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Fourier Transform Method with an Asymptotic Expansion Approach: an Application to Currency Options

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Abstract

This paper develops a Fourier transform method with an asymptotic expansion approach for option pricing. The method is applied to European currency options with a libor market model of interest rates and jump-diffusion stochastic volatility models of spot exchange rates. In particular, we derive closed-form approximation formulas of the characteristic functions of log-prices of the underlying assets and the prices of currency options based on a third order asymptotic expansion scheme; we use a jump-diffusion model with a mean-reverting stochastic variance process such as in Heston[1993]/Bates[1996] and log-normal market models for domestic and foreign interest rates. Finally, the validity of our method is confirmed through numerical examples.

Keywords : Currency option, libor market model, stochastic volatility, asymptotic expansion, Fourier transform

1 Introduction

This paper proposes new approximation formulas for evaluation of the characteristic functions of log-prices of forward foreign exchange rates and of the prices of European currency options under stochastic volatility processes of spot exchange rates in stochastic interest rates environment. In particular, we use models of variance processes such as in Heston[1993] or of jump-diffusion stochastic variance ones in Bates[1996], and apply a libor market model developed by Brace, Gatarek and Musiela[1998] and Miltersen, Sandmann and Sondermann[1997] to modeling term structures of interest rates. The correlations between the domestic and foreign interest rates, and between the spot exchange rate and its variance are allowed.

Currency options with maturities beyond one year become common in global currencies' markets and even smiles or skews for those maturities are frequently observed. Because it is well known that the effect of interest rates become more substantial in longer maturities, we have to take term structure models into account for the currency options. Further, stochastic volatility and/or jump-diffusion models of foreign exchange rates are necessary for calibration of smiles and skews. As for term structure models, market models become popular in matured interest rates markets since calibrations of caps, floors and swaptions are necessary and market models are regarded as most useful.

Hence, our objective is to develop a model with jump-diffusion stochastic volatility processes of exchange rates and with a libor market model of interest rates. Moreover, a closed-form formula is desirable in practice especially for calibrations which are usually done by numerical methods such as Monte Carlo simulation since they are very time consuming. Because it is difficult to obtain an exact closed-form formula, we derive a closed-form approximation formula by a Fourier transform method with an asymptotic expansion up to the third order where a spot exchange rate follows a jump-diffusion process, its variance follows a mean-reverting stochastic one, and domestic and foreign interest rates are generated by a libor market model. Some numerical examples presented later support accuracy achieved by our formulas.

In addition, we emphasize two remarkable features of our method. First, this method is essentially different from those used in preceding works on an asymptotic expansion in the following aspect; in our method, the distribution of the component, of the underlying asset, dependent on the interest rates are not approximated around a normal distribution such as in Takahashi and Takehara[2006] but around a log-normal one. Second, under the assumption made in this paper, our method can be applied not only to the stochastic variance model in Heston[1993] with which we are concerned in this paper but also to a broad class of models with stochastic volatility, jump-diffusion or even more general Levy processes where the closed-form characteristic functions are available.

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Our asymptotic expansion approach have been applied to a broad class of Itô processes appearing in finance. It started with pricing average options; Kunitomo and Takahashi[1992] derived a first order approximation and Yoshida[1992b] applied an asymptotic expansion method developed in statistics for stochastic processes. Takahashi[1995,1999] presented the second or third order schemes for pricing various options in a general Markovian setting with a constant interest rate. Kunitomo and Takahashi[2001] provided approximation formulas for pricing bond options and average options on interest rates in term structure models of HJM[1992] which is not necessarily Markovian.

Moreover, Takahashi and Yoshida[2004,2005] extended the method to dynamic portfolio problems in a general Markovian setting and proposed a new variance reduction scheme of Monte Carlo simulation with an asymptotic expansion. For mathematical validity of the method based on Watanabe[1987] in the Malliavin calculus, see Chapter 7 of Malliavin and Thalmaier[2006], Yoshida[1992a], Kunitomo and Takahashi[2003] and Takahashi and Yoshida[2004,2005].

Other applications and extensions of asymptotic expansions to numerical problems in finance are found as follows: Kawai[2003], Kobayashi,Takahashi and Tokioka[2003], Takahashi and Saito[2003], Kunitomo and Takahashi[2004], Kunitomo and Kim[2005], Muroi[2005], Takahashi[2005], Matsuoka,Takahshi and Uchida[2006], Takahashi and Uchida[2006].

Additionally, Takahashi and Takehara[2006] developed the approximation formulas for evaluation of the prices of European currency options and of the distribution of the underlying assets based on an asymptotic expansion approach up to the third order with a market model of interest rates and a general time-inhomogeneous Markovian stochastic volatility model of the spot exchange rate. Contrary to this paper, their paper took the standard approach which expanded the underlying process around a normal distribution and did not depend upon the assumption made in this paper.

The organization of the paper is as follows: After the next section describes a basic structure of our model, Section 3 derives an approximation formula by expansion of the component dependent on the interest rates around a log-normal distribution, with the assumption of independence between the interest rates and the spot exchange rate/its stochastic variance. Section 4 shows numerical examples and the final section states conclusion. Appendix A gives the concrete expressions of coefficients used in the asymptotic expansions, and Appendix B presents formulas used in Appendix A.

2 European Currency Options with a Market Model of Interest Rates and Stochastic Volatility Models of Spot Exchange Rates

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\le t\le T^*<\infty})$ be a complete probability space with a filtration satisfying the usual conditions. First we briefly state the basics of European currency options. The payoffs of call and put options with maturity $T \in (0, T^*]$ and strike rate $K > 0$ are expressed as $(S(T) - K)^+$ and $(K - S(T))^+$ respectively where $S(t)$ denotes the spot exchange rate at time $t \ge 0$ and x^+ denotes $\max(x, 0)$. In this paper we will concentrate on the valuation of a call option since the value of a put option can be obtained through the put-call parity or similar method. We also note that the spot exchange rate $S(T)$ can be expressed in terms of a foreign exchange forward(forex forward) rate with maturity T. That is, $S(T) = F_T(T)$ where $F_T(t)$, $t \in [0, T]$ denotes the time t value of the forex forward rate with maturity T. It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t, T)$ and $P_f(t, T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively.

Hence, our objective is to obtain the present value of the payoff $(F_T(T) - K)^+$. In particular, we need to evaluate:

$$V(0; T, K) \quad = \quad P_d(0, T) \times \mathbf{E}\left[(F_T(T) - K)^+\right] \tag{2.1}$$

where $V(0; T, K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, and $\mathbf{E}[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure) of numeraire of the domestic zero coupon bond maturing at T (we use a term of *the domestic terminal measure* in what follows).

Next, with a log-price of the forex forward $f_T(t) := ln(\frac{F_T(t)}{F_T(0)})$, (2.1) can be rewritten as:

$$V(0; T, K) \quad = \quad P_d(0, T) \times F_T(0)\ \mathbf{E}\left[(e^{f_T(T)} - e^k)^+\right]$$

where $k := ln(\frac{K}{F_T(0)})$ denotes a log-strike rate. Here we note that $e^{f_T(T)} = F_T(T)$ is a martingale under the domestic terminal measure.

The following proposition is well known (e.g. Heston[1993]).

Proposition 1 *Let $\Phi_T(u)$ denote a characteristic function of $f_T(T)$. Then, $V(0;T,K)$ is given by:*

$$\begin{aligned} V(0;T,K) &= P_d(0,T) \times \left[F_T(0) \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{\frac{e^{-iuk}\Phi_T(u-i)}{iu}\}du \right\} \right. \\ & \left. \qquad - K \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\{\frac{e^{-iuk}\Phi_T(u)}{iu}\}du \right\} \right] \end{aligned} \tag{2.2}$$

where $i := \sqrt{-1}$ and $Re(x)$ denotes a real part of x.

Then, we need to know the characteristic function of $f_T(T)$ under the domestic terminal measure for pricing the option. For this objective, a market model and stochastic volatility models are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t (\leq T_j)$ respectively; $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

Then $\mathbf{R}_{++}$-valued processes of domestic forward interest rates under the domestic terminal measure can be specified as; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$\begin{aligned} f_{dj}(t) &= f_{dj}(0) + \int_0^t \left\{ -f_{dj}(u)\tilde{\gamma}_{dj}'(u) \sum_{i=j+1}^N \frac{\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)} \right\} du \\ &\quad + \int_0^t f_{dj}(u)\tilde{\gamma}_{dj}'(u)dW_u \end{aligned} \tag{2.3}$$

where x' denotes the transpose of x, and W is a D dimensional Brownian motion under the domestic terminal measure; $\tilde{\gamma}_{dj}(u)$ is a function of time-parameter u. Similarly, $\mathbf{R}_{++}$-valued processes of foreign ones under the foreign terminal measure are specified as;

$$\begin{aligned} f_{fj}(t) &= f_{fj}(0) + \int_0^t \left\{ -f_{fj}(u)\tilde{\gamma}_{fj}'(u) \sum_{i=j+1}^N \frac{\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)} \right\} du \\ &\quad + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}'(u)dW_u^f \end{aligned} \tag{2.4}$$

where W^f is a D dimensional Brownian motion under the foreign terminal measure and $\tilde{\gamma}_{fj}(u)$ is also a function of u.

Finally, we assume that the spot exchange rate $S(t)$ and its variance $V(t)$ follow $\mathbf{R}_{++}$-valued stochastic processes under the domestic risk neutral measure (not under the domestic terminal measure) :

$$S(t) = S(0) + \int_0^t S(u)(r_d(u) - r_f(u))du + \int_0^t S(u)\sqrt{V(u)}\bar{\sigma}'d\hat{W}_u + \tilde{J}(t) \tag{2.5}$$

$$V(t) = V(0) + \int_0^t \hat{\mu}(V(u),u)du + \int_0^t \omega'(V(u),u)d\hat{W}_u \tag{2.6}$$

where $\hat{W}$ is a D dimensional Brownian motion under the domestic risk neutral measure and $\tilde{J}$ is some jump martingale independent of $\hat{W}$; $r_d(u)$ and $r_f(u)$ denote domestic and foreign instantaneous spot interest rates respectively; $\bar{\sigma}$ denotes a D dimensional constant vector satisfying $||\bar{\sigma}|| = 1$, and $\omega(x,u)$ is a function of x and u.

