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Growth Patterns of Two Types of Macro-Models: Limiting Behavior of One-and Two-Parameter Poisson-Dirichlet Models

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Abstract

This paper uses novel growth models composed of clusters of heterogeneous agents, and shows that limiting behavior of one- and two-parameter Poisson-Dirichlet models are qualitatively very different. As model sizes grow unboundedly, the coefficients of variations of extensive variables, such as the number of total clusters, and the numbers of clusters of specified sizes all approach zero in the one-parameter models, but not in the two-parameter models.

In the calculations of the coefficients of variations Mittag-Leffler distributions arise naturally. We show that the distributions of the numbers of the clusters in the models have power-law behavior.

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Intorduction

This paper discusses a new class of simple stochastic multi-sector growth models composed of clusters, where a cluster is a collection of agents of the same or similar characteristics in some sense. Depending on the context, these clusters may be sectors of macroeconomy, or firms of some sector of the economy, and so on. As time passes, the total number of agents in the model increases stochastically, either because a new agent (factors of productions) joins one of existing clusters or because a new cluster is created by the new agent. We focus on the total numbers of clusters, that is, on the number of distinct types of economic agents in the model, and on the number of clusters of some specified sizes.[1]

These models are not stochastic growth models familiar to economists. They are, however, growth models because innovations occur in one of existing clusters or new clusters are created by innovations which cause the size of models to grow unboundedly.

We then examine if the coefficients of variation of some extensive variables, such as the number of sectors or number of clusters of some specified size, converge to zero or remain positive in the limit of total number of units in the model tending to infinity.[2]

If the limit of the coefficient of variation is not zero, then the model behavior is sample-dependent, that is, is influenced by history. This phenomenon is called non self-averaging in the language of statistical physics.[3]

We show that the class of one-parameter Poisson-Dirichlet models of Kingman, also known as Ewens models in population genetics, denoted by $PD(\theta)$, $\theta > 0$ is self-averaging, but its extension to two-parameter Poisson-Dirichlet models by Pitman (1999), denoted by $PD(\theta, \alpha)$, where $0 < \alpha < 1, \alpha + \theta > 0$, is not, that is non self-averaging.[4]

The model

Consider an economy composed of several sectors. Different sectors are made up of different type of agents or productive units. The sectors are

[1]The models in this paper are in the spirit of a new class of stochastic processes called combinatorial stochastic processes by J. Pitman. See Pitman (2004) for an extensive exposition of this class. This new type of stochastic processes deals with random partitions of agents as in Kingman (1978a,b) or Ewens (1972). Loosely speaking, economic agents are regarded as exchangeable and probability distributions on the sets of their random partitions are studied.

[2]The term of clusters of "infinite" size refers to the ideal situation of very large economies. This type of limit is called thermodynamic limit in physics. We adopt this terminology in order to distinguish this type of limits from those where time goes to infinity as in the question of ergodicity.

[3]Sornette defines the square of the coefficient of variation as a measure of non-self averaging, Sornette (2000, 369). Coefficients of variation of self-averaging extensive variables tend to zero in the thermodynamic limits.

[4]Feng and Hoppe (1998) has a model of similar structure. Their focus, however, is not on the limiting behavior of the coefficients of variation. The constraints on the parameters come from the requirements that the probabilities remain positive. See Eq. (1). In Feng and Hoppe $\theta = \beta - \alpha$ where β is the birth rate of the pure birth process in their model, hence $\alpha + \theta$ is positive.

thus heterogeneous. Counting the sizes of sectors in some basic units, when the economy is of size n, there are K_n sectors, that is, K_n types of agents or productive units are in the model. The number K_n as well as the sizes of individual sectors, n_i, $i = 1, \ldots, K_n$, are random variables, where $n = \sum_i n_i$

We focus on the coefficients of variation of K_n and of $a_j(n)$, $j = 1, \ldots K_n$, where $a_j(n)$ is the number of clusters of size j, with the total n given. By definition K_n is the sum over j of $a_j(n)$ and the total number of units in the model is given by $n = \sum_j j a_j(n)$.

