathbf{e}}_0$ and $\mathbf{A}\mathbf{X}_n$ are asymptotically orthogonal, the conditional expectation given $\tilde{\mathbf{e}}_0 = \mathbf{x}$ can be calculated as

$$-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{n}\sum_{i=1}^n\mathbf{z}_i\mathbf{z}_i^{'})\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}E[\mathbf{w}_i\mathbf{w}_i^{'}]\mathbf{x}+O_p(n^{-1/2})$$
$$= -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{C}_2^{*}\mathbf{x}+O_p(n^{-1/2}).$$

The second term of $\mathbf{e}_{2.3}$ can be also expressed as

$$\begin{aligned}&\mathbf{e}_{2.3.2}\\ =\ & -(\tilde{\mathbf{e}}_0\mathbf{q}^{'})[\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n+(1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n\\ &+\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^n\mathbf{z}_i(\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n)^2-\tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)-\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbf{w}_i\mathbf{z}_i^{'})\tilde{\mathbf{e}}_0].\end{aligned}$$

Hence its conditional expectation given $\tilde{\mathbf{e}}_0 = \mathbf{x}$ can be expressed as

$$-(1-\delta)L\mathbf{q}^{'}\mathbf{Q}\mathbf{q}\mathbf{x}+\mathbf{x}(\mathbf{q}^{'}\mathbf{x})^2-\delta\kappa_3\mathbf{x}\mathbf{q}^{'}\mathbf{Q}\mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^n\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)+O_p(n^{-1/2}).$$

By combining these terms we have

$$(3.37)\qquad E[\mathbf{e}_{2.3}|\tilde{\mathbf{e}}_0=\mathbf{x}] = \mathbf{QQ}^*\mathbf{QC}_2^*\mathbf{x}+\mathbf{xx}^{'}\mathbf{C}_1^*\mathbf{x}-(1-\delta)L\mathbf{tr}(\mathbf{C}_1^*\mathbf{Q})\mathbf{x} -\delta\kappa_3\mathbf{x}(\mathbf{q}^{'}\mathbf{QD}^{'}\mathbf{m}_3)+O_p(n^{1/2}) ,$$

where we define $\mathbf{Q}^*=\mathbf{D}^{'}\mathbf{MC}^{-1}\mathbf{MC}^{-1}\mathbf{MD}$.

Secondly, we shall evaluate the terms involving $\mathbf{e}_{2.2}$. For this purpose we notice that we only need to investigate two terms as

$$(3.38)\qquad \mathbf{e}_{2.2.1}=-\mathbf{Q}[\mathbf{A}_1]\mathbf{MDe}_1^{(0)} ,$$

and

$$(3.39)\qquad \mathbf{e}_{2.2.2}=-\mathbf{Q}[\mathbf{A}_1]\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{w}_i^{'}+u_i\mathbf{q}^{'})\tilde{\mathbf{e}}_0$$

because it is straightforward to show that the rest of terms $\mathbf{e}_{2.2}-\mathbf{e}_{2.2.1}-\mathbf{e}_{2.2.2}=O_p(n^{-1/2})$. The second term can be further rewritten as

$$\begin{aligned}\mathbf{e}_{2.2.2} &= -\{2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{QD}^{'}\mathbf{MC}^{-1}+\mu_3\mathbf{QD}^{'}\mathbf{MC}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)\mathbf{C}^{-1}\\ &\quad +\mathbf{Q}_n(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{C}^{-1}+\mathbf{Qq}(\mathbf{X}_n^{'}-\lambda_0^{'}\mathbf{C}_n)\mathbf{C}^{-1}\}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{w}_i^{'}+u_i\mathbf{q}^{'})\tilde{\mathbf{e}}_0\\ &\quad +O_p(n^{-1/2}) .\end{aligned}$$

Then the conditional expectation is given by

$$\begin{aligned}(3.40)\qquad & E[\mathbf{e}_{2.2.2}|\tilde{\mathbf{e}}_0=\mathbf{x}]\\ &= -2\mathbf{q}^{'}\mathbf{xx}(\mathbf{q}^{'}\mathbf{x})-\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}E(\mathbf{w}_i\mathbf{w}_i^{'})\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{z}_i\mathbf{x}-\mathbf{Qq}E[\mathbf{X}_n^{'}\mathbf{C}^{-1}\mathbf{X}_n|\mathbf{x}]\\ &\quad -\delta\kappa_3\mathbf{QD}^{'}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)](\mathbf{q}^{'}\mathbf{x})+\delta\mathbf{Qq}E[\mathbf{X}_n^{'}\mathbf{AX}_n|\mathbf{x}]\mathbf{q}^{'}\tilde{\mathbf{e}}_0+O_p(n^{-1/2}) .\end{aligned}$$

Now we use the fact that $\mathbf{X}_n^{'}\mathbf{C}^{-1}\mathbf{X}_n$ can be decocmposed into

$$\mathbf{X}_n^{'}\mathbf{C}^{-1/2}[\bar{\mathbf{P}}_E+\mathbf{P}_E]\mathbf{C}^{-1/2}\mathbf{X}_n=\mathbf{X}_n^{'}\mathbf{AX}_n+\tilde{\mathbf{e}}_0^{'}\mathbf{Q}^{-1}\tilde{\mathbf{e}}_0$$

and also we have the relation

$$\begin{aligned}& E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{z}_ju_j)|\mathbf{x}]\\ &= \delta E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{X}_n)(\mathbf{X}_n^{'}\mathbf{Az}_i)|\mathbf{x}]+O_p(n^{-1/2})\\ &= \delta\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)+O_p(n^{-1/2}) .\end{aligned}$$

Hence the conditional expectation can be rewriten as

$$(3.41)\quad E[\mathbf{e}_{2.2.2}|\mathbf{x}] = -2\mathbf{xx}^{'}\mathbf{C}_1^*\mathbf{x}-\mathbf{tr}(\mathbf{MC}^{-1})\mathbf{QC}_2^*\mathbf{x}-\mathbf{QC}_1^*\mathbf{x}[L\mathbf{I}_K+\mathbf{x}^{'}\mathbf{Q}^{-1}\mathbf{x}] +\delta L\mathbf{QC}_1^*\mathbf{x}-\delta\kappa_3\mathbf{QD}^{'}\mathbf{m}_3(\mathbf{q}^{'}\mathbf{x})+O_p(n^{-1/2}) .$$

On the other hand, the first term of $\mathbf{e}_{2.2}$ can be expressed as

$$\begin{aligned}\mathbf{e}_{2.2.1} &= -\Big\{2(\mathbf{q}'\tilde{\mathbf{e}}_0)\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1} + \delta\mu_3\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'(\mathbf{z}_i'\lambda_0)\mathbf{C}^{-1} \\ &\qquad +\mathbf{Q}_n(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i')\mathbf{C}^{-1} + \mathbf{Q}\mathbf{q}(\mathbf{X}_n' - \delta\lambda_0'\mathbf{C}_n)\mathbf{C}^{-1}\Big\}\mathbf{M}\mathbf{D} \\ &\times \Big\{\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i')\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{z}_i'\mathbf{A}\mathbf{X}_n + \delta\mu_3\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{A}\mathbf{X}_n)^2 \\ &\qquad -\tilde{\mathbf{e}}_0(\mathbf{q}'\tilde{\mathbf{e}}_0) - \mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i')\tilde{\mathbf{e}}_0\Big\} + O_p(n^{-1/2}) .\end{aligned}$$

Given $\tilde{\mathbf{e}}_0 = \mathbf{x}$, the conditional expectation is given by

$$\begin{aligned} & E[\mathbf{e}_{2.2.1}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\ =\ & [-2(1-\delta)L(\mathbf{q}'\mathbf{x})\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{q} - 2\delta\kappa_3(\mathbf{q}'\mathbf{x})\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}'\mathbf{m}_3] \\ & -\kappa_3\mathbf{Q}\mathbf{D}'\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'E[(\mathbf{z}_i'\lambda_0)\mathbf{C}^{-1}\mathbf{M}\mathbf{D}|\mathbf{x}][(1-\delta)L\mathbf{Q}\mathbf{q} + \delta\kappa_3\mathbf{Q}\mathbf{D}'\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{A}\mathbf{z}_i)] \\ & -\mathbf{Q}\mathbf{q}E[\mathbf{X}_n'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}((1-\delta)\mathbf{Q}\mathbf{q}L + \delta\kappa_3\mathbf{Q}\mathbf{D}'\mathbf{m}_3)|\mathbf{x}] \\ & +2(\mathbf{q}'\mathbf{x})^2\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{x} + \mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}E[\mathbf{w}_i\mathbf{w}_i']\mathbf{z}_i'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{z}_i\mathbf{x} \\ & +\mathbf{Q}\mathbf{q}(\mathbf{q}'\mathbf{e}_0)E[\mathbf{X}_n'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0 - \delta\mathbf{X}_n'\mathbf{A}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0|\mathbf{x}] .\end{aligned}$$

In order to evaluate each terms of the above expression, we use the relations

$$\begin{aligned} E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'(\mathbf{z}_i'\lambda_0)\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}|\tilde{\mathbf{e}}_0 = \mathbf{x}] &= E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{X}_n'\mathbf{A}\mathbf{z}_i|\mathbf{x}] \\ &= \mathbf{O}_p(n^{-1/2}) , \\ E[\mathbf{X}_n'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0|\tilde{\mathbf{e}}_0 = \mathbf{x}] &= E[\mathbf{X}_n'\mathbf{A}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0 + \mathbf{X}_n'\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0|\mathbf{x}] \\ &= \tilde{\mathbf{e}}_0'\mathbf{Q}^{-1}\tilde{\mathbf{e}}_0 , \\ E[\mathbf{X}_n'\mathbf{A}\tilde{\mathbf{e}}_0|\tilde{\mathbf{e}}_0 = \mathbf{x}] &= \mathbf{O}_p(n^{-1/2}) ,\end{aligned}$$

and then we have the conditional expectation

$$(3.42)\qquad \begin{aligned} E[\mathbf{e}_{2.2.1}|\tilde{\mathbf{e}}_0 = \mathbf{x}] &= 2(\mathbf{q}'\mathbf{x})^2\mathbf{x} + \mathbf{Q}\mathbf{C}_2^*\mathbf{tr}(\mathbf{Q}\mathbf{D}'\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{C}^{-1}\mathbf{M}\mathbf{D})\mathbf{x} \\ &\quad +\mathbf{Q}\mathbf{q}\mathbf{q}'\mathbf{x}\mathbf{x}'\mathbf{Q}^{-1}\mathbf{x} - 3(1-\delta)L\mathbf{Q}\mathbf{C}_1^*\mathbf{x} \\ &\quad -2\delta\kappa_3(\mathbf{q}'\mathbf{x})\mathbf{Q}\mathbf{D}'\mathbf{m}_3 - \delta\kappa_3\mathbf{Q}\mathbf{q}\mathbf{x}'\mathbf{D}'\mathbf{m}_3 .\end{aligned}$$

Hence we can obtain the conditional expectation $E[\mathbf{e}_{2.2}|\tilde{\mathbf{e}}_0 = \mathbf{x}] = E[\mathbf{e}_{2.2.1}|\mathbf{x}] + E[\mathbf{e}_{2.2.2}|\mathbf{x}]$ in (3.41) and (3.42) up to $O_p(n^{-1/2})$.

