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Long-run Behavior of Macroeconomic Models with Heterogeneous Agents: Asymptotic Behavior of One- and Two-Parameter Poisson-Dirichlet Distributions

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Abstract

This paper discusses asymptotic behavior of one- and two-parameter Poisson-Dirichlet models, that is, Ewens models and its two parameter extensions by Pitman, and show that their asymptotic behavior are very different.

The paper shows asymptotic properties of a class of one- and two-parameter Poisson-Dirichlet distribution models are drastically different.

Convergence behavior is expressed in terms of generalized Mittag-Leffler distributions in the statistics literature. The coefficients of variations of suitably normalized number of clusters and of clusters of specific sizes do not vanish in the two-parameter version, but they do in one-parameter Ewens models.

Key Words:Two-parameterPoisson-Dirichlet distributions; Mittag-Leffler distributions; Non-self averaging phenomena, Power laws.

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Introduction

In old industrial organization literature, several tests and measures of degrees of industrial concentration have been used to to decide if a given industry is monopolistic or not.[1] One such test uses Herfindahl, or Herfindahl-Hirschman index of concentration. It is defined as the sum of squares of fractions of shares, i.e.,

$$H = \sum_i x_i^2,$$

where $x's$ are the fractions of "shares" of markets or sales by sectors or firms. By definition x_i is positive, and sum to one, $\sum_i x_i = 1$. As we discuss shortly, this literasture used a rudimentary version of the size-biased sampling scheme as a test on oligopoly. This meassure of concentration is used in both domestic and foregin trade context. It is sometimes (mistakenly) called Gini-index.[2] The question is that of distribution of fractions of the numbers of clusters, and the numbers of agents by types.

A simple application of shares of market by two types of agents, using one-parameter Poisson-Dirichlet distribution (also called Ewens distribution, Ewens (1972, 1979, 1990)) has been made by Aoki (2000a, 2000b).

This paper develops further the original ideas in these papers by applying some of the results from two-parameter Poisson-Dirichlet distributions in the recent combinatorial stochastic process literature, in Kingman (1993), Carlton (1999), Holst (2001), Pitman (1999, 2002), and Pitman and Yor (1996), among others.

In physics literature, Mekjian and Chase (1997) have used two-parameter models. They refer to the work by Pitman (1996). There are other works in the physics literature, in particular the papers by Derrida-Flyvbjerg mentioned in footnote 2, and Derrida (1994a, 1997).[3] There are other papers in the physics literature that deal with random partitions. Higgs (1995) have noted the similarities of some physical distributions and power laws, and mention population genetics papers by Ewens in particular. There are many papers on stick-breading version of the residual allocation processes, such as Krapivvsky, Grosse, and B. Nadin (2002). They have not touch on connections with the two-parameter Poisson-Dirichlet distributions, however.

In macroeconomic and finance modelings, agents of different characteristics or strategies are of different types and form separate clusters and affect aggregate behavior. In this paper, we therfore explore more broadly economic implications of long-run relations that may exist among non-self averaging economic or financial variables.

[1] See for example Scherer (1980) which describes many case studies.

[2] Sometimes it is called Gini-Simpson index of divesity. See Hirschman (1960) about the origin and mis-attribution of this notion to Herfindahl. In the population genetics literature H is called homozygosity. See Ewens (1972). Interestingly, the same measure has been used by Derrida-Flyvbjerk (1989) in discussing relative sizes of basins of attractions of Kaufman random maps and ramdom dynamics in statistics and physics. These, however, involve a sigle parameter θ in their statistical description. See also Aldous (1985).

[3] Derrida (1994b) has added some material on residual allocation models.

In this paper we use the coefficients of variations rather than the notion of non-self averaging in physics, because the former notion more correctly reflects the long-run or asymptotic sample dependence of some phenomena of interest. More specifically this paper shows that components of partition vectors in $PD(\alpha, \theta)$ with positive α have non-vanishing coefficients of variations (non-self averaging in the physics terminology when means do not diverge), while in $PD(\theta)$ they do not.

