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Abstract

This paper formulates and analyzes a dynamic optimization problem of bond portfolios within Markovian Heath-Jarrow-Morton term structure models. In particular, we investigate optimal yield curve strategies analytically and numerically, and provide theoretical justification for a typical strategy which is recommended in practice for an expected change in the shape of the yield curve. In the numerical analysis, we utilize a new technique based on the *asymptotic expansion approach* in order to increase efficiency in computation.

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I Introduction

Recently, the size of the world bond market has grown rapidly and bond portfolios have become increasingly important in asset allocation for both firms and individuals. For example, portfolios for individual pension funds such as individual retirement accounts (IRAs) and 401(k) plans prevail in financial markets. Managing bond portfolios is important because bonds are major instruments for pension funds. In practice, bond strategies such as interest-rate expectations strategies and yield curve strategies are well-known in active bond portfolio management.[1] Interest-rate expectation strategies are premised on expectations for the future level of interest rates. If the interest rate is expected to rise (fall), the duration of bond portfolios should be shortened (lengthened). On the other hand, yield curve strategies are based on expected changes in the shape of the yield curve. The types of yield curve shifts are broadly classified into parallel shifts, twists and butterfly shifts. The choice among bullet, barbell and ladder strategies depends upon the expectation for these yield curve shifts. Until recently, however, only a few studies have analyzed the dynamic optimal bond portfolio problem theoretically (see Brennan, Schwartz, and Lagnado (1997); Brennan and Xia (2000); Campbell and Viceira (2001); Liu (1999); Sørensen (1999); and Wachter (2001)). Moreover, most research on the *asset allocation puzzle*, as illustrated by Canner, Mankiw, and Weil (1997), show an inconsistency between the theoretical result for the static mean-variance framework and the practical result for popular investment advice on the optimal choice of bonds and stocks in a portfolio. Furthermore, although the change in the yield curve shape and level is practically important for the choice of bonds, existing studies in asset allocation do not pay much attention to the term structure of interest rates, and consequently do not effectively utilize term structure models developed in the valuation of bonds and interest-rate derivatives.

This paper first formulates a dynamic optimal bond portfolio problem based on a certain class of term structure models. In particular, we investigate optimal yield curve strategies in the Markovian Heath-Jarrow-Morton (HJM) framework, and provide theoretical justification for a typical strategy which is recommended in practice when a change in the shape of the yield curve is expected. The reason for taking the HJM framework is that models within the class generate the observed current term structure by construction, which is a desirable feature especially in liquid bond markets. Moreover, in the numerical analysis, we employ a new technique based on the *asymptotic expansion approach* developed by Kunitomo and Takahashi (2001, 2003c), Takahashi (1995, 1999), and Takahashi and Yoshida (2003, 2004). In particular, we utilize the analytical approximation obtained by the asymptotic expansion approach, and further develop the method where the analytical approximation is used in order to increase efficiency of Monte Carlo simulations in computation of optimal bond portfolios.

The paper is organized as follows. Section II formulates the dynamic optimal bond portfolio problem under the single-factor HJM model. Then, in Section III, we analyze optimal bond portfolios in the Markovian class of HJM models proposed by Ritchken and Sankarasubramanian (1995). In the final section, we summarize our main results. We also examine an investor's portfolio choice problem in the case of a Gaussian two-factor term structure model in Appendix A. In Appendix B, we briefly explain the asymptotic expansion scheme.

II Optimal Bond Portfolios in the Heath-Jarrow-Morton Framework

In this section, we shortly review the single-factor HJM term structure model of interest rates and then formulate an optimal bond portfolio problem for an investor.

A The Single-Factor Heath-Jarrow-Morton Model

Let W^P be an $\mathbb{R}$-valued Brownian motion defined on a given filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}\}_{t\geq 0})$. In the single-factor HJM model, the dynamics of the forward rate $f(t,T)$ for every fixed maturity T is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW^P(t), \tag{1}$$

where $\{\alpha(t,T) : 0 \leq t \leq T\}$ and $\{\sigma(t,T) : 0 \leq t \leq T\}$ are adapted processes satisfying certain technical conditions.[2] Assuming that the market price of interest rate risk θ exists through the no-arbitrage condition, we have

$$\alpha(t,T) = \sigma(t,T)(\nu(t,T) - \theta(t)) \tag{2}$$

for $t \in [0,T]$, where

$$\nu(t,T) \triangleq \int_t^T \sigma(t,u)du. \tag{3}$$

Here, "$\triangleq$" means "is defined by." We then define

$$W^Q(t) = W^P(t) - \int_0^t \theta(s)ds.$$

Then, W^Q is an $\mathbb{R}$-valued Brownian motion under an equivalent martingale measure Q by Girsanov's theorem.

We next consider a portfolio choice problem between a money market account (or savings account) and a zero coupon bond under the single-factor HJM model. Let $B(t)$ be the value of the money market account at time t, of which process is given by $dB(t) = r(t)B(t)dt$ with the conventional initial condition $B(0) = 1$. Then, $B(t)$ is represented by

$$B(t) = \exp\left(\int_0^t f(0,u)du + \int_0^t \int_0^u \alpha(s,u)dsdu + \int_0^t \int_0^u \sigma(s,u)dW^P(s)du\right), \tag{4}$$

since the short rate $r(t) = f(t,t)$. On the other hand, the price of the zero coupon bond with maturity T at time t denoted by $P(t,T)$ is expressed as

$$P(t,T) = \exp\left(-\int_t^T f(0,u)du - \int_0^t \int_t^T \alpha(s,u)duds - \int_0^t \int_t^T \sigma(s,u)dudW^P(s)\right), \tag{5}$$

where $P(T,T)$ is equal to unity by convention. Hereafter, we call it *T-maturity zero coupon bond.* Then, it is known that the dynamics of $P(t,T)$ is given by

$$dP(t,T) = P(t,T)\left(\mu(t,T)dt - \nu(t,T)dW^P(t)\right), \tag{6}$$

where $\mu(t,T) \triangleq r(t) + \theta(t)\nu(t,T)$.[3]

B Optimal Portfolio Problem

We formulate an investor's portfolio choice problem in the following. We suppose that the investor has a CRRA utility function over his or her terminal wealth and seeks to maximize his or her expected utility. In this setting, what proportions of the investor's wealth should be allocated in the money market account and the zero coupon bond?

The investor's utility function is defined by $U(V(T^0)) = V(T^0)^{1-\gamma}/(1-\gamma)$, where $V(t)$ is the investor's wealth at time t, T^0 is the terminal date of the investment horizon, and $\gamma > 0$ is the coefficient of relative risk aversion. Let w denote the proportion of wealth invested in the zero coupon bond. Then, given $V(0) > 0$, the investor's problem is formulated as follows:

$$\max_{(V(T^0),w)} \mathrm{E}^P[U(V(T^0))],$$

subject that $(V(T^0), w)$ is budget feasible.

There are two approaches to this optimal portfolio problem: one is the stochastic control approach initiated by Merton (1969, 1971) and Samuelson (1969); and the other is the martingale approach by Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987). We adopt the second approach in this paper. In the martingale approach, the optimal wealth discounted by the value of the money market account is a martingale under an equivalent martingale measure, and the state-price deflator plays an important role. The state-price deflator is defined by $\eta_1(t)/B(t)$, where

$$\eta_1(t) \triangleq \exp\left(\int_0^t \theta(s)dW^P(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right), \tag{7}$$

or equivalently

$$d\eta_1(t) = \eta_1(t)\theta(t)dW^P(t)$$

with the initial condition $\eta_1(0) = 1$. With this preparation, the investor's problem is reformulated as

$$\max_{V(T^0)} \mathrm{E}^P\left[\frac{1}{1-\gamma}V(T^0)^{1-\gamma}\right], \quad \text{subject that} \quad \mathrm{E}^Q\left[\frac{V(T^0)}{B(T^0)}\right] = V(0).$$

Furthermore, the constraint is rewritten by

$$\mathrm{E}^P\left[\frac{\eta_1(T^0)V(T^0)}{B(T^0)}\right] = V(0).$$

Let λ be a Lagrange multiplier. From the first-order condition, the optimal wealth is given by

$$V^*(T^0) = \lambda^{-\frac{1}{\gamma}}\left(\frac{\eta_1(T^0)}{B(T^0)}\right)^{-\frac{1}{\gamma}}.$$

Hence,

$$V^*(t) = B(t)\,\mathrm{E}^Q\left[\frac{V^*(T^0)}{B(T^0)}\middle|\,\mathscr{F}_t\right] = \lambda^{-\frac{1}{\gamma}}\frac{B(t)}{\eta_1(t)}\,\mathrm{E}^P\left[\left(\frac{\eta_1(T^0)}{B(T^0)}\right)^{1-\frac{1}{\gamma}}\middle|\,\mathscr{F}_t\right].$$

By Eq. (4), Eq. (5), Eq. (7) and the no-arbitrage condition,

$$V^*(t) = \lambda^{-\frac{1}{\gamma}}\left(\frac{\eta_1(t)}{B(t)}\right)^{-\frac{1}{\gamma}} P(t,T^0)^{1-\frac{1}{\gamma}}G(t,T^0), \tag{8}$$

where

$$G(t,T) \triangleq \mathrm{E}^P\left[\exp\left(\left(1-\frac{1}{\gamma}\right)\left(\int_t^T(\theta(s)-\nu(s,T))dW^P(s)\right.\right.\right.$$
$$\left.\left.\left.\left.-\frac{1}{2}\int_t^T(\theta(s)-\nu(s,T))^2ds\right)\right)\right|\mathscr{F}_t\right].$$

λ is determined to satisfy $V^*(0)=V(0)$, that is, $\lambda = G(0,T^0)^\gamma P(0,T^0)^{\gamma-1}/V(0)^\gamma$.

The optimal weight of the zero coupon bond, w^*, is a trading strategy which replicates V^* in Eq. (8). It can be easily seen that w^* is divided into three components: the strategies replicating $\eta_1(t)^{-1/\gamma}$, $P(t,T^0)^{1-1/\gamma}$ and $G(t,T^0)$. When $\gamma=1$ or $T^0=t$, the latter two components are zero and only the component replicating $\eta_1(t)^{-1/\gamma}$ remains because $P(t,T^0)^{1-1/\gamma} = G(t,T^0)=1$. We also remark that the trading strategy replicating $P(t,T^0)^{1-1/\gamma}G(t,T^0)$ corresponds to the intertemporal hedging portfolio named by Merton (1973) and that the trading strategy replicating $\eta_1(t)^{-1/\gamma}$ corresponds to the mean-variance efficient portfolio. The latter is easily seen as follows. If a single zero coupon bond is available, the weight of the zero coupon bond in the portfolio is equal to $(\mu(t,T)-r(t))/(\gamma\nu(t,T)^2)$ by Eq. (6) and Eq. (7), which is the product of the coefficient of relative risk tolerance and the risk premium per instantaneous conditional variance of the rate of return of the zero coupon bond.