Thirdly, we shall evaluate all terms involving $\mathbf{e}_{2.1}$ which are more complicated than other terms. We notice that it is asymptotically equivalent to

$$(3.43)\qquad \mathbf{e}_{2.1}^* = \mathbf{e}_{2.1}(A) + \mathbf{e}_{2.1}(B) + \mathbf{e}_{2.1}(C) + \mathbf{e}_{2.1}(D) ,$$

where we use the notations

$$\begin{aligned}
\mathbf{e}_{2.1}(A) &= -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{C}_n^{(2)}\mathbf{A}\mathbf{X}_n \ , \\
\mathbf{e}_{2.1}(B) &= \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{C}_n^{(1)}\mathbf{C}^{-1}\mathbf{C}_n^{(1)}\mathbf{A}\mathbf{X}_n \ , \\
\mathbf{e}_{2.1}(C) &= -\mathbf{Q}\mathbf{E}_n^{(1)}\mathbf{C}^{-1}\mathbf{C}_n^{(1)}\mathbf{A}\mathbf{X}_n \ , \\
\mathbf{e}_{2.1}(D) &= \mathbf{Q}\mathbf{E}_n^{(2)}\mathbf{A}\mathbf{X}_n \ .
\end{aligned}$$

Because the above terms contain $p_i^{(2)}$ $(i=1,\cdots,n)$, we need to use the explicit expression of λ_1, which is the solution of the equation

$$\begin{aligned}
(3.44) \qquad & \lambda_0 + \frac{1}{\sqrt{n}}\lambda_1 + O_p(n^{-1}) \\
= \ & \{\mathbf{C}_n^{-1} + \frac{1}{\sqrt{n}}[-\mathbf{C}_n^{-1}\mathbf{C}_n^{(1)}\mathbf{C}_n^{-1}]\} \\
& \times\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i u_i - \mathbf{M}_n\mathbf{D}\mathbf{e}_0] + \frac{1}{\sqrt{n}}[-\mathbf{M}_n\mathbf{D}\mathbf{e}_1 - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0})\mathbf{e}_0]\}.
\end{aligned}$$

In the above representation we have used the representation of residuals as $u_i(\hat{\theta}) = y_{1i} - \mathbf{y}_{2i}^{'}\hat{\beta} - \mathbf{z}_{1i}^{'}\hat{\gamma} = u_i - (\frac{1}{\sqrt{n}})[\mathbf{z}_i^{'}\mathbf{D} + (\mathbf{v}_{2i}^{'},\mathbf{0}^{'})]\hat{\mathbf{e}}$. Then by using (3.33), we have the representation as

$$\begin{aligned}
\lambda_1 &= -\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{e}_1^{(0)} - \mathbf{C}^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{v}_{2i}^{'},\mathbf{0}^{'})\mathbf{e}_0 - \mathbf{C}^{-1}\mathbf{C}_n^{(1)}\mathbf{A}\mathbf{X}_n + O_p(n^{-1/2}) \\
&= -\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\{\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n + (1-\delta)\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n \\
&\quad +\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2 - \tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) - \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0\} \\
&\quad -\mathbf{C}^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{w}_i^{'} + u_i\mathbf{q}^{'})\tilde{\mathbf{e}}_0 \\
&\quad -\mathbf{C}^{-1}[-2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{C}_n - \mu_3\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\lambda_0)]\mathbf{A}\mathbf{X}_n \ + \ O_p(n^{-1/2}) \\
&= -\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 - \mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n \\
&\quad +\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 \\
&\quad +\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + (\mathbf{q}^{'}\tilde{\mathbf{e}}_0)[2\mathbf{A}\mathbf{X}_n - \mathbf{C}^{-1}\mathbf{X}_n] - (1-\delta)\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{q}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n \\
&\quad -\delta\mu_3\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2 + \delta\mu_3\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2 \\
&\quad + O_p(n^{-1/2}).
\end{aligned}$$

By using the relations

$$\mathbf{C}^{-1}\mathbf{X}_n = \mathbf{A}\mathbf{X}_n + \mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{X}_n,$$

and

$$2\mathbf{A}\mathbf{X}_n - \mathbf{C}^{-1}\mathbf{X}_n = \mathbf{A}\mathbf{X}_n - \mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0,$$

we have

$$\begin{aligned}\lambda_1 &= -\mathbf{A}(\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{z}_i\mathbf{w}_i^{'})\tilde{\mathbf{e}}_0 - \mathbf{C}^{-1}\mathbf{MDQ}(\frac{1}{\sqrt{n}}\sum_{i=1}^n \mathbf{w}_i\mathbf{z}_i^{'})\mathbf{AX}_n \\ &\quad +(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{AX}_n - (1-\delta)\mathbf{C}^{-1}\mathbf{MDQqX}_n^{'}\mathbf{AX}_n \\ &\quad +\delta\mu_3\mathbf{A}\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{AX}_n)^2 + O_p(n^{-1/2}).\end{aligned}$$

Then the first term of $\mathbf{e}_{2.1}$ is expressed as

$$\begin{aligned}\mathbf{e}_{2.1}(A) &= -\mathbf{QD}^{'}\mathbf{MC}^{-1}\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^{'}[((\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0)^2 \\ &\quad -2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_1 - 2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0 p_i^{(1)} + u_i^2 p_i^{(2)}]\Big\}\mathbf{AX}_n\ .\end{aligned} \tag{3.45}$$

Then we need to evaluate each terms by using the expression for $\mathbf{e}_1$ and λ_1 . By using the explicit representation of $p_i^{(1)}$ $(i=1,\cdots,n)$, it can be written as

$$\begin{aligned}&-\mathbf{QD}^{'}\mathbf{MC}^{-1}\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^{'}[(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0 + \mathbf{w}_i^{'}\tilde{\mathbf{e}}_0 + u_i\mathbf{q}^{'}\tilde{\mathbf{e}}_0)^2 \\ &\quad -2(u_i\mathbf{z}_i^{'}\mathbf{De}_1 + u_i\mathbf{w}_i^{'}\mathbf{e}_1 + u_i^2\mathbf{q}^{'}\mathbf{e}_1) + 2u_i^2(\mathbf{z}_i^{'}\lambda_0)(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0 + \mathbf{w}_i^{'}\tilde{\mathbf{e}}_0 + u_i\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + u_i^2p_i^{(2)}]\Big\}\mathbf{AX}_n \\ =&\ -\mathbf{QD}^{'}\mathbf{MC}^{-1}\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^{'}[(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0)^2 + (\mathbf{w}_i^{'}\tilde{\mathbf{e}}_0)^2 + u_i^2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)^2 \\ &\quad -2u_i^2\mathbf{q}^{'}\mathbf{e}_1 + 2u_i^2(\mathbf{z}_i^{'}\lambda_0)(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0 + \mathbf{w}_i^{'}\tilde{\mathbf{e}}_0 + u_i\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + u_i^2p_i^{(2)}]\Big\}\mathbf{AX}_n + O_p(n^{-1/2})\ .\end{aligned}$$

We use the relations that

$$E\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^{'}[(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0)^2 + (\mathbf{w}_i^{'}\tilde{\mathbf{e}}_0)^2 + u_i^2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)^2]\mathbf{AX}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}\Big\} = O_p(n^{-1/2})$$

because the random vector $\mathbf{AX}_n$ is asymptotically uncorrelated with $\tilde{\mathbf{e}}_0$ and we have the convergence in probability as

$$2\mathbf{QD}^{'}\mathbf{MC}^{-1}(\frac{1}{n}\sum_{i=1}^n u_i^2\mathbf{z}_i\mathbf{z}_i^{'})\mathbf{AX}_n(\mathbf{q}^{'}\mathbf{e}_1) \xrightarrow{p} \mathbf{O}\ .$$

Hence for $\mathbf{e}_{2.1}(A)$ we only need to evaluate the conditional expectation of the last four terms as

$$-\mathbf{QD}^{'}\mathbf{MC}^{-1}\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^{'}[2\delta(\mathbf{z}_i^{'}\mathbf{AX}_n)(u_i^2\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0 + u_i^2\mathbf{w}_i^{'}\tilde{\mathbf{e}}_0 + u_i^3\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + u_i^2p_i^{(2)}]\Big\}\mathbf{AX}_n$$

up to the order of $O_p(n^{-1/2})$. For the last term involving $p_i^{(2)}$, we use the stochastic expansion of λ_1 and its main part becomes

$$\Big\{\frac{1}{n}\sum_{i=1}^n \mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{AX}_n)u_i^2[\mathbf{z}_i^{'}\lambda_0(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0 + \mathbf{w}_i^{'}\tilde{\mathbf{e}}_0 + u_i\mathbf{q}^{'}\tilde{\mathbf{e}}_0) - \mathbf{z}_i^{'}u_i\lambda_1 + u_i^2(\mathbf{z}_i\lambda_0)^2]\Big\}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n} \mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)u_i^2(\mathbf{z}_i^{'}\lambda_0)(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0+\mathbf{w}_i^{'}\tilde{\mathbf{e}}_0+\mathbf{q}^{'}\tilde{\mathbf{e}}_0) \\
&\quad -[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)u_i^3][-\mathbf{A}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'})\mathbf{e}_0-\mathbf{C}^{-1}\mathbf{MDQ}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'})\mathbf{A}\mathbf{X}_n \\
&\qquad +\mathbf{q}^{'}\mathbf{e}_0\mathbf{A}\mathbf{X}_n-(1-\delta)\mathbf{C}^{-1}\mathbf{MDQ}\mathbf{X}_n^{'}\mathbf{A}\mathbf{X}_n+\delta\mu_3\mathbf{A}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2] \\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^3u_i^4 \ .
\end{aligned}
$$

Here we illustrate the typical argument of order evaluations. For the final term, we write

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^3u_i^4 \\
&= \frac{1}{n}\sum_{i=1}^{n}E(u_i^4)\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^3+\frac{1}{n}\sum_{i=1}^{n}[u_i^4-E(u_i^4)]\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^3
\end{aligned}
$$

and the second term is of order $O_p(n^{-1/2})$. Then we take the conditional expectations of each terms. By applying Lemma 4.3, we find that the first term in the above equation is of order $O_p(n^{-1/2})$. Also we use the relation

$$
\begin{aligned}
&E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iu_i^2(\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0)(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2|\tilde{\mathbf{e}}_0=\mathbf{x}] \\
&= E\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i[\sigma^2+(u_i^2-\sigma^2)](\mathbf{z}_i^{'}\mathbf{D}\tilde{\mathbf{e}}_0)(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2|\tilde{\mathbf{e}}_0=\mathbf{x}\} \\
&= \sigma^2E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{D}\mathbf{x})]+\ O_p(n^{-1/2}) \ .
\end{aligned}
$$

Then we can show

$$
\begin{aligned}
E[\mathbf{QD}^{'}\mathbf{MC}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iu_i^2\mathbf{w}_i^{'}\tilde{\mathbf{e}}_0(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2|\mathbf{x}] &= \mathbf{QD}^{'}\mathbf{MC}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iE[u_i^2\mathbf{w}_i^{'}]\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i\mathbf{x} \ , \\
E[\mathbf{QD}^{'}\mathbf{MC}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iu_i^3\mathbf{q}^{'}\tilde{\mathbf{e}}_0(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2|\mathbf{x}] &= \mu_3\mathbf{QD}^{'}\mathbf{MC}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{q}^{'}\mathbf{x}\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i \ .
\end{aligned}
$$

where we have ignored the terms of $O_p(n^{-1/2})$. Then after tedious calculations of conditional expectations, we can obtain

$$
E[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_iu_i^2p_i^{(2)}|\tilde{\mathbf{e}}_0=\mathbf{x}]=\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i[E(u_i^2\mathbf{w}_i^{'})\mathbf{x}+\sigma^2\mathbf{z}_i^{'}\mathbf{D}\mathbf{x}]+O_p(n^{-1/2}) \ .
$$

Then together with the conditional expectations of other terms we can derive

$$
\begin{aligned}
(3.46) E[\mathbf{e}_{2.1}(A)|\tilde{\mathbf{e}}_0=\mathbf{x}] &= -3\delta\mathbf{QD}^{'}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)\mathbf{z}_i^{'}]\mathbf{D}\mathbf{x} \\
&\quad -3\delta\mathbf{QD}^{'}\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)]E[u_i^2\mathbf{w}_i^{'}]\mathbf{x} \\
&\quad -2\delta\mu_3\mathbf{QD}^{'}\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)]\mathbf{q}^{'}\mathbf{x}+O_p(n^{-1/2}).
\end{aligned}
$$

Similarly, the second term of $\mathbf{e}_{2.1}$ is expressed as

$$(3.47)\qquad \mathbf{e}_{2.1}(B) = \mathbf{QD}^{'}\mathbf{MC}^{-1}\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}[-2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0 + u_i p_i^{(1)}]\} \times\mathbf{C}^{-1}\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}[-2u_i(\mathbf{y}_{2i}^{'},\mathbf{z}_{1i}^{'})\mathbf{e}_0 + u_i p_i^{(1)}]\}\mathbf{AX}_n .$$