Number of Clusters in two-parameter Poisson-Dirichlet Distributions

The probabilities of new types entering models in $PD(\theta)$, and the number of clusters have been discussed in Aoki (2002, Sec.10.8, App. A.5), for example. In the two-parameter Poisson-Dirichlet distribution the conditional probabilities for the number of clusters in a sample of size n, K_n is given by

$$\Pr(K_{n+1} = k+1 | K_1, \ldots, K_n = k) = \frac{k\alpha + \theta}{n + \theta}, \tag{1}$$

and

$$\Pr(K_{n+1} = k | K_1, \ldots, K_n = k) = \frac{n - k\alpha}{n + \theta}, \tag{2}$$

where the random variable K_n is the number of different types of agents present in a sample of size n. Eq.(1) means that the $(n+1)$th entrant is a new type. Eq.(2) means that it is one of the previously existing types. Hence the number of clusters does not change.

Let the probability for $K_n = k$ be denoted by $q_{\alpha\theta}(n, k)$. From (1) and (2) it can be recursively computed using the two conditional probability equations above

$$q_{\alpha\theta}(n+1, k) = \frac{(n - k\alpha)}{(n + \theta)} q_{\alpha\theta}(n, k) + \frac{\theta + (k-1)\alpha}{n + \theta} q_{\alpha\theta}(n, k-1), \tag{3}$$

for $1 \le k \le n$. The expressions for the boundary $K_n = 1$ for all n, and that of $K_n = n$ are given by the expression

$$q_{\alpha\theta}(n, 1) = \frac{(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{(\theta+1)(\theta+2)\cdots(\theta+n-1)},$$

and

$$q_{\alpha\theta}(n, n) = \frac{(\theta+\alpha)(\theta+2\alpha)\cdots(\theta+(n-1)\alpha)}{(\theta+1)(\theta+2))\cdots(\theta+n-1)}.$$

These expressions generalize the recurrence relation for the one-parameter $PD(\theta)$. In the one-parameter case, $\theta/(\theta+n)$ is a probability that the (n+1)th agent that enter the model is a new type, and $n/(\theta+n)$ is the probability that the next agent is one of the types already in the model.

In the one-parameter case, $q_\theta(n, k) := P(K_n = k)$ is governed by the recurrence relation

$$q_\theta(n+1, k) = \frac{n}{n+\theta} q_{n,k} + \frac{\theta}{\theta+n} q_{n,k-1}.$$

The solution of this recurrence equation is expressible as

$$q_{n,k} = \frac{c(n,k)\theta^k}{\theta^{[n]}},$$

where $\theta^{[n]} := \theta(\theta+1)\cdots(\theta+n-1) = \frac{\Gamma(\theta+n)}{\Gamma(\theta)}$, and $c(n,k)$ is the unsigned (signless) Stirling number of the first kind. It satisfies the recursion

$$c(n+1,k) = nc(n,k) + c(n,k-1).$$

Since $q_{n,k}$ sums to one with respect to k we have

$$\theta^{[n]} = \sum_{k=1}^{n} c(n,k)\theta^k. \tag{4}$$

See Aoki (2002, p.208) for example on the Stirling numbers, and their combinatorial interpretations.

In the two-parameter $PD(\alpha,\theta)$ case, the probability of the number of clusters is given by

$$P_{\alpha,\theta}(K_n = k) = \frac{\theta^{[k,\alpha]}}{\alpha^k \theta^{[n]}} c(n,k;\alpha), \tag{5}$$

where

$$\theta^{[k,\alpha]} := \theta(\theta+\alpha)(\theta+2\alpha)\cdots(\theta+(k-1)\alpha),$$

and the expression $c(n,k;\alpha)$ generalizes the signless Stirling number of the first kind of one-parameter situation. This is called generalized Stirling number of the first kind. See Charalambides (2002).

