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# A Factor Allocation Approach to Optimal Bond Portfolio

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## **Abstract**

This paper proposes a new method to a bond portfolio problem in a multi-period setting. In particular, we apply a factor allocation approach to constructing the optimal bond portfolio in a class of multi-factor Gaussian yield curve models. In other words, we consider a bond portfolio problem in terms of a factors' allocation problem. Thus, we can obtain clear interpretation about the relation between the change in the shape of a yield curve and dynamic optimal strategy, which is usually hard to be obtained due to high correlations among individual bonds.

We first present a closed form solution of the optimal bond portfolio in a class of the multi-factor Gaussian term structure model. Then, we investigate the effects of various changes in the term structure on the optimal portfolio strategy through series of comparative statics.

# 1 Introduction

In recent years, the fixed income security market has grown rapidly and research for the trading strategies are getting sophisticated. However, there are few researches on dynamic optimal portfolio with term structure models. In particular, the changes in a yield curve shape are rarely reflected in the optimal portfolio strategy. Moreover, the optimal strategies for bond portfolio problems are usually hard to be interpreted because of high correlations among individual bonds. As is often reported in principal component analysis (PCA) for the change in term structure, most of the variations of spot yields with different maturities can be explained by three common factors.

Thus, to avoid high correlations, it seems better to consider portfolio problems not in terms of bonds but in terms of common factors. There are various term structure models to evaluate bonds and interest rate derivatives. By combining these models with portfolio optimization, especially, applying multi-factor yield curve models to optimal bond portfolio problems, we can analyze bond portfolio in terms of factors, which enables us to interpret optimal strategies intuitively. For example, because spot rates can be expressed as a linear combination of factors in a multi-factor Gaussian model, we can easily obtain factors exposures in a bond optimal portfolio.

Hence, we introduce a factor allocation approach. First, we decompose each bond's return into several factors' return and then the allocation of bonds can be converted into that of factors. Using this idea, we can easily analyze the effects of the parameter changes in a yield curve on the optimal portfolio.

In this paper, we use a multi-factor Gaussian term structure model with stochastic mean and power utility function for terminal wealth. Then, we derive a closed form solution of the optimal strategy for a dynamic bond portfolio problem using a result of Takahashi and Yoshida [2004]; they presented an explicit expression of optimal portfolio in a general Markovian setting based on Ocone and Karatzas [1991].

There are several previous works related to this theme. Sørensen [1999] and Brennan and Xia [2000] consider the dynamic portfolio optimization problem in a structure with a power utility, single/two-factor stochastic interest rate model and a constant market price of risk in the complete market. Sørensen [1999] uses a Vasicek one-factor model for discrete time optimal portfolio allocation utilizing the quasi-dynamic programming approach. Brennan and Xia [2000] investigates the optimal stock-bond mix along with a two-factor interest rate model and derive comparative statics with respect to the risk aversion. Liu [2006] solves the dynamic portfolio problem using a single factor model of interest rates in a general structure such as a stochastic interest rate, a stochastic market price of risk and a stochastic volatility. Most studies, however, have not focused on the term structure and the optimal bond portfolio.

Kobayashi, Takahashi and Tokioka [2005] mainly uses a general single-factor HJM model, stochastic market price of risk, stochastic volatility and the power utility. They first formulate a dynamic optimal bond portfolio problem based on the general setting utilizing the asymptotic expansion scheme and investigate the effect of the change in the market price of risk. Korn and Koziol [2006] uses multi-factor term structure models of the Vasicek type to analyze bond portfolio

optimization with the static Mean-Variance approach.

We consider the dynamic bond portfolio problem and present a closed-form optimal strategy with a model including Sørensen [1999], Brennan and Xia [2000] and Korn and Koziol [2006] as special cases and our result includes the solution of Brennan and Xia [2000] in the case in which all securities consist of bonds. Furthermore utilizing the term structure model effectively, we propose the idea of a factor allocation; the bond portfolio problem is reinterpreted as the factor allocation problem, which enables us to investigate the relation between the term structure and the optimal bond portfolio clearly. In particular, we implement comparative statics in detail with respect to parameters which affect the shape of the term structure.

The organization of this paper is as follows. In section 2, after we briefly introduce a dynamic optimization problem for bond portfolio in a class of Gaussian term structure models, we derive a closed form solution of optimal strategy. In section 3, we implement series of comparative statics to investigate the effects of changes in the yield curve shape on the optimal portfolio strategy. Finally, section 4 states conclusion.

## **2 DYNAMIC FACTOR ALLOCATION PROBLEM**

In this section, we discuss the dynamic portfolio optimization under a multi-factor Gaussian term structure model with stochastic mean. First, we describe the dynamic portfolio problem combined with a multi-factor Gaussian model by employing the result from Takahashi and Yoshida [2004]. By specifying the processes

of the state variable, the market price of risk and the instantaneous interest rate  $r(t)$ , we derive a closed form solution for optimal bond portfolio strategy under the general version of the multi-factor Gaussian term structure model with stochastic mean, which is one of our main contributions in this paper.

## 2.1 Dynamic Portfolio Problem in a General Markovian Setting

### 2.1.1 Description of the financial market

First, we describe the financial market. Assume the market is complete. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T < \infty}, \mathcal{P})$  denote a probability space with a filtration and assume it satisfies usual conditions.  $T (< \infty)$  denotes a fixed time horizon of the economy.

We suppose the instantaneous spot rate  $r(t)$  to be  $r(t) = r(\mathbf{X}_t)$ , that is, it can be represented as some function of the  $m$ -dimensional state variable  $\mathbf{X}_t := (X_{1,t}, \dots, X_{m,t})'$ , whose variation is governed by;

$$d\mathbf{X}_t = \mathbf{V}_0(\mathbf{X}_t) dt + \sum_{j=1}^m \mathbf{V}_j(\mathbf{X}_t) d\hat{W}_{j,t} \quad \mathbf{X}_0 = \mathbf{x}, \quad (2.1)$$

here we use the notation of  $x'$  as the transpose of  $x$ .  $\mathbf{V}_j(\cdot)$ ,  $j = 0, 1, \dots, m$  are  $(m \times 1)$  column vectors respectively and set a  $m \times m$  matrix as  $\mathbf{V}(\cdot) := (\mathbf{V}_1(\cdot), \dots, \mathbf{V}_m(\cdot))$ .  $\mathbf{V}_j(\cdot)$ ,  $j = 0, 1, \dots, m$  are some functions of  $\mathbf{X}_t$  and satisfy the regularity conditions.  $\hat{\mathbf{W}}_t := (\hat{W}_{1,t}, \hat{W}_{2,t}, \dots, \hat{W}_{m,t})'$ ,  $0 \leq t \leq T$  is a  $\mathbf{R}^m$ -valued Brownian motion whose components are independent Brownian motions defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T < \infty}, \mathcal{P})$ . Let  $P(t, T_i)$ ,  $i = 1, 2, \dots, m$  and  $P_0(t)$  denote the

prices at time  $t \in [0, T]$  of zero coupon bonds with the maturity  $T_i$  and that of money market account, respectively. Here we suppose the stochastic differential equation governing the movement of zero coupon bond's price,  $P(t, T_i)$ , and that of money market account as

$$\begin{aligned} dP(t, T_i) &= P(t, T_i)[\beta_i(t, \mathbf{X}_t) dt + \hat{\boldsymbol{\sigma}}_i(t) d\hat{\mathbf{W}}_t], \quad P(0, T_i) = p_i, \quad i = 1, 2, \dots, m \\ dP_0(t) &= r(\mathbf{X}_t)P_0(t) dt, \quad P_0(0) = 1 \end{aligned} \quad (2.2)$$

where  $\hat{\boldsymbol{\sigma}}_i(t) := (\hat{\sigma}_{i,1}(t), \dots, \hat{\sigma}_{i,m}(t))$ .

As we assume the financial market is complete, there uniquely exists a stochastic process,  $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$ , which is given as the solution of the equation below;

$$\boldsymbol{\beta}(t, \mathbf{X}_t) - r(\mathbf{X}_t)\mathbf{1} = \hat{\boldsymbol{\sigma}}(t)\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t) \quad (2.3)$$

where  $\boldsymbol{\beta}(t, \mathbf{X}_t)$  is a  $(m \times 1)$  vector whose  $i$ th element is  $\beta_i(t, \mathbf{X}_t)$ ,  $\hat{\boldsymbol{\sigma}}(t)$  is a  $(m \times m)$  matrix whose  $i$ th row is  $\hat{\boldsymbol{\sigma}}_i(t)$  and  $\mathbf{1}$  is a  $(m \times 1)$  unit vector. This variable,  $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$ , is often referred as the market price of risk.