Then the conditional expectation can be reduced to

$$\begin{aligned} & E[\mathbf{e}_{2.1}(B)|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\ = \; & 2\delta\mathbf{QD}^{'}\mathbf{MC}^{-1}E[\mathbf{C}_n(\mathbf{q}^{'}\mathbf{e}_0)\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i u_i^3(\mathbf{z}_i^{'}\mathbf{AX}_n)^2 \\ & +\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}u_i^3(\mathbf{z}_i^{'}\mathbf{AX}_n)\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}u_i^2(\mathbf{z}_i^{'}\mathbf{AX}_n)\mathbf{q}^{'}\mathbf{e}_0|\mathbf{x}] \\ = \; & 4\delta\mu_3(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{QD}^{'}\mathbf{MC}^{-1}\mathbf{m}_3 \; + \; O_p(n^{-1/2}) . \end{aligned}$$

For the third term, we write

$$(3.48)\qquad \mathbf{e}_{2.1}(C) = -\mathbf{QE}_n^{(1)}\mathbf{C}^{-1}\mathbf{C}_n^{(1)}\mathbf{AX}_n ,$$

where we use the fact that

$$\mathbf{QE}_n^{(1)}\mathbf{C}^{-1} = [\mathbf{Q}(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}) + \mathbf{Qq}(\mathbf{X}_n^{'} - \lambda_0^{'}\mathbf{C}_n)]\mathbf{C}^{-1} + O_p(n^{-1/2}) ,$$

and

$$\mathbf{C}_n^{(1)}\mathbf{AX}_n = [-2(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\mathbf{C}_n - \delta\mu_3\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{AX}_n)]\mathbf{AX}_n \; + \; O_p(n^{-1/2}) .$$

Hence the conditional expectation of $\mathbf{e}_{2.1}(C)$ is given by

$$\begin{aligned} & E[\mathbf{e}_{2.1}(C)|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\ = \; & 2\mathbf{Qq}(\mathbf{q}^{'}\mathbf{x})(1-\delta)E[\mathbf{X}_n^{'}\mathbf{C}^{-1}\mathbf{CAX}_n|\mathbf{x}] + \mathbf{Qq}(\mathbf{X}_n^{'} - \delta\lambda_0^{'}\mathbf{C})\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{AX}_n)^2 + O_p(n^{-1/2}) , \\ = \; & 2(1-\delta)L\mathbf{Qqq}^{'}\mathbf{x} + \delta\mu_3\mathbf{Qqe}_0^{'}\mathbf{D}^{'}\mathbf{m}_3 + O_p(n^{-1/2}) . \end{aligned}$$

The fourth term of $\mathbf{e}_{2.1}$ is expressed as

$$(3.49)\qquad \mathbf{e}_{2.1}(D) = \mathbf{Q}\{\mathbf{D}^{'}\frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\mathbf{z}_i\mathbf{z}_i^{'} + \frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\begin{pmatrix}\mathbf{v}_{2i}\\ \mathbf{0}\end{pmatrix}\mathbf{z}_i^{'}\}\mathbf{AX}_n$$

Since the first term of $\mathbf{e}_{2.1}(D)$ is similar to the last term of (3.45) and we can utilize the expression of λ_1 in (3.44), it is easy to find that the conditional expectation, which is given by

$$E[\mathbf{QD}^{'}\frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{AX}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}]$$

$$
\begin{aligned}
&= \mathbf{QD}' E\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{AX}_n)(\delta\mathbf{z}_i'\lambda_0)(\mathbf{z}_i'\mathbf{De}_0+\mathbf{w}_i'\mathbf{e}_0+u_i\mathbf{q}'\mathbf{e}_0)\\
&\quad -\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{AX}_n)[-\tilde{\mathbf{e}}_0'\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{w}_j\mathbf{z}_j'\mathbf{A}-\mathbf{X}_n'\mathbf{A}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbf{z}_j\mathbf{w}_j'\mathbf{QD}'\mathbf{MC}^{-1}+\mathbf{q}'\tilde{\mathbf{e}}_0\mathbf{X}_n'\mathbf{A}\\
&\quad -(1-\delta)\mathbf{X}_n'\mathbf{AX}_n\mathbf{q}'\mathbf{QD}'\mathbf{MC}^{-1}+\delta\mu_3\frac{1}{n}\sum_{j=1}^{n}\mathbf{z}_j(\mathbf{z}_j'\mathbf{AX}_n)^2\mathbf{A}]\mathbf{z}_iu_i+\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{AX}_n)(\mathbf{z}_i'\lambda_0)^2u_i^2|\mathbf{x}\Big)\\
&= \delta E\Big\{\mathbf{QD}'[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{Az}_i)\mathbf{z}_i']\mathbf{Dx}+\mathbf{QD}'[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'(\mathbf{z}_i'\mathbf{AX}_n)^2\mathbf{AX}_n|\mathbf{x}\Big\}+O_p(n^{-1/2})\\
&= \delta\mathbf{QD}'[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{Az}_i)\mathbf{z}_i']\mathbf{Dx}\ +\ O_p(n^{-1/2})\ .
\end{aligned}
$$

For the second term of $\mathbf{e}_{2.1}(D)$, we rewrite

$$
\begin{aligned}
&\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}p_i^{(2)}\begin{pmatrix}\mathbf{v}_{2i}\\ \mathbf{0}\end{pmatrix}\mathbf{z}_i'\mathbf{AX}_n\\
=\ &\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{w}_i+\mathbf{q}u_i)\{\lambda_0'\mathbf{z}_i(\mathbf{z}_i'\mathbf{D}+\mathbf{w}_i'+\mathbf{q}u_i')\tilde{\mathbf{e}}_0-\lambda_1'\mathbf{z}_iu_i+(\lambda_0'\mathbf{z}_i)u_i^2\}\mathbf{z}_i'\mathbf{AX}_n\\
=\ &\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{w}_i+\mathbf{q}u_i)(\mathbf{z}_i'\mathbf{D}+\mathbf{w}_i'+\mathbf{q}'u_i)\tilde{\mathbf{e}}_0(\mathbf{z}_i'\lambda_0)(\mathbf{z}_i'\mathbf{AX}_n)\\
&-\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{w}_i+\mathbf{q}u_i)(\mathbf{z}_i'u_i)\lambda_1(\mathbf{z}_i'\mathbf{AX}_n)\\
&+\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{w}_i+\mathbf{q}u_i)(\mathbf{z}_i'\lambda_0)^2u_i^2(\mathbf{z}_i'\mathbf{AX}_n)+O_p(n^{-1/2}).
\end{aligned}
$$

For the sake of exposition, we denote each term of the above expression as $\mathbf{e}_{2.1.1}(D)$, $\mathbf{e}_{2.1.2}(D)$, $\mathbf{e}_{2.1.3}(D)$, respectively. Then

$$
\begin{aligned}
E[\mathbf{e}_{2.1.1}(D)|\tilde{\mathbf{e}}_0=\mathbf{x}] &= E[\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{C}_2^*+\mathbf{qq}'u_i^2)\tilde{\mathbf{e}}_0\lambda_0'\mathbf{z}_i\mathbf{z}_i'\mathbf{AX}_n|\mathbf{x}]\\
&= \delta\mathbf{QC}_2^*\mathbf{x}\mathbf{tr}(\mathbf{MA})+\delta L\mathbf{QC}_1^*\mathbf{x}+O_p(n^{-1/2}).
\end{aligned}
\tag{3.50}
$$

Also we find that

$$
\begin{aligned}
E[\mathbf{e}_{2.1.3}(D)|\tilde{\mathbf{e}}_0=\mathbf{x}] &= E[\mathbf{Q}\frac{1}{n}\sum_{i=1}^{n}\mathbf{q}u_i^3(\mathbf{z}_i'\lambda_0)^2\mathbf{z}_i'\mathbf{AX}_n|\tilde{\mathbf{e}}_0=\mathbf{x}]\\
&= \delta^2\mu_3\mathbf{Qq}E[\frac{1}{n}\sum_{i=1}^{n}(\mathbf{z}_i'\mathbf{AX}_n)^3|\tilde{\mathbf{e}}_0=\mathbf{x}]+O_p(n^{-1/2})\ .
\end{aligned}
$$

But the random vector $\mathbf{AX}_n$ is asymptotically normal, and its limiting distribution is uncorrelated with that of $\tilde{\mathbf{e}}_0$. Hence we can show

$$
E[\mathbf{e}_{2.1.3}(D)|\tilde{\mathbf{e}}_0=\mathbf{x}]=O_p(n^{-1/2}). \tag{3.51}
$$

Then we evaluate the conditional expectation of $\mathbf{e}_{2.1.2}(D)$. Because the pairs of random vectors $(\mathbf{w}_i', u_i)$ are uncorrelated, we have the convergence in probability

$$
\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_iu_iz_i^{(j)}z_i^{(k)}\xrightarrow{p}\mathbf{0}\ .
$$

Therefore as for the remaining conditional expectation terms we use the explicit expression for λ_1 in (3.44) and we find

$$E[\mathbf{e}_{2.1.2}(D)|\tilde{\mathbf{e}}_0 = \mathbf{x}] = -\delta \mathbf{Q}\mathbf{q}E[\lambda_1^{'}(\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}u_i^2)\mathbf{A}\mathbf{X}_n|\mathbf{x}] + O_p(n^{-1/2}) = -\delta L\mathbf{Q}\mathbf{q}^{'}\mathbf{x} + O_p(n^{-1/2}) \ . \tag{3.52}$$

Hence we summarize the conditional expectation

$$E[\mathbf{e}_{2.1}(D)|\tilde{\mathbf{e}}_0 = \mathbf{x}] = \delta\mathbf{Q}\mathbf{D}^{'}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)\mathbf{z}_i^{'}]\mathbf{D}\mathbf{x} + \delta\mathbf{Q}\mathbf{C}_2^{*}\mathbf{x}\ \mathbf{tr}(\mathbf{M}\mathbf{A}) + \delta L\mathbf{Q}\mathbf{C}_1^{*}\mathbf{x} - \delta L\mathbf{Q}\mathbf{q}^{'}\mathbf{x} + O_p(n^{-1/2}) \ . \tag{3.53}$$

Finally, we can derive the conditional expectation of $E[\mathbf{e}_{2.1}|\tilde{\mathbf{e}}_0 = \mathbf{x}]$ by collecting the conditional expectations of $E[\mathbf{e}_{2.1}(A)|\mathbf{x}]$, $E[\mathbf{e}_{2.1}(B)|\mathbf{x}]$, $E[\mathbf{e}_{2.1}(C)|\mathbf{x}]$, and $E[\mathbf{e}_{2.1}(D)|\mathbf{x}]$. At the first glance of these many terms, it looks formidable to calculate them. However, the resulting formulas are relatively simple because it has turned out that many terms have disappeared in the conditional expectation formulas eventually.

4. Asymptotic Expansions when $\mu_3 = 0$

When we can ignore the effects of the third order moments of the disturbance, the asymptotic expansions of estimators for an arbitrary δ $(0 \leq \delta \leq 1)$ can be simplified greatly.

Assumption II :
For the third order moments of disturbances in (2.3) and (2.4) we assume $E[u_i^3] = \mu_3 = 0$ and $E[u_i^2\mathbf{w}_i] = \mathbf{0}$ for $i = 1, \cdots, n$.

It is immediate that the conditions in Assumption II can be relaxed as

$$\frac{1}{n}\sum_{i=1}^{n} z_i^{(j)}z_i^{(k)}z_i^{(l)}u_i^3 = o_p(\frac{1}{\sqrt{n}}) \tag{4.1}$$

and the similar conditions on the third order moments on $\{u_i^2\mathbf{w}_i\}$. In this section we shall derive the asymptotic distribution functions of estimators under **Assumption I** and **Assumption II**.

4.1 Conditional Expectation Formulas

We prepare some useful formulas on the conditional expectaions and their proofs are given in Appendix A. They are the result of straightforward calculations, but we shall give the derivations of Lemma 4.1 and Lemma 4.3 in Appendix A for the espository purpose.

