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Kazuya Kamiya
University of Tokyo

Noritsugu Morishita
Kaichi High School

Takashi Shimizu
Kansai University

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Kazuya Kamiya†, Noritsugu Morishita‡ and Takashi Shimizu§

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Abstract

This paper investigates Zhou [4]'s money search model, where money is divisible, agents can hold any amount of money, and production of goods is costly, and presents a sufficient condition, expressed in terms of exogenously given parameters, for the existence of single-price equilibria.

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1 Introduction

In their influential paper, Kiyotaki and Wright [3] successfully constructed a matching model where money has value as a medium of exchange. However, they made several simplifying assumptions, such as indivisibility of money and an inventory constraint of money holdings. This paper investigates Zhou [4]'s money search model, where money is divisible, agents can hold any amount of money, and production of goods is costly, and presents a sufficient condition, expressed in terms of exogenously given parameters, for the existence of single-price equilibria.

In Kiyotaki and Wright [3], it is assumed that money is indivisible and that agents can hold at most one unit of money. These assumptions crucially limit the applicability of the model. Subsequently, Green and Zhou [1] presented a model where money is divisible and agents can hold any amount of money, and show the existence of a stationary equilibrium. However, agents can costlessly produce goods in their model.

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†Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: kkamiya@e.u-tokyo.ac.jp)
‡Kaichi High School, 186 Tokuriki-Nishi, Iwatsuki, Saitama 339-0004, JAPAN
§Faculty of Economics, Kansai University, 3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680 JAPAN (E-mail: tshimizu@ipcku.kansai-u.ac.jp)

This assumption introduces an unrealistic feature of equilibria; there is no upper bound of money holdings in stationary equilibria because of costless production.

Introducing production cost into Green and Zhou's model, Zhou [4] eliminated the unrealistic feature in the model. In other words, every single-price equilibrium has an upper bound of money holdings beyond which agents are not willing to hold. She also presented a sufficient condition, expressed in terms of endogenously determined variables, for the existence of equilibria. However, it is next to impossible to convert it into a condition in terms of exogenously given parameters, since we need to solve high order polynomial equations. It is worthwhile noting that she could successfully presented a sufficient condition, expressed in terms of exogenously given parameters, only for a simple case: the case that the upper bound of money holdings is one unit. The existence of the other type of single-price equilibria has not been shown yet.

In this paper, we present a sufficient condition, expressed in terms of exogenously given parameters, for the existence of single-price equilibria with an arbitrary upper bound of money holdings in Zhou's model, and show that there always exists a region of parameters in which there is a single-price equilibrium. As is known, it becomes too hard to solve Bellman equations, as matching models with money become complicated. Kamiya and Shimizu [2] suggested a way to overcome this difficulty in matching model with divisible money. They showed that there generically exists a continuum of stationary equilibria in such models and at least one of the endpoints is typically easily obtained. Thus following equilibria from the endpoint, we can analyze some properties of equilibria. Following this line, we present a sufficient condition.

The plan of this paper is as follows. We present Zhou [4]'s model in Section 2 and main results in Section 3.

2 The Model

We first present Zhou [4]'s model. There is a continuum of agents of which measure is one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Only one unit of indivisible and perishable good i can be produced by a type $i - 1 \pmod{k}$ agent with production cost $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. We assume $u > c$. There is completely divisible and durable fiat money of which nominal stock is $M > 0$. Time is continuous, and pairwise random matchings take place according to Poisson process with a parameter $\mu > 0$. Let $\gamma > 0$ be the discount rate common to all agents.

Since the consumption goods are perishable and there is no double coincidence of wants, all trade should involve fiat money as a medium of exchange. Each agent is characterized by her type and the amount of money she holds. We assume that, in any matching, a partner's type is observable, but not her money holding, and that an agent knows the distribution of money holdings of the economy. Production and transaction occur according to a seller-posting-price protocol as follows. When a type i agent who has fiat money (potential buyer) meets a type $i-1$ agent (potential seller) who can produce the buyer's desired consumption good, the seller posts an offer first. Then the buyer decides to accept or reject it. Production and transaction occur if and only if the offer is accepted.

We will focus on stationary equilibria where the strategy that agents with an identical money holding and an identical type choose is symmetric and time-invariant. Therefore, we will hereafter discuss a generic type i.