Thus we can rewrite (2.2) as;

$$\begin{aligned} dP(t, T_i) &= P(t, T_i)[(r(\mathbf{X}_t) + \hat{\boldsymbol{\sigma}}_i(t)\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)) dt + \hat{\boldsymbol{\sigma}}_i(t) d\hat{\mathbf{W}}_t] \\ P(0, T_i) &= p_i, \quad i = 1, 2, \dots, m \end{aligned} \quad (2.4)$$

Given the opportunity set described in (2.2), investors will allocate their wealth among these  $(m + 1)$  assets. Let  $\phi_i(t)$  denotes the number of units of asset  $i$  held at time  $t$ . From the budget constraint, the stochastic differential equation below

should hold <sup>1</sup>;

$$d\mathcal{X}(t) = \boldsymbol{\phi}(t)' d\mathbf{P}(t) + (1 - \boldsymbol{\phi}(t)'\mathbf{1}) dP_0(t) - c(t)dt, \quad \mathcal{X}(0) = x > 0 \quad (2.5)$$

where  $\mathcal{X}(t)$  and  $c(t)$  denote investors' total wealth and their non-negative consumption rate respectively,  $\boldsymbol{\phi}(t)$  is a  $(m \times 1)$  vector whose  $i$ th element is  $\phi_i(t)$ ,  $\mathbf{1}$  is a  $(m \times 1)$  unit vector and  $\mathbf{P}$  is a  $(m \times 1)$  vector whose  $i$ th element is  $P(t, T_i)$ .

It is useful to replace  $\boldsymbol{\phi}(t)$  from (2.5) by a new variable,  $\boldsymbol{\pi}(t)$ , a  $(m \times 1)$  vector whose  $i$ th element is  $\pi_i(t)$ , which denotes the amount of money invested for  $i$ th asset at time  $t$ . Substituting for  $dP(t, T_i)/P(t, T_i)$  from (2.4), we can rewrite (2.5) as

$$\begin{aligned} d\mathcal{X}(t) &= [r(\mathbf{X}_t)\mathcal{X}(t) - c(t)] dt + \boldsymbol{\pi}(t)'[\hat{\boldsymbol{\sigma}}(t)\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t) dt + \hat{\boldsymbol{\sigma}}(t) d\hat{\mathbf{W}}_t], \\ \mathcal{X}(0) &= x > 0, \end{aligned} \quad (2.6)$$

which is often referred as the budget-constraint dynamics.

Finally, we define  $\mathcal{A}(x)$  as the set of stochastic processes  $(\boldsymbol{\pi}, c)$  such that given  $\mathcal{X}(0) = x > 0$ , for all  $t \in [0, T]$ ,  $\mathcal{X}(t) \geq 0$  (a.s.).

### 2.1.2 Optimal portfolio problem

Here we analyze optimal portfolio selection. First, we specify optimal portfolio problem for investors;

**ASSUMPTION 1** *We assume investors seek to maximize an objective function*

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<sup>1</sup>For a detailed explanation of this budget constraint, see Merton [1971]



as below;

$$\sup_{(\boldsymbol{\pi}, c) \in \mathcal{A}(x)} \mathbf{E}^{\mathcal{P}}[U(\mathcal{X}(T))], \quad (2.7)$$

subject to the budget-constraint dynamics, (2.6), here  $\mathbf{E}^{\mathcal{P}}[\cdot]$  denotes the expectation operator under  $\mathcal{P}$  and  $U$  denotes a utility function.

Note that, with the objective function above, optimal consumption rule turn out to be  $c(t) = 0$ .

Furthermore, as assumed in Sørensen [1999], Brennan and Xia [2000], Kobayashi, Takahashi and Tokioka [2005], Liu [2006], We assume a utility function in (2.7) is specified as so-called power utility, that is;

**ASSUMPTION 2** *We assume investors have a utility function as follows; for  $\delta < 1, \delta \neq 0$ ,*

$$U(x) := \frac{x^\delta}{\delta}.$$

In this Markovian setting, Takahashi and Yoshida [2004] provides the following result based on Ocone and Karatzas [1991].

**THEOREM 2.1** *Under the same conditions as in Section 4 in Takahashi and Yoshida [2004], the optimal proportion of zero coupon bonds in wealth denoted by*

$\boldsymbol{\pi}(t)'/\mathcal{X}(t)$  are given as follows;

$$\begin{aligned}
\frac{\boldsymbol{\pi}(t)'}{\mathcal{X}(t)} &= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)' \hat{\boldsymbol{\sigma}}(t)^{-1} + \frac{\delta}{1-\delta} \frac{1}{\mathbf{E}_t^{\mathcal{P}}[(H_{t,T})^{-\delta/(1-\delta)}]} \times \\
&\times \mathbf{E}_t^{\mathcal{P}} \left[ (H_{t,T})^{-\delta/(1-\delta)} \left( \int_t^T \partial r(\mathbf{X}_u) \mathbf{Y}_u \mathbf{V}(\mathbf{X}_t) du + \right. \right. \\
&+ \sum_{j=1}^m \int_t^T \partial \hat{\theta}_j(t, \mathbf{X}_u) \mathbf{Y}_u \mathbf{V}(\mathbf{X}_t) d\hat{W}_{j_u} + \\
&\left. \left. + \sum_{j=1}^m \int_t^T \hat{\theta}_j(t, \mathbf{X}_u) \partial \hat{\theta}_j(t, \mathbf{X}_u) \mathbf{Y}_u \mathbf{V}(\mathbf{X}_t) du \right) \right] \hat{\boldsymbol{\sigma}}(t)^{-1}
\end{aligned} \tag{2.8}$$

where  $\mathbf{E}_t^{\mathcal{P}}[\cdot]$  denotes the conditional expectation at time  $t$ ,  $H_{t,T}$  is defined by;

$$H_{t,T} := \exp \left( - \int_t^T \hat{\boldsymbol{\theta}}(t, \mathbf{X}_u)' d\hat{\mathbf{W}}_u - \frac{1}{2} \int_t^T |\hat{\boldsymbol{\theta}}(t, \mathbf{X}_u)|^2 du - \int_t^T r(\mathbf{X}_u) du \right),$$

and

$$\begin{aligned}
\partial r(\mathbf{X}_u) &= \left( \frac{\partial r}{\partial X_1}, \dots, \frac{\partial r}{\partial X_m} \right) \\
\partial \hat{\theta}_j(t, \mathbf{X}_u) &= \left( \frac{\partial \hat{\theta}_j}{\partial X_1}, \dots, \frac{\partial \hat{\theta}_j}{\partial X_m} \right),
\end{aligned}$$

and  $\mathbf{Y}_u$  follows the  $(m \times m)$  matrix valued stochastic differential equation; for  $u \in [t, T]$ ,

$$d\mathbf{Y}_u = \partial \mathbf{V}_0(\mathbf{X}_u) \mathbf{Y}_u du + \sum_{j=1}^m \partial \mathbf{V}_j(\mathbf{X}_u) \mathbf{Y}_u d\hat{W}_{j,u} \quad \mathbf{Y}_t = \mathbf{I} \tag{2.9}$$

where  $\partial \mathbf{V}_k(\mathbf{X}_u)$  is a  $(m \times m)$  matrix such that  $\partial \mathbf{V}_k(\mathbf{X}_u) = (\partial V_k^i / \partial X_{j,u})_{1 \leq i, j \leq m}$ ,  $V_k^i$  denotes the  $i$ th element of  $\mathbf{V}_k$ ,  $k = 0, \dots, m$  which appeared in (2.1) and  $\mathbf{I}$  denotes  $(m \times m)$  identity matrix.

*Proof.* See Section 4.1 in Takahashi and Yoshida [2004].  $\square$

In the next subsection, we will obtain a closed form solution of the optimal portfolio by specifying  $\mathbf{X}_t$ ,  $\hat{\boldsymbol{\theta}}(t, \mathbf{X}_t)$  and  $r(\mathbf{X}_t)$ .

## 2.2 Optimal Portfolio in Multi-factor Gaussian model

In this subsection, we first specify the process of  $\mathbf{X}_t$ .

**ASSUMPTION 3** *We assume the state variable  $\mathbf{X}_t$  follows the process;*

$$\begin{aligned} dX_{1,t} &= \alpha_1(X_{2,t} + X_{3,t} + \cdots + X_{m,t} - X_{1,t}) dt + \sigma_1 dW_{1,t} \\ dX_{i,t} &= \alpha_i(\bar{X}_i - X_{i,t}) dt + \sigma_i dW_{i,t}, \quad (i = 2, 3, \dots, m) \end{aligned} \tag{2.10}$$

where,  $\alpha_1 > 0, \alpha_i \geq 0, \bar{X}_i \geq 0, i = 2, 3, \dots, m, \sigma_i \geq 0, i = 1, 2, \dots, m$  are all constants and  $\mathbf{W}_t := (W_{1,t}, W_{2,t}, \dots, W_{m,t})', 0 \leq t \leq T$  is a  $\mathbf{R}^m$ -valued correlated Brownian motion defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T < \infty}, \mathcal{P})$  with  $dW_i dW_j = \rho_{ij} dt$  (if  $i = j, \rho_{ij} \equiv 1$ ). The correlated Brownian motion  $\mathbf{W}_t$  can be expressed by using an independent Brownian motion  $\hat{\mathbf{W}}_t$  as  $\mathbf{W}_t = \mathbf{C}\hat{\mathbf{W}}_t$ , where  $\mathbf{C}$  is some lower triangular matrix obtained by Cholesky decomposition.