Lemma 4.1 : Let the random vectors $\tilde{\mathbf{e}}_0, \mathbf{X}_n$, and $\mathbf{Y}_n$ be defined as in Section 3. Then

$$E[\mathbf{Y}_n\mathbf{A}\mathbf{X}_n|\tilde{e}_0 = \mathbf{x}] = \mu_3\mathbf{m}_3 + O_p(n^{-1/2}). \tag{4.2}$$

Lemma 4.2 : Let a set of random vectors $\mathbf{X} = (X_i)$ and $\mathbf{Z} = (Z_i)$ be normally distributed and the conditional expectation of X_i given $\mathbf{Z}$ be $E(X_i|\mathbf{Z})$. Then

$$(4.3)\quad E[X_iX_jX_k|\mathbf{Z}] = Cov(X_i,X_j|\mathbf{Z})E(X_k|\mathbf{Z}) + Cov(X_j,X_k|\mathbf{Z})E(X_i|\mathbf{Z}) + Cov(X_k,X_i|\mathbf{Z})E(X_j|\mathbf{Z}) + E(X_i|\mathbf{Z})E(X_j|\mathbf{Z})E(X_k|\mathbf{Z})$$

and

$$\begin{aligned}(4.4)\quad & E[X_iX_jX_kX_l|\mathbf{Z}] \\ = & \ Cov(X_i,X_j|\mathbf{Z})Cov(X_k,X_l|\mathbf{Z}) + Cov(X_i,X_k|\mathbf{Z})Cov(X_j,X_l|\mathbf{Z}) \\ & + Cov(X_i,X_l|\mathbf{Z})Cov(X_j,X_k|\mathbf{Z}) \\ + & \ Cov(X_i,X_j|\mathbf{Z})E(X_k|\mathbf{Z})E(X_l|\mathbf{Z}) + Cov(X_i,X_k|\mathbf{Z})E(X_j|\mathbf{Z})E(X_l|\mathbf{Z}) \\ & + Cov(X_i,X_l|\mathbf{Z})E(X_j|\mathbf{Z})E(X_k|\mathbf{Z}) + Cov(X_j,X_k|\mathbf{Z})E(X_i|\mathbf{Z})E(X_l|\mathbf{Z}) \\ & + Cov(X_j,X_l|\mathbf{Z})E(X_i|\mathbf{Z})E(X_k|\mathbf{Z}) + Cov(X_k,X_l|\mathbf{Z})E(X_i|\mathbf{Z})E(X_j|\mathbf{Z}) \\ + & \ E(X_i|\mathbf{Z})E(X_j|\mathbf{Z})E(X_k|\mathbf{Z})E(X_l|\mathbf{Z}) \ .\end{aligned}$$

The above formulas have been used in our derivations by setting $\mathbf{Z} = \tilde{e}_0$, which is he leading term. In order to evaluate the conditional expectation operations appeared in the stochastic expansions of estimators, we also need the next formula.

Lemma 4.3 : Let $\mathbf{u}_n = (u_i)$ and v_n be $p \times 1$ random vector and a random variable with $E(u_i) = 0, E(v_n) = 0, E(u_iu_j) = \delta(i,j), E(v_n^2) = 1$ and they have finite fourth order moments. Assume that they are sums of i.i.d. random vectors and asymptotically normally distributed and admit the asymptotic expansion of their distribution function up to $O_p(n^{-1})$. Then

$$\begin{aligned}(4.5)\ & E[v_n|\mathbf{u}_n] \\ = & \ \rho^{'}\mathbf{u}_n + \frac{1}{6\sqrt{n}}\{3\sum_{l,l'=1}^{p}\beta_{l,l',v}h_2(u_l,u_{l'}) - 3\sum_{l',l''=1}^{p}[\sum_{l=1}^{p}\beta_{l,l',l''}\rho_l]h_2(u_{l'},u_{l''})\} + O_p(n^{-1}) \ ,\end{aligned}$$

where $\beta_{l,l',v} = E(u_lu_{l'}v_n)$, $\beta_{l,l',l''} = E(u_lu_{l'}u_{l''})$, $h_2(u_l,u_{l'}) = u_lu_{l'} - \delta(l,l^{'})$ ($\delta(l,l^{'}) = 1$ if $l = l^{'}$ and $\delta(l,l^{'}) = 0$ if $l \neq l^{'}$), and $\rho = Cov(v,\mathbf{u}_n)$.

In particular, if $E(u_iu_ju_k) = 0$ $(i \neq j \neq k)$, then $\beta_{l,l',l''} = 0$ and the second term in the order of $O_p(n^{-1/2})$ in (4.5) vanishes.

Now we shall evaluate the conditional expectations of $\mathbf{e}_1^{(1)}$ given $\mathbf{e}_0 = \mathbf{x}$. This term plays an important role and makes some complications of our analyses. We first note that the conditional expectations of $\mathbf{e}_{1.2}^{(1)}$ and $\mathbf{e}_{1.3}^{(1)}$ given $\tilde{\mathbf{e}}_0 = \mathbf{x}$ can be evaluated easily. By using Lemma 4.3, the conditional expectations are given by

$$\begin{aligned}(4.6)\quad & E[\mathbf{e}_{1.2}^{(1)}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\ & = E[\mathbf{QD}^{'}\mathbf{MC}^{-1}\mathbf{Y}_n^{'}\mathbf{AX}_n(\mathbf{q}^{'}\tilde{\mathbf{e}}_0) + \tilde{\mathbf{e}}_0^{'}\mathbf{qQD}^{'}\mathbf{MC}^{-1}\mathbf{Y}_n^{'}\mathbf{AX}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\ & = \kappa_3(\mathbf{q}^{'}\mathbf{x})\mathbf{QD}^{'}\mathbf{m}_3 + \kappa_3\mathbf{xq}^{'}\mathbf{QD}^{'}\mathbf{m}_3 + O_p(n^{-1/2})\end{aligned}$$

and

$$(4.7) \quad E[\mathbf{e}_{1.3}^{(1)}|\tilde{\mathbf{e}}_0 = \mathbf{x}]$$
$$= \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}E[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{w}_i^{'}|\tilde{\mathbf{e}}_0 = \mathbf{x}]\mu_3\mathbf{m}_3 + \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_iE(u_i^2\mathbf{w}_i^{'})\mathbf{x}$$
$$= \mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{m}_3E(u_i^2\mathbf{w}_i^{'})\mathbf{x} + O_p(n^{-1/2}) \ .$$

Next we evaluate the conditional expectation of $\mathbf{e}_{1.1}^{(1)}$ which has many terms. We notice that in $\mathbf{e}_{1.1}^{(1)}$ two terms associated with the random matrices $\mathbf{C}_n^{(1)}$ and $\mathbf{E}_n^{(1)}$ have been cancelled out. Then we try to evaluate each remaining terms of order $O_p(n^{-1/2})$ and the terms in the first two lines are given by

$$E\{\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n\mathbf{z}_i^{'}(\mathbf{C}^{-1}-\mathbf{A})E[\mathbf{Y}_n|\mathbf{X}_n]\mathbf{A}\mathbf{X}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}\}$$
$$+ \quad E\{\mathbf{Q}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}(\mathbf{C}^{-1}-\mathbf{A})E[\mathbf{Y}_n|\mathbf{X}_n]\mathbf{A}\mathbf{X}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}\}$$
$$+ \quad E\{\mathbf{Q}\mathbf{q}\tilde{\mathbf{e}}_0^{'}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}E[\mathbf{Y}_n|\mathbf{X}_n]\mathbf{A}\mathbf{X}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}\} \ .$$

Similarly, the terms in the third line are given by

$$E\{-\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n\mathbf{z}_i^{'}\mathbf{C}^{-1}E[\mathbf{Y}_n|\mathbf{X}_n]\mathbf{A}\mathbf{X}_n|\tilde{\mathbf{e}}_0 = \mathbf{x}\}$$
$$+ \quad E\{-\delta\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}E[\mathbf{Y}_n|\mathbf{X}_n]\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i|\tilde{\mathbf{e}}_0 = \mathbf{x}\} \ .$$

The most important terms in $O_p(n^{-1/2})$ are the next two terms in the fourth line because they are dependent on the fourth order moments of $\{u_i\}$, which are given by

$$2\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}u_i(\mathbf{z}_i^{'}\mathbf{D}+\mathbf{w}_i^{'})\tilde{e}_0]\mathbf{A}\mathbf{X}_n$$
$$+\delta\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)(u_i^3-\mu_3-\sigma^2u_i)]\mathbf{A}\mathbf{X}_n$$

up to $O_p(n^{-1/2})$. For the simplicity let the fourth order cumulant be defined by $\kappa = [E(u_i^4)-3\sigma^4)]/\sigma^4$. Then the conditional expectations of the second term and the first term are given by

$$(4.8) \qquad \delta\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}E[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n)^2(u_i^3-\mu_3-\sigma^2u_i)|\tilde{\mathbf{e}}_0 = \mathbf{x}]$$
$$= \delta\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)E[u_i^4-\sigma^2u_i^2]\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{x} + O_p(n^{-1/2})$$
$$= \delta(2+\kappa)\mathbf{Q}\mathbf{D}^{'}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i)]\mathbf{D}\mathbf{x} + O_p(n^{-1/2}),$$

and

$$(4.9)\ 2\mathbf{QD}'\mathbf{MC}^{-1}E[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'u_i(\mathbf{z}_i'\mathbf{D}+\mathbf{w}_i')\tilde{e}_0]\mathbf{AX}_n|\tilde{\mathbf{e}}_0=\mathbf{x}]$$
$$= 2\mathbf{QD}'\mathbf{MC}^{-1}\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iu_i\mathbf{z}_i'(\mathbf{z}_i'\mathbf{Dx})\mathbf{Az}_i+\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_iE[u_i\mathbf{w}_i']\mathbf{xz}_i'\mathbf{AX}_n\}+O_p(n^{-1/2})\ .$$

For the expression of $\mathbf{e}_1^{(1)}$, there are many remaining terms in the last three lines, which are given by

$$E\{-\mathbf{Q}[(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i')\mathbf{C}^{-1}+\mathbf{q}(\mathbf{X}_n'\mathbf{C}^{-1}-\delta\mathbf{X}_n'\mathbf{A})]E[\mathbf{Y}_n|\mathbf{X}_n]\mathbf{AX}_n|\tilde{\mathbf{e}}_0=\mathbf{x}\}$$
$$+\quad E\{-\mathbf{QD}'\mathbf{MC}^{-1}\mathbf{Y}_n\mathbf{C}^{-1}\mathbf{MDQ}[(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i')\mathbf{AX}_n+(1-\delta)\mathbf{qX}_n'\mathbf{AX}_n]$$
$$+\quad E\{-\delta\mathbf{D}'[(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_iu_i)(\mathbf{z}_i'\mathbf{AX}_n)^2-\delta(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{z}_iu_i)(\mathbf{z}_i'\mathbf{AX}_n)^2|\tilde{\mathbf{e}}_0=\mathbf{x}\}$$

Finally we use **Assumption II** and many terms involving the third order moments disappear and we have only two terms involving the fourth order moments terms in (4.7) and (4.8). Hence under **Assumption II** in this section the conditional expectation of $\mathbf{e}_1^{(1)}$ is given by

$$(4.10)\qquad E[\mathbf{e}_1^{(1)}|\tilde{\mathbf{e}}_0=\mathbf{x}]=[2+\delta(2+\kappa)]\mathbf{QD}'\mathbf{FDx}+O_p(n^{-1/2}),$$

where we define

$$\mathbf{F}=\text{plim}_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i'\mathbf{Az}_i)\mathbf{z}_i'\ .$$

We note that (4.10) is one of key aspects on the semi-parametric estimation of a single structural equation.

4.2 Asymptotic Expansions of Density Functions

Since there are many terms appeared in the stochastic expansion of $\hat{\mathbf{e}}$, at first it looks formidable to evaluate all terms and to derive the asymptotic expansions of the distribution functions. Under the assumptions of this section, however, it is possible to derive them in a compact form.