Time runs continuously. Over time, one of the existing sectors grows by one unit at rate which is proportional to $(n_i - \alpha)/(n+\theta)$, $i = 1, \ldots K_n$, where α is a parameter between 0 and 1, and θ is another parameter, $\theta + \alpha > 0$. A new unit joins the existing clusters, increasing the number of clusters by one. Given that $K_n = k$, this creation of a new cluster occurs at the rate

$$1 - \sum_{i=1}^{k} \frac{(n_i - \alpha)}{(n + \theta)} = 1 - \frac{n - k\alpha}{n + \theta} = \frac{\theta + k\alpha}{n + \theta}.$$

Define $q_{\alpha,\theta}(n, k) := \Pr(K_n = k)$. Its recursion equation is then given by

$$q_{\alpha,\theta}(n + 1, k) = \frac{n - k\alpha}{n + \theta} q_{\alpha,\theta}(n, k) + \frac{\theta + (k - 1)\alpha}{n + \theta} q_{\alpha,\theta}(n, k - 1), \qquad (1)$$

where the expression for the boundary $K_n = 1$ for all n, and that of $K_n = n$ are given by the expression

$$q_{\alpha,\theta}(n, 1) = \frac{(1 - \alpha)(2 - \alpha)\cdots(n - 1 - \alpha)}{(\theta + 1)(\theta + 2)\cdots(\theta + n - 1)},$$

and

$$q_{\alpha,\theta}(n, n) = \frac{(\theta + \alpha)(\theta + 2\alpha)\cdots(\theta + (n - 1)\alpha)}{\theta + 1)(\theta + 2)\cdots(\theta + n - 1)}.$$

To reiterate, Eq. (1) states that the economy composed of k sectors increases in size by one unit either by one of the existing sectors growing by one unit, or by a new sector of size one emerging.

Also to express the above in another way, we have

$$\Pr(K_{n+1} = k + 1 | K_1, \ldots, K_{n-1}, K_n = k) = \frac{\theta + k\alpha}{n + \theta}. \qquad (2)$$

This equation shows that more new sectors are likely to emerge in the economy as the numbers of sectors grow.

Note that the rate of new sector creation is independent of the current number of clusters in the one-parameter models. This is the fundamental reason for the different thermodynamic behavior between the one- and two-parameter models.

In the one-parameter model the number of clusters may be expressed as

$$q_{0,\theta}(n, k) = \frac{c(n, k)\theta^k}{\theta^{[n]}}, \qquad (3)$$

where $\theta^{[n]} = \theta(\theta+1)\cdots(\theta+n-1)$, where $c(n,k)$ is the unsigned (signless) Stirling number of the first kind. It satisfies the recursion

$$c(n+1,k) = nc(n,k) + c(n,k-1).$$

Becuase $q_{0,\theta}$ sums to 1 for fixed n, we have

$$\theta^{[n]} = \sum_{k=1}^{n} c(n,k)\theta^k.$$

See Aoki (2002, 208) for a combinatorial interpretation of the Stirling number of the first kind.

Asymptotic Properties of the Number of Sectors

We next examine how the number of sectors behave as the size of the model grow unboudedly. We know that how it behaves when α is zero. It involves Stirling number of first kind, see Hoppe (1984) or Aoki (2002, 184).

In the two-parameter version Eq. (3) is replaced by a slightly different expression

$$q_{\alpha,\theta}(n,k) = \frac{\theta^{[k,\alpha]}}{\alpha^k \theta^{[n]}} c(n,k;\alpha), \tag{4}$$

where

$$\theta^{[k,\alpha]} := \theta(\theta+\alpha)\cdots(\theta+(k-1)\alpha).$$

The expression $c(n,k;\alpha)$ generalizes $c(n,k)$, and $\theta^{[n]}$ is now expressible as

$$\theta^{[n]} = \sum_k S_\alpha(n,k)\theta^{[k,\alpha]},$$

where

$$S_\alpha(n,k) := \frac{c(n,k;\alpha)}{\alpha^k}$$

satisfies a recursion

$$S_\alpha(n=1,k) = (n-k\alpha)S_\alpha(n,k) + S_\alpha(n,k-1), \tag{5}$$

where $S_\alpha(0,0)=1$, $S_\alpha(n,0)=0$, and $S_\alpha(0,k)=0$, for positive k.