Under this setting, (2.1) is reduced to

$$d\mathbf{X}_t = \boldsymbol{\alpha}(\bar{\mathbf{X}} - \mathbf{X}_t) dt + \mathbf{V} d\hat{\mathbf{W}}_t \quad \mathbf{X}_0 = \mathbf{x}, \tag{2.11}$$

where  $\boldsymbol{\alpha}$  is a  $(m \times m)$  diagonal matrix whose  $i$ th element is  $\alpha_i$ ,  $\bar{\mathbf{X}} := (\bar{X}_1, \bar{X}_2, \dots, \bar{X}_m)'$ ,  $\bar{X}_1 := X_{2,t} + X_{3,t} + \cdots + X_{m,t}$ ,  $\mathbf{V} := \boldsymbol{\sigma}\mathbf{C}$  and  $\boldsymbol{\sigma}$  is a  $(m \times m)$  diagonal matrix whose  $i$ th element is  $\sigma_i$ .

Next, we put assumptions on  $\boldsymbol{\theta}(t, \mathbf{X}_t)$  and  $r(\mathbf{X}_t)$  as follows;

**ASSUMPTION 4** (1) *We assume the market price of risk  $\boldsymbol{\theta}(t, \mathbf{X}_t)$  is given by;*

$$\boldsymbol{\theta}(t, \mathbf{X}_t) := \boldsymbol{\theta} = (\theta_1, \dots, \theta_m)',$$

where  $\theta_i, i = 1, 2, \dots, m$  are all constants.

(2) *We also assume the instantaneous spot rate  $r(t)$  is expressed as  $r(t) = r(\mathbf{X}_t) := X_{1,t}$ .*

Then, the dynamics of  $\mathbf{X}_t$  under a risk neutral measure  $\mathcal{Q}$  is given;

$$d\mathbf{X}_t = [\boldsymbol{\alpha}(\bar{\mathbf{X}} - \mathbf{X}_t) - \boldsymbol{\sigma}\boldsymbol{\theta}] dt + \mathbf{V} d\hat{\mathbf{W}}_t^* \quad \mathbf{X}_0 = \mathbf{x}, \quad (2.12)$$

where  $\mathbf{W}_t^*$  denotes a  $\mathbf{R}^m$ -valued Brownian motion under  $\mathcal{Q}$  given as;

$$\mathbf{W}_t^* = \mathbf{W}_t + \boldsymbol{\theta}t.$$

This model belongs to the class of the multi-factor affine model and the general version of stochastic mean model including Vasicek[1977], Hull and White [1990], Balduzzi et al. [2000], He [2001] and Takahashi and Sato [2001] as special cases.

From Assumption 4 and (2.12),  $r(u)$  is expressed as follows;

$$\begin{aligned}
r(u) &= e^{-\alpha_1(u-t)}X_{1,t} + \sum_{i=2}^m \left( \alpha_1 \int_t^u e^{-\alpha_i(s-t)-\alpha_1(u-s)} ds \right) X_{i,t} \\
&\quad - \frac{\sigma_1 \theta_1}{\alpha_1} (1 - e^{-\alpha_1(u-t)}) \\
&\quad + \sum_{i=2}^m \left( \alpha_1 \int_t^u \int_t^s e^{-\alpha_i(s-\tau)-\alpha_1(u-s)} d\tau ds \right) (\alpha_2 \bar{X}_i - \sigma_i \theta_i) \\
&\quad + \sigma_1 \int_t^u e^{-\alpha_1(u-\tau)} dW_{1,\tau} \\
&\quad + \sum_{i=2}^m \sigma_i \alpha_1 \int_t^u \int_t^s e^{-\alpha_i(s-\tau)-\alpha_1(u-s)} ds dW_{i,\tau}
\end{aligned} \tag{2.13}$$

Under the no-arbitrage condition, the zero coupon price at time  $t$  with the maturity  $T_i$ ,  $P(t, T_i)$  is obtained by;

$$P(t, T_i) = E_t^{\mathcal{Q}} \left[ \exp \left( - \int_t^{T_i} r(u) du \right) \right]$$

where  $E_t^{\mathcal{Q}}[\cdot]$  is the conditional expectation operator under  $\mathcal{Q}$  given information at time  $t$ . Then the bond price at time  $t$  with the maturity  $T_i$  is given by;

$$P(\mathbf{X}_t, T_i) := P(t, T_i) = \exp \left( a_{i,0}(\tau_i) + \sum_{j=1}^m a_{i,j}(\tau_i) X_{j,t} \right)$$

where  $a_{i,0}(\tau_i)$  is some deterministic function with respect to  $\tau_i := T_i - t$  and  $a_{i,j}(\tau_i)$  is given by;

$$a_{i,1}(\tau_i) := -\frac{1 - e^{-\alpha_1 \tau_i}}{\alpha_1}, \quad a_{i,j}(\tau_i) := -\alpha_1 \int_t^T \int_t^u e^{-\alpha_i(s-t)-\alpha_1(u-s)} ds du.$$

From *Itô's lemma*, we obtain;

$$\begin{aligned} \frac{dP(\mathbf{X}_t, T_i)}{P(\mathbf{X}_t, T_i)} &= \sum_{j=1}^m a_{i,j}(\tau_i) dX_{j,t} + \frac{\partial P(\mathbf{X}_t, T_i)}{\partial t} / P(\mathbf{X}_t, T_i) dt \\ &+ \frac{1}{2} \sum_{k,j=1}^m a_{i,k}(\tau_i) a_{i,j}(\tau_i) \rho_{kj} \sigma_k \sigma_j dt. \end{aligned} \quad (2.14)$$

From the right hand side in (2.14), we can see that the return of each zero coupon bond can be decomposed into 3 terms; factor variation, carry and factor convexity. Especially, from the first term, we obtain factor exposure of the  $i$ th bond for each factor  $j = 1, \dots, m$  as  $a_{i,j}(\tau_i)$ . Here we define a  $(m \times 1)$  vector,  $\mathbf{a}_i(\tau_i)$ , by  $\mathbf{a}_i(\tau_i) := (a_{i,1}(\tau_i), a_{i,2}(\tau_i), \dots, a_{i,m}(\tau_i))'$  and then the first term in (2.14) can be rewritten as  $\mathbf{a}_i(\tau_i)' d\mathbf{X}_t$ . Therefore, with (2.11),  $\hat{\boldsymbol{\sigma}}_i(t)$  in (2.2) can be rewritten as  $\hat{\boldsymbol{\sigma}}_i(t) = \mathbf{a}_i(\tau_i)' \mathbf{V}$ . Finally, set factor exposure matrix  $\mathbf{A}$  as  $\mathbf{A} := (a_{i,j}(\tau_i))_{1 \leq i, j \leq m}$  then  $\hat{\boldsymbol{\sigma}}$  appeared in (2.6) and (2.8) can be rewritten as

$$\hat{\boldsymbol{\sigma}} = \mathbf{A} \mathbf{V}. \quad (2.15)$$

Then, by adding Assumption 3 and 4, the optimal portfolio strategy in Theorem 2.1 reduces to the following proposition;

**PROPOSITION 2.1** *Under the same conditions as in Theorem 2.1 and Assumption 3-4, the optimal proportion of zero coupon bonds are given as follows;*

$$\frac{\boldsymbol{\pi}(t)'}{\mathcal{X}(t)} = \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta} \mathcal{Y}' \mathbf{V} \hat{\boldsymbol{\sigma}}^{-1} \quad (2.16)$$

where  $\mathcal{Y}$  is defined as below;

$$\mathcal{Y} := \begin{pmatrix} \frac{1-e^{-\alpha_1(T-t)}}{\alpha_1} \\ \alpha_1 \int_t^T \int_t^u e^{-\alpha_2(s-t)-\alpha_1(u-s)} ds du \\ \vdots \\ \alpha_1 \int_t^T \int_t^u e^{-\alpha_m(s-t)-\alpha_1(u-s)} ds du \end{pmatrix} \quad (2.17)$$

*Proof.*  $\partial \hat{\theta}_j(\mathbf{X}_u) \equiv 0$ ,  $j = 1, \dots, m$  follows from Assumption 4(1). As  $\partial r(\mathbf{X}_u) = (1, 0, \dots, 0)$  from Assumption 4(2),  $\partial r(\mathbf{X}_u) \mathbf{Y}_u$  reduces to the first row of  $\mathbf{Y}_u$ , which is obtained by;

$$\left( e^{-\alpha_1(u-t)}, \alpha_1 \int_t^u e^{-\alpha_2(s-t)-\alpha_1(u-s)} ds, \dots, \alpha_1 \int_t^u e^{-\alpha_m(s-t)-\alpha_1(u-s)} ds \right).$$

Finally using  $\mathbf{V}(\mathbf{X}_t) = \mathbf{V}$  and  $\int_t^T \partial r(\mathbf{X}_u) \mathbf{Y}_u du = \mathcal{Y}'$ , (2.8) turns out to be;

$$\begin{aligned} \frac{\boldsymbol{\pi}(t)'}{\mathcal{X}(t)} &= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta} \frac{1}{\mathbf{E}_t^{\mathcal{P}}[(H_{t,T})^{-\delta/(1-\delta)}]} \times \\ &\times \mathbf{E}_t^{\mathcal{P}} \left[ (H_{t,T})^{-\delta/(1-\delta)} \left( \int_t^T \partial r(\mathbf{X}_u) \mathbf{Y}_u \mathbf{V} du \right) \right] \hat{\boldsymbol{\sigma}}^{-1} \\ &= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta} \frac{1}{\mathbf{E}_t^{\mathcal{P}}[(H_{t,T})^{-\delta/(1-\delta)}]} \times \\ &\times \mathbf{E}_t^{\mathcal{P}}[(H_{t,T})^{-\delta/(1-\delta)}] \left( \int_t^T \partial r(\mathbf{X}_u) \mathbf{Y}_u du \right) \mathbf{V} \hat{\boldsymbol{\sigma}}^{-1} \\ &= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\sigma}}^{-1} + \frac{\delta}{1-\delta} \mathcal{Y}' \mathbf{V} \hat{\boldsymbol{\sigma}}^{-1}. \end{aligned}$$

□

The first term on the right hand side of (2.16) represents so-called Mean-Variance portfolio in a continuous-time setting; hence we call it MV term. The second term is specific to a multi-period setting and represents the intertemporal hedging demand defined by Merton [1971]; we call it IR-hedging term.