By applying Lemma 4.3, the conditional expectation of $\mathbf{e}_0^{(1)}$ in (3.4) given $\mathbf{X}_n$ can be calculated as

$$(4.11)\qquad E[\mathbf{e}_0^{(1)}|\mathbf{X}_n]$$
$$= -\mu_3\mathbf{QD}'\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\mathbf{AX}_n\mathbf{X}_n'\mathbf{C}^{-1}\mathbf{z}_i]$$
$$+\frac{1}{6\sqrt{n}}\{-3\mathbf{QD}'\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}(E(u_i^4)-\sigma^4)\sum_{l,l'}z_i^lz_i^{l'}h_2(x_l,x_{l'})]\mathbf{AX}_n$$
$$+3\mu_3^2\mathbf{QD}'\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i'\sum_{l,l'}z_i^lz_i^{l'}h_2(x_l,x_{l'})]\mathbf{AX}_n\}$$
$$+O_p(n^{-1/2}).$$

By decomposing $\mathbf{AX}_n\mathbf{X}_n^{'}\mathbf{C}^{-1} = \mathbf{AX}_n\mathbf{X}_n^{'}\mathbf{A} + \mathbf{AX}_n\tilde{\mathbf{e}}_0^{'}\mathbf{D}^{'}\mathbf{MC}^{-1}$ and taking the conditional expectaions, we have

$$
\begin{aligned}
(4.12)\qquad & E[\mathbf{e}_0^{(1)}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
= \quad & -\mu_3\mathbf{QD}^{'}\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)] \\
& +\frac{1}{6\sqrt{n}}\Big\{-3\mathbf{QD}^{'}\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}(E(u_i^4)-\sigma^4)\times 2\mathbf{z}_i\mathbf{z}_i{'}\mathbf{C}^{-1}\mathbf{MD}\tilde{\mathbf{e}}_0(\mathbf{z}_i^{'}\mathbf{Az}_i)] \\
& \qquad +3\mu_3^2\mathbf{QD}^{'}\mathbf{MC}^{-1}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(2\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{MD}\tilde{\mathbf{e}}_0\mathbf{z}_i^{'}\mathbf{Az}_i)]\Big\} + O_p(n^{-1/2})\ . \\
= \quad & \frac{1}{\sqrt{n}}\{-(2+\kappa)\mathbf{QD}^{'}[\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)\mathbf{z}_i^{'}]\mathbf{Dx}\} + O_p(n^{-1})
\end{aligned}
$$

under **Assumption II** in this section and we use the notation $\kappa = (E(u_i^4) - 3\sigma^4)/\sigma^4$. Also we use the relation that for a constant matrix $\mathbf{A}$ $(= (A_{jk}))$

$$
\begin{aligned}
E[\mathbf{Y}_n\mathbf{AY}_n|\tilde{\mathbf{X}}_n] &= E[(u_i^2-\sigma^2)^2][\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)\mathbf{z}_i^{'}] \\
&+ [\mu_3]^2\sum_{j,k=1}^{p}A_{jk}(\frac{1}{n})\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}z_i^{(j)}\mathbf{C}^{-1}(\mathbf{X}_n\mathbf{X}_n^{'}-\mathbf{C})\mathbf{C}^{-1}(\frac{1}{n}\sum_{i=1}^{n}z_i^{(k)}\mathbf{z}_i^{'}) \\
&= E[(u_i^2-\sigma^2)^2][\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_i)\mathbf{z}_i^{'}] + O_p(n^{-1/2})
\end{aligned}
$$

under **Assumption II**. Then by using (3.24) we have

$$
\begin{aligned}
E[\mathbf{e}_0^{(2)}|\tilde{\mathbf{e}}_0 = \mathbf{x}] &= \mathbf{QD}^{'}\mathbf{MC}^{-1}E[\mathbf{Y}_n\mathbf{AY}_n\mathbf{AX}_n|\mathbf{x}] \\
&= O_p(n^{1/2})
\end{aligned}
$$

under Assumption II because the random vector $\mathbf{AX}_n$ is asymptotically uncorrelated with $\tilde{\mathbf{e}}_0$. By using direct calculations we have

$$
\begin{aligned}
(4.13)\, & E[\mathbf{e}_0^{(1)}\mathbf{e}_0^{(1)'}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
= \quad & \mathbf{QD}^{'}\mathbf{MC}^{-1}E[\mathbf{Y}_n\mathbf{AX}_n\mathbf{X}_n^{'}\mathbf{AY}_n|\mathbf{x}]\mathbf{C}^{-1}\mathbf{MDQ} \\
= \quad & (2+\kappa)\mathbf{QD}^{'}\mathbf{FDQ} + [\kappa_3]^2\mathbf{QD}^{'}[\mathbf{m}_3\mathbf{m}_3^{'} + (\frac{1}{n})^2\sum_{i,j=1}^{n}\mathbf{z}_i(\mathbf{z}_i^{'}\mathbf{Az}_j)^2\mathbf{z}_j^{'}]\mathbf{DQ}\ + O_p(n^{-1/2})
\end{aligned}
$$

in the general case.

Let a random matrix be $\mathbf{U}_n = (u_{jk}) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}$. By applying Lemma 4.3 and using the fact that $Cov(u_{jk}, \tilde{\mathbf{e}}_0) = \mathbf{0}$, we have

$$
\begin{aligned}
(4.14)\qquad & E[u_{jk}|\mathbf{X}_n] \\
& = \frac{1}{2\sqrt{n}}\sum_{l,l^{'}=1}^{K}\{\frac{1}{n}\sum_{i=1}^{n}(\mathbf{C}^{-1/2}\mathbf{z}_i)_l(\mathbf{C}^{-1/2}\mathbf{z}_i)_l^{'}z_i^{(k)}E(u_iw_i^{(j)}) \\
& \qquad \times[(\mathbf{C}^{-1/2}\mathbf{X}_n)_l(\mathbf{C}^{-1/2}\mathbf{X}_n)_{l^{'}} - \delta(l,l^{'})]\} + O_p(n^{-1}) \\
& = \frac{1}{2\sqrt{n}}\frac{1}{n}\sum_{i=1}^{n}E(u_i^2w_i^{(j)})z_i^{(k)}[\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{X}_n\mathbf{X}_n^{'}\mathbf{C}^{-1}\mathbf{z}_i] + O_p(n^{-1/2}).
\end{aligned}
$$

Then by decomposing $\mathbf{C}^{-1} = \mathbf{A} + \mathbf{C}^{-1}\mathbf{M}\mathbf{D}^{'}\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}$ and $\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{X}_n\mathbf{X}_n^{'}\mathbf{C}^{-1}\mathbf{z}_i = 2\mathbf{z}_i^{'}\mathbf{A}\mathbf{X}_n\tilde{\mathbf{e}}_0^{'}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{z}_i + \mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\tilde{\mathbf{e}}_0\tilde{\mathbf{e}}_0^{'}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{z}_i$, and taking the conditional expectation, we have

$$(4.15)\quad E[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{z}_i^{'}|\tilde{\mathbf{e}}_0 = \mathbf{x}]$$
$$= \frac{1}{2\sqrt{n}}\frac{1}{n}\sum_{i=1}^{n}E(u_i^2\mathbf{w}_i)\mathbf{z}_i^{'}[\mathbf{z}_i^{'}\mathbf{C}^{-1}\mathbf{M}\mathbf{D}(\mathbf{x}\mathbf{x}^{'} - \mathbf{Q})\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{z}_i] + O_p(n^{-1/2})\ .$$

By using the above type of the conditional expectations formulas and lengthy calculation under **Assumption II**, we can find that

$$\begin{aligned}
& E[\mathbf{e}_0^{(1)}\mathbf{e}_1^{(0)'}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
= & -\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i E[u_i^2\mathbf{w}_i^{'}]\mathbf{Q} + \mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i(\mathbf{q}^{'}\tilde{\mathbf{e}}_0)\tilde{\mathbf{e}}_0^{'} \\
& +(1-\delta)\mu_3\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}(L+2)\frac{1}{n}\sum_{i=1}^{n}\mathbf{z}_i\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i\mathbf{q}^{'}\mathbf{Q} \\
& -\delta[\mu_3]^2\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\left((\frac{1}{n})^2\sum_{i,j}\mathbf{z}_i\mathbf{z}_j\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_i\mathbf{z}_j^{'}\mathbf{A}\mathbf{z}_j + 2(\frac{1}{n})^2\sum_{i,j}\mathbf{z}_i\mathbf{z}_j(\mathbf{z}_i^{'}\mathbf{A}\mathbf{z}_j)^2]\right)\mathbf{C}^{-1}\mathbf{M}\mathbf{D}\mathbf{Q} \\
= & O_p(n^{-1/2})\ .
\end{aligned}$$

For the convenience of notations we define $\mathbf{e}_0^* = \tilde{\mathbf{e}}_0$, $\mathbf{e}_1^* = \mathbf{e}_1^{(0)} + \mathbf{e}_0^{(1)}$, and $\mathbf{e}_2^* = \mathbf{e}_0^{(2)} + \mathbf{e}_1^{(1)} + \mathbf{e}_2$. Then by using the conditional expectation formulaa in Lemma 4.1 and Lemma 4.3, we can represent the conditional expectation in the general case as

$$(4.16)\qquad E[\mathbf{e}_1^*|\tilde{\mathbf{e}}_0 = \mathbf{x}] = (1-\delta)L\mathbf{Q}\mathbf{q} - \mathbf{x}\mathbf{x}^{'}\mathbf{q} - (1-\delta)\kappa_3\mathbf{Q}\mathbf{D}^{'}\mathbf{m}_3 + O_p(n^{-1/2}),$$

and under **Assumption II** we have

$$\begin{aligned}
(4.17)\qquad & E[\mathbf{e}_2^*|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
= & -(2+\kappa)\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{x} + [2+\delta(2+\kappa)]\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{x} \\
& +\mathbf{x}\mathbf{x}^{'}\mathbf{C}_1^*\mathbf{x} + \mathbf{Q}\mathbf{Q}^*\mathbf{Q}\mathbf{C}_2^*\mathbf{x} \\
& -(1-\delta)L[\mathbf{x}\ \mathbf{tr}(\mathbf{C}_1^*\mathbf{Q}) + 2\mathbf{Q}\mathbf{C}_1^*\mathbf{x}] - (1-\delta)\mathbf{Q}\mathbf{C}_2^*\mathbf{x}\ \mathbf{tr}(\mathbf{M}\mathbf{A}) \\
& +[-3\delta+\delta]\mathbf{Q}\mathbf{D}'\mathbf{F}\mathbf{D}\mathbf{x} + O_p(n^{-1/2}) \\
= & (\delta-1)\kappa\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{x} + \mathbf{x}\mathbf{x}^{'}\mathbf{C}_1^*\mathbf{x} + \mathbf{Q}\mathbf{Q}^*\mathbf{Q}\mathbf{C}_2^*\mathbf{x} \\
& -(1-\delta)L[\mathbf{x}\ \mathbf{tr}(\mathbf{C}_1^*\mathbf{Q}) + 2\mathbf{Q}\mathbf{C}_1^*\mathbf{x}] - (1-\delta)\mathbf{Q}\mathbf{C}_2^*\mathbf{x}\ \mathbf{tr}(\mathbf{M}\mathbf{A}) \\
& +O_p(n^{-1/2}).
\end{aligned}$$

Also the second order conditional moments of $\mathbf{e}_1^*$ under **Assumption II** can be represented as

$$\begin{aligned}
(4.18)\ & E[\mathbf{e}_1^*\mathbf{e}_1^{*'}|\tilde{\mathbf{e}}_0 = \mathbf{x}] \\
& = (2+\kappa)\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{Q} + \mathbf{x}^{'}\mathbf{C}_1^*\mathbf{x}\mathbf{x}\mathbf{x}^{'} + \mathbf{Q}\mathbf{Q}^*\mathbf{x}^{'}\mathbf{C}_2^*\mathbf{x} \\
& \quad +\mathbf{Q}\mathbf{C}_2^*\mathbf{Q}\mathbf{tr}(\mathbf{M}\mathbf{A}) + (1-\delta)^2L(L+2)\mathbf{Q}\mathbf{C}_1^*\mathbf{Q} - (1-\delta)L[\mathbf{Q}\mathbf{C}_1^*\mathbf{x}\mathbf{x}^{'} + \mathbf{x}\mathbf{x}^{'}\mathbf{C}_1^*\mathbf{Q})] \\
& \quad +O_p(n^{-1/2}).
\end{aligned}$$