Let $\eta \in \mathbb{R}_+$ denote an agent's money holding. A strategy of type i agent is defined as a pair of an offer strategy $\omega(\eta) : \mathbb{R}_+ \to \mathbb{R}_+$ and a reservation price strategy $\rho(\eta) : \mathbb{R}_+ \to \mathbb{R}_+$. The former is a price that a type i agent with money holding η offers when she meets a potential buyer. A seller with money holding η offers $\omega(\eta)$. In case that a value function is continuous from the right, it will be shown that by the perfectness condition ρ gives the maximum price that a buyer is willing to defray for the consumption good, and so it becomes a function rather than a correspondence. Of course, since the reservation price cannot exceed the buyer's money holdings, ρ should satisfy the following feasibility condition:

$$\rho(\eta) \leq \eta. \tag{1}$$

Let H, the money holding distribution, be a distribution defined on $\mathbb{R}_+$. From H, the stationary distribution of offer prices, Ω, and the stationary distribution of reservation prices, R, are defined as follows.

$$\Omega(x) = H\{o | o \leq x\}, \tag{2}$$

$$R(x) = H\{r | r < x\}. \tag{3}$$

We define R to be continuous from the left.

Let $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}_+$ be a value function. That is $\mathcal{V}(\eta)$ is the maximum value of discounted utility achievable by the agent's current money holding η. At every moment, a type i agent with money holding η meets a type $i-1$ agent with probability μ/k.

Transaction does not occur and money holding does not change if the partner's offer x exceeds the type i's reservation price r. If partner's offer price x is not more than reservation price r, then transaction occurs and the type i agent derives utility u from consumption and enters in the next trading opportunity with money holding $\eta - x$. The probability that type i with money holding η meets a type $i+1$ agent is also μ/k. Transaction does not occur if the type i's offer o is greater than the partner's reservation price. If type i's offer o does not exceed the partner's reservation price, then transaction occurs and faces the next matching opportunity with money holding $\eta + o$. Then, using γ, μ, Ω, and R, the Bellman equation for $\mathcal{V}(\eta)$ is given by

$$\gamma \mathcal{V}(\eta) = \frac{\mu}{k} \max_{r \in [0,\eta]} \int_0^r [u + \mathcal{V}(\eta - x) - \mathcal{V}(\eta)] d\Omega(x) + \frac{\mu}{k} \max_{o \in \mathbb{R}_+} [1 - R(o)][\mathcal{V}(\eta + o) - c - \mathcal{V}(\eta)]. \tag{4}$$

Some remarks on $\mathcal{V}(\eta)$ as follows. $\mathcal{V}(\eta)$ is nonnegative, since an agent can always choose $r = 0$, i.e., she can always refrain from purchase. $\mathcal{V}(\eta)$ is bounded above, since consumption opportunities occur with $1/\mu$ intervals on average and the utility should be discounted.

In terms of $\mathcal{V}(\eta)$, it is optimal to accept offer o if $u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)$. The same condition in terms of reservation price ρ is $\rho(\eta) \geq o$. Then, in case that a value function is continuous from the right, the perfectness condition with respect to reservation price is as follows:

$$\rho(\eta) = \max\{r \in [0,\eta] \big| u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta)\}. \tag{5}$$

That is, type i's reservation price is her full value for good $i+1$, and thus it is a function of η. In order to assure that (5) is actually defined, we confine our attention to the case that a value function is continuous from the right hereafter.

The economy is stationary if H is an initial stationary distribution of the process induced by the optimal trading strategy (ω, ρ). Now we define the stationary equilibrium grounded on the above. We adopt stationary perfect Bayesian Nash equilibrium as our equilibrium concept.