Let $S_\alpha(n,k) := \frac{1}{\alpha^k} c(n,k;\alpha)$. It satisfies the recursion

$$S_\alpha(n+1,k) = (n-k\alpha)S_\alpha(n,k) + S_\alpha(n,k-1).$$

Instead of (4) we have

$$\theta^{[n]} = \sum_{k=1}^{n} S_\alpha(n,k)\theta^{[k,\alpha]}. \tag{6}$$

Pitman (1999) obtained its asymptotic expression as

$$S_\alpha(n,k) \sim \frac{\Gamma(n)}{\Gamma(k)} n^{-\alpha}\alpha^{1-k} g_\alpha(x),$$

where $k \sim xn^\alpha$. Here, g_α is the Mittag-Leffler (α)function. This function is discussed in the next section.

Asymptotic Behavior of Cluster Sizes

We collect here some known asymptotoc facts about cluster sizes as $n \to \infty$.

The number of clusters K_n

$$EK_n = \frac{\theta}{\alpha}[\frac{(\theta+\alpha)^{[n]}}{\theta^{[n]}} - 1],$$

where we note that

$$\frac{(\theta+\alpha)^{[n]}}{\theta^{[n]}} = \frac{\Gamma(\theta)}{\Gamma(\theta+\alpha)}\frac{\Gamma(\theta+\alpha+n)}{\Gamma(\theta+n)}.$$

Applying the asymptotic expression for the Gamma function for large n

$$\frac{\Gamma(n+a)}{\Gamma(n)} \sim n^a,$$

to the above expression, we have an asymptotic expression,

$$E(\frac{K_n}{n^\alpha}) \sim \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}. \quad (7)$$

Yamato and Sibuya (2000) obained the asymptotic value of the variance of K_n/n^α,

$$var(K_n/n^\alpha) \sim \frac{\Gamma(\theta+1)}{\alpha^2}\gamma_{\alpha,\theta} \geq 0, \quad (8)$$

where

$$\gamma_{\alpha,\theta} := \frac{\theta+\alpha}{\Gamma(\theta+2\alpha)} - \frac{\Gamma(\theta+1)}{[\Gamma(\theta+\alpha)]^2}. \quad (9)$$

Note that this quantity vanishes for all one-parameter models,

Fact: $\gamma_{0,\theta} = 0$.

This fact is important in the long-run behavior of components of the partition vectors, to be discussed in the next subsection.

We calculate the asymptotic behavior of the coefficient of variation, $\frac{(var(K_n/n^\alpha)^{1/2}}{E(K_n/n^\alpha}$.

It is given asymptotically by $\sqrt{\frac{\gamma_{\alpha\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha)$. This ratio is zero at $\alpha = 0$. This is one of the important difference in the asymptotic behaviors of one- and two-parameter Poisson-Dirichlet models.

Actually they calculate more generally

$$limE(\frac{K_n}{n^\alpha})^r = \mu'_r,$$

where μ'_r is the $r-th$ moment of the generalized Mittag-Leffler distribution with density

$$g_{\alpha,\theta} := \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)}x^{\frac{\theta}{\alpha}}g_\alpha(x),$$

where $\theta/\alpha > -1$, and where $g_\alpha(x)$ is the Mittag-Leffler (α) density function. It is known that this function is uniquely determined by the moment conditions

$$\int_0^\infty x^p g_\alpha(x)dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)},$$

for all $p > -1$. The moments of this density satisfy the sufficient condition for the density to be uniquely determined by the set of all moments so that the method of moments applies. Note that the integral of $g_{\alpha,\theta}$ over the interval from zero to infinity is 1, as it should be. See Pollard (1946), for example, for the expression of the density. See also Blumenfeld and Mandelbrot (1997) who credit Feller (1949) as the original source.