Hereafter, the variance and jump processes are specified as in Bates[1996], that is, $V(t)$ and $\tilde{J}(t)$ are given by;

$$V(t) = V(0) + \int_0^t \kappa(\theta - V(u))du + \int_0^t \omega\bar{v}'\sqrt{V(u)}d\hat{W}_u \tag{2.7}$$

$$\tilde{J}(t) = J(t) - \lambda\eta t \tag{2.8}$$

where κ, θ and ω are all constant and denote the speed of the mean-reversion of the variance process, the level of the mean-reversion and the volatility on the variance, respectively; $\bar{v}'$ is some D dimensional constant vector denoting the correlation structure between the variance and other factors, that is, the domestic and foreign interest rates and the spot exchange rate. $J(t)$ denotes a compound Poisson process with intensity of λ and with a random jump size l whose distribution is determined as $ln(1+l) \sim N(ln(1+\eta) - \frac{1}{2}\delta^2, \delta^2)$.

We next note the following well known relations among Brownian motions under different measures;

$$\begin{aligned} W_u &= \hat{W}_u - \int_0^u \tilde{\sigma}_{dN+1}(s)ds \\ &= W_u^f + \int_0^u \{\tilde{\sigma}_{fN+1}(s) - \tilde{\sigma}_{dN+1}(s) + \sqrt{V(s)}\bar{\sigma}\}ds \end{aligned}$$

where $\tilde{\sigma}_{dN+1}(u)$ and $\tilde{\sigma}_{fN+1}(u)$ are volatilities of the domestic and foreign zero coupon bonds with the maturity T_{N+1}, that is,

$$\tilde{\sigma}_{dN+1}(u) := \sum_{i \in J_{N+1}(u)} \frac{-\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)}, \quad \tilde{\sigma}_{fN+1}(u) := \sum_{i \in J_{N+1}(u)} \frac{-\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)}$$

and $J_{j+1}(t) = \{n(t)-1, n(t), n(t)+1, \cdots, j\}$. Since $\gamma_{fi}(t) = 0$ and $\gamma_{di}(t) = 0$ for all i such that $T_i \le t$, the set of indices $J_{j+1}(t)$ can be changed into $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, which does not depend on t.

By above equations, expressions of those processes under different measures are unified into those under the same measure, the domestic terminal one:

$$\begin{aligned} f_{fj}(t) &= f_{fj}(0) + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)\left\{-\sum_{i \in \hat{J}_{j+1}} \frac{-\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)} + \sum_{i \in \hat{J}_{N+1}} \frac{-\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)}\right\}du \\ &\quad - \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)\sqrt{V(u)}\bar{\sigma}du + \int_0^t f_{fj}(u)\tilde{\gamma}_{fj}^{'}(u)dW_u \qquad (2.9) \\ V(t) &= V(0) + \int_0^t \kappa(\theta - V(u))du \\ &\quad - \int_0^t \omega\bar{v}^{'}\sqrt{V(u)} \sum_{i \in \hat{J}_{N+1}} \frac{\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)}du + \int_0^t \omega\bar{v}^{'}\sqrt{V(u)}dW_u \qquad (2.10) \end{aligned}$$

Next, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) := F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) := S(t)\frac{P_f(t,T_{N+1})}{P_d(t,T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure, and obtain its log-process, $f_{N+1}(t)$ under that measure by Itô's formula:

$$\begin{aligned} f_{N+1}(t) &= ln(\frac{F_{N+1}(t)}{F_{N+1}(0)}) \\ &= -\frac{1}{2}\int_0^t \left\|\sum_{j \in \hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(u)\tilde{\gamma}_{fj}(u)}{1+\tau_j f_{fj}(u)} - \frac{-\tau_j f_{dj}(u)\tilde{\gamma}_{dj}(u)}{1+\tau_i f_{dj}(u)}\right) + \sqrt{V(u)}\bar{\sigma}\right\|^2 du \\ &\quad + \int_0^t \left[\sum_{j \in \hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(u)\tilde{\gamma}_{fj}(u)}{1+\tau_j f_{fj}(u)} - \frac{-\tau_j f_{dj}(u)\tilde{\gamma}_{dj}(u)}{1+\tau_i f_{dj}(u)}\right) + \sqrt{V(u)}\bar{\sigma}\right]^{'} dW_u \\ &\quad + \hat{J}(t) - \lambda\eta t \qquad (2.11) \end{aligned}$$

where $\hat{J}(t)$ denotes a compound Poisson process with intensity of λ and with a Gaussian random jump size.

3 An Approximation Scheme based on an Asymptotic Expansion Approach

3.1 An independence assumption

Hereafter, we make an assumption;

[**A 1**] *Domestic and foreign interest rates are assumed to be independent of the spot exchange rate and its variance. Then,* $\tilde{\gamma}_{dj}(u)$, $\tilde{\gamma}_{fj}(u)$, $\bar{\sigma}$ *and* $\bar{v}$ *satisfy the following conditions.*
For all $u \in (0,T]$ *and* $j \in \hat{J}_{N+1}$,

$$\begin{cases} \tilde{\gamma}_{dj}(u)^{'}\bar{\sigma} = 0, \ \tilde{\gamma}_{fj}(u)^{'}\bar{\sigma} = 0 \\ \tilde{\gamma}_{dj}(u)^{'}\bar{v} = 0, \ \tilde{\gamma}_{fj}(u)^{'}\bar{v} = 0 \end{cases} \tag{3.1}$$

Under this assumption, the equation (2.11) can be decomposed as:

$$f_{N+1}(t) = Y(t) + Z(t) + \hat{J}(t) - \lambda\eta t \tag{3.2}$$

where

$$\begin{aligned} Y(t) &= -\frac{1}{2}\int_0^t \left\| \sum_{j\in\hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(u)\tilde{\gamma}_{fj}(u)}{1+\tau_j f_{fj}(u)} - \frac{-\tau_j f_{dj}(u)\tilde{\gamma}_{dj}(u)}{1+\tau_i f_{dj}(u)} \right) \right\|^2 du \\ &\quad + \int_0^t \left[\sum_{j\in\hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(u)\tilde{\gamma}_{fj}(u)}{1+\tau_j f_{fj}(u)} - \frac{-\tau_j f_{dj}(u)\tilde{\gamma}_{dj}(u)}{1+\tau_i f_{dj}(u)} \right) \right]^{'} dW_u \qquad (3.3) \\ Z(t) &= -\frac{1}{2}\int_0^t V(u)du + \sqrt{V(u)}\bar{\sigma}^{'} dW_u \qquad (3.4) \end{aligned}$$

Note that $Y(t)$, $Z(t)$ and $\hat{J}(t)$ are independent and that $Y(t)$ depends only on the domestic and foreign interest rates (in what follows we sometimes call $Y(t)$ *the interest-rate part* of the forex forward to emphasize this property) and that $Z(t)$ does only on the variance of the spot exchange rate (we sometimes call $Z(t)$ *the volatility part* as well). Moreover, under the same assumption, the equations (2.9) and (2.10) are simplified as follows:

$$\begin{aligned} f_{fj}(t) &= f_{fj}(0) + \int_0^t f_{fj}(u)\tilde{\gamma}^{'}_{fj}(u) \left\{ -\sum_{i\in\hat{J}_{j+1}} \frac{-\tau_i f_{fi}(u)\tilde{\gamma}_{fi}(u)}{1+\tau_i f_{fi}(u)} + \sum_{i\in\hat{J}_{N+1}} \frac{-\tau_i f_{di}(u)\tilde{\gamma}_{di}(u)}{1+\tau_i f_{di}(u)} \right\} du \\ &\quad + \int_0^t f_{fj}(u)\tilde{\gamma}^{'}_{fj}(u)dW_u \qquad (3.5) \\ V(t) &= V(0) + \int_0^t \kappa(\theta - V(u))du + \int_0^t \omega\bar{v}^{'}\sqrt{V(u)}dW_u \qquad (3.6) \end{aligned}$$

Let $\Phi_{N+1}(t,u)$ denote the characteristic function of $f_{N+1}(t)$. Then, $\Phi_{N+1}(t,u)$ can be also decomposed as;

$$\Phi_{N+1}(t,u) = \Phi_Y(t,u)\Phi_Z(t,u)\Phi_{\hat{J}}(t,u)\exp\{-iu\lambda\eta t\} \tag{3.7}$$

where $\Phi_Y(t,u)$, $\Phi_Z(t,u)$ and $\Phi_{\hat{J}}(t,u)$ denote the characteristic functions of $Y(t)$, $Z(t)$ and $\hat{J}(t)$, respectively.

For evaluation of European currency options, an explicit expression of $\Phi_{N+1}(t,u)$ is necessary. However, the process of $Y(t)$ is too complicated to obtain the analytical expression of $\Phi_Y(t,u)$ while that of $\Phi_Z(t,u)$ is well known (see Section 6.3.2 in Brigo and Mercurio[2006] or Section 25.5 in Björk[2004]) . Then, we suggest to utilize an asymptotic expansion for the approximation of $\Phi_Y(t,u)$.

Before concentrating on the approximation of the interest-rate part of the forex forward, we state the expression of $\Phi_Z(t,u)$(see Duffie, Pan and Singleton[1999] for details) and $\Phi_{\hat{J}}(t,u)$:

$$\begin{aligned} \Phi_Z(t,u) &= \left\{ \cosh\frac{\xi t}{2} + \frac{\kappa - i\rho\omega u}{\xi}\sinh\frac{\xi t}{2} \right\}^{-\frac{2\kappa\theta}{\omega^2}} \\ &\quad \times \exp\left\{ \frac{\kappa\theta(\kappa - i\rho\omega u)t}{\omega^2} - \frac{(u^2+iu)V(0)}{\xi\cosh\frac{\xi t}{2} + (\kappa - i\rho\omega u)} \right\} \qquad (3.8) \\ \Phi_{\hat{J}}(t,u) &= \exp\left\{ \lambda t \left(\exp\left\{ \left(ln(1+\eta) - \frac{1}{2}\delta^2 \right) iu - \frac{1}{2}\delta^2 u^2 \right\} \right) \right\} \qquad (3.9) \end{aligned}$$

where $\rho := \bar{\sigma}^{'}\bar{v}$ and $\xi := \sqrt{\omega^2(u^2+iu) + (\kappa - i\rho\omega u)^2}$.