Definition 1 $< H, R, \Omega, \omega, \rho, \mathcal{V} >$, where $\mathcal{V}$ is continuous from the right, is said to be a *stationary equilibrium* if

1. H is stationary under trading strategies ω and ρ, and the distribution of offer prices Ω and that of reservation prices R are derived from H by (2) and (3),

2. $\int \eta dH = M$, and

3. given the distributions H, R and Ω, the reservation price strategy ρ and the offer strategy ω satisfy the feasibility condition (1) and the perfectness condition (5), respectively, and the value function $\mathcal{V}$, together with ρ and ω, solves the Bellman equation (4). Therefore,

$$\mathcal{V}(\eta) = \frac{1}{\phi+2}\left[\int_0^{\rho(\eta)} \{u + \mathcal{V}(\eta - x)\} d\Omega(x) + \{1 - \Omega(\rho(\eta))\} \mathcal{V}(\eta) \right.$$
$$\left. + R(\omega(\eta))\mathcal{V}(\eta) + \{1 - R(\omega(\eta))\} \ \{\mathcal{V}(\eta + \omega(\eta)) - c\}\right]$$

holds, where $\phi = k\gamma/\mu$.

Remark 1 The equilibrium concept in Zhou [4] is slightly different from ours. In addition to our equilibrium conditions, she requires "weak undominatedness" on equilibrium strategies.

3 The Main Result

To begin with, we define the concept of a single-price equilibrium of which existence we are going to show.

Definition 2 $< H, R, \Omega, \omega, \rho, \mathcal{V} >$ is said to be a *single-price equilibrium (SPE)* with some price $p > 0$ if

1. it is a stationary equilibrium, and

2. with probability one, for a meeting between a buyer and a seller, either trade occurs with price p or trade does not occur.

In what follows, we focus on a stationary distribution H such that its support is the set $\{0, p, 2p, \dots\}$ for some $p > 0$. Thus H can be expressed by $h_n = H(\{np\}), n = 0, 1, \dots$, the measure of the set of agents with money holding np. Of course, h satisfies $\sum_n h_n = 1$ and $h_n \geq 0$ for all n.

Now we are ready to present the main theorem.

Theorem 1 Let N be an arbitrary positive integer. Then there exist a $\underline{\phi}$ such that, for any $\phi > \underline{\phi}$,

$$(\phi+1)^N < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}$$

holds, and for any u and c satisfying

$$(\phi+1)^N < \frac{u}{c} < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}, \tag{6}$$

there exists a single-price equilibrium with some $p > 0$ in which the upper bound of money holdings is Np, i.e, $h_N > 0$ and $h_n = 0$ for all $n > N$.

For a given N, we specify a strategy which is shown to be a single-price equilibrium strategy as follows:

- a seller with η, $0 \leq \eta < Np$, offers p,
- a seller with η, $\eta \geq Np$, offers ∞, i.e., she offers no trade, and
- the reservation price of a buyer with η, $\eta \geq p$, is more than or equal to p.

Clearly, the upper bound of money holdings is Np, and the second condition for a single-price equilibrium is satisfied. Thus it suffices to prove the existence of $\mathcal{V}$ and H which, together with the above strategy, satisfy the first condition, i.e., the condition for a stationary equilibrium.

Note that Zhou [4] and Kamiya and Shimizu [2] show the existence of a SPE with $N = 1$. In this paper, we investigate SPEs with $N \geq 2$ as well.

Below, we prove the existence of $\mathcal{V}$ and H which, together with the above strategy, satisfy the condition for a stationary equilibrium. The proof is divided into the following five steps.

Step 1: Let $V_n = \mathcal{V}(np), n = 0, 1, \ldots$. Then $(h_0, \ldots, h_N)$ and $(V_0, \ldots, V_N)$ should satisfy the following equations in stationary equilibria:

$$F_0 = h_0 + \cdots + h_N - 1 = 0, \tag{7}$$

$$F_n = h_{n-1}(1-h_0) + h_{n+1}(1-h_N) - h_n(1-h_0+1-h_N) = 0, \quad n = 1, \ldots, N-1, \tag{8}$$

$$G_0 = V_0 - \frac{1}{\phi+2}\{(1-h_0)(V_1-c) + h_0V_0 + V_0\} = 0, \tag{9}$$

$$G_n = V_n - \frac{1}{\phi+2}\{(1-h_0)(V_{n+1}-c) + h_0V_n + (1-h_N)(u+V_{n-1}) + h_NV_n\} = 0, \quad n = 1, \ldots, N-1, \tag{10}$$

$$G_N = V_N - \frac{1}{\phi+2}\{V_N + (1-h_N)(u+V_{N-1}) + h_NV_N\} = 0. \tag{11}$$

The first equation (7) simply says that the total measure is one. The second equation (8) is the condition for stationarity of money holdings distribution. The last three equations (9), (10), and (11) are the Bellman equation, i.e., the condition that the specified strategy indeed realizes the value.