Mittag-Leffler distributions

Pitman (2002, Sec. 3) has stronger result:

$$K_n/n^\alpha \to \mathcal{L}, a.s.,$$

where the expression $\mathcal{L}$ has the density

$$\frac{d}{ds}P_{\alpha,\theta}(\mathcal{L} \in ds) = g_{\alpha,\theta}$$

where letting $\eta = \frac{\theta}{\alpha}$ we define

$$g_{\alpha,\theta}(s) := \frac{\Gamma(\theta+1)}{\Gamma(\eta+1)} s^\eta g_\alpha(s),$$

where $s > 0$, and where $g_\alpha = g_{\alpha,0}$ is the Mittag-Leffler density

$$g_\alpha(s) = \frac{1}{\pi}\sum_{k=1}^{\infty}[\frac{\Gamma(k\alpha)}{\Gamma(k)} sin(k\pi\alpha)(-s)^{k-1}].$$

We note that

$$\mu_1' = E_{\alpha,\theta}(\mathcal{L}) = \Gamma(\theta+1)/\alpha\Gamma(\theta+\alpha),$$

and

$$\mu_2' = E_{\alpha,\theta}(\mathcal{L}^2) = \Gamma(\theta+1)(\theta+\alpha)/\alpha^2\Gamma(\theta+2\alpha).$$

Hence variance of $\mathcal{L}$ is given as $\mu_2' - (\mu_1')^2 = [\Gamma(\theta+1)/\alpha^2]\gamma_{\alpha\theta}$.

For the record we have

Fact

$$E(\frac{K_n}{n^\alpha}) = \frac{\Gamma(\theta = 1}{\alpha\Gamma(\alpha+\theta)}, var_{\alpha,\theta}\{\frac{K_n}{n^\alpha}\} = var_{\alpha,\theta}\mathcal{L} = \frac{\Gamma(\theta+1)}{\alpha^2}\gamma_{\alpha,\theta}. \qquad (10)$$

The partition vector a

Denote the partition vector by $\mathbf{a} = (a_1, a_2, \ldots)$, where we recall that a_i is the number of distinct clusters of size i, hence $\sum_i a_i = K_n$, and $\sum_i ia_i = n$. Yamato and Sibuya obtain the limit of the first component, a_1

$$limE[\frac{a_1}{n^\alpha}] = \frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)},$$

and

$$lim\, var(\frac{a_1}{n^\alpha}) = \Gamma(\theta+1)\gamma_{\alpha,\theta} \geq 0.$$

In fact all a_j/n^α have asymptotically non-vanishing coefficients of variations, that is, are all non-self averaging, as well as ja_j/n^α, where ja_j is the total number of agents in the clusters of size j. Note that they are all zero with $\alpha = 0$, that is the asymptotic coefficients of variations of a_j/n^α are all zero in $PD(\theta)$ models. In two-parameter models they are given by

$$\sqrt{a_1/n^\alpha}/E(a_1/n^\alpha) \sim \sqrt{\gamma_{\alpha\theta}/\Gamma(\theta=1)}\Gamma(\theta+\alpha).$$

Fact Combining the above with (7) we have

$$\frac{Ea_1}{EK_n} \to \alpha.$$

The expressions for a_i/n^α, $i \geq 1$ are all non-zero with $0 < \alpha < 1$.

Sibuya (2005) used Formula 6.1.41 in Abramovitz and Stegun (1965) to obtain asymptotic expression

$$E(\frac{a_j}{n^\alpha}) \approx \frac{(1-\alpha)^{[j-1]}}{j!}\frac{\Gamma(\theta+1)}{\Gamma(\theta+\alpha)} + O(n^{-1}).$$