Next, we consider the characteristic function of the standardized estimator $\hat{\mathbf{e}}$ in order to derive the asymptotic expansion of the distribution. We shall calculate

$$
\begin{aligned}
(4.19) \qquad C(t) &= E[\exp(i\mathbf{t}^{'}\mathbf{x})] \\
&+\frac{1}{\sqrt{n}}E[i\mathbf{t}^{'}E(\mathbf{e}_1^*|\mathbf{x})\exp(i\mathbf{t}^{'}\mathbf{x})] \\
&+\frac{1}{2n}E\{2i\mathbf{t}^{'}E(\mathbf{e}_2^*|\mathbf{x})\exp(i\mathbf{t}^{'}\mathbf{x})+i^2\mathbf{t}^{'}E(\mathbf{e}_1^*\mathbf{e}_1^{*'}|\mathbf{x})\mathbf{t}\exp(i\mathbf{t}^{'}\mathbf{x})\}+O(\frac{1}{n\sqrt{n}})\ ,
\end{aligned}
$$

where $\mathbf{x}=\tilde{\mathbf{e}}_0$, $\mathbf{t}=(t_i)$ is a $p\times 1$ vector of real variables and $i^2=-1$. By modifying the Fourier Inversion Formulae developed by *Appendix B* of Fujikoshi et. al. (1982), we can invert the characteristic function in (4.19). Although the intermediate computations are quite tedious but straightforward. We first consider the asymptotic expansion of the density function of $\tilde{\mathbf{e}}_0$ and we know that its limiting distribution as $n\to+\infty$ is normal. By expanding its characteristic function $E[exp(i\mathbf{t}^{'}\tilde{\mathbf{e}}_0)]$ and inverting it as the standard practice in the asymptotic theory, we have

$$
\begin{aligned}
(4.20) \qquad & \phi_{\mathbf{Q}}^*(\xi) \\
&=\phi_{\mathbf{Q}}(\xi)\Big\{1+\frac{1}{6\sqrt{n}}\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}h_3(\xi_l,\xi_{l^{'}},\xi_{l^{''}}) \\
&\quad+\frac{1}{24n}\sum_{l,l^{'},l^{''},l^{'''}=1}^{p}\beta_{l,l^{'},l^{''},l^{'''}}h_4(\xi_l,\xi_{l^{'}},\xi_{l^{''}},\xi_{l^{'''}}) \\
&\quad+\frac{1}{72n}\sum_{l,l^{'},l^{''},m,m^{'},m^{''}=1}^{p}\beta_{l,l^{'},l^{''}}\beta_{m,m^{'},m^{''}}h_6(\xi_l,\xi_{l^{'}},\xi_{l^{''}},\xi_m,\xi_{m^{'}},\xi_{m^{''}})\Big\} \\
&+O(n^{-3/2}),
\end{aligned}
$$

where $\phi_{\mathbf{Q}}(\xi)$ is the p-dimensional normal density function with means $\mathbf{0}$ and the variance-covariance matrix $\mathbf{Q}$,

$$
\begin{aligned}
\beta_{l,l^{'},l^{''}} &= \mu_3\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}z_{il}^*z_{il^{'}}^*z_{il^{''}}^*\ , \\
\beta_{l,l^{'},l^{''},l^{'''}} &= \mu_4(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}z_{il}^*z_{il^{'}}^*z_{il^{''}}^*z_{il^{'''}}^*) \\
&\quad -\sum_{l,l^{'},l^{''},l^{'''}}\sigma^4[\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}z_{il}^*z_{il^{'}}^*][\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}z_{il^{''}}^*z_{il^{'''}}^*],
\end{aligned}
$$

where $\mu_4=E[u_i^4]$, $\mathbf{z}_i^*=(z_i^{(l)*})=\mathbf{Q}\mathbf{D}^{'}\mathbf{M}\mathbf{C}^{-1}\mathbf{z}_i$ $(i=1,\cdots,n)$, and $\sum_{l,l^{'},l^{''},l^{'''}}$ means the combinations of two pairs such as $(l,l^{'})$ and $(l^{''},l^{'''})$ (it is 3 when $l=l^{'}=l^{''}=l^{'''}$, for instance). Also $h_3(x_l,x_{l^{'}},x_{l^{''}})$ and $h_4(x_l,x_{l^{'}},x_{l^{''}},x_{l^{'''}})$ are the Hermitian polynomials as

$$
(4.21) \qquad h_3(x_l,x_{l^{'}},x_{l^{''}}) = (-1)^3\frac{1}{\phi_{\mathbf{Q}}(x)}\frac{\partial^3\phi_{\mathbf{Q}}(x)}{\partial x_l\partial x_{l^{'}}\partial x_{l^{''}}},
$$

$$
(4.22) \qquad h_4(x_l,x_{l^{'}},x_{l^{''}},x_{l^{'''}}) = (-1)^4\frac{1}{\phi_{\mathbf{Q}}(x)}\frac{\partial^4\phi_{\mathbf{Q}}(x)}{\partial x_l\partial x_{l^{'}}\partial x_{l^{''}}\partial x_{l^{'''}}}.
$$

We notice that (4.20) is common for all efficient estimators and then it does not make any effects on the comparisons of alternative efficient estimators. When the third order moments of disturbances are zeros, the terms of $O_p(n^{-1/2})$ on the right-hand side vanish (i.e. $\beta_{l,l',l''} = 0$) and we only have extra terms in the order of n^{-1} . In theses cases we can directly use the Fourier inversion formulae reported in Appendix B because only terms on the effects of non-normality of disturbance terms appear as $\mathbf{QD}^{'}\mathbf{FDx}$ in the order of $O_p(n^{-1})$ and the resulting expressions become considerably simplified. Also we notice that when the disturbance terms are homoscedastic as in Assumption I, we have $\mathbf{C} = \sigma^2\mathbf{M}$, $\mathbf{Q}^* = \mathbf{D}^{'}\mathbf{MC}^{-1}\mathbf{MC}^{-1}\mathbf{MD} = \sigma^{-2}\mathbf{Q}^{-1}$, and $\mathbf{tr}(\mathbf{MA}) = \sigma^{-2}L$. Then we can state our main result after lengthy but straightforward computations by using the formulas given in Appendix B.

Theorem 4.1 : Under **Assumption I** and **Assumption II**, the asymptotic expansion of the joint density function of $\hat{\mathbf{e}}^*$ for the class of modified MEL estimators as $n \to \infty$ is given by

$$
\begin{aligned}
& f(\xi) = \phi^*_{\mathbf{Q}}(\xi) \\
& + \frac{1}{\sqrt{n}}\phi_{\mathbf{Q}}(\xi)(\mathbf{q}^{'}\xi)[p+1+(1-\delta)L - \xi^{'}\mathbf{Q}^{-1}\xi] \\
& + \frac{1}{2n}\phi_{\mathbf{Q}}(\xi)\Big(\xi^{'}\mathbf{C}_1\xi\{[p+1+(1-\delta)L-\xi^{'}\mathbf{Q}^{-1}\xi]^2 + p + 1 - 3\xi^{'}\mathbf{Q}^{-1}\xi + 2(1-\delta)^2 L\} \\
& \qquad + \mathbf{tr}(\mathbf{C}_1\mathbf{Q})[(1-\delta)L][2-(1-\delta)(L+2)] \\
& \qquad + \xi^{'}\mathbf{C}_2\xi\{L[1-2(1-\delta)] - p - 2 + \xi^{'}\mathbf{Q}^{-1}\xi\} + \mathbf{tr}(\mathbf{C}_2\mathbf{Q})\{L[2(1-\delta)-1]\} \\
& \qquad + [2+(2\delta-1)\kappa][\xi^{'}\mathbf{D}^{'}\mathbf{FD}\xi - \mathbf{tr}(\mathbf{D}^{'}\mathbf{FDQ})]\Big) \\
& + o(\frac{1}{n}) \ ,
\end{aligned}
$$

where ξ is a $p \times 1$ ($p = G_1 + K_1$) vector, $\phi^*_{\mathbf{Q}}(\xi)$ and $\mathbf{F}$ are given by (4.20) and (4.10), respectively, and $\mathbf{C}_1 = \mathbf{C}_1^*$ $(= \mathbf{q}\mathbf{q}^{'})$, $\mathbf{C}_2 = \sigma^{-2}\mathbf{C}_2^*$ $(= \sigma^{-2}E[\mathbf{w}_i\mathbf{w}_i^{'}])$, and $\kappa = [E(u_i^4) - 3\sigma^4)/\sigma^4$.

When the disturbanceterms are normally distributed all terms except the leading term vanish in (4.20) and $\phi^*_{\mathbf{Q}}(\mathbf{x}) = \phi_{\mathbf{Q}}(\mathbf{x})$. There is an interesting observation in Theorem 4.1 that if we further drop the last term

$$[2+(2\delta-1)\kappa][\xi^{'}\mathbf{D}^{'}\mathbf{FD}\xi - \mathbf{tr}(\mathbf{D}^{'}\mathbf{FDQ})]$$

and the disturbance terms are normally distributed, the resulting formulas are identical to those for the limited information maximum likelihood (LIML) estimator and the two stage least squares (TSLS) estimator, which have been reported by Fujikoshi et. al. (1982).

By using the asymptotic expansion of the density function, we can evaluate the asymptotic mean and the asymptotic mean squared errors of the modified MEL estimator. We summarize the resulting formulas.

Theorem 4.2 : Under the assumptions of Theorem 4.1, the asymptotic bias and the asymptotic mean squared errors of $\hat{\mathbf{e}}^*$ for the modified (MEL) estimators based on the asymptotic expansion of the density function as $n \to \infty$ up to $O(n^{-1})$ are given by

$$ABIAS_n(\hat{\mathbf{e}}) = [(1-\delta)L - 1]\mathbf{Qq} \ , \tag{4.23}$$

and

$$(4.24) \quad AMSE_n(\hat{\mathbf{e}})$$
$$= \mathbf{Q} + \frac{1}{n}\Big\{\mathbf{Q}\mathbf{C}_1\mathbf{Q}[6-6(1-\delta)L+(1-\delta)^2L(L+2)]$$
$$+\mathbf{Q}\mathbf{tr}(\mathbf{C}_1\mathbf{Q})[3-2(1-\delta)L]+\mathbf{Q}\mathbf{tr}(\mathbf{C}_2\mathbf{Q})+[L+2-2L(1-\delta)]\mathbf{Q}\mathbf{C}_2\mathbf{Q}$$
$$+[2+(2\delta-1)\kappa]\mathbf{Q}\mathbf{D}'\mathbf{F}\mathbf{D}\mathbf{Q}\Big\} \ ,$$

where we use the notations in Theorem 4.1.

4.3 A Simple Case

We notice that the exact density functions of estimators and their asysmptotic expansions are quite complicated in the general case. Hence it is interesting to derive the asymptotic expansions of the distribution functions of estimators in the simplest case when $G_1 = 1$. We take the estimator on the coefficient of an endogenerous variables in the right hand side and standardize

$$(4.25) \qquad \mathbf{P}\Big(\frac{\sqrt{\mathbf{\Pi}_{22}'\mathbf{A}_{22.1}\mathbf{\Pi}_{22}}}{\sigma}(\hat{\beta}-\beta)\le x\Big)$$

since its limiting distribution is the standard normal.