As observed in (2.14), the dynamics of a single bond's return can be decomposed into exposures to several factors' return. Thus the allocation to bonds is converted into the allocation to factors. While it is difficult to interpret the optimal portfolio in terms of bonds because of high correlation among them, we can interpret the optimal portfolio easier in terms of factors.

Therefore we consider factor exposure of the optimal strategy in Proposition 2.1. The process of the instantaneous return of the optimal bonds' portfolio can be described as follows;

$$\begin{aligned}
\frac{d\mathcal{X}(t)}{\mathcal{X}(t)} &= \left(1 - \sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)}\right) r(\mathbf{X}_t) dt + \sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)} \frac{dP(\mathbf{X}_t, T_i)}{P(\mathbf{X}_t, T_i)} \\
&= \left(1 - \sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)}\right) r(\mathbf{X}_t) dt + \sum_{j=1}^m \left(\sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)} a_{i,j}(\tau_i)\right) dX_{j,t} \\
&\quad + \sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)} \frac{\partial P(\mathbf{X}_t, T_i)}{\partial t} / P(\mathbf{X}_t, T_i) dt + \frac{1}{2} \sum_{k,j=1}^m \sum_{i=1}^m \frac{\pi_i(t)}{\mathcal{X}(t)} a_{i,k}(\tau_i) a_{i,j}(\tau_i) \rho_{kj} \sigma_k \sigma_j dt.
\end{aligned} \tag{2.18}$$

The factor exposure to  $X_{j,t}$  is given by the coefficient of  $dX_{j,t}$  in the second term on the right hand side of (2.18). Then, we obtain the following corollary.

**COROLLARY 2.1** *Factor exposure of the optimal portfolio strategy in Proposition 2.1 can be obtained as follows;*

$$\frac{\boldsymbol{\pi}(t)' \mathbf{A}}{\mathcal{X}(t)} = \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \mathbf{V}^{-1} + \frac{\delta}{1-\delta} \boldsymbol{\vartheta}'. \tag{2.19}$$



*Proof.* Using (2.15) and the result of Proposition 2.1, we can calculate as follows;

$$\begin{aligned}
\frac{\boldsymbol{\pi}(t)' \mathbf{A}}{\mathcal{X}(t)} &= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \hat{\boldsymbol{\sigma}}^{-1} \mathbf{A} + \frac{\delta}{1-\delta} \mathcal{Y}' \mathbf{V} \hat{\boldsymbol{\sigma}}^{-1} \mathbf{A} \\
&= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \mathbf{V}^{-1} \mathbf{A}^{-1} \mathbf{A} + \frac{\delta}{1-\delta} \mathcal{Y}' \mathbf{V} \mathbf{V}^{-1} \mathbf{A}^{-1} \mathbf{A} \\
&= \frac{1}{1-\delta} \hat{\boldsymbol{\theta}}' \mathbf{V}^{-1} + \frac{\delta}{1-\delta} \mathcal{Y}'.
\end{aligned}$$

□

Thus using this corollary, we can convert the optimal portfolio into its factor exposure and we can reinterpret the bond allocation as the factor allocation. This idea is useful for investigating the relationship between the change in term structure and that in the factor exposure of the optimal portfolio strategy.

In concluding this section, one final remark on Corollary 2.1 deserves mention. In this multi-factor model, the optimal allocation to each factor does not depend on the components of portfolio. That is, being current term structure equal, the factor allocation is also invariant even though the optimal bond allocation varies as we change component of the bond portfolio. Thus we can say that the essence of the portfolio problem lies in the factor allocation rather than the bond allocation. Once we obtain the optimal factor allocation from Corollary 2.1, then we can also get the optimal allocation for any bond portfolio through its factor exposure matrix. Therefore, we can see that the optimal allocation of bonds is essentially obtained by that of factors. When the term structure varies, the change will be reflected by the factor allocation directly and the bond portfolio will also change through the change of factor allocation. Therefore the effect of the term structure variation on the bond portfolio is indirect and thus it is easier to analyze factor

allocation directly.

The factor allocation depends on parameters such as speed parameters of mean-reversion  $k_i, i = 1, \dots, m$  which are strongly related with the shape of the yield curve. One of our objectives is to analyze how the changes in the shape of yield curve affect the IR-hedging term and hence the whole portfolio strategy. To investigate this issue, we introduce a concrete 3-factor model in the next section and implement series of comparative statics.

### 3 Numerical Analysis

So far, we have outlined the dynamic portfolio problem with multi-factor model. In this section, we introduce a 3-factor model as an example of the multi-factor stochastic mean model in (2.12) and we implement series of comparative statics based upon the model. The result brings clear interpretation of the optimal portfolio, which is also one of our main contributions in this paper.

#### 3.1 3-factor Model

In this subsection, we concentrate on the case where the risk-neutral dynamics of  $X_1, X_2$  and  $X_3$  are given by;

$$\begin{aligned} dX_1 &= [\alpha_1(X_2 + X_3 - X_1) - \theta_1\sigma_1] dt + \sigma_1 dW_1^* \\ dX_2 &= [\alpha_2(\bar{X}_2 - X_2) - \theta_2\sigma_2] dt + \sigma_2 dW_2^* \\ dX_3 &= [\alpha_3(\bar{X}_3 - X_3) - \theta_3\sigma_3] dt + \sigma_3 dW_3^*. \end{aligned}$$

Here, we assume  $\alpha_1 > \alpha_i, i = 2, 3, \alpha_2 > 0, \alpha_3 \geq 0, \theta_i \leq 0, i = 1, 2, 3$  and the other conditions are the same in the previous section.

Then  $r(u), \mathbf{V}$  and  $\mathcal{Y}$  in this model are expressed as follows;

$$\begin{aligned}
r(u) &= e^{-\alpha_1(u-t)} X_{1,t} + \frac{\alpha_1}{\alpha_1 - \alpha_2} (e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) X_{2,t} \\
&+ \frac{\alpha_1}{\alpha_1 - \alpha_3} (e^{-\alpha_3(u-t)} - e^{-\alpha_1(u-t)}) X_{3,t} \\
&- (1 - e^{-\alpha_1(u-t)}) \frac{\sigma_1 \theta_1}{\alpha_1} \\
&+ \left\{ (1 - e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1 - \alpha_2} (e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) \right\} \left( \bar{X}_2 - \frac{\sigma_2 \theta_2}{\alpha_2} \right) \\
&+ \left\{ (1 - e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1 - \alpha_3} (e^{-\alpha_3(u-t)} - e^{-\alpha_1(u-t)}) \right\} \left( \bar{X}_3 - \frac{\sigma_3 \theta_3}{\alpha_3} \right) \quad (\text{when } \alpha_3 \neq 0) \tag{3.1} \\
&+ \sigma_1 \int_t^u e^{-\alpha_1(u-s)} dW_{1,s} \\
&+ \sigma_2 \frac{\alpha_1}{\alpha_1 - \alpha_2} \int_t^u \left( e^{-\alpha_2(u-s)} - e^{-\alpha_1(u-s)} \right) dW_{2,s} \\
&+ \sigma_3 \frac{\alpha_1}{\alpha_1 - \alpha_3} \int_t^u \left( e^{-\alpha_3(u-s)} - e^{-\alpha_1(u-s)} \right) dW_{3,s}
\end{aligned}$$