3.2 An asymptotic expansion approach

An asymptotic expansion approach describes the processes of forward rates as $f_{dj}^{(\epsilon)}(t)$ and $f_{fj}^{(\epsilon)}(t)$ respectively, both explicitly depend upon a parameter $\epsilon \in (0,1]$, and expands the processes around $\epsilon = 0$, that is asymptotic expansions are made around deterministic processes.

First, the processes of $f_{dj}^{(\epsilon)}(t)$ and $f_{fj}^{(\epsilon)}(t)$ are redefined as follows; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$\begin{aligned} f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon^2 \int_0^t \left\{ -f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u) \sum_{i=j+1}^{N} \frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)} \right\} du \\ &\quad + \epsilon \int_0^t f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u) dW_u \end{aligned} \tag{3.10}$$

$$\begin{aligned} f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) + \epsilon^2 \int_0^t \left\{ -f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) \sum_{i=j+1}^{N} \frac{\tau_i f_{fi}^{(\epsilon)}(u)\gamma_{fi}(u)}{1+\tau_i f_{fi}^{(\epsilon)}(u)} \right\} du \\ &\quad + \epsilon \int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) dW_u^f \end{aligned} \tag{3.11}$$

where $\tilde{\gamma}_{dj}(t)$ and $\tilde{\gamma}_{fj}(t)$ in the previous section are replaced by $\epsilon\gamma_{dj}(t)$ and $\epsilon\gamma_{fj}(t)$, respectively.

Hence, the processes of $f_{fj}^{(\epsilon)}(t)$ and $Y^{(\epsilon)}(t)$ under the domestic terminal measure are expressed as follows:

$$\begin{aligned} f_{fj}^{(\epsilon)}(t) &= f_{fj}^{(\epsilon)}(0) + \epsilon^2 \int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) \left\{ -\sum_{i\in \hat{J}_{j+1}} \frac{-\tau_i f_{fi}^{(\epsilon)}(u)\gamma_{fi}(u)}{1+\tau_i f_{fi}^{(\epsilon)}(u)} + \sum_{i\in \hat{J}_{N+1}} \frac{-\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)} \right\} du \\ &\quad + \epsilon \int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}^{'}(u) dW_u \end{aligned} \tag{3.12}$$

$$\begin{aligned} Y^{(\epsilon)}(t) &= -\frac{\epsilon^2}{2} \int_0^t \left\| \sum_{j\in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}(u)} - \sum_{j\in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}(u)} \right\|^2 du \\ &\quad + \epsilon \int_0^t \left[\sum_{j\in \hat{J}_{N+1}} \frac{-\tau_j f_{fj}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}(u)} - \sum_{j\in \hat{J}_{N+1}} \frac{-\tau_j f_{dj}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}(u)} \right]^{'} dW_u \end{aligned} \tag{3.13}$$

Of course, variables and functions such as $F_{N+1}^{(\epsilon)}(t)$, $f_{N+1}^{(\epsilon)}(t)$, $\Phi_{N+1}^{(\epsilon)}(t,u)$ or $\Phi_{Y}^{(\epsilon)}(t,u)$ explicitly depend on ϵ.

Next, we expand forward rates' processes up to the second order of $\epsilon(\epsilon^2$-order) around $\epsilon = 0$ to obtain the third order asymptotic expansion of $Y^{(\epsilon)}(t)$, the interest-rate part of the forex forward. These expansions can be obtained by differentiating the right hand side of the equations (3.10), (3.12) and (3.13) with respect to ϵ and setting $\epsilon = 0$. The result is stated as the following lemma.

Lemma 1 *The asymptotic expansions of domestic and foreign forward rates are given as follows:*

$$f_{dj}^{(\epsilon)}(t) = f_{dj}(0) + \epsilon A_{dj}^{(1)}(t) + \epsilon^2 A_{dj}^{(2)}(t) + o(\epsilon^2) \tag{3.14}$$

$$f_{fj}^{(\epsilon)}(t) = f_{fj}(0) + \epsilon A_{fj}^{(1)}(t) + \epsilon^2 A_{fj}^{(2)}(t) + o(\epsilon^2) \tag{3.15}$$

where

$$\begin{aligned}
A_{dj}^{(1)}(t) &:= \frac{\partial f_{dj}^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} = f_{dj}(0)\int_0^t \gamma_{dj}^{'}(u)dW(u),\\
A_{dj}^{(2)}(t) &:= \frac{1}{2}\frac{\partial^2 f_{dj}^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0} = f_{dj}(0)\int_0^t \gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u)du + \int_0^t A_{dj}^{(1)}(u)\gamma_{dj}^{'}(u)dW_u\\
A_{fj}^{(1)}(t) &:= \frac{\partial f_{fj}^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} = f_{fj}(0)\int_0^t \gamma_{fj}^{'}(u)dW(u),\\
A_{fj}^{(2)}(t) &:= \frac{1}{2}\frac{\partial^2 f_{fj}^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0}\\
&= f_{fj}(0)\int_0^t \gamma_{fj}^{'}(u)\left\{\sum_{i\in \hat{J}_{j+1}} -\left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)}\right)\gamma_{fi}(u) + \sum_{i\in \hat{J}_{N+1}}\left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u)\right\}du\\
&\quad + \int_0^t A_{fj}^{(1)}(u)\gamma_{fj}^{'}(u)dW_u
\end{aligned}$$

(*Proof*)
Only (3.14) is shown. (3.15) is obtained similarly. Differentiating the equation (3.10) with respect to ϵ once and twice, we have:

$$\begin{aligned}
\frac{\partial f_{dj}^{(\epsilon)}(t)}{\partial \epsilon} &= 2\epsilon\int_0^t\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du\\
&+ \epsilon^2\int_0^t\frac{\partial}{\partial\epsilon}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du\\
&+ \int_0^t f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)dW_u + \epsilon\int_0^t\left\{\frac{\partial}{\partial\epsilon}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 f_{dj}^{(\epsilon)}(t)}{\partial \epsilon^2} &= 2\int_0^t\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du\\
&+ 4\epsilon\int_0^t\frac{\partial}{\partial\epsilon}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du\\
&+ \epsilon^2\int_0^t\frac{\partial^2}{\partial\epsilon^2}\left\{-f_{dj}^{(\epsilon)}(u)\gamma_{dj}^{'}(u)\sum_{i=j+1}^{N}\frac{\tau_i f_{di}^{(\epsilon)}(u)\gamma_{di}(u)}{1+\tau_i f_{di}^{(\epsilon)}(u)}\right\}du\\
&+ 2\int_0^t\left\{\frac{\partial}{\partial\epsilon}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u + \epsilon\int_0^t\left\{\frac{\partial^2}{\partial\epsilon^2}f_{dj}^{(\epsilon)}(u)\right\}\gamma_{dj}^{'}(u)dW_u.
\end{aligned}$$

Then, setting $\epsilon = 0$, we obtain $A_{dj}^{(1)}(t)$ and $A_{dj}^{(2)}(t)$. □

We next define the following variables:

$$\begin{cases}
\sigma_X(u) := \sum_{i\in\hat{J}_{N+1}} g_{fi}^{(0)}(u) - \sum_{i\in\hat{J}_{N+1}} g_{di}^{(0)}(u),\\
g_{fi}^{(0)}(u) := \left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)}\right)\gamma_{fi}(u),\ g_{di}^{(0)}(u) := \left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)}\right)\gamma_{di}(u)\\
g_{fi}^{(1)}(u) := \left(\frac{-\tau_i}{(1+\tau_i f_{fi}(0))^2}\right)\gamma_{fi}(u),\ g_{di}^{(1)}(u) := \left(\frac{-\tau_i}{(1+\tau_i f_{di}(0))^2}\right)\gamma_{di}(u)\\
g_{fi}^{(2)}(u) := \left(\frac{2\tau_i^2}{(1+\tau_i f_{fi}(0))^3}\right)\gamma_{fi}(u),\ g_{di}^{(2)}(u) := \left(\frac{2\tau_i^2}{(1+\tau_i f_{di}(0))^3}\right)\gamma_{di}(u)
\end{cases} \tag{3.16}$$

Then, the asymptotic expansion of the interest-rate part of the forex forward up to the third order of ϵ(ϵ^3-order) can be derived.

Proposition 2 *The asymptotic expansion of* $Y^{(\epsilon)}(t)$ *up to the third order is expressed as follows:*

$$Y^{(\epsilon)}(t) = \epsilon A_t^{(1)} + \epsilon^2 A_t^{(2)} + \epsilon^3 A_t^{(3)} + o(\epsilon^3) \tag{3.17}$$

where

$$A_t^{(1)} := \int_0^t \sigma_X(u)' dW_u, \tag{3.18}$$

$$A_t^{(2)} := -\frac{1}{2}\int_0^t ||\sigma_X(u)||^2 du + \int_0^t [\sum_{i\in\hat{J}_{N+1}} g_{fi}^{(1)}(u)A_{fi}^{(1)}(u) - \sum_{i\in\hat{J}_{N+1}} g_{di}^{(1)}(u)A_{di}^{(1)}(u)]' dW_u \tag{3.19}$$

and

$$\begin{aligned} A_t^{(3)} := & -\sum_{i,j\in\hat{J}_{N+1}} \int_0^t (g_{fi}^{(1)}(u))' A_{fi}^{(1)}(u) g_{fj}^{(0)}(u) du + \sum_{i,j\in\hat{J}_{N+1}} \int_0^t (g_{fi}^{(1)}(u))' A_{fi}^{(1)}(u) g_{dj}^{(0)}(u) du \\ & + \sum_{i,j\in\hat{J}_{N+1}} \int_0^t (g_{di}^{(1)}(u))' A_{di}^{(1)}(u) g_{fj}^{(0)}(u) du - \sum_{i,j\in\hat{J}_{N+1}} \int_0^t (g_{di}^{(1)}(u))' A_{di}^{(1)}(u) g_{dj}^{(0)}(u) du \\ & + \sum_{i\in\hat{J}_{N+1}} \int_0^t A_{fi}^{(2)}(u)(g_{fi}^{(1)}(u))' dW_u + \frac{1}{2}\sum_{i\in\hat{J}_{N+1}} \int_0^t (A_{fi}^{(1)}(u))^2 (g_{fi}^{(2)}(u))' dW_u \\ & - \sum_{i\in\hat{J}_{N+1}} \int_0^t A_{di}^{(2)}(u)(g_{di}^{(1)}(u))' dW_u - \frac{1}{2}\sum_{i\in\hat{J}_{N+1}} \int_0^t (A_{di}^{(1)}(u))^2 (g_{di}^{(2)}(u))' dW_u \end{aligned} \tag{3.20}$$