The key is that the stationarity conditions for money holdings distribution at state 0 and N are redundant.[1] For, from (7) and (8), we obtain

$$h_n = h_0\left(\frac{1-h_0}{1-h_N}\right)^n, \quad n = 1, \ldots, N, \tag{12}$$

and therefore, it is verified that it satisfies the stationary conditions for $n = 0$ and $n = N$, i.e.,

$$h_1(1-h_N) - h_0(1-h_0) = 0,$$
$$h_{N-1}(1-h_0) - h_N(1-h_N) = 0.$$

Then the number of equations is one less than the number of variables. Thus if the regularity condition holds, then the set of solution is one-dimensional.

Below, we prove the existence of $(h_0, \ldots, h_N)$ and $(V_0, \ldots, V_N)$ satisfying the above equations (7)-(11). Then we extend $V = (V_0, \ldots, V_N)$ to $\mathcal{V}$ and show that they satisfy

[1] This feature holds in general. See Kamiya and Shimizu [2] for the details.

all conditions for a stationary equilibrium.

Step 2: Next, we prove the existence of $(V_0, \ldots, V_N)$ with $(h_0, \ldots, h_N) = (1, 0, \ldots, 0)$ satisfying equations (7)-(11). Note that at $(h_0, \ldots, h_N) = (1, 0, \ldots, 0)$, $M = \int \eta dH = \sum_{n=0}^{N} ph_n$ cannot be satisfied. Therefore, by showing the regularity and the one-dimensional structure, we obtain $(h_0, \ldots, h_N)$ and $(V_0, \ldots, V_N)$ satisfying equations (7)-(11), where $(h_0, \ldots, h_N)$ is close to $(1, 0, \ldots, 0)$. It of course satisfies $M = \int \eta dH = \sum_{n=0}^{N} ph_n$ for some $p > 0$.

First, substituting $(h_0, \ldots, h_N) = (1, 0, \ldots, 0)$ into equations (9)-(11), we obtain $(V_0, \ldots, V_N)$ as follows:

$$V_n = \frac{1}{\phi}\left\{1 - \frac{1}{(\phi+1)^n}\right\} u, \quad n = 0, \ldots, N. \tag{13}$$

Note that $(V_0, \ldots, V_N) \neq (0, \ldots, 0)$, i.e., money has value.

Next, we show the regularity, i.e., the regularity of the Jacobian matrix of equations (7)-(11) at $(h_0, \ldots, h_N) = (1, 0, \ldots, 0)$. We can easily obtain

$$\left[\frac{D(F_0, \ldots, F_{N-1})}{D(h_1, \ldots, h_N)}\right]_{h_0=1} = \begin{vmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ -1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & \cdots & \cdots & -1 & 1 \end{vmatrix} = N,$$

and

$$\left[\frac{D(G_0, \ldots, G_N)}{D(V_0, \ldots, V_N)}\right]_{h_0=1} = \begin{vmatrix} \frac{\phi}{\phi+2} & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{1}{\phi+2} & \frac{\phi+1}{\phi+2} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{1}{\phi+2} & \frac{\phi+1}{\phi+2} & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & \cdots & \cdots & -\frac{1}{\phi+2} & \frac{\phi+1}{\phi+2} \end{vmatrix} = \frac{\phi(\phi+1)^N}{(\phi+2)^{N+1}}.$$

Since $\partial F_i / \partial V_j = 0$ for any i and j, we obtain the Jacobian as follows:

$$\begin{aligned} \left[\frac{D(F_0, \ldots, F_{N-1}, G_0, \ldots, G_N)}{D(h_1, \ldots, h_N, V_0, \ldots, V_N)}\right]_{h_0=1} &= \left[\frac{D(F_0, \ldots, F_{N-1})}{D(h_1, \ldots, h_N)}\right]_{h_0=1} \times \left[\frac{D(G_0, \ldots, G_N)}{D(V_0, \ldots, V_N)}\right]_{h_0=1} \\ &= N\frac{\phi(\phi+1)^N}{(\phi+2)^{N+1}} \neq 0. \end{aligned}$$