$$\begin{aligned}
r(u) &= e^{-\alpha_1(u-t)} X_{1,t} + \frac{\alpha_1}{\alpha_1 - \alpha_2} (e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) X_{2,t} + (1 - e^{-\alpha_1(u-t)}) X_{3,t} \\
&- (1 - e^{-\alpha_1(u-t)}) \frac{\sigma_1 \theta_1}{\alpha_1} \\
&+ \left\{ (1 - e^{-\alpha_1(u-t)}) - \frac{\alpha_1}{\alpha_1 - \alpha_2} (e^{-\alpha_2(u-t)} - e^{-\alpha_1(u-t)}) \right\} \left( \bar{X}_2 - \frac{\sigma_2 \theta_2}{\alpha_2} \right) \\
&- \left\{ (u-t) - \frac{1 - e^{-\alpha_1(u-t)}}{\alpha_1} \right\} \sigma_3 \theta_3 \quad (\text{when } \alpha_3=0) \tag{3.2} \\
&+ \sigma_1 \int_t^u e^{-\alpha_1(u-s)} dW_{1,s} \\
&+ \sigma_2 \frac{\alpha_1}{\alpha_1 - \alpha_2} \int_t^u \left( e^{-\alpha_2(u-s)} - e^{-\alpha_1(u-s)} \right) dW_{2,s} \\
&+ \sigma_3 \int_t^u \left( 1 - e^{-\alpha_1(u-s)} \right) dW_{3,s}
\end{aligned}$$

$$\mathbf{V} = \begin{pmatrix} \sigma_1 & \mathbf{0} \\ & \sigma_2 \\ \mathbf{0} & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 \\ \rho_{12} \sqrt{1 - \rho_{12}^2} \\ \rho_{13} \quad x \quad y \end{pmatrix}$$

where

$$x := \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}}$$

$$y := \sqrt{\frac{1 - (\rho_{12}^2 + \rho_{23}^2 + \rho_{13}^2) + 2\rho_{12}\rho_{23}\rho_{13}}{1 - \rho_{12}^2}}.$$

$$\mathcal{Y} := \begin{pmatrix} \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \\ \frac{\alpha_1}{\alpha_1 - \alpha_2} \left[ \frac{1 - e^{-\alpha_2(T-t)}}{\alpha_2} - \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \right] \\ \frac{\alpha_1}{\alpha_1 - \alpha_3} \left[ \frac{1 - e^{-\alpha_3(T-t)}}{\alpha_3} - \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \right] \end{pmatrix} \quad (\text{when } \alpha_3 \neq 0) \quad (3.3)$$

$$\mathcal{Y} := \begin{pmatrix} \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \\ \frac{\alpha_1}{\alpha_1 - \alpha_2} \left[ \frac{1 - e^{-\alpha_2(T-t)}}{\alpha_2} - \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \right] \\ (T - t) - \frac{1 - e^{-\alpha_1(T-t)}}{\alpha_1} \end{pmatrix} \quad (\text{when } \alpha_3 = 0). \quad (3.4)$$

## 3.2 Properties of the 3-factor model

In this subsection, we briefly review properties of the 3-factor model introduced above. First, we check the characteristics of factors and parameters such as  $\alpha_i$  and  $\theta_i$ . Here we present the comparative statics of the 3-factor model with  $\theta_1 = \theta_2 = \alpha_3 = 0$  which will be used for the analysis in the next subsection.

As explained by He [2001],  $X_1$  captures the short rate controlled by the central bank,  $X_2$  represents the movements of the yield curve slope and  $X_3$  tracks the movements of the long-term interest rate. We investigate how the change of each factor affects the term structure. In Exhibits 1 – 3, the horizontal axis and the vertical axis represent spot rate maturities and changes in spot rates when each factor  $X_i$  increases by 20 basis points (bps) respectively. From Exhibits 1 – 3, we can confirm  $X_1$  affects mainly the short sector in the term structure and  $X_2$  and  $X_3$  affect mainly the middle and the long term sectors respectively.

The positive constants  $\alpha_1$  and  $\alpha_2$  control the speed of mean-reversion of  $X_1$  and  $X_2$  respectively. The faster (slower) the reversion is, the shorter (longer) the shock to the economic system (or new information) stays in the market, and therefore the steeper (flatter) the yield curve becomes. Thus,  $\alpha_1$  and  $\alpha_2$  affect the slope of the short - middle term sector and that of the middle - long term sector respectively. We investigate how the changes of parameters in the model affect the term structure in Exhibits 4 – 7 where the horizontal axis and the vertical axis represent spot rate maturities and changes in spot rates respectively. We can confirm above from Exhibit 4, Exhibit 5 and Exhibit 6 which show the effects of each parameter's change. Exhibit 7 shows the increase of the market price of risk

pushes up the spot rates of the long term sector.

### 3.3 Comparative Statics

Next we show the results from series of comparative statics based on this model. From Corollary 2.1, we can see how the changes in the shape of yield curve, investor's preference and so on affect the portfolio strategy. The relative risk aversion (RRA given by  $1 - \delta$ ) affects both. The MPR  $\theta_i$ , the volatility  $\sigma_i$  and the correlation  $\rho_{ij}$  affect only the MV term. Investment Horizon  $T$  and the speed of mean-reversion  $\alpha_i$  affect only the IR-hedging term.

From Corollary 2.1, we can also see the hedging demand for  $X_1$  and  $X_2$  will converge to 0 as  $\alpha_1, \alpha_2 \rightarrow \infty$ . It is well-known that when the state variables are deterministic, the dynamic parts will disappear. As  $\alpha_1, \alpha_2 \rightarrow \infty$ ,  $X_1$  and  $X_2$  become almost deterministic because these two factors follow mean-reverting processes. Therefore the hedge demand for  $X_1$  and  $X_2$  will disappear. We can observe the same result for  $X_3$  in (3.3) as  $\alpha_1, \alpha_3 \rightarrow \infty$ . On the other hand, the hedging demand for  $X_3$  in (3.4) is the increasing function of investment horizon  $T$  because when  $\alpha_3 = 0$ ,  $X_3$  does not have a mean-reversion property.

Based upon these analysis, we implement series of comparative statics. First, the initial parameters are reported in Exhibit 8. We use estimates of these parameters from He [2001] and set the components of the portfolio, the investment horizon and the RRA as in Exhibit 9. The optimal portfolio strategy and its factor allocation in this setting are given in Exhibit 10.

Two points deserve mention. First, the IR-hedging term is not negligible and

affects the whole strategy. Second, although the allocation to the long-term bond (30-year bond) is quite small in the MV term because of its high volatility, its allocation in the IR-hedging term is not small because it is necessary to hedge mainly against  $X_3$ . This is important because if we use Mean-Variance scheme only, the position will be a simple spread strategy without 30-year bond which cannot hedge against yield curve risks other than the spread risk. Thus our approach is more preferable and practical because we can hedge against several yield curve risks.

Next, we shift several parameters to investigate further details of properties of this model. The results are shown in Exhibits 11 - 21 where we take the values of each parameter in a horizontal axis and the portfolio weight for each bond or the allocation to each factor in a vertical axis, respectively. First, we see the effect of the variation of  $\alpha_1$ . The results are shown in Exhibit 11 and Exhibit 12. From these results, we can see several points. First, as mentioned before, the effects on each bond's allocation are very complicated and unclear because of high correlations among bonds; from Exhibit 11, it is difficult to see how the change of  $\alpha_1$  affects. In contrast, Exhibit 12 shows a clear message; there is no change in the MV term of factor allocation and thus only the IR-hedging term can reflect the steepening and flattening. As  $\alpha_1$  becomes larger, the allocation to  $X_1$  in the IR-hedging term decreases. This is because the larger  $\alpha_1$  becomes, the faster  $X_1$  reverts and hence  $X_1$  becomes less uncertain. Therefore, the steepening (flattening) in the short term sector makes investors reduce (increase) the hedging position to  $X_1$ , and increase (reduce) hedging positions against  $X_2$  and  $X_3$  because these factors become riskier than  $X_1$ .

From Exhibit 14 we can see the same is true for  $\alpha_2$ ; the increase of  $\alpha_2$  decreases

the exposure to  $X_2$ . Moreover, this case shows the advantage of the factor allocation approach. It is difficult to interpret how  $\alpha_2$  affects each bond from Exhibit 13. On the other hand, from Exhibit 14, we can interpret the effect intuitively as follows; steepening (flattening) in the middle term sector makes investors reduce (increase) only their hedge exposures to  $X_2$  while exposures to  $X_1$  and  $X_3$  stay constant.

Next we consider the effect of the change of  $\theta_3$ . In Exhibit 15, the factor allocation in the IR-hedging term unchanges while the exposures to  $X_2$  and  $X_3$  in the MV term change. The decrease of  $\theta_3$  makes the instantaneous expected return of  $X_3$  increase and therefore  $X_3$  becomes more attractive in Mean-Variance basis. Thus the exposure to  $X_3$  increase as  $\theta_3$  decreases. Moreover, because the correlation between  $X_2$  and  $X_3$  is set to be negative, investors can increase the exposure to  $X_3$  further by using the exposure to  $X_2$  for hedging. As a result, the exposure to  $X_2$  also increases. We can confirm this from Exhibit 18. The exposure to  $X_3$  with  $\rho_{23} \neq 0$  is larger than that with  $\rho_{23} = 0$ .

From Corollary 2.1, we can confirm  $\sigma_i, i = 1, 2, 3$  affect only the MV term. We can see this from Exhibit 16 and Exhibit 17. The increase of  $\sigma_i$  will reduce only the allocation to  $X_i$  and others stay constant.