(*Proof*)
We first note that

$$Y^{(\epsilon)}(t) = \epsilon \frac{\partial Y^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0} + \frac{\epsilon^2}{2}\frac{\partial^2 Y^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0} + \frac{\epsilon^3}{6}\frac{\partial^3 Y^{(\epsilon)}(t)}{\partial \epsilon^3}|_{\epsilon=0} + o(\epsilon^3),$$

and set $A_t^{(1)} := \frac{\partial Y^{(\epsilon)}(t)}{\partial \epsilon}|_{\epsilon=0}$, $A_t^{(2)} := \frac{1}{2}\frac{\partial^2 Y^{(\epsilon)}(t)}{\partial \epsilon^2}|_{\epsilon=0}$ and $A_t^{(3)} := \frac{1}{6}\frac{\partial^3 Y^{(\epsilon)}(t)}{\partial \epsilon^3}|_{\epsilon=0}$. As for (3.18), differentiating the equation (3.13) with respect to ϵ once, we have:

$$\begin{aligned} \frac{\partial Y^{(\epsilon)}(t)}{\partial \epsilon} = & -\epsilon \int_0^t \left|\left| \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}^{(\epsilon)}(u)} \right|\right|^2 du \\ & -\frac{\epsilon^2}{2}\int_0^t \frac{\partial}{\partial\epsilon} \left|\left| \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}^{(\epsilon)}(u)} \right|\right|^2 du \\ & + \int_0^t \left[\sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}^{(\epsilon)}(u)} \right]' dW_u \\ & + \epsilon \int_0^t \frac{\partial}{\partial\epsilon} \left[\sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{fj}^{(\epsilon)}(u)\gamma_{fj}(u)}{1+\tau_j f_{fj}^{(\epsilon)}(u)} - \sum_{j\in\hat{J}_{N+1}} \frac{-\tau_j f_{dj}^{(\epsilon)}(u)\gamma_{dj}(u)}{1+\tau_i f_{dj}^{(\epsilon)}(u)} \right]' dW_u \end{aligned}$$

Then, setting $\epsilon = 0$, noting the definitions of $g_{fi}^{(0)}(u)$, $g_{di}^{(0)}(u)$ and $\sigma_X(u)$ in (3.16), we obtain the expression of $A_t^{(1)}$, that is (3.18).

Although tedious calculations are necessary, (3.19) and (3.20) can be obtained in the similar manner; we first differentiate the equation (3.13) with respect to ϵ twice and three times. Then, setting $\epsilon = 0$, and substituting the expressions of $A^{(1)}_{dj}(t)$, $A^{(2)}_{dj}(t)$, $A^{(1)}_{fj}(t)$ and $A^{(2)}_{fj}(t)$ given in Lemma 1, and noting the definitions of $g^{(1)}_{fi}(u)$, $g^{(1)}_{di}(u)$, $g^{(2)}_{fi}(u)$, $g^{(2)}_{di}(u)$ and $\sigma_X(u)$ in (3.16), we obtain the expressions of $A^{(2)}_t$ and $A^{(3)}_t$. $\square$

Next, we define a random variable $X^{(\epsilon)} := \frac{Y^{(\epsilon)}(T_{N+1})}{\epsilon}$. First, we note that $\Phi^{(\epsilon)}(u) := \Phi^{(\epsilon)}_{N+1}(T_{N+1}, u)$ which is necessary for pricing options is given by;

$$\Phi^{(\epsilon)}(u) = \Phi^{(\epsilon)}_Y(u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\} \tag{3.21}$$

$$= \Phi^{(\epsilon)}_X(\epsilon u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\} \tag{3.22}$$

where $\Phi^{(\epsilon)}_Y(u) := \Phi^{(\epsilon)}_Y(T_{N+1}, u)$, $\Phi_Z(u) := \Phi_Z(T_{N+1}, u)$ in (3.8), $\Phi_{\hat{J}}(u) := \Phi_{\hat{J}}(T_{N+1}, u)$ in (3.9) and $\Phi^{(\epsilon)}_X(u)$ denotes the characteristic function of $X^{(\epsilon)}$.

Second, we also note that $X^{(\epsilon)}$ is expanded up to the second order as follows:

$$X^{(\epsilon)} = g_1 + \epsilon g_2 + \epsilon^2 g_3 + o(\epsilon^2), \tag{3.23}$$

where $g_1 := A^{(1)}_{T_{N+1}}$, $g_2 := A^{(2)}_{T_{N+1}}$ and $g_3 := A^{(3)}_{T_{N+1}}$.

Finally, note that the first order term g_1 follows a normal distribution with mean 0 and variance Σ:

$$\Sigma := \int_0^{T_{N+1}} ||\sigma_X(u)||^2 du. \tag{3.24}$$

Using the following theorem, we will obtain an approximation of $\Phi^{(\epsilon)}(u)$, the characteristic function of $f^{(\epsilon)}_{N+1}(T_{N+1})$.

Theorem 1 *Under the assumption of $\Sigma > 0$, an asymptotic expansion of $\phi^{(\epsilon)}_X(x)$, the density function of $X^{(\epsilon)}$, is given by*

$$\phi^{(\epsilon)}_X(x) = \Bigg[1 + D^{(\epsilon)}_1\frac{x}{\Sigma} + D^{(\epsilon)}_2\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) + D^{(\epsilon)}_3\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) + D^{(\epsilon)}_4\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right)$$
$$+ D^{(\epsilon)}_5\left(\frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3}\right)\Bigg] \times \phi_{0,\Sigma}(x) + o(\epsilon^2) \tag{3.25}$$

where

$$\phi_{\mu,\Sigma}(x) := \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{(x-\mu)^2}{2\Sigma}}$$

and

$$\begin{cases} D^{(\epsilon)}_1 := \epsilon C_{2,1} \\ D^{(\epsilon)}_2 := \epsilon^2 C_{3,1} + \frac{1}{2}\epsilon^2 C_{4,1},\ D^{(\epsilon)}_3 := \epsilon C_{2,2} \\ D^{(\epsilon)}_4 := \epsilon^2 C_{3,2} + \frac{1}{2}\epsilon^2 C_{4,2},\ D^{(\epsilon)}_5 := \frac{1}{2}\epsilon^2 C_{4,3}. \end{cases} \tag{3.26}$$

All of $C_{2,1}, C_{2,2}, C_{3,1}, C_{3,2}, C_{4,1}, C_{4,2}$ and $C_{4,3}$ are constants and are defined in Appendix A.

(*Proof*)
Substituting $d = 1$, $\phi^{(\epsilon)}(x) \equiv 1$, and $B = (-\infty, x]$ in Theorem 3.4 of Kunitomo and Takahashi[2003], we can obtain an asymptotic expansion of the probability distribution function of $X^{(\epsilon)}$:

$$\begin{aligned} P\left(\{X^{(\epsilon)} \leq x\}\right) = & \int_{-\infty}^x \phi_{0,\Sigma}(z)dz \\ & + \epsilon\int_{-\infty}^x -\frac{\partial}{\partial z}\left\{\mathbf{E}[g_2|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz \\ & + \epsilon^2\int_{-\infty}^x -\frac{\partial}{\partial z}\left\{\mathbf{E}[g_3|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz \\ & + \frac{1}{2}\epsilon^2\int_{-\infty}^x \frac{\partial^2}{\partial z^2}\left\{\mathbf{E}[g_2^2|g_1 = z]\phi_{0,\Sigma}(z)\right\}dz + o(\epsilon^2). \end{aligned}$$

Then, differentiating both sides of the equation above with respect to x, we have:

$$\begin{aligned}\phi_X^{(\epsilon)}(x) &= \phi_{0,\Sigma}(x)\\ &-\epsilon\frac{\partial}{\partial x}\{\mathbf{E}[g_2|g_1=x]\phi_{0,\Sigma}(x)\}\\ &-\epsilon^2\frac{\partial}{\partial x}\{\mathbf{E}[g_3|g_1=x]\phi_{0,\Sigma}(x)\}\\ &+\frac{1}{2}\epsilon^2\frac{\partial^2}{\partial x^2}\left\{\mathbf{E}[g_2^2|g_1=x]\phi_{0,\Sigma}(x)\right\}+o(\epsilon^2).\end{aligned}$$

Finally, noting that $\mathbf{E}[g_2|g_1=x], \mathbf{E}[g_3|g_1=x]$, and $\mathbf{E}[g_2^2|g_1=x]$ are polynomials of x(see Appendix A for details.):

$$\mathbf{E}[g_2|g_1=x] = C_{2,1}+C_{2,2}(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma}) \tag{3.27}$$

$$\mathbf{E}[g_3|g_1=x] = C_{3,1}\frac{x}{\Sigma}+C_{3,2}(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}) \tag{3.28}$$

$$\mathbf{E}[g_2^2|g_1=x] = C_{4,1}+C_{4,2}(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma})+C_{4,3}(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}), \tag{3.29}$$

we obtain the result. □

With the density function in (3.25), the characteristic function of $X^{(\epsilon)}$ can be calculated.

Corollary 1 *Under the assumption of $\Sigma > 0$, an asymptotic expansion of $\Phi_X^{(\epsilon)}(u)$, the characteristic function of $X^{(\epsilon)}$, is given by*

$$\Phi_X^{(\epsilon)}(u) = \left[1+D_1^{(\epsilon)}iu+D_2^{(\epsilon)}(iu)^2+D_3^{(\epsilon)}(iu)^3+D_4^{(\epsilon)}(iu)^4+D_5^{(\epsilon)}(iu)^6\right]\times\Phi_{0,\Sigma}(u)+o(\epsilon^2) \tag{3.30}$$

where

$$\Phi_{\mu,\Sigma}(u) := e^{i\mu u-\frac{\Sigma}{2}u^2};$$

$D_1^{(\epsilon)}$, $D_2^{(\epsilon)}$, $D_3^{(\epsilon)}$, $D_4^{(\epsilon)}$ and $D_5^{(\epsilon)}$ are given in the equation (3.26).