We state the asymptotic behavior of K_n/n^α and a_j/n^α as

Proposition: As in (10)

$$lim\, var_{\alpha,\theta}(K_n/n^\alpha) = var_{\alpha,\theta}(\mathcal{L}),$$

and

$$lim\, var_{\alpha,\theta}(a_j/n^\alpha) = \alpha^2 var_{\alpha,\theta}(\mathcal{L}) = \Gamma(\theta+1)\gamma_{\alpha,\theta},$$

and we have

$$a_j/n^\alpha \sim \sqrt{\gamma_{\alpha\theta}}j!\Gamma(\theta+\alpha)/(1-\alpha)^{j-1}.$$

They also show that covariances of components of the partition vectors are non-self averaging with positive α values:

Fact:

$$limCov(\frac{a_i}{n^\alpha}, \frac{a_j}{n^\alpha}) = \Gamma(\theta+1)\gamma_{\alpha,\theta} \times \frac{(1-\alpha)^{[i-1]}}{i!}\frac{(1-\alpha)^{[j-1]}}{j!} > 0, : \alpha > 0.$$

The correlation coefficient between them is given by

$$\rho_{i,j} \sim \frac{(1-\alpha)^{[i-1]}}{i!}\frac{(1-\alpha)^{[j-1]}}{j!}.$$

It is also known that

$$\frac{j!a_j/n^\alpha}{\alpha(1-\alpha)^{[j-1]}} \to^d \mathcal{L}. \quad (11)$$

We have

$$E(\frac{a_j}{n^\alpha}|K_n = k) \sim \frac{(1-\alpha)^{[j-1]}}{j!}(1-j/n)^{-(1+\alpha)} \times \xi,$$

where ξ depends on $g_{\alpha,\theta}$.

The number of clusters, K_n, is spread among the components of the partition vector, $a_i, i = 1, 2, \ldots, n$ at the proportion $\alpha(1-\alpha)^{[j-1]}/j!$, $0 < \alpha < 1$. Devroye (1993) calls this Sibuya distribution.

We also note that

$$Lim\frac{E(a_i)}{E(K_n)} = \frac{\alpha^2}{\Gamma(\theta+\alpha)\gamma_{\alpha,\theta}}.$$

We note that a_j/K_n is self-averaging for all $j = 1, \ldots, n$. Yamato and Sibuya also examined the clusters of size k or less

$$K[1,k] := a_1 + a_2 + \cdots + a_k,$$

and the number of agents in $K[1,k]$, denoted by $N[1,k]$ and obtained their limiting expressions as

$$\frac{K[1,k]}{n^\alpha} \to^d \{1 - \frac{(1-\alpha)^{[k]}}{k!}\}\mathcal{L},$$

and

$$\frac{N[1,k]}{n^\alpha} \to^d \alpha\frac{(2-\alpha)^{[k-1]}}{(k-1)!}\mathcal{L},$$

Sibuya also notes that

$$\{\frac{a_1}{n^\alpha}, \frac{2a_2}{n^\alpha} \cdots \frac{.ka_k}{n^\alpha}\}$$

converges in distribution to a sequence of random variables depending on $\mathcal{L}$ as

$$\{1, \frac{(1-\alpha)}{1!}, \ldots, \frac{(1-\alpha)^{[k-1]}}{(k-1)!}\}.$$

In $PD(\alpha)$ it is known that

$$\frac{K_n - \theta ln(n)}{\sqrt{\theta ln(n)}} \to N(0,1).$$

Hence $(K_n/ln(n))$ is self-averaging.

Almost sure convergence

Denote by $a_j(n)$ the number of clusters of size j when there are n agents in the model. We noted earlier that $\sum_{j=1}^n ja_j(n) = n$, and $K_n := \sum_j^n a_j(n)$ is the total number of clusters formed by the total of n agents.