Next we can refer to Exhibit 19 and Exhibit 20 to see the effect of the investment horizon. Two points deserve mention. First, the MV term has no horizontal effect and this effect is specific to the dynamic term. Second, from Exhibit 19 the effect on the allocation to bonds is not monotone. The increase of the investment horizon increases the uncertainty in the future but at the same time it also reduces the duration of each bond and that makes uncertainty decrease. Which of them



is dominant depends on circumstances. On the other hand, the effect on the factor allocation is monotone increasing from Exhibit 20. This is simply because uncertainty becomes larger as the investment horizon increases.

Finally, we consider the effect of the variation of the RRA. The results are given in Exhibit 21. When the  $RRA=1$ , the investor has a log utility and the optimal strategy is the same with that of Mean-Variance approach. From Exhibit 21 we can see several points. First, as for  $X_1$  the MV term has no exposure to  $X_1$  therefore the IR-hedging term is dominant. If the  $RRA < 1$ , the investor is less risk averse and takes an aggressive position, which corresponds to positive exposures in the IR-hedging term. If the  $RRA > 1$ , on the other hand, the investor is more risk averse and takes a cautious position; the opposite position with the MV term.

We summarize the results of the comparative statics in Exhibit 22.

## 4 Conclusion

In this paper, we analyzed the dynamic fixed-income portfolio optimization with a multi-factor Gaussian yield curve model. The main results obtained in this paper are summarized as follows.

First, we combined the dynamic portfolio optimization and a multi-factor Gaussian term structure model to obtain an analytical expression of the optimal bond allocation; it enables us to examine the model parameters' effects efficiently.

Second, by introducing the idea of factor allocation, we can easily interpret optimal portfolio which is usually hard to be understood by considering the portfolio in terms of bonds.

Third, we investigate how the change in the term structure affects the optimal portfolio through series of comparative statics.

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EXHIBIT 4:  $\alpha_1$  effect; +0.5

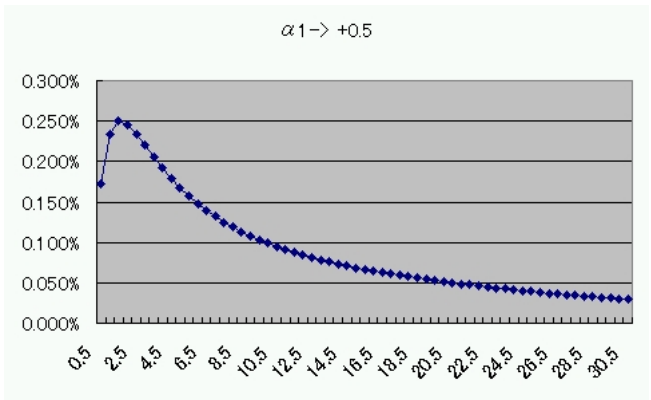


EXHIBIT 5:  $\alpha_2$  effect; +0.5,  $X_2(0) < 0$

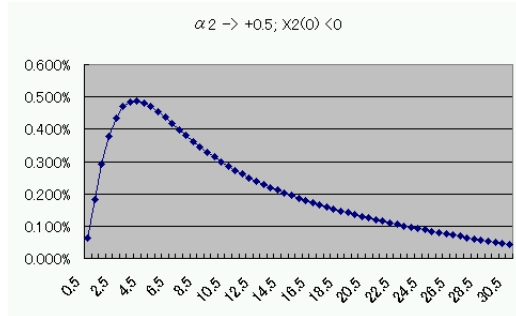


EXHIBIT 1:  $X_1$  effect; +20 bps

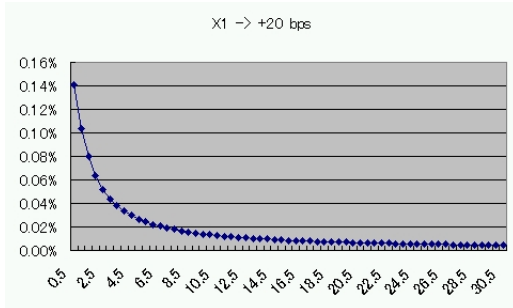


EXHIBIT 6:  $\alpha_2$  effect; +0.5,  $X_2(0) > 0$

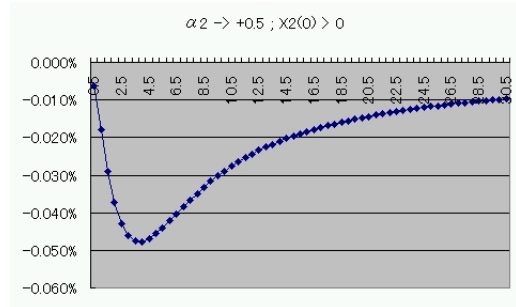


EXHIBIT 2:  $X_2$  effect; +20 bps

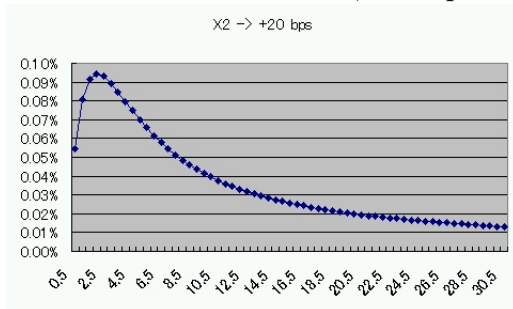


EXHIBIT 7:  $\theta_3$  effect; -0.7

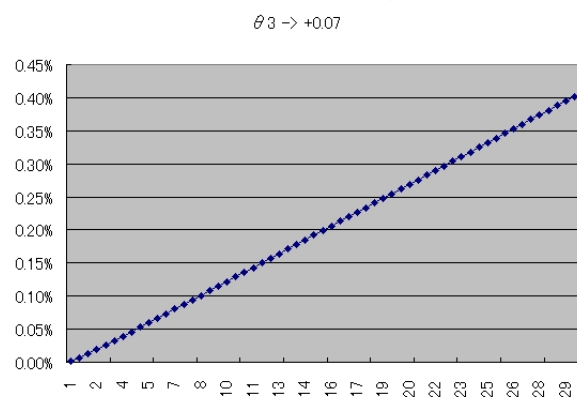


EXHIBIT 3:  $X_3$  effect; +20 bps

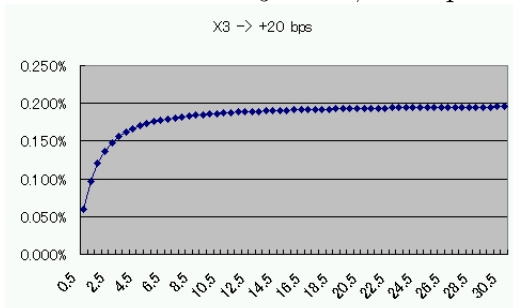


EXHIBIT 8: Model Parameter Set

$\alpha_1$	1.50	$\sigma_1$	0.50%
$\alpha_2$	0.50	$\sigma_2$	1.50%
$\alpha_3$	0	$\sigma_3$	1.25%
$\theta_1$	0	$\rho_{12}$	0
$\theta_2$	0	$\rho_{13}$	0
$\theta_3$	-0.125	$\rho_{23}$	-0.3

EXHIBIT 9: Portfolio Parameter Set

Investment Horizon	1
RRA	4
Zero Coupon Bonds	2
	7
	30

EXHIBIT 10: Initial Result

	<b>MV</b>		<b>IR Hedge</b>		<b>Total</b>
2	0.882		0.937	2	1.820
7	-0.836	+	-0.353	7	-1.189
30	-0.002		0.045	30	0.043
				Cash	0.673
				Total	1

factor allocation

	<b>MV</b>	<b>IR Hedge</b>	<b>Total</b>
$X_1$	0.0000	-0.3884	-0.3884
$X_2$	0.7661	-0.3027	0.4635
$X_3$	4.1570	-0.3616	3.7954