Finally, we derive an approximation formula for valuation of the European call option written on $F_{N+1}^{(\epsilon)}(T_{N+1})$.

Let $V(0;T_{N+1},K)$ be a value of the option with maturity T_{N+1} and strike rate K at time 0.
First, note that

$$\frac{V(0;T_{N+1},K)}{P_d(0,T_{N+1})} = F_{N+1}(0)\ \mathbf{E}\left[e^{f^{(\epsilon)}}1_{\{f^{(\epsilon)}>k\}}\right]-K\ \mathbf{E}\left[1_{\{f^{(\epsilon)}>k\}}\right]$$

where the notation $f^{(\epsilon)} := f_{N+1}(T_{N+1})$ is used.
Then $Q(A) := \mathbf{E}\left[\frac{1}{\mathbf{E}[e^{f^{(\epsilon)}}]}e^{f^{(\epsilon)}}1_A\right]$ is defined as a probability measure on $(\Omega,\mathcal{F})$, and hence under the measure Q, we obtain;

$$\frac{V(0;T_{N+1},K)}{P_d(0,T_{N+1})} = F_{N+1}(0)\mathbf{E}[e^{f^{(\epsilon)}}]\ \mathbf{E}^Q\left[1_{\{f^{(\epsilon)}>k\}}\right]-K\ \mathbf{E}\left[1_{\{f^{(\epsilon)}>k\}}\right] \tag{3.31}$$

where $\mathbf{E}^Q[\cdot]$ denotes an expectation operator under Q.

Next, Let $\Phi^{(\epsilon)}(u) := \mathbf{E}\left[e^{iuf^{(\epsilon)}}\right]$ and $\hat{\Phi}_Q^{(\epsilon)}(u) := \mathbf{E}^Q\left[e^{iuf^{(\epsilon)}}\right]$. Note that

$$\begin{aligned}\Phi^{(\epsilon)}(u) &= \Phi_Y^{(\epsilon)}(u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\}\\ &= \Phi_X^{(\epsilon)}(\epsilon u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\}\end{aligned}$$

and that $\Phi_Q^{(\epsilon)}(u)$ can be expressed as;

$$\Phi_Q^{(\epsilon)}(u) = \mathbf{E}\left[\frac{1}{\mathbf{E}[e^{f^{(\epsilon)}}]}e^{i(u-i)f^{(\epsilon)}}\right] = \frac{1}{\mathbf{E}[e^{f^{(\epsilon)}}]}\Phi^{(\epsilon)}(u-i).$$

We easily notice that $\mathbf{E}[e^{f^{(\epsilon)}}]$ can be decomposed as $\mathbf{E}[e^{f^{(\epsilon)}}]$=$\mathbf{E}[e^{\epsilon X^{(\epsilon)}}]$ $\times\mathbf{E}[e^{Z(T_{N+1})}]$ $\times\mathbf{E}[e^{\hat{J}(T_{N+1})-\lambda\eta T_{N+1}}]$ under the independence assumption and $\mathbf{E}[e^{Z(T_{N+1})}]=\mathbf{E}[e^{\hat{J}(T_{N+1})-\lambda\eta T_{N+1}}]=1$. Moreover, by $\mathbf{E}[e^{\epsilon X^{(\epsilon)}}]=\Phi_X^{(\epsilon)}(-\epsilon i)$, we obtain

$$\Phi_Q^{(\epsilon)}(u)=\frac{\Phi^{(\epsilon)}(u-i)}{\Phi_X^{(\epsilon)}(-\epsilon i)}=\frac{\Phi_X^{(\epsilon)}(u-i)\Phi_Z(u-i)\Phi_{\hat{J}}(u-i)\exp\{-i(u-i)\lambda\eta T_{N+1}\}}{\Phi_X^{(\epsilon)}(-\epsilon i)}.$$

Then, the first and second terms of the right hand side of (3.31) can be evaluated through the Levy's inversion formula as:

$$\begin{aligned}\frac{V(0;T_{N+1},K)}{P_d(0,T_{N+1})} &= F_{N+1}(0)\mathbf{E}[e^{f^{(\epsilon)}}]\left\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\Phi_Q^{(\epsilon)}(u)}{iu}\}du\right\}\\ &\quad -K\left\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\Phi^{(\epsilon)}(u)}{iu}\}du\right\}\\ &= F_{N+1}(0)\Phi_X^{(\epsilon)}(-\epsilon i)\left\{\frac{1}{2}+\frac{1}{\pi}\frac{1}{\Phi_X^{(\epsilon)}(-\epsilon i)}\int_0^\infty Re\{\frac{e^{-iuk}\Phi^{(\epsilon)}(u-i)}{iu}\}du\right\}\\ &\quad -K\left\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\Phi^{(\epsilon)}(u)}{iu}\}du\right\}. \end{aligned} \tag{3.32}$$

However, we do not have the exact close-form expression $\Phi_X^{(\epsilon)}(u)$ while $\Phi_Z(u)$ and $\Phi_{\hat{J}}(u)$ are given analytically. Thus, we approximate it by $\hat{\Phi}_X^{(\epsilon)}(u)$ which is defined as:

$$\hat{\Phi}_X^{(\epsilon)}(u)=\left[1+D_1^{(\epsilon)}iu+D_2^{(\epsilon)}(iu)^2+D_3^{(\epsilon)}(iu)^3+D_4^{(\epsilon)}(iu)^4+D_5^{(\epsilon)}(iu)^6\right]\times\Phi_{0,\Sigma}(u).$$

Then, by substituting $\hat{\Phi}_X^{(\epsilon)}(u)$ for $\Phi_X^{(\epsilon)}(u)$ in the equation (3.32), we can provide $\hat{V}(0;T_{N+1},K)$, the approximation of the option value as follows:

$$\begin{aligned}\frac{\hat{V}(0;T_{N+1},K)}{P_d(0,T_{N+1})} &:= F_{N+1}(0)\left\{\frac{1}{2}\hat{\Phi}_X^{(\epsilon)}(-\epsilon i)+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\hat{\Phi}^{(\epsilon)}(u-i)}{iu}\}du\right\}\\ &\quad -K\left\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\hat{\Phi}^{(\epsilon)}(u)}{iu}\}du\right\}\end{aligned}$$

where $\hat{\Phi}^{(\epsilon)}(u):=\hat{\Phi}_X^{(\epsilon)}(\epsilon u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\}$.
This result is summarized as the following theorem.

Theorem 2 *Assume $\Sigma>0$. Let $\hat{V}(0;T_{N+1},K)$ be an approximated value of $V(0;T_{N+1},K)$ which denotes the exact value of the option with maturity T_{N+1} and strike rate K. Then, $\hat{V}(0;T_{N+1},K)$ is given by:*

$$\begin{aligned}\hat{V}(0;T_{N+1},K) &:= P_d(0,T_{N+1})\left[F_{N+1}(0)\left\{\frac{1}{2}\hat{\Phi}_X^{(\epsilon)}(-\epsilon i)+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\hat{\Phi}^{(\epsilon)}(u-i)}{iu}\}du\right\}\right.\\ &\quad \left. -K\left\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\{\frac{e^{-iuk}\hat{\Phi}^{(\epsilon)}(u)}{iu}\}du\right\}\right].\end{aligned} \tag{3.33}$$

Here, $k:=ln(\frac{K}{F_{N+1}(0)})$ and $\hat{\Phi}^{(\epsilon)}(u):=\hat{\Phi}_X^{(\epsilon)}(\epsilon u)\Phi_Z(u)\Phi_{\hat{J}}(u)\exp\{-iu\lambda\eta T_{N+1}\}$; $\Phi_Z(u)=\Phi_Z(T_{N+1},u)$ is given by (3.8), $\Phi_{\hat{J}}(u)=\Phi_{\hat{J}}(T_{N+1},u)$ is given by (3.9), and $\hat{\Phi}_X^{(\epsilon)}(u)$ is defined as follows;

$$\hat{\Phi}_X^{(\epsilon)}(u)=\left[1+D_1^{(\epsilon)}iu+D_2^{(\epsilon)}(iu)^2+D_3^{(\epsilon)}(iu)^3+D_4^{(\epsilon)}(iu)^4+D_5^{(\epsilon)}(iu)^6\right]\times\Phi_{0,\Sigma}(u).$$

4 Numerical Examples

In this section, we examine the effectiveness of our method through some numerical examples. The approximate option prices by our method are compared with their estimates by Monte Carlo simulations.

First of all, the processes of domestic and foreign forward interest rates and of a volatility of the spot exchange rate are specified. We suppose $D=4$, that is the dimension of a Brownian motion is set to be four; it represents the

Table 1: Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.2	0.05	0.2
case (ii)	0.02	0.5	0.05	0.2
case (iii)	0.05	0.2	0.02	0.5

uncertainty of domestic and foreign interest rates, the spot exchange rate, and its variance. We note that correlations between domestic and foreign interest rates and between the spot exchange rate and its variance are allowed. For simplicity we also suppose $\lambda = 0$ which implies no jump part.

The parameters in our model are set as follows:
For the process of the stochastic variance, we set $V(0) = \theta = 0.015$ and $\kappa = 0.5$ in the equation (2.7); ω is set to be zero (i.e. the variance is set to be constant) in former examples below(indicated by "C.V."), or $\omega = 0.1$ in the latter(indicated by "S.V."). $\bar{v}$ is a four dimensional vector given below.
We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities have flat structures and are constant over time: that is, for all $j, f_{dj}(0) = f_d,\ f_{fj}(0) = f_f,\ \ \gamma_{dj}(t) = \gamma_d^* \bar{\gamma}_d 1_{[0,T_j)}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \bar{\gamma}_f 1_{[0,T_j)}(t)$. Here, γ_d^* and γ_f^* are constant scalars, and $\bar{\gamma}_d$ and $\bar{\gamma}_f$ denote four dimensional constant vectors. We consider three different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 1. Moreover, given correlation parameters $\tilde{\rho}$ and ρ which denote the correlations between domestic and foreign interest rates and between the spot exchange rate and its variance respectively, the constant vectors $\bar{\gamma}_d$, $\bar{\gamma}_f$, $\bar{\sigma}$ and $\bar{v}$ can be determined to satisfy $||\bar{\gamma}_d|| = ||\bar{\gamma}_f|| = ||\bar{\sigma}|| = ||\bar{v}|| = 1$, $\bar{\gamma}_d' \bar{\gamma}_f = \tilde{\rho}$, $\bar{\sigma}' \bar{v} = \rho$ and the independence assumption.