In the univariate case we use the notation

$$Q_{11} = \sigma^2\left(\mathbf{\Pi}_{22}'\mathbf{M}_{22.1}\mathbf{\Pi}_{22}\right)^{-1}$$

as the $(1,1)-$element of $\mathbf{Q}$ and we partition a $[1+(p-1)]\times[1+(p-1)]$ matrix as

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix} .$$

The right-hand side of $\phi^*(x)$ for the standardized estimator in (4.25) can be simplified and it is in the form of

$$(4.26)\phi(x)\{1+\frac{1}{\sqrt{n}}[\beta_3(x^3-3x)]+\frac{1}{n}[\frac{\beta_4}{24}(x^4-6x^2+3)+\frac{\beta_3^2}{72}(x^6-15x^4+45x^2-15)]\},$$

where β_3 and β_4 are the third and fourth order culumants in (4.20) by replacing z_i^{**} $(= Q_{11}^{-1/2}z_i^{(1)*})$ for $\mathbf{z}_i^*$ $(i=1,\cdots,n)$ and $\phi(x)$ is the density function of the standard normal distribution.

For any $p-$dimensional normal density $\phi_{\mathbf{Q}}(\xi)$, we partition any $[1+(p-1)]\times[1+(p-1)]$ matrix $\mathbf{B}$ and p $(=[1+(p-1)])$ vector $\xi' = (\xi_1,\xi_2')'$, we can write

$$(4.27) \qquad \int_{\mathbf{R}^{p-1}}[\xi'\mathbf{B}\xi-\mathbf{tr}(\mathbf{B}\mathbf{Q})]\phi_{\mathbf{Q}}(\xi)d\xi_2 = E_{\xi_2|\xi_1}\{\mathbf{tr}[\mathbf{B}(\xi\xi'-\mathbf{Q})]\phi_{Q_{11}}(\xi_1)\} \ .$$

By using the fact that $\xi_2|\xi_1$ follows the $(p-1)-$dimensional (conditional) normal distribution $N_{p-1}[\mathbf{Q}_{21}Q_{11}^{-1}\xi_1,\mathbf{Q}_{22.1}]$ and $\mathbf{Q}_{22.1}=\mathbf{Q}_{22}-\mathbf{Q}_{21}Q_{11}^{-1}\mathbf{Q}_{21}$, we can evaluate (4.26) explicitly and it is rewritten as

$$(4.28) \quad \mathbf{tr}\left[\mathbf{B}\begin{pmatrix} \xi_1^2-Q_{11} & \xi_1^2Q_{11}^{-1}\mathbf{Q}_{12}-\mathbf{Q}_{12} \\ \mathbf{Q}_{21}Q_{11}^{-1}\xi_1^2-\mathbf{Q}_{21} & \mathbf{Q}_{21}Q_{11}^{-1}\xi_1^2Q_{11}^{-1}\mathbf{Q}_{12}+\mathbf{Q}_{22.1}-\mathbf{Q}_{22}\end{pmatrix}\right]\phi_{Q_{11}}(\xi_1)$$
$$= \mathbf{tr}[Q_{11}^{-1}(1,\mathbf{0}')\mathbf{Q}\mathbf{C}\mathbf{Q}\begin{pmatrix}1\\ \mathbf{0}\end{pmatrix}Q_{11}^{-1}(\xi_1^2-Q_{11})]\phi_{Q_{11}}(\xi_1).$$

When the third order moments of disturbances are zeros, the asymptotic expansion of the density function of the standardized estimator can be simplified. We set the standardized form as (4.25) and notice that two matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ appeared in the asymptotic expansions have non-zero elements only in the upper-left parts. By evaluating the integrations with respect to the $(p-1)$ last elements of ξ in Theorem 4.1, the asymptotic expansion of the density becomes

$$\begin{aligned}(4.29)\qquad & \phi^*(x)+\frac{1}{\sqrt{n}}\phi(x)(-\alpha_* x)[2+(1-\delta)L-x^2] \\ & +\frac{1}{2n}\phi(x)\{\alpha_*^2x^2[(2+(1-\delta)L-x^2)^2+2-3x^2+2(1-\delta)^2L] \\ & +\alpha_*^2[(1-\delta)L][2-(1-\delta)(L+2)] \\ & +\eta_*\left(x^2[L(1-2(1-\delta))-3+x^2]+L[2(1-\delta)-1]\right) \\ & +\gamma_*[2+(2\delta-1)\kappa](x^2-1)\} \\ & +O_p(n^{-3/2})\ ,\end{aligned}$$

where $\phi^*(\cdot)$ is given by (4.26) with $\beta_3=0$, $\alpha_*=-Cov(\mathbf{v}_{2i},u_i)/|\mathbf{\Omega}|^{1/2}$, $\eta_*=|\Omega|/\sigma^2$, and

$$\gamma_*=(1,\mathbf{0})Q_{11}^{-1}\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{Q}\left(\begin{array}{c}1\\ \mathbf{0}\end{array}\right)\ .$$

If we further assume the normal disturbances, then the formula can be further simplified. By setting $\delta=1$ for the MEL estimator, we have

$$\begin{aligned}(4.30)\qquad & \mathbf{P}\left(\frac{\sqrt{\mathbf{\Pi}_{22}^{'}\mathbf{A}_{22.1}\mathbf{\Pi}_{22}}}{\sigma}(\hat{\beta}_{MEL}-\beta)\le x)\right) \\ = \ & \Phi(x)+\{-\frac{\alpha}{\mu}x^2-\frac{1}{2\mu^2}[(\gamma+L)x+(1-2\alpha^2)x^3+\alpha^2x^5]\}\phi(x) \\ & +O(\mu^{-3}),\end{aligned}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and the density function of the standard normal distribuntion, respectively, and

$$(4.31)\qquad \mu^2 = (1+\alpha^2)\frac{\mathbf{\Pi}_{22}^{'}\mathbf{A}_{22.1}\mathbf{\Pi}_{22}}{\omega_{22}}\ ,$$

$$(4.32)\qquad \alpha = \frac{\omega_{22}}{|\Omega|^{1/2}}(\beta-\frac{\omega_{12}}{\omega_{22}})\ ,$$

$$(4.33)\qquad \gamma = 2\frac{1+\alpha^2}{\omega_{22}}(1,\mathbf{0})Q_{11}^{-1}\mathbf{Q}\mathbf{D}^{'}\mathbf{F}\mathbf{D}\mathbf{Q}Q_{11}^{-1}\left(\begin{array}{c}1\\ \mathbf{0}\end{array}\right)\ .$$

Also under the normal disturbances and we set $\delta=0$ for the GMM estimator, the resulting asymptotic expansion of the distribution function becomes

$$\begin{aligned}(4.34)\qquad & \mathbf{P}\left(\frac{\sqrt{\mathbf{\Pi}_{22}^{'}\mathbf{A}_{22.1}\mathbf{\Pi}_{22}}}{\sigma}(\hat{\beta}_{GMM}-\beta)\le x)\right) \\ = \ & \Phi(x)+\{-\frac{\alpha}{\mu}[x^2-L]-\frac{1}{2\mu^2}[(\gamma+L^2\alpha^2)x+(1-2(L+1)\alpha^2)x^3+\alpha^2x^5]\}\phi(x) \\ & +O(\mu^{-3})\ .\end{aligned}$$

Furthermore if we set $\gamma = 0$ in the above expressions, the resulting formulas in (4.30) and (4.34) are identical to those for the limited information maximum likelihood (LIML) estimator and the two stage least squares (TSLS) estimator obtained by Anderson (1974), and Anderson and Sawa (1973), respectively.

5. Related Problems

5.1 Asymptotic Bias and MSE in the General Case

It is rather straightforward to derive the asymptotic bias of the estimator. By using the explicit expressions for $\mathbf{e}_1^{(0)}$ and $\mathbf{e}_0^{(1)}$ in (4.17), we have

$$(5.1)\quad ABIAS_n(\hat{\mathbf{e}}) = \frac{1}{\sqrt{n}}E[\mathbf{e}_1^{(0)} + \mathbf{e}_0^{(1)}] + O(n^{-1})$$
$$= \frac{1}{\sqrt{n}}\Big\{[(1-\delta)L - 1]\mathbf{Q}\mathbf{q} - (1-\delta)\kappa_3\mathbf{Q}\mathbf{D}^{'}\mathbf{m}_3\Big\} + O(n^{-1}).$$

On the other hand, it is quite tedious to obtain the explicit formula of the additional terms in the asymptotic MSE of the modified estimators in the general case. In principle it can be calculated by

$$(5.2)\quad AM_n(\hat{\mathbf{e}}\hat{\mathbf{e}}^{'})$$
$$= E\Big\{[\tilde{\mathbf{e}}_0 + \frac{1}{\sqrt{n}}(\mathbf{e}_0^{(1)} + \mathbf{e}_1^{(0)}) + \frac{1}{n}(\mathbf{e}_0^{(2)} + \mathbf{e}_1^{(1)} + \mathbf{e}_2)]$$
$$\times[\tilde{\mathbf{e}}_0 + \frac{1}{\sqrt{n}}(\mathbf{e}_0^{(1)} + \mathbf{e}_1^{(0)}) + \frac{1}{n}(\mathbf{e}_0^{(2)} + \mathbf{e}_1^{(1)} + \mathbf{e}_2)]^{'}\Big\} + O(n^{-1})\ .$$

Let $AMSE^{*}(\hat{\mathbf{e}})$ be $AM(\hat{\mathbf{e}}\hat{\mathbf{e}}^{'})$ in Theorem 4.2. Then there are eight additional terms for an arbitrary δ $(0 \leq \delta \leq 1)$ when we cannot ignore the effects of third order moments of disturbance terms. For the MEL estimator case, however, there are only four additional terms. Although it is straightforward to write down those terms, we have omitted to report the details since they are complicated and may not be useful at the present stage of our investigation.

5.2 Related Applications and Remarks

In this paper we have given the details of derivations of the asymptotic expansions of the density functions for the class of semi-parametric estimators including the MEL estimator and the GMM estimator. There have been related works and applications of the asymptotic expansions of the density functions of the alternative estimators reported in this paper. Kunitomo and Matsushita (2003) have investigated the finite sample properties of the distribution functions the MEL and GMM estimators, and have given extensive tables when $G_1 = 1$. In the more general case, however, it would not be possible to investigate the finite sample properties and the asymptotic expansion method should be useful for comparing different estimators.

There can some modifications of the MEL and GMM estimators. Kunitomo (2002) has introduced a class of modified MEL estimator by using the asymptotic expansion method although he did not give the details of the derivations. Also the problem should be closely related with the higher order efficiency of estimation. Takeuchi and Morimune (1985) gave the classic result on the econometric estimation problem while Newey and Smith (2001) have one conjecture in the semi-parametric framework. However, it needs some careful investigations and our related results shall be reported in a further occasion.

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Appendices

In Appendix A, we give the proofs of two lemmas used in Section 4. In Appendix B we gather some useful formulae on the Fourier inversion because it seems they are not readily available.