This function generalizes the power-series relation for $\theta^{[n]}$ in terms of the Stirling number of the first kind to that of this generalized Stirling numbers.

The Coefficients of Variaton

The number of clusters of model of size n

Yamato and Sibuya (2000) have calculated moments of the number of clusters, K_n^r , $r = 1,2,\ldots$ recursively. For example they derive a recursion relation

$$E(K_{n+1}) = \frac{\theta}{n+\theta} + (1+\frac{\alpha}{n+\theta})E(K_n)$$

from which they obtain an asymptotic relation

$$E[\frac{K_n}{n^\alpha}] \sim \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)} \tag{6}$$

by applying the asymptotic expression of the Gamma function

$$\frac{\Gamma(n+a)}{\Gamma(n)} \sim n^a. \tag{7}$$

They also obtain the expression for the variance of K_n/n^α as

$$var(K_n/n^\alpha) \sim \frac{\Gamma(\theta+1)}{\alpha^2}\gamma(\alpha,\theta), \tag{8}$$

where

$$\gamma(\alpha,\theta) := (\theta+\alpha)/(\Gamma(\theta+\alpha)) - \Gamma(\theta+1)/[\Gamma(\theta+\alpha)]^2. \tag{9}$$

Note that the expression $\gamma(\alpha,\theta)$ is zero when α is zero, and positive otherwise. This is the fundamental difference between the two classes of models discussed in this paper as we see next. The expression for the coefficient of variation of K_n normalized by n^α then is given by

$$limC.V.(K_n/n^\alpha) = \frac{\Gamma(\theta+\alpha)}{\Gamma(\theta+1)}\sqrt{\gamma(\alpha,\theta)}. \tag{10}$$

We state this result as

Proposition The limit of the coefficient of varition is positive with positive α, and it is zero only with $\alpha = 0$.

In other words, models with $0 < \alpha < 1$ are non-self-averaging. Past events does not influence the path of the growth of the one-parameter model, but past events affect the growth of the two-parameter models, i.e., the model exhibits non ergodic growth paths.

The components of the pattern vector a

Let $a_j(n)$ be the number of sectors of size j when the size of the economy is n. From the definitions, note that $K_n = \sum_j a_j(n)$, and $\sum_j ja_j(n)_n$, where j ranges from 1 to n.

The expected value of the number of clusters of size j, given the total size of model n is

$$E(a_j) = \frac{n!}{j!(n-j)!}\frac{(\theta+\alpha)^{[n-j]}}{(1-\alpha)^{[j-1]}(\theta+1)^{[n-1]}}.$$

The results in Yamato and Sibuya can be used to show that the limit of the coefficient of variation of $a_j(n)/n^\alpha$ as n goes to infinity has the same limiting behavior as K_n/n^α, i.e., zero for $\alpha = 0$, and positive for $0 < \alpha < 1$. Yamato and Sibuya (2000) have shown that

$$\frac{a_j(n)}{K_n} \to P_{\alpha,j}$$

a.s. where

$$P_{\alpha,j} = \frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)}.$$

The coefficients of variations of $a_j(n)$ remains positive with positive αs.

Mittag-Leffler Distributions

In this section we match the moments of the random variable K_n/n^α with those of the generalized Mittag-Leffler distribution, and deduce that K_n/n^α has what is called as the generalized Mittag-Leffler distribution. This is an example of the method of moments. See Durrett (2005) or Feller, vol II (1966).