Therefore, by the inverse function theorem, there is a one-dimensional solution set containing

$$((h_0,\dots,h_N),(V_0,\dots,V_N)) = \left((1,0,\dots,0),\left(0,\frac{1}{\phi}\left\{1-\frac{1}{(\phi+1)}\right\}u,\dots,\frac{1}{\phi}\left\{1-\frac{1}{(\phi+1)^N}\right\}u\right)\right).$$

Let the set be Θ. We consider $((h_0,\dots,h_N),(V_0,\dots,V_N)) \in \Theta$ as a function of h_0 when h_0 is close to one.

Step 3: Next, we show that the solution $((h_0,\dots,h_N),(V_0,\dots,V_N)) \in \Theta$ corresponding to $h_0 = 1-\varepsilon$, where ε is a small positive number, satisfies $h_n \in (0,1), n = 0,\dots,N$.

From (12), we obtain

$$h_N(1-h_N)^N = h_0(1-h_0)^N.$$

Differentiating both sides by h_0, we obtain

$$\frac{dh_N}{dh_0}\left[(1-h_N)^N - Nh_N(1-h_N)^{N-1}\right] = (1-h_0)^N - Nh_0(1-h_0)^{N-1}.$$

Then we obtain

$$\left[\frac{dh_N}{dh_0}\right]_{h_0=1} = 0.$$

Applying this argument recursively, we obtain the higher order derivatives of h_N with respect to h_0 as follows:

$$\left[\frac{d^i h_N}{d(h_0)^i}\right]_{h_0=1} = \begin{cases} 0 & \text{if } i < N, \\ (-1)^N (N)! & \text{if } i = N. \end{cases} \tag{14}$$

Then for $n = 1,\dots,N-1$,

$$\left[\frac{d^i}{d(h_0)^i}\left\{\left(\frac{1-h_0}{1-h_N}\right)^n\right\}\right]_{h_0=1} = \begin{cases} 0 & \text{if } i < n, \\ (-1)^n (n)! & \text{if } i = n \end{cases}$$

hold, and therefore, again from (12), we obtain

$$\left[\frac{d^i h_n}{d(h_0)^i}\right]_{h_0=1} = \begin{cases} 0 & \text{if } i < n, \\ (-1)^n (n)! & \text{if } i = n. \end{cases} \tag{15}$$

By Taylor's theorem, (14) and (15) imply

$$h_n = (1-h_0)^n + o(1-h_0)^n \geq 0, \quad n = 1,\dots,N,$$

where o denotes Landau's "ou". This assures that when h_0 is slightly smaller than 1, $h_n > 0, n = 1, \ldots, N$, hold. Thus $M = \int \eta dH = \sum_{n=0}^{N} ph_n$ is satisfied for some $p > 0$.

Step 4: Next, we extend $((V_0, \ldots, V_N), (h_0, \ldots, h_N)) \in \Theta$ to $(\mathcal{V}, H)$. Clearly, defining $h_n = 0$ for $n = N+1, N+2, \ldots,$ we obtain H. By the strategy specified above, the Bellman equation for $n = N+1, N+2, \ldots,$ is as follows.

$$V_n = \frac{1}{\phi + 2} \left\{ V_n + (1 - h_N)(u + V_{n-1}) + h_N V_n \right\}, \quad n = N+1, N+2, \ldots. \tag{16}$$

Then V_n for $n = N+1, N+2, \ldots,$ can be obtained recursively. Note that in case of $(h_0, h_1, \ldots, h_N) = (1, 0, \ldots, 0)$,

$$V_n = \frac{1}{\phi} \left\{ 1 - \frac{1}{(\phi+1)^n} \right\} u, \quad n = N+1, N+2, \ldots. \tag{17}$$

Let $\lfloor x \rfloor$ denote the integer part of a real number x, and the value function $\mathcal{V}(\eta)$ is defined as

$$\mathcal{V}(\eta) = V(\lfloor \eta/p \rfloor).$$

Step 5: Lastly, we search for a sufficient condition for the incentive of choosing the specified strategy with $(\mathcal{V}, H)$ obtained in Step 4. Below, we present a sufficient condition in case that h_0 is slightly smaller than 1. There are four cases.