Few properties on the optimal bond portfolio are known, however, since it is generally difficult to derive the trading strategy replicating $G(t,T^0)$ explicitly. When every $(\theta(s)-\nu(s,T))$ for $s\in[t,T]$ is deterministic, $G(t,T)$ is explicitly solved as

$$G(t,T)=\exp\left(-\frac{1}{2\gamma}\left(1-\frac{1}{\gamma}\right)\int_t^T(\theta(s)-\nu(s,T))^2ds\right)$$

by the formula of the moment generating function for the normal distribution. Thus, the component replicating G is zero. It is also known that V^* becomes equal to the value of $V(0)/P(0,T^0)$ units of the T^0-maturity zero coupon bond when γ approaches infinity.[4] In the following section, we will characterize the optimal bond portfolios by specifying a certain class of term structure models in the HJM framework.

III Optimal Portfolios and Yield Curve Strategies within Markovian Heath-Jarrow-Morton Models

We suppose that the diffusion coefficient in Eq. (1) has a functional form;

$$\sigma(t,T) = \sigma_r(r(t),t) \exp\left(-\int_t^T \kappa(x)dx\right). \tag{9}$$

Then, given that the market price of risk is a function of the short rate r and the time parameter t, $\theta(t) = \theta(r(t),t)$, the dynamics of the term structure is described by the following two-dimensional state variable Markovian process (r, ϕ) as in Ritchken and Sankarasubramanian (1995):

$$dr(t) = \left(\kappa(t)(f(0,t) - r(t)) + \phi(t) - \sigma_r(r(t),t)\theta(r(t),t) + \frac{\partial}{\partial T} f(0,t)\right) dt + \sigma_r(r(t),t)dW^P(t) \tag{10a}$$

and

$$d\phi(t) = (\sigma_r(r(t),t)^2 - 2\kappa(t)\phi(t))dt, \tag{10b}$$

where the second state variable is $\phi(t) \triangleq \int_0^t \sigma(s,t)^2 ds$. The volatility of the rate of return of the zero coupon bond is given by

$$\nu(t,T) = \sigma_r(r(t),t)\int_t^T \exp\left(-\int_t^u \kappa(x)dx\right)du \tag{11}$$

from Eq. (3). We can apply Ocone and Karatzas's (1991) result because markets are complete. Then, the optimal weight of the zero coupon bond satisfies the following formula:

$$\begin{aligned}\nu(t,T)w^*(t) &= \frac{1}{\gamma}\theta(r(t),t) + \left(1-\frac{1}{\gamma}\right)\frac{\mathrm{E}^P\left[\xi(t,T^0)^{1-\frac{1}{\gamma}}\int_t^{T^0} D_t r(s)ds \middle| \mathscr{F}_t\right]}{\mathrm{E}^P\left[\xi(t,T^0)^{1-\frac{1}{\gamma}}\middle| \mathscr{F}_t\right]} \\ &+\left(1-\frac{1}{\gamma}\right)\frac{\mathrm{E}^P\left[\xi(t,T^0)^{1-\frac{1}{\gamma}}\left(\int_t^{T^0}\theta(r(s),s)\left(D_t\theta(r(s),s)\right)ds - \int_t^{T^0} D_t\theta(r(s),s)dW^P(s)\right)\middle| \mathscr{F}_t\right]}{\mathrm{E}^P\left[\xi(t,T^0)^{1-\frac{1}{\gamma}}\middle| \mathscr{F}_t\right]},\end{aligned} \tag{12}$$

where $\xi(t,T)$ denotes the ratio of two state-price deflators, that is:

$$\xi(t,T) = \frac{\frac{\eta_1(T)}{B(T)}}{\frac{\eta_1(t)}{B(t)}} = \exp\left(\int_t^T \theta(r(s),s)dW^P(s) - \frac{1}{2}\int_t^T \theta(r(s),s)^2 ds - \int_t^T r(s)ds\right),$$

or

$$d_T\xi(t,T) = \xi(t,T)\left(-r(T)dT + \theta(r(T),T)dW^P(T)\right)$$

with the initial condition $\xi(t,t) = 1$. Here, D_t denotes the Malliavin derivative (see Ikeda and Watanabe (1989) and Nualart (1995) for details). The right-hand side of Eq. (12) exhibits the investor's demand which consists of three parts. We call the first term on the right-hand side of Eq. (12) the *MV (mean-variance) term*, because it generates the mean-variance efficient demand. The sum of the second and third terms yields the intertemporal hedging demand. They represent hedge demand against the fluctuation of the short rate and against that of the market price of risk. Hence, we will call the second and the third terms the *IR (interest rate)-hedging term* and the *MPR (market price of risk)-hedging term* respectively (see Detemple, Garcia, and Rindisbacher (2000)).

A The Case of the Extended Vasiček Model

We start with a well-known Gaussian single-factor model. In this case, we specify $\sigma_r(r(t),t) = \sigma > 0$ and $\kappa(x) = \kappa > 0$ in Eq. (9). Hence,

$$\sigma(t,T) = \sigma \exp(-\kappa(T-t)). \tag{13}$$

Then, when the market price of risk is a positive constant:

$$\theta(r(t),t) = \theta > 0, \tag{14}$$

the second state variable ϕ is deterministic, and the dynamics of the term structure is degenerated into the following one-dimensional Markovian process:

$$dr(t) = (\psi^Q(t) - \kappa r(t))dt + \sigma dW^Q(t), \tag{15}$$

where

$$\psi^Q(t) \triangleq \frac{\partial}{\partial T} f(0,t) + \frac{\sigma^2(1-\exp(-2\kappa t))}{2\kappa} + \kappa f(0,t).$$

This short rate process is known as the extended Vasiček model by Hull and White (1990). From Eq. (13), Eq. (11) leads to

$$\nu(t,T) = \frac{\sigma(1-\exp(-\kappa(T-t)))}{\kappa}. \tag{16}$$

Further, we have

$$D_t\theta(s) = 0 \tag{17a}$$

and

$$D_t r(s) = \sigma \exp(-\kappa(s-t)) \tag{17b}$$

from Eq. (14) and Eq. (15).

We consider a portfolio choice problem among the $\bar{T}_1$-maturity zero coupon bond, the $\bar{T}_2$-

maturity zero-coupon bond with $\bar{T}_1 > \bar{T}_2$, and the money market account under a constraint such that the duration of the bond portfolio at time t should be equal to $(\bar{T} - t)$ for a given $\bar{T}$. Here, two zero coupon bonds represent long-term and intermediate-term bonds whereas the money market account stands for a short-term bond, and the constraint we impose is relatively common in practical bond portfolio management.

Because the dynamics of the term structure expressed in Eq. (15) is generated by a single factor, the bond markets are still complete even with the duration constraint. Let w_1 and w_2 be the proportions of wealth invested in the $\bar{T}_1$- and $\bar{T}_2$-maturity zero coupon bonds. From Eq. (16) and Eq. (17), Eq. (12) becomes

$$\nu(t, \bar{T}_1) w_1(t) + \nu(t, \bar{T}_2) w_2(t) = \frac{1}{\gamma}\theta + \left(1 - \frac{1}{\gamma}\right) \nu(t, \bar{T}^0).$$

The duration constraint on the bond portfolio is expressed as

$$w_1(t)(\bar{T}_1 - t) + w_2(t)(\bar{T}_2 - t) = \bar{T} - t.$$

Gathering these equations, it follows that the optimal proportions should satisfy

$$\begin{bmatrix} \nu(t, \bar{T}_1) & \nu(t, \bar{T}_2) \\ \bar{T}_1 - t & \bar{T}_2 - t \end{bmatrix} \begin{bmatrix} w_1^*(t) \\ w_2^*(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma}\theta + \left(1 - \frac{1}{\gamma}\right) \nu(t, T^0) \\ \bar{T} - t \end{bmatrix}. \tag{18}$$

We can show the matrix on the left-hand side of the above equation is nonsingular as follows. Recall that the volatility of the rate of return of a zero coupon bond (3) is the sum of diffusion coefficients of forward rates and that the diffusion coefficient (13) exponentially decreases as the term to maturity $(T-t)$ increases. Thus, Eq. (16) yields

$$\frac{\partial}{\partial T}\left(\frac{\nu(t,T)}{T-t}\right)=\frac{\sigma}{\kappa}\frac{(\kappa(T-t)+1)\exp(-\kappa(T-t))-1}{(T-t)^2}<0 \tag{19}$$

for $T>t$. This reduces to

$$\Delta_1 \triangleq \begin{vmatrix}\nu(t,\bar{T}_1) & \nu(t,\bar{T}_2)\\ \bar{T}_1-t & \bar{T}_2-t\end{vmatrix}=\left(\bar{T}_1-t\right)\left(\bar{T}_2-t\right)\left(\frac{\nu(t,\bar{T}_1)}{\bar{T}_1-t}-\frac{\nu(t,\bar{T}_2)}{\bar{T}_2-t}\right)<0 \tag{20}$$

for $\bar{T}_1>\bar{T}_2>t$.

Hence, the optimal proportions are expressed as

$$\begin{bmatrix} w_1^*(t)\\ w_2^*(t)\\ w_3^*(t)\end{bmatrix}=\frac{1}{\gamma}\begin{bmatrix}\frac{\theta\left(\bar{T}_2-t\right)}{\Delta_1}\\ -\frac{\theta\left(\bar{T}_1-t\right)}{\Delta_1}\\ 1+\frac{\theta\left(\bar{T}_1-\bar{T}_2\right)}{\Delta_1}\end{bmatrix}+\left(1-\frac{1}{\gamma}\right)\begin{bmatrix}\frac{\nu(t,T^0)\left(\bar{T}_2-t\right)}{\Delta_1}\\ -\frac{\nu(t,T^0)\left(\bar{T}_1-t\right)}{\Delta_1}\\ 1+\frac{\nu(t,T^0)\left(\bar{T}_1-\bar{T}_2\right)}{\Delta_1}\end{bmatrix}$$
$$+\frac{1}{\Delta_1}\begin{bmatrix}-\nu(t,\bar{T}_2)\left(\bar{T}-t\right)\\ \nu(t,\bar{T}_1)\left(\bar{T}-t\right)\\ -\left(\nu(t,\bar{T}_1)-\nu(t,\bar{T}_2)\right)\left(\bar{T}-t\right)\end{bmatrix},$$

where $w_3 \triangleq 1-(w_1+w_2)$ denotes the proportion of wealth invested in the money market account. We can see the market price of risk θ has effects on the optimal portfolio only through the change in the mean-variance efficient portfolio, and from the sign of Eq. (20) they are

$$\frac{\partial w_1^*(t)}{\partial \theta} < 0, \quad \frac{\partial w_2^*(t)}{\partial \theta} > 0, \quad \frac{\partial w_3^*(t)}{\partial \theta} < 0. \tag{21}$$

We are investigating an investor's subjective equilibrium in the above analysis. Eq. (21) is regarded as the so-called bullet strategy in terms of the yield curve strategies. The signs in Eq. (21) can be explained by analogy to the Rybczynski theorem in international trade theory. The right-hand side of Eq. (18), which represents the optimal amount of risk the investor takes and the duration of the bond portfolio, can be regarded as endowments of two factors. The increase in θ makes the investor take more risk. With the constant duration of the bond portfolio, the Rybczynski theorem states that the weight of the more "risk-intensive" zero coupon bond should be increased and that the weight of the less "risk-intensive" zero coupon bond should be decreased. The degree of risk-intensity is measured by the ratio of risk to the term to maturity, $\nu(t,T)/(T-t)$, that can be also interpreted as the yield volatility. Then, Eq. (19) indicates that the zero coupon bond with a shorter maturity is more risk-intensive. Therefore, the optimal weight of the shorter $\bar{T}_2$-maturity zero coupon bond increases when θ increases, and the optimal weight of the longer $\bar{T}_1$-maturity zero coupon bond decreases. Moreover, as the extent of the increase in the weight of the $\bar{T}_2$-maturity zero coupon bond is larger than that of the decrease in the weight of the $\bar{T}_1$-maturity zero coupon bond, the optimal weight of the money market account decreases when θ increases.