EXHIBIT 11: The effect of  $\alpha_1$  variation;  
bonds

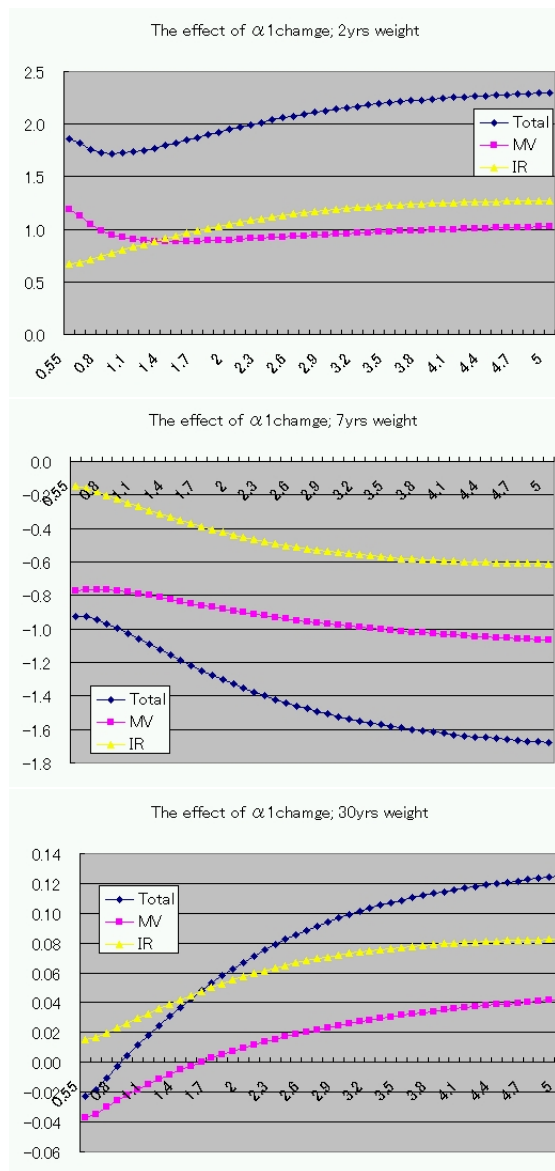


EXHIBIT 12: The effect of  $\alpha_1$  variation;  
factor allocation

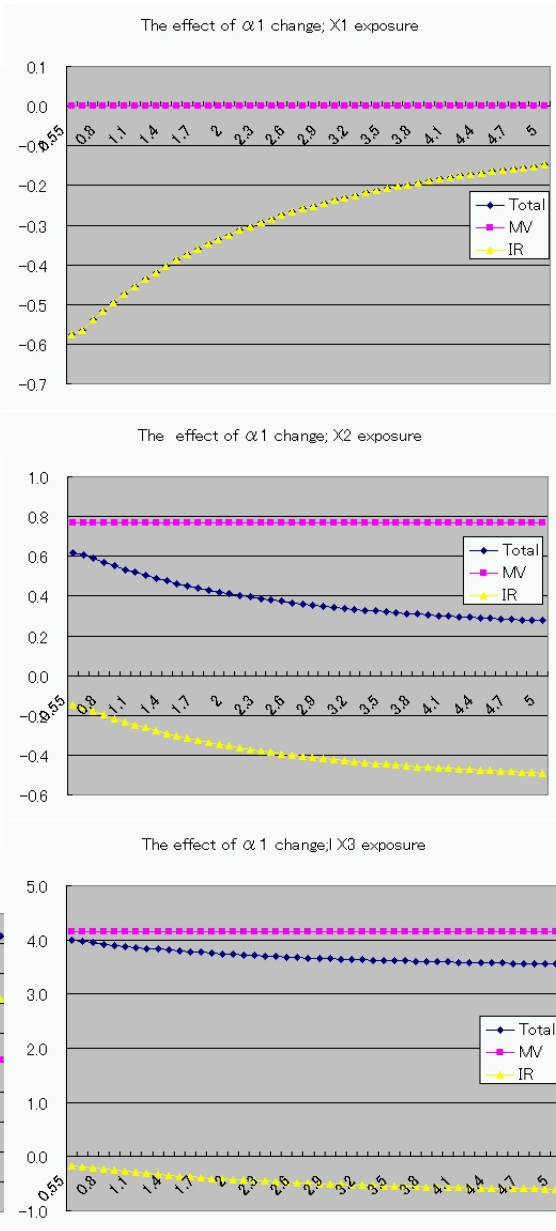


EXHIBIT 13: The effect of  $\alpha_2$  variation;  
bonds

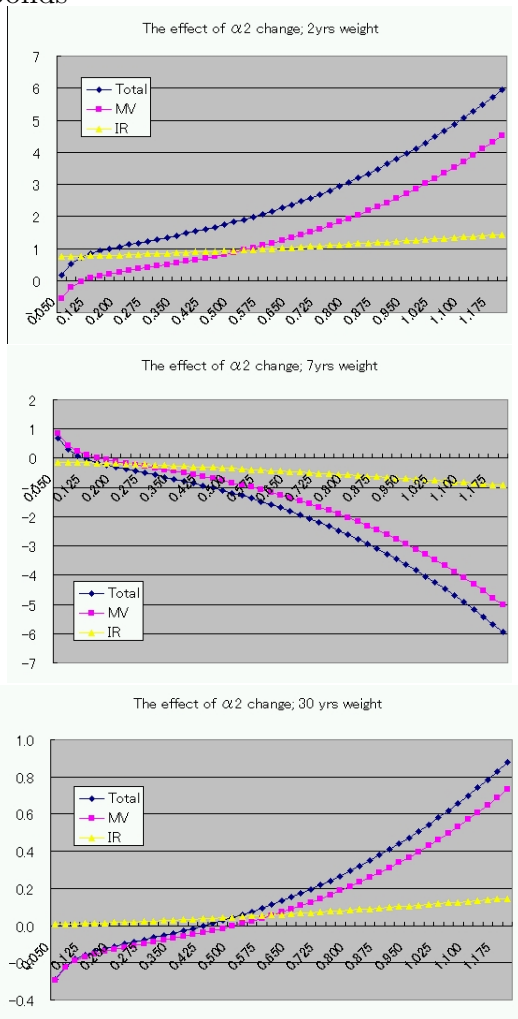


EXHIBIT 14: The effect of  $\alpha_2$  variation;  
exposure to  $X_2$

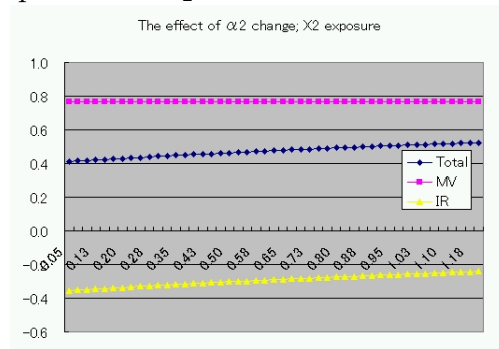


EXHIBIT 15: The effect of  $\theta_3$  variation; factor allocation

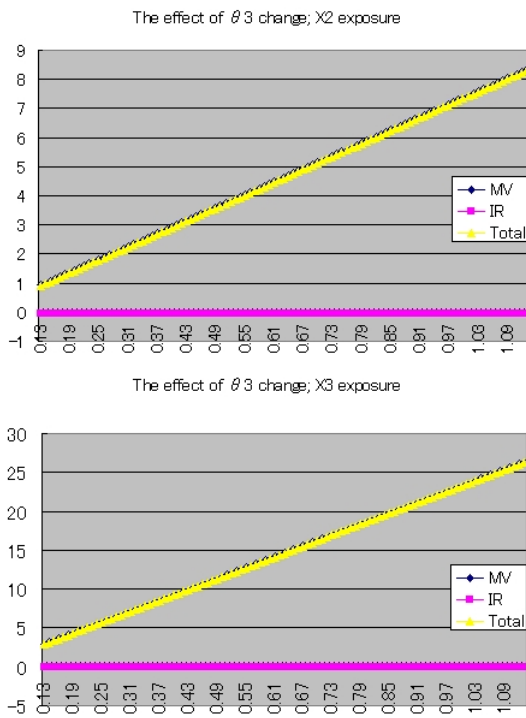


EXHIBIT 16: The effect of  $\sigma_2$  variation;  $X_2$

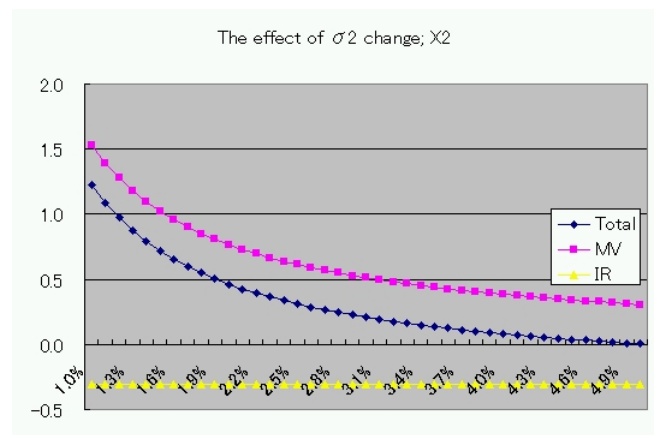


EXHIBIT 17: The effect of  $\sigma_3$  variation;  $X_3$

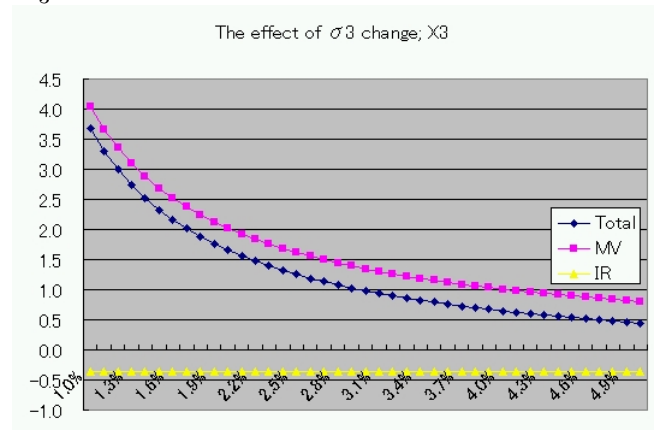




EXHIBIT 18:  $\theta_3$  and  $\rho_{23}$

$\theta_3$	0.2	0.3	0.4	0.5	0.6	0.7	0.8phantom000
Factor Allocation							
<b>MV term</b>							
$X_2 (\rho_{23} = -0.38)$	1.1103	1.6655	2.2207	2.7758	3.3310	3.8862	4.4413
$X_3 (\rho_{23} = -0.38)$	6.0246	9.0369	12.0492	15.0615	18.0738	21.0861	24.0984
$X_2 (\rho_{23} = 0)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$X_3 (\rho_{23} = 0)$	5.1546	7.7320	10.3093	12.8866	15.4639	18.0412	20.6186
$X_2 (\rho_{23} = 0.38)$	-1.1103	-1.6655	-2.2207	-2.7758	-3.3310	-3.8862	-4.4413
$X_3 (\rho_{23} = 0.38)$	6.0246	9.0369	12.0492	15.0615	18.0738	21.0861	24.0984