Case 1: Incentive for agents with ip to offer p instead of ∞ for $i = 0, \ldots, N-1$.

A necessary and sufficient condition is

$$I_{1\infty}^i = [(1 - h_0)(V_{i+1} - c) + h_0 V_i] - V_i \geq 0.$$

Note that $((h_0, \ldots, h_N), (V_0, \ldots, V_N)) \in \Theta$ is a function of h_0 when h_0 is close to one, then

$$\begin{aligned} \left[I_{1\infty}^i\right]_{h_0=1} &= 0, \text{ and} \\ \left[\frac{dI_{1\infty}^i}{dh_0}\right]_{h_0=1} &= \left[-(V_{i+1} - V_i - c) + (1-h_0)\frac{dV_{i+1}}{dh_0} - (1-h_0)\frac{dV_0}{dh_0}\right]_{h_0=1} \\ &= -\left[V_{i+1} - V_i - c\right]_{h_0=1} \end{aligned}$$

hold. By Taylor's theorem, the condition is equivalent to

$$I_{1\infty}^i(h_0) = [V_{i+1} - V_i - c]_{h_0=1}(1 - h_0) + o(1 - h_0) \geq 0. \tag{18}$$

Then substituting (13) into the first term of the RHS, it is positive for all i if and only if

$$\frac{u}{c} > (\phi + 1)^N.$$

Thus if this condition is satisfied, then the incentive is satisfied for $h_0 = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Case 2: Incentive for agents with ip to offer p instead of jp for $i = 0, \ldots, N-1$ and $j = 2, \ldots, N$.

Since the money holdings of agents who can accept jp are at least jp, then a sufficient condition is

$$I_{1j}^i = [(1 - h_0)(V_{i+1} - c) + h_0 V_i] - \left[\sum_{k=j}^{N} h_k (V_{i+j} - c) + \left(1 - \sum_{k=j}^{N} h_k \right) V_i \right] \geq 0.$$

Then from (14) and (15),

$$\begin{aligned} \left[I_{1j}^i \right]_{h_0=1} &= 0, \text{ and} \\ \left[\frac{dI_{1j}^i}{dh_0} \right]_{h_0=1} &= - \left[V_{i+1} - V_i - c \right]_{h_0=1} \end{aligned}$$

hold. By Taylor's theorem, the above condition is equivalent to

$$I_{1j}^i(h_0) = \left[V_{i+1} - V_i - c \right]_{h_0=1} (1 - h_0) + o(1 - h_0) \geq 0. \tag{19}$$

Then substituting (13) into the first term of the RHS, it is positive for all i if and only if

$$\frac{u}{c} > (\phi + 1)^N.$$

Thus if this condition is satisfied, then the incentive is satisfied for $h_0 = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Before investigating Case 3, we prove the following lemma:

Lemma 1 For $i = N, N+1, \ldots,$ and $j = 1, 2, \ldots, N$,

$$0 < V_{i+j+1} - V_{i+1} < V_{i+j} - V_i$$

unless $h_N = 1$.

Proof:
From (16), we obtain

$$V_n = \frac{1-h_N}{\phi}u - \left(\frac{1-h_N}{\phi+1-h_N}\right)^{n-N}\left(\frac{1-h_N}{\phi}u - V_N\right), \quad n = N, N+1, \ldots.$$

Note that $\left(\frac{1-h_N}{\phi}u - V_N\right)$ is strictly positive. Therefore we obtain

$$V_{i+j+1} - V_{i+1} = \left(\frac{1-h_N}{\phi+1-h_N}\right)^{i+1-N}\left[1 - \left(\frac{1-h_N}{\phi+1-h_N}\right)^j\right]\left(\frac{1-h_N}{\phi}u - V_N\right) > 0,$$

$$(V_{i+j} - V_i) - (V_{i+j+1} - V_{i+1}) =$$
$$\left(\frac{1-h_N}{\phi+1-h_N}\right)^{i-N}\left(\frac{\phi}{\phi+1-h_N}\right)\left[1 - \left(\frac{1-h_N}{\phi+1-h_N}\right)^j\right]\left(\frac{1-h_N}{\phi}u - V_N\right) > 0.$$

∎

Case 3: Incentive for agents with ip to offer ∞ instead of jp for $i = N, \ldots$ and $j = 1, \ldots, N$.