On the other hand, a change in θ has an impact on expectation for term structure movements. From Eq. (2) and Eq. (13), the increase in θ causes a decrease in the drift coefficient of the forward rate process without changing its diffusion coefficient, and the extent of the decrease in the drift coefficient reduces as the term to maturity $(T-t)$ increases. This movement can be regarded as a combination of a downward parallel shift and a steepening twist of the forecasted forward curve. Jones (1991) found a positive correlation between these two types of yield curve shifts.

Comparing the investor's behavior in two economies of different θ's, we obtain the following result:

Proposition 1 *Consider two economies with the same current term structure and the same diffusion coefficient of the forward rate process described by Eq. (13). In the economy where a downward parallel shift and a steepening twist of the forward curve are further expected in the future, an investor will choose to hold more intermediate-term zero coupon bonds and less short-term and long-term zero coupon bonds in a portfolio with the same duration.*

This result is consistent with practical investment advice since the bullet strategy is usually recommended under the expectation of the steepening change in the yield curve. We show in the appendix that the recommendation is also valid in the case of a Gaussian two-factor model.

B A General Single-Factor Case

In this subsection, we analyze the effects of the change in the market price of risk on the shape of the yield curve and the optimal portfolio, and investigate whether Proposition 1 in Section III.A is valid in a more general class of single-factor models. The model in the previous subsection is limited in the following sense: the MPR-hedge demand is *zero* because the market price of risk is a constant (see Eq. (14)). Further, the change in the market price of risk has no effect on the IR-hedge demand in the optimal portfolio and hence it only affects the mean-variance part in the optimal portfolio.

Hereafter, we specify $\kappa(t)$ and $\sigma_r(r(t),t)$ in Eq. (9) and the market price of risk $\theta(r(t),t)$ as follows:

$$
\begin{aligned}
\kappa(t) &= \kappa, \\
\sigma_r(r(t),t) &= \sigma r(t)^{\beta_\sigma}, \\
\theta(r(t),t) &= \theta r(t)^{\beta_\theta}.
\end{aligned}
\tag{22}
$$

Then, for $s \geq t$, the dynamics of the Malliavin derivative of $(r(s),\phi(s))$, $(D_t r(s) D_t\phi(s))$, is described by the following Markovian process from Eq. (10):

$$
\begin{aligned}
& d_s \begin{bmatrix} D_t r(s) \\ D_t \phi(s) \end{bmatrix} \\
&= \left(\begin{bmatrix} -\left(\kappa + (\beta_\sigma + \beta_\theta)\sigma\theta r(s)^{\beta_\sigma+\beta_\theta-1}\right) & 1 \\ 2\beta_\sigma \sigma^2 r(s)^{2\beta_\sigma - 1} & -2\kappa \end{bmatrix} ds + \begin{bmatrix} \beta_\sigma \sigma r(s)^{\beta_\sigma - 1} & 0 \\ 0 & 0 \end{bmatrix} dW^P(s) \right) \begin{bmatrix} D_t r(s) \\ D_t \phi(s) \end{bmatrix}; \\
& \qquad \begin{bmatrix} D_t r(t) \\ D_t \phi(t) \end{bmatrix} = \begin{bmatrix} \sigma r(t)^{\beta_\sigma} \\ 0 \end{bmatrix}.
\end{aligned}
$$

We also notice that

$$
D_t\theta(r(s),s) = \beta_\theta \theta r(s)^{\beta_\theta - 1}(D_t r(s)).
$$

Utilizing these equations, we are able to evaluate the IR-hedging and MPR-hedging terms on the right-hand side of Eq. (12) by numerical methods. Hence, we can examine the effects of a change in the market price of risk on optimal portfolio weights, including IR-hedge and MPR-hedge components, in a more general setting than in Section III.A.

Without loss of generality, we suppose that available bonds for investment are zero coupon bonds with maturities $\bar{T}_1$ and $\bar{T}_2$ ($\bar{T}_1 > \bar{T}_2 > t$), and the money market account resembles the one in the previous subsection. Furthermore, we put the constraint that the duration of the portfolio is equal to $\bar{T} - t$. Under this setting, if the total value on the right-hand side of Eq. (12) is denoted by $\nu_{V^*}(t, T)$, the optimal portfolio weights should satisfy

$$\begin{bmatrix} \nu(t, \bar{T}_1) & \nu(t, \bar{T}_2) \\ \bar{T}_1 - t & \bar{T}_2 - t \end{bmatrix} \begin{bmatrix} w_1^*(t) \\ w_2^*(t) \end{bmatrix} = \begin{bmatrix} \nu_{V^*}(t, T^0) \\ \bar{T} - t \end{bmatrix}.$$

By Eq. (9) and Eq. (22), the volatility coefficient of the instantaneous forward rate is expressed as $\sigma(t, T) = \sigma r(t)^{\beta_\sigma} \exp(-\kappa(T - t))$, and it is an exponentially decreasing function of the term to maturity $(T - t)$ as in the case of Eq. (13). Thus, based on the intuitive explanation in Section III.A or by simple calculation, if the total risk the investor takes in the portfolio, $\nu_{V^*}(t, T)$, is an increasing function of θ, it is easily shown that the signs of partial derivatives of the optimal portfolio weights with respect to θ are the same as given by Eq. (21). On the other hand, as for the change in the shape of the term structure, we confirm by Eq. (2) that the yield curve shifts downward and steepens as θ increases.

[Insert Table I and Table II around here.]

Figure 1 and Figure 2 show the results of our numerical analysis. The values of the parameters are specified, based on Chan, Karolyi, Longstaff, and Sanders (1992) and Pearson and Sun (1994), and they are reported in Table I and Table II except θ. In the figures, θ moves from 0 to 1.5, which includes the range of the estimates for θ (0.2944−1.0815) in Pearson and Sun (1994). We analyze not only the total demand, but also each component of the total demand such as the mean-variance demand, the IR-hedge demand and the MPR-hedge demand. We note that the mean-variance portfolio is analytically obtained while the IR-hedge demand and the MPR-hedge demand should be evaluated numerically. We employ a new numerical computation technique based on the *asymptotic expansion approach* developed by Kunitomo and Takahashi (2001, 2003c), Takahashi (1995, 1999), Takahashi and Saito (2003), and Takahashi and Yoshida (2003, 2004). The methodology is summarized in Appendix B. In the comparative statics of the parameter θ used for the discussion below, we first compare the values of the IR-hedging and the MPR-hedging terms on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion, and those computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme. In the figures, the former values are labeled "IRHA " (Eq. (A.12b) in Appendix B) and "MPRHA" (Eq. (A.12c) in Appendix B), and the latter values are "IRH" and "MPRH." Moreover, "MV" in the figures indicates the values of the MV term which can be analytically obtained, and "TD" and "TDA" are total values on the right-hand side of Eq. (12), ν_{V^*}, which are obtained by the Monte Carlo simulations and by the asymptotic expansion respectively. That is, TD = MV + IRH + MPRH and TDA = MV + IRHA + MPRHA are satisfied. Then, we observe from Figure 3, which is abstracted from Figure 1, that the difference in the values of the IR-hedging term is small while the difference in the values of the MPR-hedging term is relatively large, although for the purpose of comparative statics the evaluation based on the asymptotic expansion is enough. Further, in order to obtain more precise values for the MPR-hedging term efficiently, we propose another new method where the analytical approximation obtained by the asymptotic expansion is effectively used in a Monte Carlo simulation. We can see easily that this method gives much faster convergence than a naive Monte Carlo simulation from Figure 4. In the figure, we plot the errors of the values "MPRHAmc" obtained by this method and those by the naive Monte Carlo simulations for various numbers of trials against the value by the Monte Carlo simulation where the number of trials is 10,000,000. For comparison, we also display the error level of MPRHA. The parameters used in the figure are given by Table I and $\theta = 1.0815$. Finally we emphasize that our new method can be applied to a broad class of problems because the asymptotic expansion is a unified method (see Appendix B). Thus, both the analytical approximation itself and the asymptotic expansion with the Monte Carlo simulation are powerful numerical schemes for a problem which requires a substantial burden in computation. For example, the Monte Carlo simulation with asymptotic method is successfully applied to pricing of swap options in the HJM framework (Takahashi (2003), Kunitomo and Takahashi (2003a)) and pricing of options under a jump-diffusion process (Kunitomo and Takahashi (2003a, 2003b)).

We return to the comparative statics on the right-hand side of Eq. (12) of the parameter θ in the market price of risk. First, we easily notice that the MV term is an increasing function of the parameter θ. The IR-hedging term decreases while the MPR-hedging term increases as the parameter θ increases: as θ increases, the speed of mean reversion toward the long-run mean picks up, which reduces the fluctuation of the short rate r. Hence, the IR-hedging term which generates the hedging demand against the fluctuation of the short rate decreases. On the other hand, the increase in θ makes the fluctuation of the market price of risk itself increase, which dominates the effect caused by the increase in the adjustment speed of the short rate. As a result, the MPR-hedging term increases. In this way, the change in θ has the opposite effect on the IR-hedging and MPR-hedging terms. However, the total effect is essentially determined by the MV term because the change in θ gives much greater positive effect on it. Therefore, the total value on the right-hand side of Eq. (12) ν_{V^*} is indeed an increasing function of θ.

Thus, in the relatively broad class of term structure models, the relation between the change in the optimal portfolio weights and the change in the shape of the yield curve is consistent to the one stated in Proposition 1. In other words, Proposition 1, which is derived under the assumption that the dynamics of the yield curve is described by the extended Vasiček model, is valid as a first approximation for the general case.