Appendix A : Proof of Lemmas

[A-1] : Proof of Lemma 4.1
Let $X_1 = (\mathbf{Y}_n)_{ij}$, $X_2 = (\mathbf{A}\mathbf{X}_n)_k$ and $X_3 = (\tilde{\mathbf{e}}_0)_l$. Since the limiting distribution of random vector $(X_1, X_2, X_3)^{'}$ is normal, the conditional distribution of $(X_1, X_2)^{'}$ given X_3 is also asymptotically normal. Then we have

$$E[X_1X_2|X_3] \;\cong\; E[X_1|X_3]E[X_2|X_3] + [Cov(X_1, X_2) - \frac{Cov(X_1, X_3)Cov(X_2, X_3)}{Var(X_3)}].$$

Because X_2 and X_3 are asymptotically orthogonal, $E[X_2|X_3] \cong 0$ and $Cov(X_2, X_3) \cong 0$. Also by noting that

$$\text{(A.3)} \qquad Cov(X_1, X_2) \cong \frac{1}{n}\sum_{\alpha=1}^{n} z_\alpha^{(i)} z_\alpha^{(j)} (\mathbf{A}\mathbf{z}_\alpha)_k E[u_\alpha^3],$$

we have the result. (Q.E.D)

[A-2] : Proof of Lemma 4.3
Let $\mathbf{z}_n = (\mathbf{u}_n^{'}, v_n)^{'}$ be a $(p+1) \times 1$ random vector which is a sum of i.i.d. random vectors $\mathbf{z}_j^{(n)}$:

$$\text{(A.4)} \qquad \mathbf{z}_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} \mathbf{z}_j^{(n)}$$

and $E[\mathbf{z}_j^{(n)}] = \mathbf{0}$, $E[\mathbf{z}_j^{(n)}\mathbf{z}_j^{(n)'}] = \mathbf{\Sigma} > \mathbf{0}$. Then under a set of moment conditions the characteristic function of $\mathbf{z}_n$ can be expressed as

$$\begin{aligned} \text{(A.5)} \qquad \varphi(\mathbf{t}) &= E[e^{i\mathbf{t}^{'}\mathbf{z}_n}] \\ &= \prod_{j=1}^{n} E[e^{i\sum_{k=1}^{p+1} t_j z_{jk}^{(n)}}] \\ &= e^{-\frac{1}{2}\mathbf{t}^{'}\mathbf{\Sigma}\mathbf{t}}\{1 + \frac{1}{6\sqrt{n}}\sum_{l,l^{'},l^{''}=1}^{p+1} \beta_{l,l^{'},l^{''}}(it_l)(it_{l^{'}})(it_{l^{''}})\} + O(n^{-1}) , \end{aligned}$$

where $\beta_{l,l^{'},l^{''}}$ are the third order moments of $\mathbf{z}_j^{(n)}$. Then the density function of $\mathbf{z}_n$ has a representation

$$\text{(A.6)} \qquad f_n(\mathbf{z}) = \phi_\Sigma(\mathbf{z})\Big\{1 + \frac{1}{6\sqrt{n}}\sum_{l,l^{'},l^{''}=1}^{p+1} \beta_{l,l^{'},l^{''}} h_3(z_l, z_{l^{'}}, z_{l^{''}})\Big\} + O(n^{-1}) ,$$

where $h_3(z_l, z_{l^{'}}, z_{l^{''}})$ are the third-order Hermitian polynomials and we set a $(p+1) \times (p+1)$ variance-covariance matrix of $\mathbf{z}_n$ as

$$\Sigma = \begin{pmatrix} \mathbf{I}_p & \rho \\ \rho^{'} & 1 \end{pmatrix}$$

for the mathematical convenience. Let $f_n(\mathbf{u}_n)$ be the marginal density and $f_n(v_n|\mathbf{u}_n)$ be the conditional density, which can be represented as

$$
\begin{aligned}
\text{(A.7)} \quad & f_n(v_n|\mathbf{u_n}) \\
& = \phi(v|\rho^{'}\mathbf{u}_n, 1-\rho^{'}\rho) \\
& \quad \times\Big\{1+\frac{1}{6\sqrt{n}}[\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}h_{3,\cdot}(u_l,u_{l^{'}},u_{l^{''}})+3\sum_{l,l^{'}}^{p}\beta_{l,l^{'},p+1}h_{3,\cdot}(u_l,u_{l^{'}},v_n) \\
& \quad +3\sum_{l=1}^{p}\beta_{l,p+1,p+1}h_{3,\cdot}(u_l,v_n,v_n)+\beta_{p+1,p+1,p+1}h_3(v_n,v_n,v_n) \\
& \quad -\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}h_3(u_l,u_{l^{'}},u_{l^{''}})]\Big\} \\
& +O(n^{-1}) \;,
\end{aligned}
$$

where $\phi(v|\rho^{'}\mathbf{u}_n, 1-\rho^{'}\rho)$ is the conditional density function and $h_{3,\cdot}(\cdot)$ and $h_3(\cdot)$ are the third order Hermitian polynomials. Then the conditional expectation is given by

$$
\begin{aligned}
& E[v_n|\mathbf{u}_n] \\
= & \int v_n f_n(v_n|\mathbf{u}_n)dv_n \\
= & \rho^{'}\mathbf{u}_n+\frac{1}{6\sqrt{n}}\Big\{\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}\int v(-1)^3\frac{\partial^3 f_n(\mathbf{u}_n,v)}{\partial u_l\partial u_{l^{'}}\partial u_{l^{''}}}\frac{1}{f_n(\mathbf{u}_n)}dv \\
& +3\sum_{l,l^{'}=1}^{p}\beta_{l,l^{'},p+1}(-1)^3\frac{\partial^2}{\partial u_l\partial u_{l^{'}}}\int v\frac{\partial f_n(\mathbf{u}_n,v)}{\partial v}\frac{1}{f_n(\mathbf{u}_n)}dv \\
& +3\sum_{l=1}^{p}\beta_{l,p+1,p+1}(-1)^3\frac{\partial}{\partial u_l}\int v\frac{\partial^2 f_n(\mathbf{u}_n,v)}{\partial v^2}\frac{1}{f_n(\mathbf{u}_n)}dv \\
& +\beta_{p+1,p+1,p+1}(-1)^3\int v\frac{\partial^3 f_n(\mathbf{u}_n,v)}{\partial v^3}\frac{1}{f_n(\mathbf{u}_n)}dv \\
& -(\rho^{'}\mathbf{u}_n)\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}h_3(u_l,u_{l^{'}},u_{l^{''}})\Big\}+O(n^{-1}).
\end{aligned}
$$

By using the integral by parts calculations, the third term and the fourth term of the right-hand side of $O(n^{-1/2})$ are zeros. Hence

$$
\begin{aligned}
& E[v_n|\mathbf{u}_n] \\
& = \rho^{'}\mathbf{u}_n \\
& \quad +\frac{1}{6\sqrt{n}}\Big\{(-1)\sum_{l,l^{'},l^{''}=1}^{p}\beta_{l,l^{'},l^{''}}[\frac{\partial^3 f_n(\cdot)}{\partial u_l\partial u_{l^{'}}\partial u_{l^{''}}}(\rho^{'}\mathbf{u}_n f_n(\mathbf{u}_n))]/f_n(\mathbf{u}_n)
\end{aligned}
$$

$$+3\sum_{l,l'=1}^{p}\beta_{l,l',p+1}[\frac{\partial^2}{\partial u_l \partial u_{l'}} f_n(\mathbf{u}_n)]/f_n(\mathbf{u}_n) - \rho^{'}\mathbf{u}_n \sum_{l,l',l''=1}^{p}\beta_{l,l',l''}h_3(u_l,u_{l'},u_{l''})\Big\} + O(n^{-1}).$$

$$= \rho^{'}\mathbf{u}_n$$

$$+\frac{1}{6\sqrt{n}}\Big\{3\sum_{l,l'=1}^{p}\beta_{l,l',p}h_2(u_l,u_{l'}) - \sum_{l,l',l''=1}^{p}\beta_{l,l',l''}[\rho^{'}\mathbf{u}_n h_3(u_l,u_{l'},u_{l''})]$$

$$+\sum_{l,l',l''=1}^{p}\beta_{l,l',l''}[\rho^{'}\mathbf{u}_n h_3(u_l,u_{l'},u_{l''}) - \rho_l h_2(u_{l'},u_{l''}) - \rho_{l'} h_2(u_{l'},u_{l''}) - \rho_{l''} h_2(u_{l'},u_{l''})]\Big\}$$

$$+O(n^{-1}) ,$$

where $h_2(u_l, u_{l'})$ are the second order Hermite polynomials. Since two terms in the above expressions on the right-hand side are cancelled out, we have the desired result. (Q.E.D.)

Appendix B : Useful Formulas

This appendix gives the useful formulas, which correspond to the inversion of the characteristic function from the conditional expectations given $\mathbf{z}$ and $\mathbf{z}$ follows the p-dimensional normal distribution $N_p(\mathbf{0},\mathbf{Q})$. All inversion results we have needed in Sections 3 and 4 can be reduced the Fourier Inversion formulas for the density function as

(A.8) $$\mathcal{F}^{-1}\{h(-i\mathbf{t})\mathcal{E}[g(\mathbf{z})\exp(i\mathbf{t}^{'}\mathbf{z})]\} = h(\frac{\partial}{\partial \xi})g(\xi)\phi_{\mathbf{Q}}(\xi)$$

for any polynomials $h(\cdot)$ and $g(\cdot)$, where $\mathbf{t} = (t_i)$ is a $p \times 1$ vector, $i^2 = -1$, and the differentiation vector

$$\frac{\partial}{\partial \xi'} = (\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_p}) .$$

The method adopted here was developed by Fujikoshi et. al. (1982) and they were given by Anderson et. al. (1986). We present the useful results because it seems they are not readily available.

Lemma B.1 : Let $\alpha^{'} = (\alpha_1, \cdots, \alpha_p)$ be a $1 \times p$ constant vector and $\mathbf{A}$ be a symmetric constant matrix and $p = G_1 + K_1$. Then

(A.9) $$\frac{\partial}{\partial \xi'}[\alpha\phi_{\mathbf{Q}}(\xi)] = -\alpha\mathbf{Q}^{-1}\xi\phi_{\mathbf{Q}}(\xi) ,$$

(A.10) $$\frac{\partial}{\partial \xi'}[\xi(\alpha^{'}\xi)\phi_{\mathbf{Q}}(\xi)] = (\alpha^{'}\xi)(p+1-\xi^{'}\mathbf{Q}^{-1}\xi)\phi_{\mathbf{Q}}(\xi) ,$$

(A.11) $$\frac{\partial}{\partial \xi'}[\mathbf{Q}\mathbf{A}\xi\phi_{\mathbf{Q}}(\xi)] = [\mathbf{tr}(\mathbf{A}\mathbf{Q}) - \xi^{'}\mathbf{A}\xi)]\phi_{\mathbf{Q}}(\xi) ,$$

(A.12) $$\frac{\partial}{\partial \xi'}[\xi\xi^{'}\mathbf{A}\phi_{\mathbf{Q}}(\xi)] = (\xi^{'}\mathbf{A}\xi)(\alpha^{'}\xi)(p+2-\xi^{'}\mathbf{Q}^{-1}\xi)\phi_{\mathbf{Q}}(\xi) .$$

Lemma B.2 : Let $\mathbf{A}$ and other notations be the same as Lemma B.1. Then we have the following relations.

(A.13)
$$\mathbf{tr}\frac{\partial^2}{\partial\xi\partial\xi'}[\mathbf{QAQ}\phi_{\mathbf{Q}}(\xi)]$$
$$= [\xi^{'}\mathbf{A}\xi - \mathbf{tr}(\mathbf{AQ})]\phi_{\mathbf{Q}}(\xi) \ ,$$

(A.14)
$$\mathbf{tr}\frac{\partial^2}{\partial\xi\partial\xi'}[\mathbf{Q}\xi^{'}\mathbf{A}\xi\phi_{\mathbf{Q}}(\xi)]$$
$$= [2\mathbf{tr}(\mathbf{AQ}) - (p+4-\xi^{'}\mathbf{Q}^{-1}\xi)\xi^{'}\mathbf{A}\xi]\phi_{\mathbf{Q}}(\xi) \ ,$$

(A.15)
$$\mathbf{tr}\frac{\partial^2}{\partial\xi\partial\xi'}[\mathbf{QA}\xi\xi^{'}\phi_{\mathbf{Q}}(\xi)]$$
$$= [(p+1-\xi^{'}\mathbf{Q}^{-1}\xi)(\mathbf{tr}(\mathbf{AQ}) - \xi^{'}\mathbf{A}\xi)) - 2\xi^{'}\mathbf{A}\xi]\phi_{\mathbf{Q}}(\xi) \ ,$$

(A.16)
$$\mathbf{tr}\frac{\partial^2}{\partial\xi\partial\xi'}[\xi\xi^{'}\xi^{'}\mathbf{A}\xi\phi_{\mathbf{Q}}(\xi)]$$
$$= \xi^{'}\mathbf{A}\xi[(p+1-\xi^{'}\mathbf{Q}^{-1}\xi)^2 + 3(p+1) + 2 - 5\xi^{'}\mathbf{Q}^{-1}\xi)]\phi_{\mathbf{Q}}(\xi) \ ,$$

where $\mathbf{tr}\frac{\partial^2}{\partial\xi\partial\xi'}[\,\cdot\,]$ stands for $\sum_i\sum_j \partial^2/\partial\xi_i\partial\xi_j[\,\cdot\,]_{ij}$.