Finally, we make another assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

This section shows numerical examples for call option prices calculated with Monte Carlo simulations, with our approximation formulas of the second and third orders and in addition with the approximation formula introduced by Takahashi, Takehara and Yamazaki[2006](TTY[2006]), with different maturities of five and ten years in different cases("C.V." and "S.V.") for the variance of the spot exchange rate; each estimate based on the Monte Carlo simulation is obtained by 1,000,000 trials with *antithetic variables method*. As for the correlations, we suppose $\tilde{\rho} = 0.5$ and $\rho = -0.5$. Moreover, these consist of results under three different scenarios (i)-(iii) in Table 1 for term structures of interest rates. We set $S(0) = 100$, and $K = F_{N+1}(0) \times 0.5, \cdots, F_{N+1}(0) \times 1.5$.

Tables 2-13 and Figures 1-12 show the differences of the second/third order approximations and of those in TTY[2006] against the estimates by Monte Carlo in the cases "C.V." (listed in Tables 2-4 and Figures 1-3 with a maturity of five year and in Tables 8-10 and Figures 7-9 with ten year, respectively) or "S.V." (listed in Tables 5-7 and Figures 4-6 with a maturity of five year and in Tables 11-13 and Figures 10-12 with ten year, respectively): "difference"("diff.") and "relative difference"("rel.diff.") are defined by (the approximate value)-(the estimate by Monte Carlo) and (difference)/(the estimate by Monte Carlo)$\times$ 100(%), respectively. Colored cells in these tables indicate which is the closest to the estimates by Monte Carlo simulations at that moneyness in three approximations done by the asymptotic expansion up to the second and third orders and by the method introduced in TTY[2006].

To begin with, we note that the figures of the differences between the estimates by Monte Carlo simulations and our approximations in the "C.V." case generally look quite similar to those in the "S.V." case except for only those in the case (i) of five year, while the stochastic structure of the variance of the spot exchange rate in each case differs and so do the option prices. This may seem natural since we make use of the exact characteristic function for the volatility part in our procedure. Thus, it comes to substantial in pricing these options how the characteristic function of the interest-rate part is approximated. This aspect implies importance of our method.

Furthermore, there are the following two features in our numerical examples.

First, through almost all experiments, the third order terms improve the approximation by the second order, even with a maturity of five year where every approximation seems to work well. Compared with the largest differences in the second order of -0.0447(case(iii),ATM), -0.0569(case(iii),ATM), -0.4176(case(iii),20%ITM) and -0.4430(case(iii),20%ITM) in the cases with "C.V.", 5y, with "S.V.", 5y, "C.V.", 10y and with "S.V.", 10y, respectively, in the third order they are -0.0358(case(ii),50%OTM), -0.0280(case(ii),50%OTM), -0.3659(case(ii),50%OTM) and -0.3287(case(ii),50%OTM) respectively. In most of these experiments, the same result holds for the approximation by TTY[2006]; their largest differences in the same cases above are -0.0427(case(iii),ATM), -0.0568(case(iii),ATM), -0.4455(case(ii),50%OTM) and -0.4651(case(ii),50%OTM) respectively, which are the worst in these three approximations.

Second, in comparison with other two approximations, the approximation by the third order seems to work significantly well especially around ATM. In contrast to the performances of others at ATM, the largest differences of the third order approximation in "C.V.", 5y, with "S.V.", 5y, "C.V.", 10y and with "S.V.", 10y, are only 0.0155(case(i)), -0.0096(case (iii)), -0.0994(case (iii)) and -0.0949(case (iii)) respectively. We can find this feature with a glance at figures. For a practical purpose, This may be an advantage of our method since the liquidity of options is in general the highest at/around ATM.

5 Conclusion

This paper proposed approximation formulas based on a Fourier transform method with an asymptotic expansion to evaluate currency options with a libor market model of domestic and foreign interest rates and jump-diffusion stochastic volatility processes of spot exchange rates by expanding the interest-rate part. Then, the distribution of the component of the underlying asset dependent on the interest rates are not approximated around a normal distribution such as in Takahashi and Takehara[2006] but around a log-normal one.

We also provided numerical examples to investigate the accuracy of the approximations for option prices with maturities of five and ten year; in general, satisfactory results were obtained for the approximation up to the third order.

Finally, we state our research plans as follows: we may use higher order asymptotic expansions and will also utilize asymptotic expansion formulas for extended models where a stochastic volatility structure of interest rates or a more general stochastic structure of the volatility of the spot exchange rate are allowed. Especially, to any model where the analytical characteristic function of the volatility part is known, the same procedure in this article can be applied under the independence assumption.

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6 Appendix

A Coefficients in the Asymptotic Expansion

This section presents the expressions of coefficients $C_{2,1}$, $C_{2,2}$, $C_{3,1}$, $C_{3,2}$, $C_{4,1}$, $C_{4,2}$ and $C_{4,3}$ in Theorem 3.1. First, we show them as a relatively compact form:

$$\begin{cases} C_{2,1} := \bar{C} \\ C_{2,2} := \sum_{i \in \hat{J}_{N+1}} (a^f_{2i} - a^d_{2i}) \\ C_{3,1} := -\sum_{i,j \in \hat{J}_{N+1}} \left\{ (a^f_{3i,j} + a^d_{3i,j}) - (b_{3i,j} + c_{3i,j}) \right\} + \sum_{i \in \hat{J}_{N+1}} \left\{ (d^f_{3i} + f^f_{3i}) - (d^d_{3i} + f^d_{3i}) \right\} \\ C_{3,2} := \sum_{i \in \hat{J}_{N+1}} \left\{ (e^f_{3i} + g^f_{3i}) - (e^d_{3i} + g^d_{3i}) \right\} \\ C_{4,1} := a_4 + \sum_{i,j \in \hat{J}_{N+1}} (e^f_{4i,j} + e^d_{4i,j} - h_{4i,j}) \\ C_{4,2} := b_4 + \sum_{i,j \in \hat{J}_{N+1}} (d^f_{4i,j} + d^d_{4i,j} - g_{4i,j}) \\ C_{4,3} := \sum_{i,j \in \hat{J}_{N+1}} (c^f_{4i,j} + c^d_{4i,j} - f_{4i,j}) \end{cases} \tag{A.1}$$

Subsections A.1 and A.2 below provide the expressions for the terms on the right hand side of (A.1) in detail. For the derivation of the coefficients with supersprict 'd', since they are obtained by a similar calculation in this section, it is omitted and will be given upon request.

Here, it is stressed that all coefficients are expressed by the form of only nine functionals defined in Appendix B, and that this seems to make it easy for us to implement our method.

A.1 *The second order*

In this subsection, we concentrate on the second order scheme. First, we note that g_1 and g_2 are expressed as

$$\begin{aligned} g_1 &= A^{(1)}_{T_{N+1}} = \int_0^{T_{N+1}} \sigma_X(u)' dW_u, \\ g_2 &= A^{(2)}_{T_{N+1}} \\ &= -\frac{1}{2}\int_0^{T_{N+1}} ||\sigma_X(u)||^2 du + \sum_{i \in \hat{J}_{N+1}} \left(\int_0^{T_{N+1}} g^{(1)}_{fi}(u) A^{(1)}_{fi}(u)' dW_u - \int_0^{T_{N+1}} g^{(1)}_{di}(u) A^{(1)}_{di}(u)' dW_u \right). \end{aligned}$$

Let $T \equiv T_{N+1}$, $F(0) \equiv F_{N+1}(0)$.

Then,

$$\mathbf{E}[g_2|g_1 = x] = \bar{C} + \mathbf{E}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} A^{(1)}_{fi}(u)(g^{(1)}_{fi}(u))' dW_u | g_1 = x\right] - \mathbf{E}\left[\int_0^T \sum_{i \in \hat{J}_{N+1}} A^{(1)}_{di}(u)(g^{(1)}_{di}(u))' dW_u | g_1 = x\right]$$

where $\bar{C}$ is a constant and defined by $\bar{C} := -\frac{1}{2}\int_0^T ||\sigma_X(u)||^2 du = -\frac{1}{2}\Sigma$. To evaluate the right hand side of the equation above, we utilize a formula associated with conditional expectations of Gaussianity: The formulas used in this and the following subsections are listed in Appendix B. In particular, applying (2) in Appendix B, we can evaluate each term in $\mathbf{E}[g_2|g_1 = x]$ as follows:

$$\begin{aligned} \mathbf{E}\left[\int_0^T A^{(1)}_{fi}(u)(g^{(1)}_{fi}(u))' dW_u | g_1 = x\right] &= \left(\frac{-\tau_i f_{fi}(0)}{(1 + \tau_i f_{fi}(0))^2}\right) I^2_2(\gamma_{fi}, \gamma_{fi}; T) \times \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &=: a^f_{2i}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \end{aligned}$$

Then, $C_{2,1}$ and $C_{2,2}$ are defined by

$$\begin{aligned} C_{2,1} &= \bar{C} \\ C_{2,2} &= \sum_{i\in \hat{J}_{N+1}} (a^f_{2i} - a^d_{2i}) \end{aligned}$$

A.2 *The third order*

A.2.1 *Computation of* $\mathbf{E}[g_3|g_1 = x]$

We first note that

$$\begin{aligned} g_3 \; = \; A^{(3)}_T \;\; = \;\; & \sum_{i,j\in \hat{J}_{N+1}} \Bigg(-\int_0^T (g^{(1)}_{fi}(u))' A^{(1)}_{fi}(u) g^{(0)}_{fj}(u)du + \int_0^T (g^{(1)}_{fi}(u))' A^{(1)}_{fi}(u) g^{(0)}_{dj}(u)du \\ & \qquad + \int_0^T (g^{(1)}_{di}(u))' A^{(1)}_{di}(u) g^{(0)}_{fj}(u)du - \int_0^T (g^{(1)}_{di}(u))' A^{(1)}_{di}(u) g^{(0)}_{dj}(u)du \Bigg) \\ & + \sum_{i\in \hat{J}_{N+1}} \left(\int_0^T A^{(2)}_{fi}(u)(g^{(1)}_{fi}(u))' dW_u + \frac{1}{2}\int_0^T (A^{(1)}_{fi}(u))^2 (g^{(2)}_{fi}(u))' dW_u \right) \\ & - \sum_{i\in \hat{J}_{N+1}} \left(\int_0^T A^{(2)}_{di}(u)(g^{(1)}_{di}(u))' dW_u + \frac{1}{2}\int_0^T (A^{(1)}_{di}(u))^2 (g^{(2)}_{di}(u))' dW_u \right). \end{aligned}$$