IV Conclusion

This paper formulates and analyzes a dynamic optimization problem of bond portfolios. We assume that the investor is a CRRA utility maximizer over terminal wealth without any liabilities. We then focus on the relation between the investor's portfolio choice and a forecasted change in the shape of the yield curve. The main results obtained in this paper are summarized as follows. First, we characterized the optimal bond portfolio in the general single-factor HJM term structure framework. Second, we solved the optimal bond portfolios analytically or numerically within a certain class of Markovian HJM models. In the numerical analysis, we utilized a new technique based on the *asymptotic expansion approach* in order to obtain an analytical approximation and to increase efficiency in the Monte Carlo simulation. Finally, we provided a theoretical foundation of the bullet strategy, which is a yield curve strategy recommended in practice when the yield curve is expected to steepen. Moreover, we can utilize our model and technique to investigate the interactions between optimal strategies and the market price of risk under the other expectations for change in a yield curve such as yield curve inversion.[5] When the interest rates are much higher than a normal level, they are expected to decrease. Under the expectations hypothesis, long-term interest rates are lower than short-term interest rates, and hence a yield curve inversion is likely to be the case. We can proceed to analysis in such a situation if $f(0,T)$ and $\partial f(0,T)/\partial T$ in our model (10) are suitably set. This is a topic for future extension.

Appendices

We first consider an investor's portfolio choice problem with a Gaussian two-factor term structure model of interest rates, and examine the optimality of yield curve strategies. Then, we briefly explain the asymptotic expansion scheme used in Section III.B. We only give the outline of the scheme; see Takahashi and Yoshida (2004) for the details of the analytical formulae in the scheme.

Appendix A Optimal Bond Portfolios with a Gaussian Two-Factor Model

Let W_1^P and W_2^P be Wiener processes on a filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}\}_{t\geq 0})$ with constant instantaneous correlation ρ. Consider the two-factor term structure model developed by Hull and White (1994):

$$dr(t) = \left(\psi^P(t) + u(t) - ar(t)\right) dt + \sigma_1 dW_1^P(t), \tag{A.1}$$

$$du(t) = -bu(t)dt + \sigma_2 dW_2^P(t),$$

where a, b, σ_1 and σ_2 are positive constants, and $\psi^P(t)$ is a deterministic function of the time parameter t. We set $u(0) = 0$ without loss of generality. Let the market price of risks attached to W_1^P and W_2^P be two positive constants, θ_1 and θ_2, and put

$$d\boldsymbol{W}^Q(t) = d\boldsymbol{W}^P(t) - \boldsymbol{\theta}dt,$$

$$\text{where} \quad d\boldsymbol{W}^P(t) \triangleq \begin{bmatrix} dW_1^P(t) \\ dW_2^P(t) \end{bmatrix}, \quad d\boldsymbol{W}^Q(t) \triangleq \begin{bmatrix} dW_1^Q(t) \\ dW_2^Q(t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\theta} \triangleq \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

Then, W_1^Q and W_2^Q are Wiener processes under an equivalent martingale measure Q. The value of the money market account at time t is given by $B(t) = \exp\left(\int_0^t r(v)dv\right)$, and the state-price deflator is defined by $\eta_2(t)/B(t)$, where

$$\eta_2(t) \triangleq \exp\left(\int_0^t \boldsymbol{\theta}^\top d\boldsymbol{W}^P(s) - \frac{1}{2}\int_0^t \boldsymbol{\theta}^\top \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \boldsymbol{\theta} ds\right), \tag{A.2}$$

or equivalently

$$d\eta_2(t) = \eta_2(t)\boldsymbol{\theta}^\top d\boldsymbol{W}^P(t)$$

with the initial condition $\eta_2(0) = 1$. Here, $x^\top$ denotes the transpose of x.

We consider a portfolio choice problem among the $\bar{T}_1$-maturity zero coupon bond, the $\bar{T}_2$-maturity zero coupon bond with $\bar{T}_1 > \bar{T}_2$, and the money market account as in Section III. However, we impose no restriction on the duration of the bond portfolio. In this setting, the T-maturity zero coupon bond price is given by

$$P(t,T) = \exp\left(A(t,T) - B(t,T)r(t) - C(t,T)u(t)\right), \tag{A.3}$$

where

$$\begin{aligned} B(t,T) &\triangleq \frac{1-\exp(-a(T-t))}{a}, \\ C(t,T) &\triangleq \frac{1}{a-b}\left(\frac{\exp(-a(T-t))}{a} - \frac{\exp(-b(T-t))}{b}\right) + \frac{1}{ab}, \end{aligned}$$

and $A(t,T)$ is a deterministic function of t and T. We note that B and C are positive for $T > t$. Then, the dynamics of $P(t,T)$ is described by

$$dP(t,T) = P(t,T)\left(\mu(r(t),t,T)dt - \nu_1(t,T)dW_1^P(t) - \nu_2(t,T)dW_2^P(t)\right), \tag{A.4}$$

where

$$\mu(r(t),t,T) \triangleq r(t) + \theta_1\nu_1(t,T) + \theta_2\nu_2(t,T),$$

$$\begin{aligned}\nu_1(t,T) &\triangleq \sigma_1 B(t,T)\\ &= \frac{1-\exp(-a(T-t))}{a}\sigma_1 > 0,\end{aligned}$$

and

$$\begin{aligned}\nu_2(t,T) &\triangleq \sigma_2 C(t,T)\\ &= \left(\frac{1}{a-b}\left(\frac{\exp(-a(T-t))}{a} - \frac{\exp(-b(T-t))}{b}\right) + \frac{1}{ab}\right)\sigma_2 > 0.\end{aligned}$$

Since $\nu_2(t,T)/\nu_1(t,T)$ increases as T increases, $\begin{bmatrix}\nu_1(t,\bar{T}_1) & \nu_1(t,\bar{T}_2)\\ \nu_2(t,\bar{T}_1) & \nu_2(t,\bar{T}_2)\end{bmatrix}$ is nonsingular for $\bar{T}_1 > \bar{T}_2 > t$ and its determinant

$$\Delta_2 \triangleq \begin{vmatrix}\nu_1(t,\bar{T}_1) & \nu_1(t,\bar{T}_2)\\ \nu_2(t,\bar{T}_1) & \nu_2(t,\bar{T}_2)\end{vmatrix} < 0. \tag{A.5}$$

This nonsingularity means that two Wiener processes can be replicated by the asset price processes. Thus, markets are complete in this economy. Following the technique in Section II.B, the investor's optimal wealth is expressed as

$$V^*(t) = \lambda^{-\frac{1}{\gamma}}\left(\frac{\eta_2(t)}{B(t)}\right)^{-\frac{1}{\gamma}} P(t,T^0)^{1-\frac{1}{\gamma}} H(t,T^0), \tag{A.6}$$

where λ is a Lagrange multiplier and $H(t,T)$ is a deterministic function which depends only on t and T. Let $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^\top$ be the vector of the weights of the $\bar{T}_1$- and $\bar{T}_2$-maturity zero coupon bonds. From Eq. (A.2), Eq. (A.4) and Eq. (A.6), the vector of the optimal weights $\boldsymbol{w}^*$ should satisfy

$$\begin{bmatrix} \nu_1(t,\bar{T}_1) & \nu_1(t,\bar{T}_2) \\ \nu_2(t,\bar{T}_1) & \nu_2(t,\bar{T}_2) \end{bmatrix} \boldsymbol{w}^*(t) = \frac{1}{\gamma}\boldsymbol{\theta} + \left(1 - \frac{1}{\gamma}\right) \begin{bmatrix} \nu_1(t,T^0) \\ \nu_2(t,T^0) \end{bmatrix}.$$

In addition, let w_3 be the weight of the money market account. Then, the optimal weights are solved as

$$\begin{aligned} \begin{bmatrix} w_1^*(t) \\ w_2^*(t) \\ w_3^*(t) \end{bmatrix} &= \frac{1}{\gamma} \begin{bmatrix} \frac{\theta_1\nu_2(t,\bar{T}_2) - \theta_2\nu_1(t,\bar{T}_2)}{\Delta_2} \\ \frac{\theta_2\nu_1(t,\bar{T}_1) - \theta_1\nu_2(t,\bar{T}_1)}{\Delta_2} \\ 1 + \frac{\theta_1(\nu_2(t,\bar{T}_1) - \nu_2(t,\bar{T}_2)) - \theta_2(\nu_1(t,\bar{T}_1) - \nu_1(t,\bar{T}_2))}{\Delta_2} \end{bmatrix} \\ &\quad + \left(1 - \frac{1}{\gamma}\right) \begin{bmatrix} \frac{\nu_1(t,T^0)\nu_2(t,\bar{T}_2) - \nu_2(t,T^0)\nu_1(t,\bar{T}_2)}{\Delta_2} \\ \frac{\nu_1(t,\bar{T}_1)\nu_2(t,T^0) - \nu_2(t,\bar{T}_1)\nu_1(t,T^0)}{\Delta_2} \\ 1 + \frac{\nu_1(t,T^0)(\nu_2(t,\bar{T}_1) - \nu_2(t,\bar{T}_2)) - \nu_2(t,T^0)(\nu_1(t,\bar{T}_1) - \nu_1(t,\bar{T}_2))}{\Delta_2} \end{bmatrix}. \end{aligned} \tag{A.7}$$

The vector in the first term on the right-hand side of the above equation stands for the mean-variance portfolio, while the vector in the second term represents the intertemporal hedging portfolio. A similar result is obtained by Brennan and Xia (2000) in a different context. Eq. (A.7) shows that the optimal bond portfolio depends on the market price of risks; θ_1 and θ_2 have effects on the optimal portfolio only through the changes of the mean-variance portfolio, and the intertemporal hedging portfolio is invariant with these market price of risks. From

Eq. (A.4) and Eq. (A.5), we obtain

$$\frac{\partial w_1^*(t)}{\partial \theta_1} < 0, \qquad \frac{\partial w_2^*(t)}{\partial \theta_1} > 0, \qquad \frac{\partial w_3^*(t)}{\partial \theta_1} < 0,$$
$$\frac{\partial w_1^*(t)}{\partial \theta_2} > 0, \qquad \frac{\partial w_2^*(t)}{\partial \theta_2} < 0, \qquad \frac{\partial w_3^*(t)}{\partial \theta_2} > 0.$$

The signs in the above equations can be explained by analogy to the Rybczynski theorem as in Section III.A. In this model, two factors correspond to two kinds of risks represented by Wiener processes, and the degree of factor intensity is measured by $\nu_1(t,T)/\nu_2(t,T)$, which is a decreasing function of T.

Moreover, as in Section III.A, the changes of θ_1 and θ_2 have impacts on expectations for term structure movements. From Eq. (A.1) and Eq. (A.3), the dynamics of the forward rate is given by

$$\begin{aligned} df(t,T) = \big(&\sigma_1^2 B(t,T)B_T(t,T) + \sigma_2^2 C(t,T)C_T(t,T) + \rho\sigma_1\sigma_2\left(B_T(t,T)C(t,T)\right.\\ &\left.+B(t,T)C_T(t,T)\right) - \sigma_1\theta_1 B_T(t,T) - \sigma_2\theta_2 C_T(t,T)\big)\,dt \\ &+ \sigma_1 B_T(t,T)dW_1^P(t) + \sigma_2 C_T(t,T)dW_2^P(t), \end{aligned}$$

where

$$B_T(t,T) = \exp(-a(T-t)) > 0$$

and

$$C_T(t,T) = \frac{-\exp(-a(T-t)) + \exp(-b(T-t))}{a-b} > 0.$$

Here, $B_T(t,T)$ and $C_T(t,T)$ mean the derivatives of $B(t,T)$ and $C(t,T)$ with respect to the maturity T. It can be seen that the magnitudes of both θ_1 and θ_2 have effects on the drift coefficient. Because $B_{TT}(t,T) = -a\exp(-a(T-t)) < 0$, and $C_{TT}(t,T) = (a\exp(-a(T-t)) - b\exp(-b(T-t)))/(a-b)$ is positive (negative) when T is smaller (larger) than $t + (\log a - \log b)/(a-b)$, an increase in θ_1 gives a combination of a downward parallel shift and a steepening twist in the forecasted forward curve, while an increase in θ_2 gives a combination of a downward parallel shift and a positive butterfly (or less humpedness).