The generalized Mittag-Leffler distribution has the density

$$g_{\alpha,\theta}(x) := \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{\theta/\alpha} g_\alpha(x),$$

where $\theta + \alpha > 0$, and where $g_\alpha(x)$ is the Mittag-Leffler density uniquely determined by

$$\int_0^\infty x^p g_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)},$$

for all $p > -1$. The explicit expression of g_α is given by Pollard (1948) who calculcated the invers Laplace transform of $exp(-s^\alpha)$,

$$exp(-s^\alpha) = \int_0^\infty e^{-sx} g_\alpha(x) dx.$$

Also see Podlubny (1999) who has some related expressions. See also Blumenfeld and mandelabrot (1997) for comments on Feller's contribution, or Pitman (2002, 12).

We know that

$$K_n/n^\alpha \to \mathcal{L},$$

in distribution, and in a.s., as shown in Pitman (2002, Sec. 3), Feng and Hoppe (1989), and Yamato and Sibuya (2000).

The random variable $\mathcal{L}$ has the density

$$\frac{d}{ds} P_{\alpha,\theta}(\mathcal{L} \in ds) = g_{\alpha,\theta}.$$

We have calculated above the variance of $\mathcal{L}$ and see that its variance vanishes if α is zero.

Power Laws

Pitman (2002, 73) has shown that

$$Pr(K_n = k) \sim g_{\alpha,\theta}(s) n^{-\alpha},$$

as $n \to \infty$ with $k \sim sn^\alpha$.

Note that this is a power law relation . Pitman's formula for the probability of $K_n = k$, with $k \sim sn^\alpha$ indicates that the power law n^α which is $2\alpha < 2 = 1 + \mu$ with $0 < \mu < 1$.

The moments of these one- and two-parameter models are related to those of the Mittag-Leffler distribution and its extension in a simple way. This is significant for the following reason. As Darling-Kac theorem implies, Darling and Kac (1957), any analysis involving first passages, occupation times, waiting time distributions and the like are bound to involve the Mittag-Leffler functions. In other words, Mittag-Leffler functions are generic in examining model behaviors as the model sizes grow unboundedly.

Potential Applications

Using the Laplace transform of Mittag-Leffler function, Mainardi and his associate and colleagues have discussed fractional calculus, and fractional master equations, with applications to financial problems in mind, Mainardi et al. For example see Scalas (2006).

The class of models discussed in this paper may thus turns out to be important not only in finance but also in macroeconomics. For example, we may redo Dixit (1989) or Sutton (2002) from the new point of view presented in this paper. There is also a possible connection with Derrida (1994). In finance, there are already some applications of Mittag-Leffler functions by Mainardi and his associates; Mainardi and Gorenflo (2000), and Mainardi, Raberto, Gorenflo and Scalas (2000).

Concluding Remarks

In traditional microeconomic foundations of macroeconomics one deals almost exclusively with well-posed optimization problems for the representative agents with well-defined peaks and valleys of the cost functions. It is also taken for granted that as the number of agents goes to infinity, any unpleasant fluctuations vanish, and well-defined deterministic macroeconomic relations prevail. In other words, non-self-averaging phenomena are not in the mental picture of macro or microeconomists.

We know however that as we go to problems which require agents to solve some combinatorial optimization problems, this nice mental picture may not apply. In the limit of the number of agents going to infinity, some results remain sample-dependent and deterministic results will not follow. Some of this type of phenomena have been reported in Aoki (1996, Sec. 7.1.7) and also in Aoki (1996, 225) where Derrida's random energy model was introduced to the economic audience.

What are some of the implications of economic models with non-self-averaging behavior? For one thing, it means that we cannot blindly try for larger size samples in the hope that we obtain better estimates of whatever we are trying to estimate or model.

The example in this paper is just a hint of the potential of using combinatorial stochastic processes. Some economic analysis, such as by Fab ritiis, Pammolli, and Riccaboni (2003), Amaral et al. (1998) and Sutton (2002) may be re-examined with profit. See Aoki and Yoshikawa (2006) for more systematic re-examination of macroeconomic foundations by means of tools and concepts of statistical physics and combinatorial stochastic processes.

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