By Rouault (1976, 1978)

$$\frac{a_j(n)}{K_n} \to \frac{\alpha\Gamma(j-\alpha)}{\Gamma(1-\alpha)j!}, a.s.$$

Recallint that $K_n/n^\alpha \to \mathcal{L}$,a.s., we have

$$a_j(n)/n^\alpha \to \frac{\alpha\Gamma(j-\alpha)}{\Gamma(1-\alpha)j!}\mathcal{L}, a.s.$$

wbere

$$\frac{a_j(n)}{K_n} \to \frac{\alpha}{j!}P_{\alpha,j},$$

where

$$P_{\alpha j} = \frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)},$$

for every $j = 1, 2, \ldots$ a.s. as n goes to infinity, and that $a_j(n) \sim P_{\alpha,j}\mathcal{L}n^\alpha$ in a two-parameter Poisson-Dirichlet case.

Local Limit Theorem

Suppose N independent positive random variables X_i, $i = 1, 2, \dots N$ are normalized by their sum $S_N = X_1 + \cdots + X_N$

$$x_i = X_i/S_N, i = 1, \dots N,$$

so that

$$Y_1 := \sum_i x_i = 1.$$

Suppose that the probability density of X_i is such that it has a power-law tail,

$$\rho(x) \sim Ax^{-1-\mu},$$

with $0 < \mu < 1$. Then, $S_N/N^{1/\mu}$ has a stable distribution (called Lévy distribution).

Pitman's formula for the probability of $K_n = k$, with $k \sim sn^{\alpha}$ indicates that the power law n^{α} which is $2\alpha < 2$ or $2\alpha = 1 + \mu$ with $0 < \mu < 1$, the case in Derrida.

With the 2-parameter PD distribution satisfying the power law condition, Derrida's conclusion that the Hs are non-self averaging applies to this case as well.

Estimating the Parameters

Carlton (1999) and Sibuya (2005)are the only systematic source on estimating the parameters of two-parameter Poisson-Dirichlet distributions.

With $\alpha = 0$, Ewens had shown that K_n is the sufficient statistics for θ. Carlton discusses the case where α is known and θ unknown. He derives the asymptotic distribution of the maximum likelihood estimate of θ, given n samples.

Lemma

Given α in (0,1), the maximum-likelihood estimate of θ, $\hat{\theta}_n$ is given by

$$\psi(1 + \hat{\theta}_n/\alpha) - \alpha\psi(1 + \hat{\theta}_n) \to logS, as.$$

Here ψ is the digamma function.

With θ known, and α unknown, Carlton proves

Lemma

Let $\{A_1, \dots, A_n\}$ is distributed according to the two-parameter Ewens distribution of size n. (His Eq. (4.2) on page 55.) Then,

$$\hat{\alpha}_n = \frac{logK_n}{logn} \to \alpha \; a.s.$$

Sibuya uses the conditional probability distribution of the partition vector components, given that $\sum_i a_i = k$, and expresses the distribution

$$P(\mathbf{a}| \sum a_j = k) = \frac{1}{S_{\alpha}(n,k)} \frac{n!}{\prod a_j!} \prod_j \{\frac{(1-\alpha)^{[j-1]}}{j!}\}^{a_j}$$

which is proportional to

$$exp\{-\sum \frac{j}{2(j-2)!}a_j\}\alpha + O(\alpha^2)$$

and test the hypothesis $\alpha = 0$, against the alternative hypothesis $\alpha < 0$.

Sibuya proposes the rejection region

$$\sum \frac{j}{2(j-2)!} - a_j > const.k.$$

When both parameters are unknown, the estimation problem is apparently unsolved.

Some Potential Applications

In physics literature, Derrida (1994 a, b) sketched a derivation that the expected values of $Y_k = \sum_i x_i^k, k = 2, 3, \ldots$ can be calculated for mean field spin glass models using the Parisi replica approach, and remarkably the formula is the same as the GEM model described above.

In the rest of this section we focus on economic examples.

Example Scaling of GDP growth rates was considered by Canning, Amaral, Lee, Meyer, and Stanley (1998). They showed that the standard deviation of the GDP growth rate may sclae as $Y^{-\beta}$, with β about 0.15. Here, we heuristically explain how their finding may be explained using a random partition framework.