Define $C^{(2)}_{dj}(u)$ and $C^{(2)}_{fj}(u)$ as

$$\begin{aligned} C^{(2)}_{dj}(u) &:= \int_0^u \gamma'_{dj}(s) \sum_{i=j+1}^{N} \left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)} \right) \gamma_{di}(s) ds, \\ C^{(2)}_{fj}(u) &:= \int_0^u \gamma'_{fj}(s) \left\{ \sum_{i\in \hat{J}_{j+1}} - \left(\frac{-\tau_i f_{fi}(0)}{1+\tau_i f_{fi}(0)} \right) \gamma_{fi}(s) + \sum_{i\in \hat{J}_{N+1}} \left(\frac{-\tau_i f_{di}(0)}{1+\tau_i f_{di}(0)} \right) \gamma_{di}(s) \right\} ds. \end{aligned}$$

Then, we take the expectation of each term of g_3 conditional to $g_1 = x$. To evaluate each expectation, we use formulas in Appendix B, again. we report results below;

1. Apply *formula 1.*

$$\begin{aligned} & \mathbf{E}\left[\int_0^T (g^{(1)}_{fi}(u))' A^{(1)}_{fi}(u) g^{(0)}_{fj}(u)du | g_1 = x \right] \\ = & \left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2} \right) \left(\frac{-\tau_j f_{fj}(0)}{1+\tau_j f_{fj}(0)} \right) \hat{I}^1_1(\gamma_{fi}, \gamma_{fi}, \gamma_{fj}; T) \times \frac{x}{\Sigma} \\ := & \; a^f_{3i,j} \frac{x}{\Sigma} \end{aligned}$$

2. Apply *formula 1.*

$$\begin{aligned} & \mathbf{E}\left[\int_0^T (g^{(1)}_{fi}(u))' A^{(1)}_{fi}(u) g^{(0)}_{dj}(u)du | g_1 = x \right] \\ = & \left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2} \right) \left(\frac{-\tau_j f_{dj}(0)}{1+\tau_j f_{dj}(0)} \right) \hat{I}^1_1(\gamma_{fi}, \gamma_{fi}, \gamma_{dj}; T) \times \frac{x}{\Sigma} \\ := & \; b_{3i,j} \frac{x}{\Sigma} \end{aligned}$$

3. Apply *formula 1.*

$$\begin{aligned} & \mathbf{E}\left[\int_0^T (g^{(1)}_{di}(u))' A^{(1)}_{di}(u) g^{(0)}_{fj}(u)du | g_1 = x \right] \\ = & \left(\frac{-\tau_i f_{di}(0)}{(1+\tau_i f_{di}(0))^2} \right) \left(\frac{-\tau_j f_{fj}(0)}{1+\tau_j f_{fj}(0)} \right) \hat{I}^1_1(\gamma_{di}, \gamma_{di}, \gamma_{fj}; T) \times \frac{x}{\Sigma} \\ := & \; c_{3i,j} \frac{x}{\Sigma} \end{aligned}$$

4. Apply *formulas 1,3.*

$$\begin{aligned}
&\mathbf{E}\left[\int_0^T A_{fi}^{(2)}(u)(g_{fi}^{(1)}(u))^{'}dW_u|g_1=x\right]\\
=&\left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2}\right)\left\{I_1^1(C_{fi}^{(2)}\times\gamma_{fi};T)\times\frac{x}{\Sigma}+I_3^3(\gamma_{fi},\gamma_{fi},\gamma_{fi};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right\}\\
=:&\ d_{3i}^f\frac{x}{\Sigma}+e_{3i}^f\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}$$

5. Apply *formula 4.*

$$\begin{aligned}
&\frac{1}{2}\times\mathbf{E}\left[\int_0^T (A_{fi}^{(1)}(u))^2(g_{fi}^{(2)}(u))^{'}dW_u|g_1=x\right]\\
=&\left(\frac{\tau_i^2 f_{fi}(0)^2}{(1+\tau_i f_{fi}(0))^3}\right)\left[I_1^4(\gamma_{fi},\gamma_{fi},\gamma_{fi};T)\times\frac{x}{\Sigma}+I_3^4(\gamma_{fi},\gamma_{fi},\gamma_{fi};T)\times\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)\right]\\
=:&\ f_{3i}^f\frac{x}{\Sigma}+g_{3i}^f\left(\frac{x^3}{\Sigma^3}-\frac{3x}{\Sigma^2}\right)
\end{aligned}$$

Finally, coefficients of $C_{3,1}$ and $C_{3,2}$ can be defined as follows;

$$\begin{aligned}
C_{3,1} &= -\sum_{i,j\in\hat{J}_{N+1}}\left\{(a_{3i,j}^f+a_{3i,j}^d)-(b_{3i,j}+c_{3i,j})\right\}+\sum_{i\in\hat{J}_{N+1}}\left\{(d_{3i}^f+f_{3i}^f)-(d_{3i}^d+f_{3i}^d)\right\}\\
C_{3,2} &= \sum_{i\in\hat{J}_{N+1}}\left\{(e_{3i}^f+g_{3i}^f)-(e_{3i}^d+g_{3i}^d)\right\}
\end{aligned}$$

A.2.2 *Computation of* $\mathbf{E}[g_2^2|g_1=x]$

We first note that g_2^2 is expressed as

$$\begin{aligned}
g_2^2 &= \left[\bar{C}+\sum_{i\in\hat{J}_{N+1}}\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)^{'}dW_u-\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)^{'}dW_u\right)\right]^2\\
&= \bar{C}^2+2\bar{C}\times\sum_{i\in\hat{J}_{N+1}}\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)^{'}dW_u-\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)^{'}dW_u\right)\\
&\quad+\left[\sum_{i\in\hat{J}_{N+1}}\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)^{'}dW_u-\int_0^T g_{di}^{(1)}(u)A_{di}^{(1)}(u)^{'}dW_u\right)\right]^2.
\end{aligned}$$

Next, we easily notice that $\mathbf{E}[g_2^2|g_1=x]$ consists of the following terms.

1.

$$\begin{aligned}
&\mathbf{E}\left[\bar{C}^2+2\bar{C}\int_0^T[\sum_{i\in\hat{J}_{N+1}}g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)-\sum_{i\in\hat{J}_{N+1}}g_{di}^{(1)}(u)A_{di}^{(1)}(u)]^{'}dW_u|g_1=x\right]\\
=&\ \bar{C}^2+2\bar{C}\times C_{2,2}(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma})\\
:=&\ a_4+b_4(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma})
\end{aligned}$$

2. Apply *formula 5.*

$$
\begin{aligned}
& \mathbf{E}\left[\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)dW_u\right)\left(\int_0^T g_{fj}^{(1)}(u)A_{fj}^{(1)}(u)dW_u\right)|g_1=x\right] \\
= & \left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2}\right)\left(\frac{-\tau_j f_{fj}(0)}{(1+\tau_j f_{fj}(0))^2}\right)\times \\
& \left[I_4^5(\gamma_{fi},\gamma_{fi},\gamma_{fj},\gamma_{fj};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)\right. \\
& \left.\quad +I_2^5(\gamma_{fi},\gamma_{fi},\gamma_{fj},\gamma_{fj};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^5(\gamma_{fi},\gamma_{fi},\gamma_{fj},\gamma_{fj};T)\right] \\
=: & \; c_{4i,j}^f\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+d_{4i,j}^f\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+e_{4i,j}^f
\end{aligned}
$$

3. Apply *formula 5.*

$$
\begin{aligned}
& 2\times\mathbf{E}\left[\left(\int_0^T g_{fi}^{(1)}(u)A_{fi}^{(1)}(u)dW_u\right)\left(\int_0^T g_{dj}^{(1)}(u)A_{dj}^{(1)}(u)dW_u\right)|g_1=x\right] \\
= & \; 2\left(\frac{-\tau_i f_{fi}(0)}{(1+\tau_i f_{fi}(0))^2}\right)\left(\frac{-\tau_j f_{dj}(0)}{(1+\tau_j f_{dj}(0))^2}\right)\times \\
& \left[I_4^5(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)\right. \\
& \left.\quad +I_2^5(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\times\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+I_0^5(\gamma_{fi},\gamma_{fi},\gamma_{dj},\gamma_{dj};T)\right] \\
=: & \; f_{4i,j}\left(\frac{x^4}{\Sigma^4}-\frac{6x^2}{\Sigma^3}+\frac{3}{\Sigma^2}\right)+g_{4i,j}\left(\frac{x^2}{\Sigma^2}-\frac{1}{\Sigma^2}\right)+h_{4i,j}
\end{aligned}
$$

Consequently, $C_{4,1}, C_{4,2}$ and $C_{4,3}$ are defined as;

$$
\begin{aligned}
C_{4,1} &= a_4+\sum_{i,j\in\hat{J}_{N+1}}(e_{4i,j}^f+e_{4i,j}^d-h_{4i,j}) \\
C_{4,2} &= b_4+\sum_{i,j\in\hat{J}_{N+1}}(d_{4i,j}^f+d_{4i,j}^d-g_{4i,j}) \\
C_{4,3} &= \sum_{i,j\in\hat{J}_{N+1}}(c_{4i,j}^f+c_{4i,j}^d-f_{4i,j}).
\end{aligned}
$$

B Formulas

In this section, the formulas 1.- 5. and definitions of functionals $\{I_k^l(\cdots;T)\}$ used in the previous sections are listed up for convenience. They are derived by direct calculations using Gaussianity of the processes involved, which are straightforward, but lengthy and hence omitted. $W=\{(W_t^1,\cdots,W_t^d):0\le t\}$ denotes a d-dimensional Brownian motion. Let $q_i:[0,T]\mapsto\mathbf{R}^d, i=1,2,3,4,5$ be non-random functions and define Σ as

$$
\Sigma=\int_0^T q_{1t}^{'}q_{1t}dt,
$$

where $z^{'}$ is the transpose of z. Suppose $q_{1t}=\sigma_X(t)$ so that we abbreviate 'q_1' in $\{I_k^i(\cdots;T)\}$, and assume that $0<\Sigma<\infty$ and integrability in the following formulas.