Altogether, when θ_1 increases, the forecasted forward curve shifts downward and steepens. At the same time, the weight of the intermediate-term zero coupon bond in the optimal bond portfolio increases and the weights of the short-term and long-term zero coupon bonds decrease. Note that the effects of θ_1 are the same as those of θ in Section III.A. Hence, the advice that the bullet strategy is preferable under the expectation of the steepening change in the yield curve is also supported in this two-factor term structure model.

On the other hand, when θ_2 increases, the shift of the forecasted forward curve is combined with a downward parallel shift and a positive butterfly, and the optimal weights of the short-term and long-term zero coupon bonds increase and the optimal weight of the intermediate-term zero coupon bond decreases; that is, the barbell strategy is preferable. This result is explained as follows. Since the forecasted forward curve shifts downward by an increase in θ_2, the higher rate of return of a zero coupon bond is expected. Then, the investor will take more risk of W_2^P without changing the amount of risk of W_1^P. Because $\nu_2(t,T)/\nu_1(t,T)$ is an increasing function of T, the weight of the more W_2^P-intensive long-term zero coupon bond should be increased while that of the more W_1^P-intensive intermediate-term zero coupon bond should be decreased by analogy to the Rybczynski theorem. On the contrary, the investor takes too much risk of W_1^P if the investor raises the weight of the intermediate-term zero coupon bond so as to take more risk of W_2^P, betting on the positive butterfly.

Appendix B The Asymptotic Expansion Scheme

Appendix B.1 Analytical Approximation

We explain the asymptotic expansion scheme in a slightly general setting, and hence start with a basic setup of the financial market. Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $T \in (0,\infty)$ denotes some fixed time horizon of the economy. $\boldsymbol{W}^P(t) = \begin{bmatrix} W_1^P(t) & \cdots & W_d^P(t) \end{bmatrix}^\top$, $0 \le t \le T$ is an $\mathbb{R}^d$-valued Brownian motion defined on $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_t\}$, $0 \le t \le T$ stands for P-augmentation of the natural filtration, $\mathscr{F}_t^{\boldsymbol{W}^P} \triangleq \sigma(\boldsymbol{W}^P(s); 0 \le s \le t)$. $S_i(t)$, $i = 1, \cdots, m$ and $S_0(t)$ denote the prices at time $t \in [0,T]$ of the risky asset i and of the riskless asset respectively. The prices are assumed to follow the stochastic process: for $t \in [0,T]$,

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{d} b_{ij}(t)dW_j^P(t)\right), \quad i = 1, \cdots, m$$

and

$$dS_0(t) = r(t)S_0(t)dt$$

with $m \geq d$ and initial conditions $S_i(0) = s_i$ and $S_0(0) = 1$, where we suppose that $r(t)$, $a_i(t)$ and $b_{ij}(t)$, $i = 1, \cdots, m,\ j = 1, \cdots, d$ are bounded and progressively measurable with respect to $\{\mathscr{F}_t\}$. We also assume $\mathrm{Rank}(b(t)) = d$ almost surely, where $b(t)$ is an $(m \times d)$ matrix defined by $b(t) \triangleq [b_{ij}(t)]_{1 \leq i \leq m,\ 1 \leq j \leq d}$. Moreover, the vector of the market price of risks $\boldsymbol{\theta}(t)$ is assumed to exist. Then, an $\mathbb{R}^d$-valued progressively measurable bounded process $\boldsymbol{\theta}(t)$ satisfies

$$b(t)\boldsymbol{\theta}(t) = \boldsymbol{a}(t) - r(t)\mathbf{1},$$

where $\boldsymbol{a}(t) \triangleq \begin{bmatrix} a_1(t) & \cdots & a_m(t) \end{bmatrix}^\top$ and $\mathbf{1} \triangleq \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top$. Let $\boldsymbol{X}^{[\varepsilon]}(u)$ be an $\mathbb{R}^n$-valued diffusion process defined by the stochastic differential equation:

$$d\boldsymbol{X}^{[\varepsilon]}(u) = \boldsymbol{V}_0\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) du + V\left(\boldsymbol{X}^{[\varepsilon]}, \varepsilon\right) d\boldsymbol{W}^P(u); \quad \boldsymbol{X}^{[\varepsilon]}(t) = \boldsymbol{x}(t) \tag{A.8}$$

for $u \in [t, T]$, where $V\left(\boldsymbol{X}^{[\varepsilon]}, \varepsilon\right) \triangleq \begin{bmatrix} \boldsymbol{V}_1\left(\boldsymbol{X}^{[\varepsilon]}, \varepsilon\right) & \cdots & \boldsymbol{V}_d\left(\boldsymbol{X}^{[\varepsilon]}, \varepsilon\right) \end{bmatrix} \triangleq \left[V_{ij}\left(\boldsymbol{X}^{[\varepsilon]}, \varepsilon\right)\right]_{1 \leq i \leq n,\ 1 \leq j \leq d}$ is an $(n \times d)$ matrix, which depends on some parameter $\varepsilon \in (0, 1]$.

Let $Y^{[\varepsilon]}(u; t)$ be a unique solution of the $\mathbb{R}^{n \times n}$-valued stochastic differential equation:

$$dY^{[\varepsilon]}(u;t) = \sum_{j=1}^{d} \partial_{\boldsymbol{x}} \boldsymbol{V}_j\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) Y^{[\varepsilon]}(u;t) dW_j^P(u); \quad Y^{[\varepsilon]}(t;t) = I_n, \tag{A.9}$$

where for every fixed j, $\partial_{\boldsymbol{x}} \boldsymbol{V}_j\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) \triangleq \partial \boldsymbol{V}_j\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) / \partial \boldsymbol{x} \triangleq \left[\partial V_{ij}\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) / \partial x_k\right]_{1 \leq i,k \leq n}$ is an $(n \times n)$ matrix, and I_n denotes the $(n \times n)$ identity matrix. It is then known that the

Malliavin derivative of $\boldsymbol{X}^{[\varepsilon]}(u),\ u \geq t$ is expressed as

$$D_t \boldsymbol{X}^{[\varepsilon]}(u) = Y^{[\varepsilon]}(u;t) V\left(\boldsymbol{X}^{[\varepsilon]}(t), \varepsilon\right) = Y^{[\varepsilon]}(u;t) V(\boldsymbol{x}(t), \varepsilon), \quad \text{for} \quad u \geq t.$$

We also utilize the fact that for a function f of $\boldsymbol{X}^{[\varepsilon]}(u)$,

$$D_t f\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) = \partial f\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)\left(D_t \boldsymbol{X}^{[\varepsilon]}(u)\right) = \partial f\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t) V(\boldsymbol{x}(t), \varepsilon), \quad \text{for} \quad u \geq t,$$

where $\partial f\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) \triangleq \partial f\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)/\partial \boldsymbol{x}$ (see for instance Nualart (1995)).

Note that time-dependent-coefficient diffusion processes are included in the above equation if we enlarge the process to a higher-dimensional one. We also assume that the processes $r(u)$ and $\boldsymbol{\theta}(u)$ are some functions of $\boldsymbol{X}^{[\varepsilon]}(u)$, that is, $r(u) = r\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)$ and $\boldsymbol{\theta}(u) = \boldsymbol{\theta}\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)$. In the setting when the objective function is determined by a CRRA utility function over terminal wealth, the vector of the optimal proportions of wealth invested in the risky assets for the dynamic optimization problem, $\boldsymbol{w}^*(t)$, satisfies

$$\begin{aligned} b(t)^\top \boldsymbol{w}^*(t) = \frac{1}{\gamma}\boldsymbol{\theta}(\boldsymbol{x}(t)) - \left(1 - \frac{1}{\gamma}\right) & \frac{1}{\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}} \middle| \mathscr{F}_t\right]} \\ & \mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\left(\int_t^T \partial r\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t) V(\boldsymbol{x}(t), \varepsilon) du\right.\right. \\ & \quad + \sum_{j=1}^d \int_t^T \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t) V(\boldsymbol{x}(t),\varepsilon) dW_j^P(u) \\ & \left.\left. + \sum_{j=1}^d \int_t^T \theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t) V(\boldsymbol{x}(t),\varepsilon) du\right)^\top \middle| \mathscr{F}_t\right], \end{aligned} \tag{A.10}$$

where γ is the coefficient of relative risk aversion, and $\xi(t,T)$ denotes the ratio of two state-price deflators defined by

$$\xi(t,T) \triangleq \exp\left(-\int_t^T \boldsymbol{\theta}\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)^\top d\boldsymbol{W}^P(u) - \frac{1}{2}\int_t^T \left|\boldsymbol{\theta}(\boldsymbol{X}^{[\varepsilon]}(u))\right|^2 du - \int_t^T r\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) du\right).$$

Without loss of generality, we can set $t=0$. We put the MV, IR-hedging and MPR-hedging terms as follows:

$$(\text{MV term}) = \frac{1}{\gamma}\boldsymbol{\theta}(\boldsymbol{x}(t)), \tag{A.11a}$$

$$(\text{IR-hedging term}) = -\left(1-\frac{1}{\gamma}\right)\frac{\left(\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\int_t^T \partial r\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)Y^{[\varepsilon]}(u;t)du\right]V(\boldsymbol{x}(t),\varepsilon)\right)^\top}{\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\right]}, \tag{A.11b}$$

$$\begin{aligned}(\text{MPR-hedging term}) = &-\left(1-\frac{1}{\gamma}\right)\frac{1}{\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\right]}\\ &\left(\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\left(\sum_{j=1}^d\int_t^T \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)Y^{[\varepsilon]}(u;t)dW_j^P(u)\right.\right.\right.\\ &\left.\left.\left.+\sum_{j=1}^d\int_t^T \theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)\partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)Y^{[\varepsilon]}(u;t)du\right)\right]V(\boldsymbol{x}(t),\varepsilon)\right)^\top.\end{aligned} \tag{A.11c}$$

In order to obtain an approximation formula based on the asymptotic expansion around $\varepsilon=0$, we set a basic assumption, the *deterministic limit condition*:

Assumption 1

$$V(\cdot, 0) \equiv 0.$$

It follows from Assumption 1 that the limit process $\{\boldsymbol{X}(u) : t \leq u \leq T\} \triangleq \left\{\boldsymbol{X}^{[0]}(u) : t \leq u \leq T\right\}$ is a unique (deterministic) solution of the ordinary differential equation:

$$\boldsymbol{X}^{[0]}(u) = \boldsymbol{x}(t) + \int_t^u \boldsymbol{V}_0\left(\boldsymbol{X}^{[0]}(s), 0\right) ds.$$

Next, put $Y(s;t) \triangleq Y^{[0]}(s;t)$ and then clearly $Y(s;t)$ is a unique (deterministic) solution of the ordinary differential equation:

$$dY(s;t) = \partial_{\boldsymbol{x}} \boldsymbol{V}_0\left(\boldsymbol{X}^{[0]}(s), 0\right) Y(s;t) ds; \quad Y(t;t) = I_n,$$

for $s \in [t, T]$. Finally, Theorem 2 of Takahashi and Yoshida (2004) provides the following result:

Theorem 1 (Theorem 2 of Takahashi and Yoshida (2004)) *An asymptotic expansion of the optimal portfolio for an investor with a CRRA utility function is given by*

$$\begin{aligned} b(t)^\top \boldsymbol{w}^*(t) = \frac{1}{\gamma}\Bigg(\boldsymbol{\theta}(\boldsymbol{x}(t)) + (1-\gamma)\varepsilon(\partial_\varepsilon V(\boldsymbol{x}(t), 0))^\top \Bigg(\int_t^T \partial r^{[0]}(u) Y(u;t) du \\ + \frac{1}{\gamma}\sum_{j=1}^d \int_t^T \theta_j^{[0]}(u) \partial\theta_j^{[0]}(u) Y(u;t) du\Bigg)^\top\Bigg) + \boldsymbol{o}(\varepsilon), \end{aligned}$$

where $\partial_\varepsilon V\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right) \triangleq \left[\partial V_{ij}\left(\boldsymbol{X}^{[\varepsilon]}(u), \varepsilon\right)/\partial\varepsilon\right]_{1 \leq i,j \leq n}$, $f^{[0]}(u) \triangleq f(\boldsymbol{X}(u))$, $\partial f^{[0]}(u) \triangleq \partial f(\boldsymbol{X}(u))/\partial \boldsymbol{x}$, *and $\boldsymbol{o}(\varepsilon)$ represents the terms of higher orders than ε.*

This formula is obtained by taking expectation after a Taylor expansion around $\varepsilon = 0$ which is justified based on the Malliavin calculus. See Takahashi and Yoshida (2004) for further details. They also derive a formula of the higher order up to ε^2 for a CRRA utility function and a formula for general utility functions.

The MV, IR-hedging and MPR-hedging terms in the asymptotic expansion are given as follows:

$$\text{(MV term)} = \frac{1}{\gamma}\boldsymbol{\theta}(\boldsymbol{x}(t)), \tag{A.12a}$$

$$\text{(IR-hedging term)} = -\varepsilon\left(1-\frac{1}{\gamma}\right)(\partial_\varepsilon V(\boldsymbol{x}(t),0))^\top\left(\int_t^T \partial r^{[0]}(u)Y(u;t)du\right)^\top, \tag{A.12b}$$

$$\text{(MPR-hedging term)} = -\varepsilon\frac{1}{\gamma}\left(1-\frac{1}{\gamma}\right)(\partial_\varepsilon V(\boldsymbol{x}(t),0))^\top\left(\sum_{j=1}^{d}\int_t^T \theta_j^{[0]}(u)\partial\theta_j^{[0]}(u)Y(u;t)du\right)^\top. \tag{A.12c}$$

In our examples of Section III.B, for the terminal date of the investment horizon $T^0 > 0$, $d = 1$ and $m = 2$;

$$S_0(t) = B(t), \quad S_1(t) = P(t,\bar{T}_1), \quad S_2(t) = P(t,\bar{T}_2)$$

for some fixed $\bar{T}_1 > \bar{T}_2 (> t)$ and for all $t \in [0, T^0]$; $\boldsymbol{X}^{[\varepsilon]}(u) = \begin{bmatrix} X_1^{[\varepsilon]}(u) & X_2^{[\varepsilon]}(u) & X_3^{[\varepsilon]}(u)\end{bmatrix}^\top = \begin{bmatrix} r(u) & \phi(u) & u\end{bmatrix}^\top$. We also note that $\theta\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) = \theta\left(X_1^{[\varepsilon]}(u)\right)$ and $\varepsilon = \sigma$.

Appendix B.2 Numerical Computation of the MPR-hedging term

Hereafter we illustrate our scheme where the asymptotic expansion is applied to the MPR-hedging term (A.11c) to increase efficiency in Monte Carlo simulations. In computation of the MPR-hedging term, we first review an estimator by naive Monte Carlo simulation. Consider the evaluation problem of $V = \mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[\varepsilon]}(T)\right)\right]$, where $\boldsymbol{Z}^{[\varepsilon]}(T)$ is the unique solution to a system of stochastic differential equations. In a typical discretization, called the Euler-Maruyama scheme, we divide the time horizon into n equal intervals and denote the Monte Carlo approximation $V(\varepsilon, n, N)$ based on the independent N replications. Then, the Monte Carlo estimate of V can be given by

$$V(\varepsilon, n, N) = \frac{1}{N}\sum_{l=1}^{N}\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l, \tag{A.13}$$

where $[X]_l$ $(l = 1, \ldots, N)$ is the realized value of the l-th independent trial for a random variable X and $\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)$ is the discretization of $\boldsymbol{Z}^{[\varepsilon]}(T)$. As a new estimate of V based on the asymptotic expansion approach, we are proposing to use $\hat{V}(\varepsilon, n, N)$ by

$$\hat{V}(\varepsilon, n, N) = \mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[0]}(T)\right)\right] + \frac{1}{N}\sum_{l=1}^{N}\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right) - f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l,$$

where we have implicitly assumed that $\mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[\varepsilon]}(T)\right)\right]$ can be evaluated analytically. This method of estimation can be explained intuitively and is likely to improve the standard estimate $V(\varepsilon, n, N)$. When the difference between $\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l - V^*$ (i.e. the realized value of the l-th independent trial $\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l$ minus its true value) and $\left[f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l - \mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[0]}(T)\right)\right]$ (i.e. the realized value of the l-th independent trial $\left[f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l$ minus its true value) is small, then we expect that the error, $\hat{V}(\varepsilon, n, N) - V^*$, becomes small because two errors of $\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l$ and $\left[f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l$ can be canceled out. Then, we rewrite

$$\begin{aligned}\hat{V}(\varepsilon, n, N) - V^* = \frac{1}{N}\sum_{l=1}^{N}\Bigg(\left(\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l - \mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[\varepsilon]}(T)\right)\right]\right) \\ - \left(\left[f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l - \mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[0]}(T)\right)\right]\right)\Bigg) \quad \text{(A.14)}\end{aligned}$$

and we have denoted $\bar{\boldsymbol{Z}}^{[0]}(t)$ as $\bar{\boldsymbol{Z}}^{[\varepsilon]}(t)$ with $\varepsilon = 0$. From this representation we expect that the correlation between $\left[f\left(\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)\right)\right]_l$ and $\left[f\left(\bar{\boldsymbol{Z}}^{[0]}(T)\right)\right]_l$ becomes positively high due to positively high correlation between $\bar{\boldsymbol{Z}}^{[0]}(T)$ and $\bar{\boldsymbol{Z}}^{[\varepsilon]}(T)$. This type of estimate is similar to the control variate technique, which has been known in the Monte Carlo method. In the standard control variate method, however, it is often difficult to find the key quantity which is correlated with the target variable and whose expectation can be evaluated analytically (in our case $\mathrm{E}^P\left[f\left(\boldsymbol{Z}^{[0]}(T)\right)\right]$ for $f\left(\boldsymbol{Z}^{[0]}(T)\right)$), and in that situation it cannot be used in general cases. On the contrary, our estimate based on the asymptotic expansion can be applied easily to such situations. In the following, we will show this idea more concretely. See Theorem 2 and Theorem 3 of Takahashi and Yoshida (2003) for rigorous mathematical proof.

Suppose that a Markovian system of the stochastic differential equations used to compute Eq. (A.11c) in Monte Carlo simulation is given by Eq. (A.8), Eq. (A.9), $h^{[\varepsilon]}(t,T) \triangleq \xi(t,T)^{1-\frac{1}{\gamma}}$

and

$$\boldsymbol{\zeta}^{[\varepsilon]}(T) \triangleq \sum_{j=1}^{d} \int_t^T \theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t)du$$
$$+ \sum_{j=1}^{d} \int_t^T \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t)dW_j^P(u).$$

Hence, $h^{[\varepsilon]}(t,T)$ and $\boldsymbol{\zeta}^{[\varepsilon]}(T)$ satisfy the following stochastic differential equations:

$$d_u h^{[\varepsilon]}(t,u) = -\left(1-\frac{1}{\gamma}\right) h^{[\varepsilon]}(t,u) \left(\left(r\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) + \frac{1}{2\gamma}\left|\boldsymbol{\theta}\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)\right|^2\right)du\right.$$
$$\left. + \left(\boldsymbol{\theta}\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)\right)^\top d\boldsymbol{W}^P(u)\right); \quad h^{[\varepsilon]}(t,t) = 1,$$
$$d\boldsymbol{\zeta}^{[\varepsilon]}(u) = \sum_{j=1}^{d} \theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t)du + \sum_{j=1}^{d} \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) Y^{[\varepsilon]}(u;t)dW_j^P(u);$$
$$\boldsymbol{\zeta}^{[\varepsilon]}(t) = \mathbf{0}.$$

In this case the components of $\boldsymbol{Z}^{[\varepsilon]}(u)$ are given by those of $\boldsymbol{X}^{[\varepsilon]}(u)$, $Y^{[\varepsilon]}(u;t)$, $h^{[\varepsilon]}(t,u)$ and $\boldsymbol{\zeta}^{[\varepsilon]}(u)$.