We modify the model of Huang and Solomon (2001) and apply the same procedures to estimate the growth rate of real GDP.[4] View the real economy as composed of K sectors of various sizes. Stochastically one or more of the sectors experience what we call elementary events, the aggregate of which yields the real growth of the economy, leading to its random growth rates. To be simple one may assume that the individual elementary growth of sectors is random $\lambda = 1+g$, where $g = \pm\gamma$ randomly with some positive γ. Further, we adopt the mechanism of Huang and Solomon that a random number τ of this type of elementary events are experienced in a unit of calendar time. The random growth rate is the composite effects of these random elementary events.

We refer the detail of the mechanism to their paper, and mention only that the growth rate will be exponential only if the number of changes τ is less than some critical value τ_c, and change in GDP has a power law density with index $-(1+\alpha)$.

The value of α is defined to be the ratio of minimum and average real consumption in the model $q = c_{min}/c_{average}$, and is tied to α by

$$\alpha \approx 1/(1-q),$$

when K is sufficiently larger that $e^{1/q}$, due to inherent normalization conditions of densities involved.

[4] Their focus is on financial sector, not real sector. See Aoki and Yoshikawa (2006 a, b).

For example, setting $q = 0.25$ leads to $\alpha = 1.33$, and K must be such that $K >> e^4 > 55$. The value of τ_c is defined by $(N/2q)^\alpha$. With τ less than τ_c, the growth rate r can be shown to have the density

$$p(r) = Cexp(-a|r - r_m|),$$

for $r > r_m$, with a different constant for the case $r < r_m$.

The deviation of r is then related to variability of K and τ, among others. From this one can deduce that the average deviation in the growth rates is basically determined by percentage changes of the size of the largest cluster which can be related to the GDP when the productivity is assumed not to vary too much, and the conclusion follows that the standard deviation of the growth rate is $Y^{-\mu}$ with μ less than 1. See Aoki and Yoshikawa (2006a, b) for detail.

Concluding Remarks

In physics phenomena with non-vanishing coefficients of variation abound. In traditional microeconomic foundations of economics, one deals almost exclusively with well-posed optimization problems for the representative agents with well defined peaks and valleys of the cost functions. It is also taken for granted that as the number of agents goes to infinity, any unpleasant fluctuations vanish and well defined deterministic macroeconomic relations prevail. In other words, non-self-averaging phenomena are not in the mental pictures of average macro- or microeconomists.

However, we know that as we go to problems which require agents to solve some combinatorial optimization problems, this nice picture may disappear. In the limit of the number of agents going to infinity some results are sample-dependent and deterministic results will not follow. Some of this type of phenomena have been reported in Aoki (1996, Sec. 7.1.7) and also in Aoki (1996, p. 225) where Derrida's random energy model was introduced to the economic audience. Unfortunately it did not catch the attention of the economic audiences. See Mertens (2000). This paper is another attempt at exposing non-self-averaging phenomena in economics. We also mention a possibility of extending the phrase to cover existence of non-degenerate distributions with time going to infinity. What are the implications if some economic models have non-self averaging property? For one thing, it means that we cannot blindly try for larger size samples in the hope that we obtain better estimates.

The example above is just an indication of the potential of this approach of using exchangeable random partition methods. It is the opinion of this author that subjects such as in the papers by Fabritiis, Pammolli, and Riccaboni (2003), or by Amaral et al (1998) could be re-examined from the random combinatorial partition approach with profit. Another example is Sutton (2002). He modeled independent business in which the business sizes vary by partitions of integers to discuss the dependence of variances of firm growth rates. He assumed each partition is equally likely, however. Use of random partitions discussed in this paper may provide more realistic or flexible framework for the question he examined.

Finally, the key question in applications to macroeconomic or financial modelings of the random partition approach is "What are the most likely combinations of the values of $K_n = k$, a_j , and ja_j all suitably normalized ?" This question appears too complicated to answer analytically at this time. Some simulations would help.

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