Since the money holdings of agents who accept jp are at least jp, a sufficient condition is

$$\begin{aligned} I^i_{\infty j} &= V_i - \left[\sum_{k=j}^{N} h_k (V_{i+j} - c) + \left(1 - \sum_{k=j}^{N} h_k\right) V_i\right] \\ &= \left(\sum_{k=j}^{N} h_k\right)\left[c - (V_{i+j} - V_i)\right] \geq 0. \end{aligned}$$

By Lemma 1, as long as we consider an equilibrium with $h_0 = 1 - \varepsilon$ for small positive ε, it suffices to show

$$I^N_{\infty j} = \left(\sum_{k=j}^{N} h_k\right)\left[c - (V_{N+j} - V_N)\right] \geq 0, \quad j = 1, \ldots, N.$$

Then from (14) and (15),

$$\left[I^N_{\infty j}\right]_{h_0=1} = \left[\frac{dI^N_{\infty j}}{dh_0}\right]_{h_0=1} = \cdots = \left[\frac{d^{j-1} I^N_{\infty j}}{d(h_0)^{j-1}}\right]_{h_0=1} = 0, \text{ and}$$

$$\left[\frac{d^j I^N_{\infty j}}{d(h_0)^j}\right]_{h_0=1} = \left[\frac{d^j h_j}{d(h_0)^j}\left(c - (V_{N+j} - V_N)\right)\right]_{h_0=1} = (-1)^j j! \left[c - (V_{N+j} - V_N)\right]_{h_0=1}.$$

hold. By Taylor's theorem, the condition implies

$$I^N_{\infty j}(h_0) = [c - (V_{N+j} - V_N)]_{h_0=1} (1-h_0)^j + o(1-h_0)^j \geq 0. \tag{20}$$

Then substituting (17) into the first term of the RHS, it is positive for all j if and only if

$$c - \frac{1}{\phi(\phi+1)^N}\left\{1 - \frac{1}{(\phi+1)^N}\right\} u > 0,$$

which is equivalent to

$$\frac{u}{c} < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}.$$

Thus if this condition is satisfied, then the incentive is satisfied for $h_0 = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough.

Case 4: Incentive for agents with ip to accept an offer p for $i = 1, \ldots$.

For $h_0 = 1 - \varepsilon$ where $\varepsilon > 0$ is small enough, this is easily verified since

$$[u + V_{i-1} - V_i]_{h_0=1} = \left\{1 - \frac{1}{(\phi+1)^i}\right\} u > 0$$

hold.

From the above arguments on incentive conditions, if (6) holds, then, at $\eta = 0, p, \ldots, Np, \ldots$, all incentive conditions are satisfied for $h_0 = 1 - \varepsilon$, where $\varepsilon > 0$ is small enough. Note that we can take positive ε, since there is a finite number of conditions in (18), (19), and (20). The incentives at $\eta \notin \{0, p, \ldots, Np, \ldots\}$ can be easily checked.

By Steps 1-5, we have shown that there exists a single-price equilibrium with the upper bound Np if (6) holds. In case of $N = 1$, this condition becomes

$$(\phi+1) < \frac{u}{c} < (\phi+1)^2.$$

Thus we can choose $\underline{\phi} = 0$. In case of $N \geq 2$, if

$$\phi(\phi+1)^{2N} - (\phi+1)^{2N} + (\phi+1)^N = (\phi+1)^N \left\{(\phi+1)^N(\phi-1) + 1\right\} > 0 \tag{21}$$

is satisfied, then there is a nonempty set of parameters satisfying (6). We define $\underline{\phi}$ as the largest solution of the equation

$$(\phi + 1)^N(\phi - 1) + 1 = 0.$$

It is easily verified that such $\underline{\phi}$ exists and $\underline{\phi} \in (0, 1)$ for $N \geq 2$. Since the LHS of (21) goes to ∞ as $\phi \to \infty$, then for any $\phi > \underline{\phi}$ there exists a non-empty region of u/c such that there exists a single-price equilibrium with the upper bound Np.

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