The estimator based on naive Monte Carlo simulation (A.13) for the denominator of the MPR-hedging term (A.11c),

$$\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\right] = \mathrm{E}^P\left[h^{[\varepsilon]}(t,T)\right] \tag{A.15}$$

may be expressed as

$$\frac{1}{N}\sum_{l=1}^{N}\left[\bar{h}^{[\varepsilon]}(t,T)\right]_l. \tag{A.16}$$

Similarly, the estimator based on naive Monte Carlo simulation (A.13) for the numerator of the MPR-hedging term (A.11c),

$$\mathrm{E}^P\left[\xi(t,T)^{1-\frac{1}{\gamma}}\left(\sum_{j=1}^{d}\int_t^T \partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)Y^{[\varepsilon]}(u;t)dW_j^P(u)\right.\right.$$
$$\left.\left.+\sum_{j=1}^{d}\int_t^T \theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)\partial\theta_j\left(\boldsymbol{X}^{[\varepsilon]}(u)\right)Y^{[\varepsilon]}(u;t)du\right)\right] = \mathrm{E}^P\left[h^{[\varepsilon]}(t,T)\boldsymbol{\zeta}^{[\varepsilon]}(T)\right] \quad \text{(A.17)}$$

may be expressed as

$$\frac{1}{N}\sum_{l=1}^{N}\left[\bar{h}^{[\varepsilon]}(t,T)\bar{\boldsymbol{\zeta}}^{[\varepsilon]}(T)\right]_l. \quad \text{(A.18)}$$

Next, we consider modified estimators for (A.15) and (A.17) in the following. First, we note that

$$h^{[0]}(t,T) = C \times \chi^{[0]}(T),$$
$$\text{where}\quad \chi^{[0]}(T) \triangleq \exp\left(-\left(1-\frac{1}{\gamma}\right)\left(\frac{1}{2}\left(1-\frac{1}{\gamma}\right)\int_t^T\left|\boldsymbol{\theta}^{[0]}(u)\right|^2du\right.\right.$$
$$\left.\left.+\int_t^T\left(\boldsymbol{\theta}^{[0]}(u)\right)^\top d\boldsymbol{W}^P(u)\right)\right)$$

and

$$C \triangleq \exp\left(-\left(1-\frac{1}{\gamma}\right)\left(\int_t^T r^{[0]}(u) + \frac{1}{2\gamma}\int_t^T\left|\boldsymbol{\theta}^{[0]}(u)\right|^2du\right)\right).$$

A modified estimator for the denominator (A.15) is given by

$$\mathrm{E}^P\left[h^{[0]}(t,T)\right] + \frac{1}{N}\sum_{l=1}^{N}\left[\bar{h}^{[\varepsilon]}(t,T) - \bar{h}^{[0]}(t,T)\right]_l, \tag{A.19}$$

$$\text{where} \quad \mathrm{E}^P\left[h^{[0]}(t,T)\right] = C \tag{A.20}$$

because clearly $\mathrm{E}^P\left[\chi^{[0]}(T)\right] = 1$. Further, $\bar{h}^{[0]}(t,u)$ denotes the Euler-Maruyama scheme of $h^{[0]}(t,u)$:

$$d_u h^{[0]}(t,u) = -\left(1-\frac{1}{\gamma}\right) h^{[0]}(t,u)\left(\left(r^{[0]}(u) + \frac{1}{2\gamma}\left|\boldsymbol{\theta}^{[0]}(u)\right|^2\right)du + \left(\boldsymbol{\theta}^{[0]}(u)\right)^\top d\boldsymbol{W}^P(u)\right);$$
$$h^{[0]}(t,t) = 1.$$

In the similar way, a modified estimator for the numerator (A.17) is given by

$$\mathrm{E}^P\left[h^{[0]}(t,u)\boldsymbol{\zeta}^{[0]}(T)\right] + \frac{1}{N}\sum_{l=1}^{N}\left[\bar{h}^{[\varepsilon]}(t,T)\bar{\boldsymbol{\zeta}}^{[\varepsilon]}(T) - \bar{h}^{[0]}(t,T)\bar{\boldsymbol{\zeta}}^{[0]}(T)\right]_l, \tag{A.21}$$

where

$$\mathrm{E}^P\left[h^{[0]}(t,T)\boldsymbol{\zeta}^{[0]}(T)\right] = \frac{C}{\gamma}\sum_{j=1}^{d}\int_t^T \theta_j^{[0]}(u)\partial\theta_j^{[0]}(u)Y^{[0]}(u;t)du \tag{A.22}$$

and $\bar{\boldsymbol{\zeta}}^{[0]}(u)$ denotes the Euler-Maruyama scheme of $\boldsymbol{\zeta}^{[0]}(u)$:

$$d\boldsymbol{\zeta}^{[0]}(u) = \sum_{j=1}^{d}\theta_j^{[0]}(u)\partial\theta_j^{[0]}(u)Y^{[0]}(u;t)du + \sum_{j=1}^{d}\partial\theta_j^{[0]}(u)Y^{[0]}(u;t)d\boldsymbol{W}_j^P(u); \quad \boldsymbol{\zeta}^{[0]}(t) = 0.$$

In our examples of Section III.B, for the terminal date of the investment horizon $T^0 > 0$, $d = 1$ and $m = 2$;

$$S_0(t) = B(t), \quad S_1(t) = P(t, \bar{T}_1), \quad S_2(t) = P(t, \bar{T}_2)$$

for some fixed $\bar{T}_1 > \bar{T}_2 (> t)$ and for all $t \in [0, T^0]$; $\boldsymbol{X}^{[\varepsilon]}(u) = \begin{bmatrix} X_1^{[\varepsilon]}(u) & X_2^{[\varepsilon]}(u) & X_3^{[\varepsilon]}(u) \end{bmatrix}^\top = \begin{bmatrix} r(u) & \phi(u) & u \end{bmatrix}^\top$. We also note that $\theta\left(\boldsymbol{X}^{[\varepsilon]}(u)\right) = \theta\left(X_1^{[\varepsilon]}(u)\right)$ and $\varepsilon = \sigma$.

We obtain a new estimator for the MPR-hedging term, $\hat{V}_{\mathrm{MPR}}$, from Eq. (A.19) and Eq. (A.21). $V_{\mathrm{MPR}}^{\mathrm{mc}}$ is the value obtained by naive Monte Carlo simulation of the MPR-hedging term from Eq. (A.16) and Eq. (A.18) based on the discretization of the Euler-Maruyama scheme. $V_{\mathrm{MPR}}^{\mathrm{mc,ae}}$ is the value obtained by Monte Carlo simulation from $\sum_{l=1}^{N} \left[\bar{h}^{[0]}(t,T)\bar{\boldsymbol{\zeta}}^{[0]}(T)\right]_l / N$ in Eq. (A.21) divided by $\sum_{l=1}^{N} \left[\bar{h}^{[0]}(t,T)\right]_l / N$ in Eq. (A.19) based on the discretization of the Euler-Maruyama scheme. Finally, $V_{\mathrm{MPR}}^{\mathrm{ae}}$ is the analytical value obtained from Eq. (A.20) and Eq. (A.22), which is equivalent to Eq. (A.12c): the evaluation of the asymptotic expansion of the MPR-hedging term up to the ε-order. Note that C is canceled out. We expect that the error of $\hat{V}_{\mathrm{MPR}}$ is small relative to the error of $V_{\mathrm{MPR}}^{\mathrm{mc}}$ if errors of $V_{\mathrm{MPR}}^{\mathrm{mc}}$ and of $V_{\mathrm{MPR}}^{\mathrm{mc,ae}}$ cancel each other; that is, if $(V_{\mathrm{MPR}}^{\mathrm{mc}} - V_{\mathrm{MPR}}^{*})$ and $(V_{\mathrm{MPR}}^{\mathrm{mc,ae}} - V_{\mathrm{MPR}}^{\mathrm{ae}})$ take similar values as shown in Eq. (A.14). We confirm that this is the case from Figure 5 based on Figure 4; in Figure 5, the values $(\hat{V}_{\mathrm{MPR}} - V_{\mathrm{MPR}}^{*})/V_{\mathrm{MPR}}^{*}$, $(V_{\mathrm{MPR}}^{\mathrm{mc}} - V_{\mathrm{MPR}}^{*})/V_{\mathrm{MPR}}^{*}$ and $(V_{\mathrm{MPR}}^{\mathrm{mc,ae}} - V_{\mathrm{MPR}}^{\mathrm{ae}})/V_{\mathrm{MPR}}^{\mathrm{ae}}$ are labeled "Error for MPRHAmc," "Error for MPRH" and "Error for MPRH0" respectively.

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Footnotes

[1]See, for instance, Chapter 17 of Fabozzi (2000) for the details of active bond portfolio management strategies.

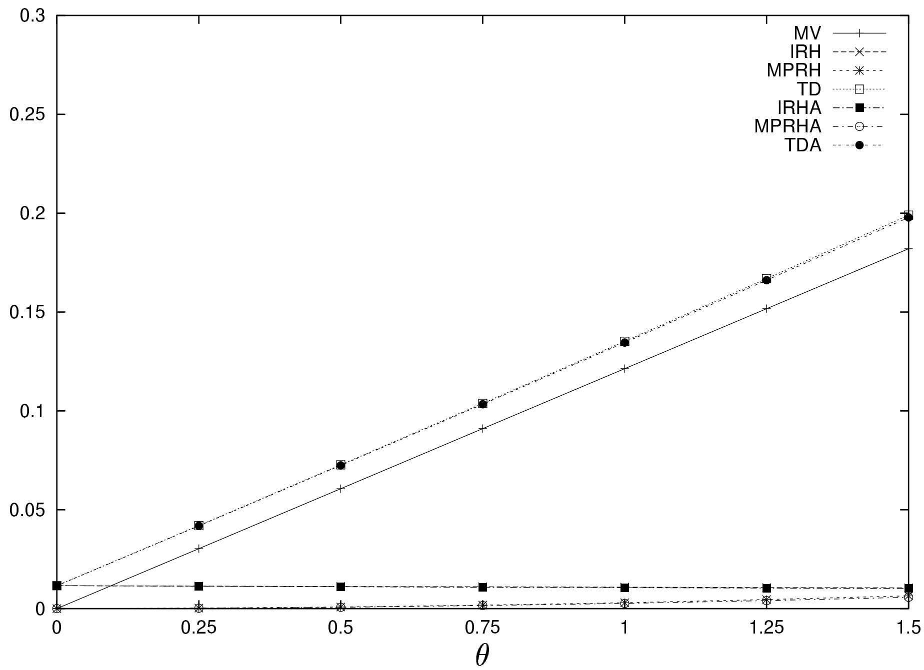
[2]See, for example, Heath, Jarrow, and Morton (1992) and Musiela and Rutkowski (1997).

[3]See, for example, Heath, Jarrow, and Morton (1992) and Musiela and Rutkowski (1997).

[4]See Sørensen (1999) and, for more general setting, Wachter (2001).

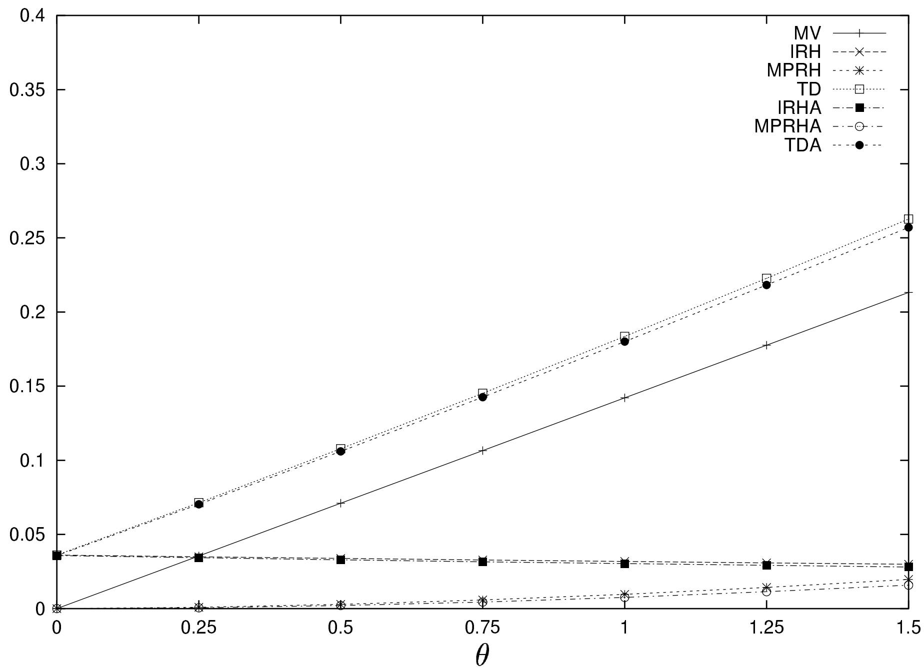
[5]We appreciate that an anonymous referee suggested this idea.