EXHIBIT 19: The effect of *Investment Horizon* variation; bonds

EXHIBIT 20: The effect of *Investment Horizon* variation; factor allocation

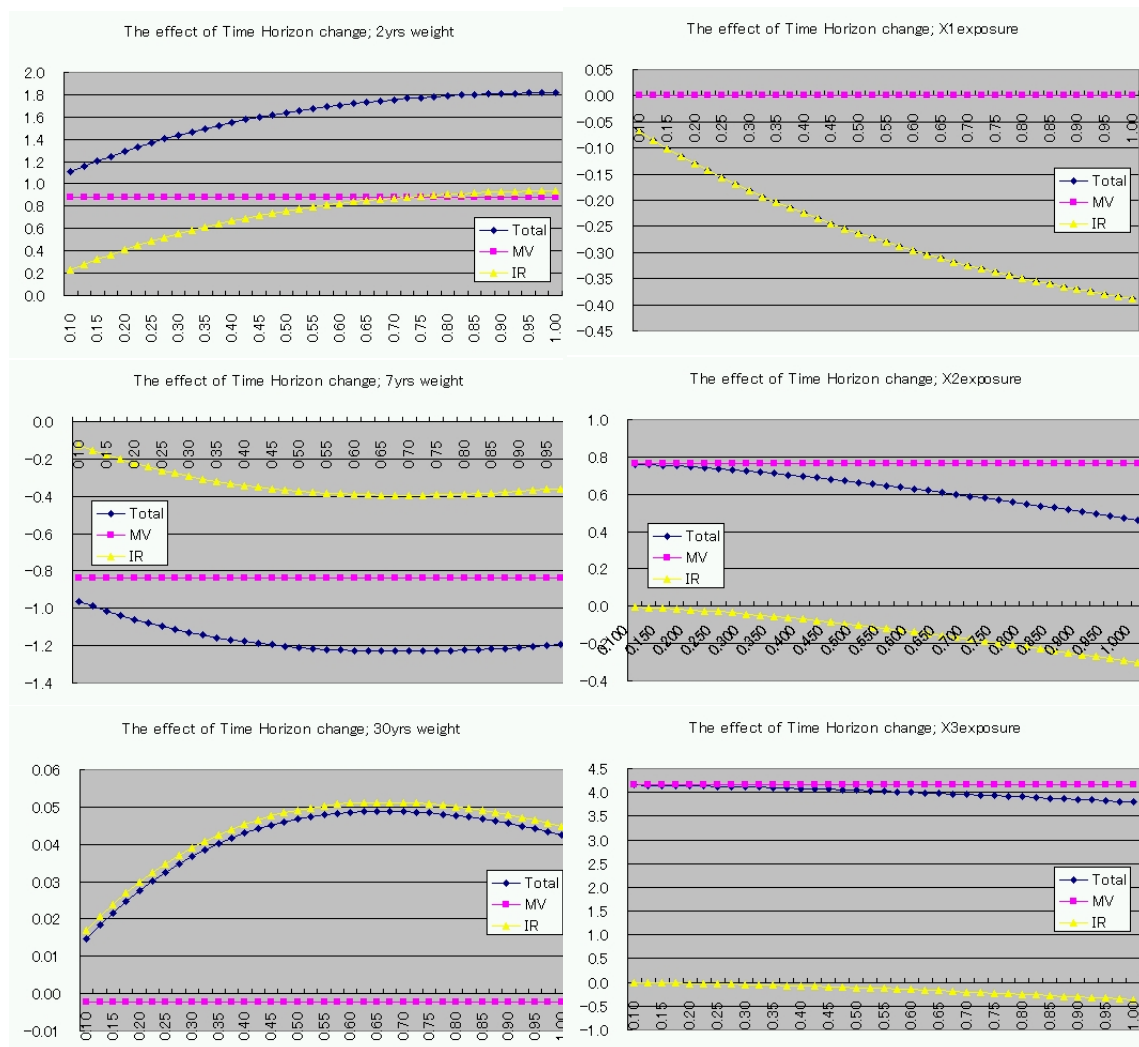
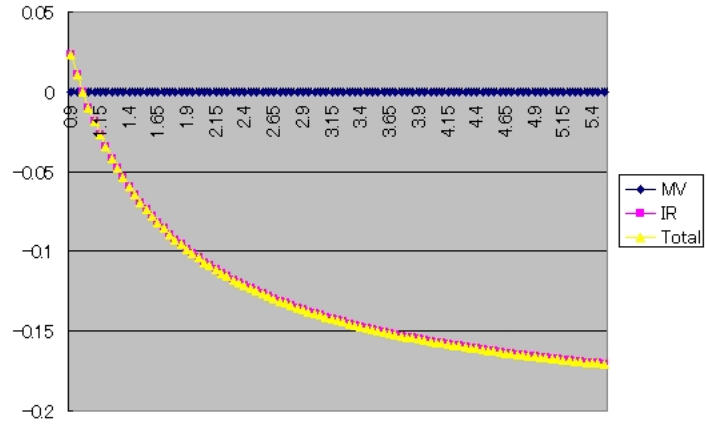
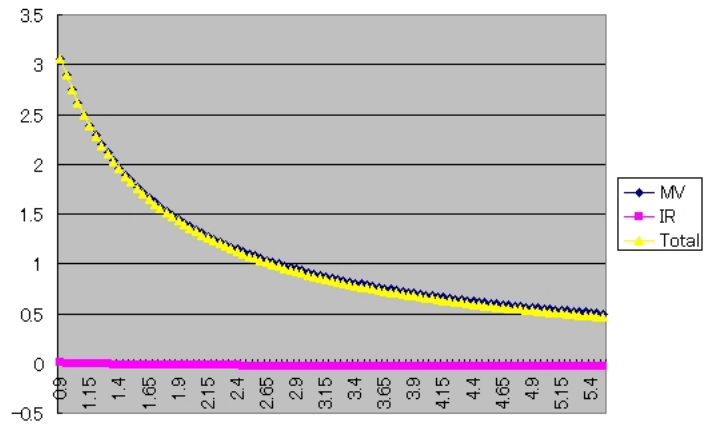


EXHIBIT 21: The effect of *RRA* variation; factor allocation

The effect of *RRA* change; X1 exposure



The effect of *RRA* change; X2 exposure



The effect of *RRA* change; X3 exposure

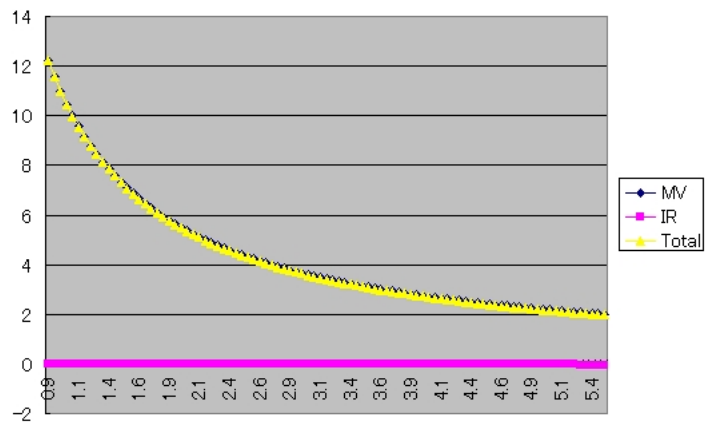


EXHIBIT 22: The summary of effects

Parameter	The effect on the portfolio strategy (the factor allocation)	
	<i>MV term</i>	<i>IR Hedge term</i>
Term Structure		
$\alpha_1$	————	$\alpha_1 \uparrow \Rightarrow X_1 \downarrow, X_2 \& X_3 \uparrow$
$\alpha_2$	————	$\alpha_2 \uparrow \Rightarrow X_2 \downarrow$
$\theta_3$	$\theta_3 \uparrow \Rightarrow X_3 \uparrow$ $\rho_{23} < 0 \Rightarrow X_3 \uparrow\uparrow, X_2 \uparrow$	————
$\sigma_1$	————	————
$\sigma_2$	$\sigma_2 \uparrow \Rightarrow X_2 \downarrow$	————
$\sigma_3$	$\sigma_3 \uparrow \Rightarrow X_3 \downarrow$	————
Investment strategy		
T (Investment Horizon)	————	$T \uparrow \Rightarrow X_1, X_2, X_3 \uparrow$
RRA (Risk Preference)	$RRA > 1, RRA \uparrow \Rightarrow X_2 \& X_3 \downarrow$	$RRA > 1, RRA \uparrow \Rightarrow X_1, X_2, X_3 \uparrow$