Figure 1: Comparative statics of the parameter θ, which leads to a combination of a downward parallel shift and a steepening twist of the forecasted forward curve, on the right-hand side of Eq. (12) with the other parameters given by Parameter Set 1.



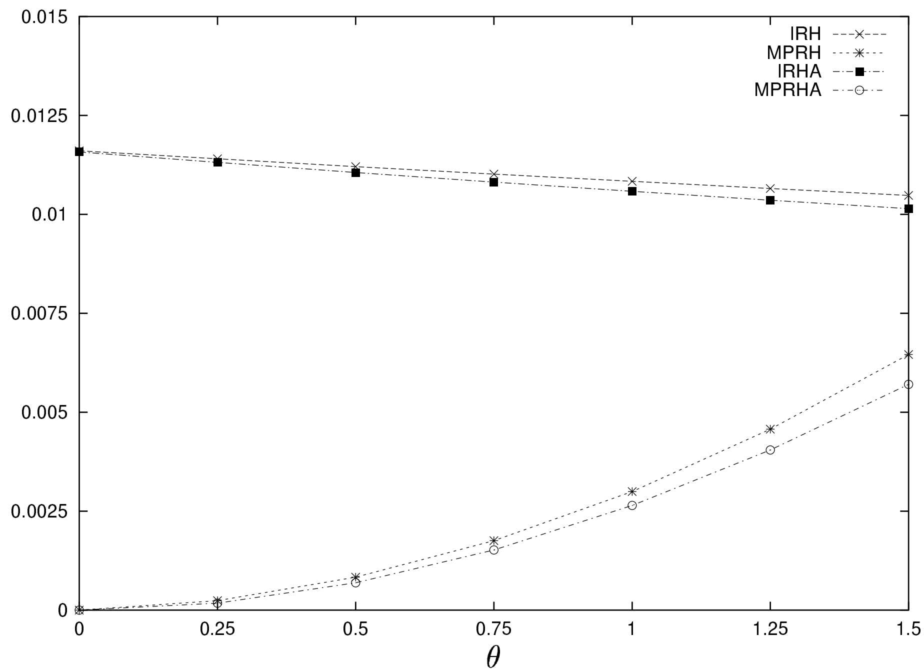
MV: The values of the MV term on the right-hand side of Eq. (12) which can be analytically obtained
IRH: The values of the IR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
MRPH: The values of the MPR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
TD: Total values on the right-hand side of Eq. (12) which are obtained by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
IRHA: The values of the IR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
MPRHA: The values of the MPR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
TDA: Total values on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion

Figure 2: Comparative statics of the parameter θ, which leads to a combination of a downward parallel shift and a steepening twist of the forecasted forward curve, on the right-hand side of Eq. (12) with the other parameters given by Parameter Set 2.



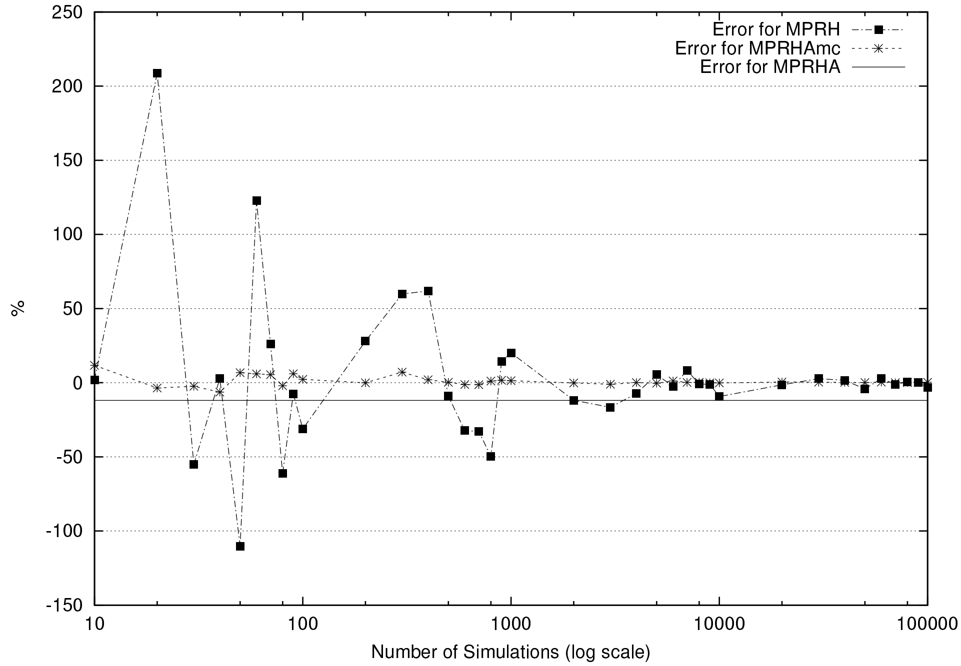
MV: The values of the MV term on the right-hand side of Eq. (12) which can be analytically obtained
IRH: The values of the IR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
MRPH: The values of the MPR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
TD: Total values on the right-hand side of Eq. (12) which are obtained by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
IRHA: The values of the IR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
MPRHA: The values of the MPR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
TDA: Total values on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion

Figure 3: Comparative statics of the parameter θ, which leads to a combination of a downward parallel shift and a steepening twist of the forecasted forward curve (The other parameters are given by Parameter Set 1.)



IRH: The values of the IR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
MRPH: The values of the MPR-hedging term on the right-hand side of Eq. (12) computed by Monte Carlo simulations where the number of trials in each simulation is 1,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
IRHA: The values of the IR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
MPRHA: The values of the MPR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion

Figure 4: Comparison of convergence for the MPR-hedging term on the right-hand side of Eq. (12) with the parameters given by Parameter Set 1 and $\theta = 1.0815$.



"Errors" are plotted against the value of the MPR-hedging term computed by the Monte Carlo simulation where the number of trials is 10,000,000 and the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme.

MPRH: The values of the MPR-hedging term on the right-hand side of Eq. (12) computed by naive Monte Carlo simulations where the number of time steps is 365 per year in discretization of the Euler-Maruyama scheme
MPRHA: The values of the MPR-hedging term on the right-hand side of Eq. (12) analytically approximated by the asymptotic expansion
MPRHAmc: The values of the MPR-hedging term on the right-hand side of Eq. (12) obtained by a new technique where the analytical approximation obtained by the asymptotic expansion is effectively used in Monte Carlo simulations.

Figure 5: Errors of the new estimator "MPRHAmc" and their components with the parameters given by Parameter Set 1 and $\theta = 1.0815$.

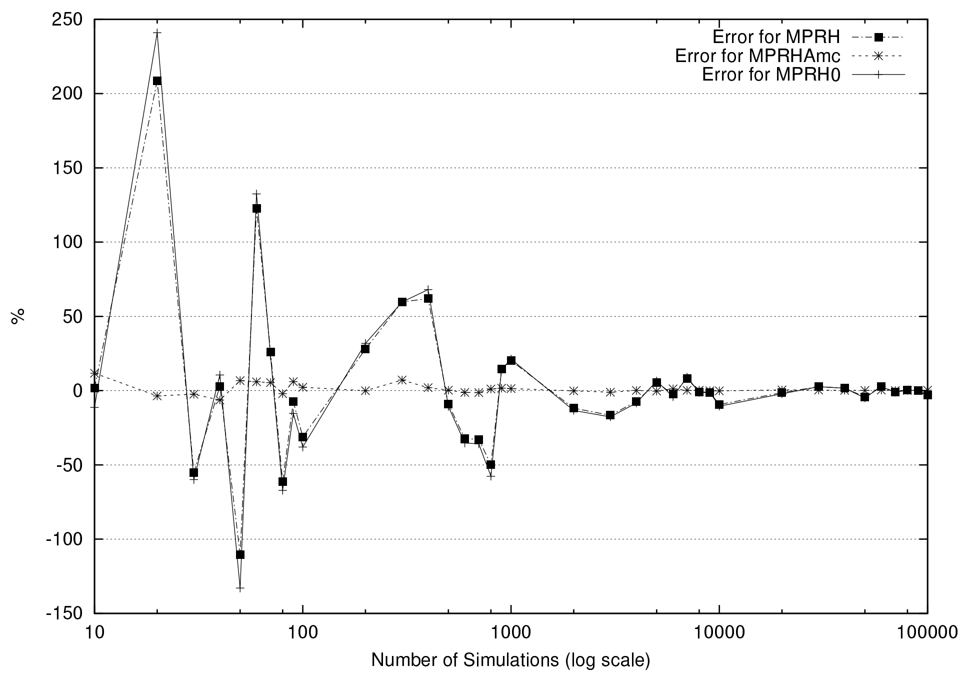


Table I: Parameter Set 1

Parameter	Value
γ	2
Investment horizon	5 years
Current forward curve	5.89% (flat)
β_σ	0.5
β_θ	0.5
κ	1.1407
σ	0.1092

The values of parameters are specified, based on Pearson and Sun (1994) using monthly prices of ten on-the-run Treasury bills, notes, and bonds from December 1971 to September 1979. We assume the flat current forward curve and $\beta_\sigma = \beta_\theta = 0.5$. The parameters κ and σ, and the current short rate $r(0)$ are calculated, comparing $dr(t) = \kappa(r(0) - r(t))dt + \sigma\sqrt{r(t)}dW^P(t)$ with the estimated Cox, Ingersoll, and Ross (1985) model of the real interest rate. γ denotes the coefficient of relative risk aversion of the investor.

Table II: Parameter Set 2

Parameter	Value
γ	2
Investment horizon	5 years
Current forward curve	8.08% (flat)
β_σ	0.5
β_θ	0.5
κ	0.2339
σ	0.08544

The values of parameters are specified, based on Chan, Karolyi, Longstaff, and Sanders (1992) using the annualized one-month U.S. Treasury bill yield from June 1964 to December 1989. We assume the flat current forward curve and $\beta_\sigma = \beta_\theta = 0.5$. The parameters κ and σ, and the current short rate $r(0)$ are calculated, comparing $dr(t) = \kappa(r(0) - r(t))dt + \sigma\sqrt{r(t)}dW^P(t)$ with the estimated Cox, Ingersoll, and Ross (1985) model. γ denotes the coefficient of relative risk aversion of the investor.