1.

$$
\begin{aligned}
\mathbf{E}\left[\int_0^T q_{2t}^{'}dW_t|\int_0^T q_{1v}^{'}dW_v=x\right] &= \left(\int_0^T q_{2t}^{'}q_{1t}dt\right)\frac{x}{\Sigma} \\
&=: I_1^1(q_2;T)\frac{x}{\Sigma}
\end{aligned}
$$

2.

$$\begin{aligned}\mathbf{E}\left[\int_0^T \int_0^t q_{2u}^{'} dW_u q_{3t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] &= \left(\int_0^T \int_0^t q_{2u}^{'} q_{1u} du q_{3t}^{'} q_{1t} dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &=: I_2^2(q_2, q_3; T)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\end{aligned}$$

3.

$$\begin{aligned}\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2u}^{'} dW_u q_{3s}^{'} dW_s q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] &= \left(\int_0^T q_{4t}^{'} q_{1t} \int_0^t q_{3s}^{'} q_{1s} \int_0^s q_{2u}^{'} q_{1u} du ds dt\right)\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) \\ &=: I_3^3(q_2, q_3, q_4; T)\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)\end{aligned}$$

4.

$$\begin{aligned}&\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2u}^{'} dW_u\right)\left(\int_0^t q_{3s}^{'} dW_s\right) q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] \\ &= \left\{\int_0^T \left(\int_0^t q_{2u}^{'} q_{1u} du\right)\left(\int_0^t q_{3s}^{'} q_{1s} ds\right) q_{4t}^{'} q_{1t} dt\right\}\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) \\ &\quad + \left(\int_0^T \int_0^t q_{2u}^{'} q_{3u} du q_{4t}^{'} q_{1t} dt\right)\frac{x}{\Sigma} \\ &=: I_3^4(q_2, q_3, q_4; T)\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) + I_1^4(q_2, q_3, q_4; T)\left(\frac{x}{\Sigma}\right)\end{aligned}$$

5.

$$\begin{aligned}&\mathbf{E}\left[\left(\int_0^T \int_0^t q_{2s}^{'} dW_s q_{3t}^{'} dW_t\right)\left(\int_0^T \int_0^r q_{4u}^{'} dW_u q_{5r}^{'} dW_r\right) | \int_0^T q_{1v}^{'} dW_v = x\right] \\ &= \left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\left(\int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{4u}^{'} q_{1u} du dr\right)\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) \\ &\quad + \left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{5r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &\quad + \left(\int_0^T q_{5t}^{'} q_{1t} \int_0^t q_{3r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &\quad + \left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2r}^{'} q_{5r} \int_0^r q_{4u}^{'} q_{1u} du dr dt\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &\quad + \left\{\int_0^T q_{3t}^{'} q_{5t} \left(\int_0^t q_{2s}^{'} q_{1s} ds\right)\left(\int_0^t q_{4u}^{'} q_{1u} du\right) dt\right\}\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &\quad + \left(\int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{3u}^{'} q_{4u} \int_0^u q_{2s}^{'} q_{1s} ds du dr\right)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) \\ &\quad + \int_0^T \int_0^t q_{2u}^{'} q_{4u} du q_{3t}^{'} q_{5t} dt \\ &=: I_4^5(q_2, q_3.q_4, q_5; T)\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}\right) + I_2^5(q_2, q_3, q_4, q_5; T)\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) + I_0^5(q_2, q_3, q_4, q_5; T)\end{aligned}$$

Finally, we define

6.

$$\hat{I}_1^1(q_2, q_3, q_4; T) := \int_0^T q_{3t}^{'} q_{4t} \int_0^t q_{2u}^{'} q_{1u} du dt.$$

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	39.0934	31.4731	24.3071	17.9655	12.7332	8.6896	5.7388	3.6927	2.3288	1.4471	0.8893
TTY[2006]	39.0933	31.4740	24.3109	17.9749	12.7425	8.6973	5.7459	3.6947	2.3240	1.4364	0.8758
diff.	−0.0001	0.0009	0.0038	0.0094	0.0093	0.0077	0.0071	0.0020	−0.0048	−0.0107	−0.0135
rel.diff.	0.00%	0.00%	0.02%	0.05%	0.07%	0.09%	0.12%	0.05%	−0.21%	−0.74%	−1.52%
A.E.(2nd)	39.0932	31.4740	24.3108	17.9748	12.7423	8.6971	5.7458	3.6947	2.3239	1.4364	0.8758
diff.	−0.0002	0.0009	0.0037	0.0093	0.0091	0.0075	0.0070	0.0020	−0.0049	−0.0107	−0.0135
rel.diff.	0.00%	0.00%	0.02%	0.05%	0.07%	0.09%	0.12%	0.05%	−0.21%	−0.74%	−1.52%
A.E.(3rd)	39.0936	31.4755	24.3147	17.9810	12.7501	8.7051	5.7527	3.6994	2.3264	1.4365	0.8737
diff.	0.0002	0.0024	0.0076	0.0155	0.0169	0.0155	0.0139	0.0067	−0.0024	−0.0106	−0.0156
rel.diff.	0.00%	0.01%	0.03%	0.09%	0.13%	0.18%	0.24%	0.18%	−0.10%	−0.73%	−1.75%

Table 2: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(i), C.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	39.0943	31.4737	24.3136	17.9824	12.7600	8.7219	5.7770	3.7281	2.3595	1.4709	0.9072
TTY[2006]	39.0933	31.4744	24.3119	17.9768	12.7449	8.7001	5.7487	3.6972	2.3260	1.4381	0.8770
diff.	−0.0010	0.0007	−0.0017	−0.0056	−0.0151	−0.0218	−0.0283	−0.0309	−0.0335	−0.0328	−0.0302
rel.diff.	0.00%	0.00%	−0.01%	−0.03%	−0.12%	−0.25%	−0.49%	−0.83%	−1.42%	−2.23%	−3.33%
A.E.(2nd)	39.0930	31.4733	24.3102	17.9752	12.7447	8.7016	5.7518	3.7013	2.3303	1.4419	0.8803
diff.	−0.0013	−0.0004	−0.0034	−0.0072	−0.0153	−0.0203	−0.0252	−0.0268	−0.0292	−0.0290	−0.0269
rel.diff.	0.00%	0.00%	−0.01%	−0.04%	−0.12%	−0.23%	−0.44%	−0.72%	−1.24%	−1.97%	−2.97%
A.E.(3rd)	39.0946	31.4801	24.3262	18.0010	12.7770	8.7347	5.7803	3.7212	2.3403	1.4420	0.8714
diff.	0.0003	0.0064	0.0126	0.0186	0.0170	0.0128	0.0033	−0.0069	−0.0192	−0.0289	−0.0358
rel.diff.	0.00%	0.02%	0.05%	0.10%	0.13%	0.15%	0.06%	−0.19%	−0.81%	−1.96%	−3.95%

Table 3: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(ii), C.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	45.3077	36.4864	28.1980	20.8684	14.8114	10.1247	6.7025	4.3174	2.7200	1.6832	1.0278
TTY[2006]	45.3031	36.4739	28.1737	20.8323	14.7694	10.0820	6.6619	4.2845	2.6955	1.6665	1.0164
diff.	−0.0046	−0.0125	−0.0243	−0.0361	−0.0420	−0.0427	−0.0406	−0.0329	−0.0245	−0.0167	−0.0114
rel.diff.	−0.01%	−0.03%	−0.09%	−0.17%	−0.28%	−0.42%	−0.61%	−0.76%	−0.90%	−0.99%	−1.11%
A.E.(2nd)	45.3035	36.4752	28.1756	20.8338	14.7694	10.0800	6.6580	4.2797	2.6905	1.6620	1.0127
diff.	−0.0042	−0.0112	−0.0224	−0.0346	−0.0420	−0.0447	−0.0445	−0.0377	−0.0295	−0.0212	−0.0151
rel.diff.	−0.01%	−0.03%	−0.08%	−0.17%	−0.28%	−0.44%	−0.66%	−0.87%	−1.08%	−1.26%	−1.47%
A.E.(3rd)	45.3055	36.4837	28.1957	20.8663	14.8101	10.1216	6.6939	4.3048	2.7031	1.6619	1.0015
diff.	−0.0022	−0.0027	−0.0023	−0.0021	−0.0013	−0.0031	−0.0086	−0.0126	−0.0169	−0.0213	−0.0263
rel.diff.	0.00%	−0.01%	−0.01%	−0.01%	−0.01%	−0.03%	−0.13%	−0.29%	−0.62%	−1.27%	−2.56%

Table 4: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(iii), C.V.**

Figure 1:Differences in the **case(i), C.V., 5y**

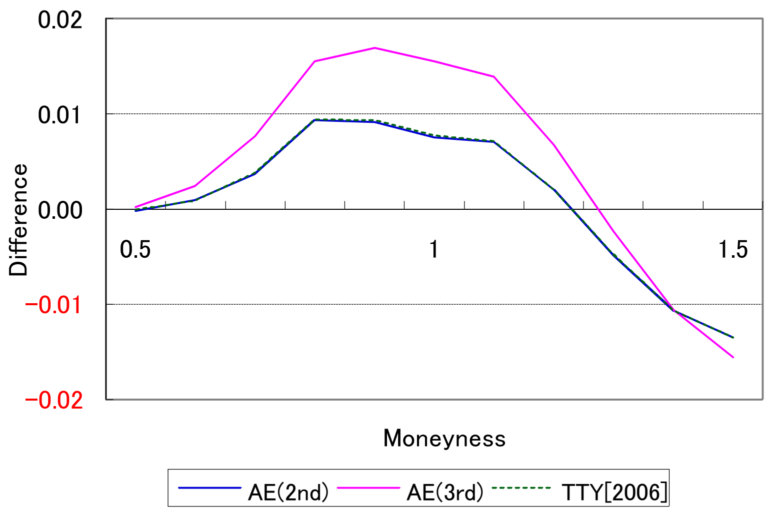


Figure 2:Differences in the **case(ii), C.V., 5y**

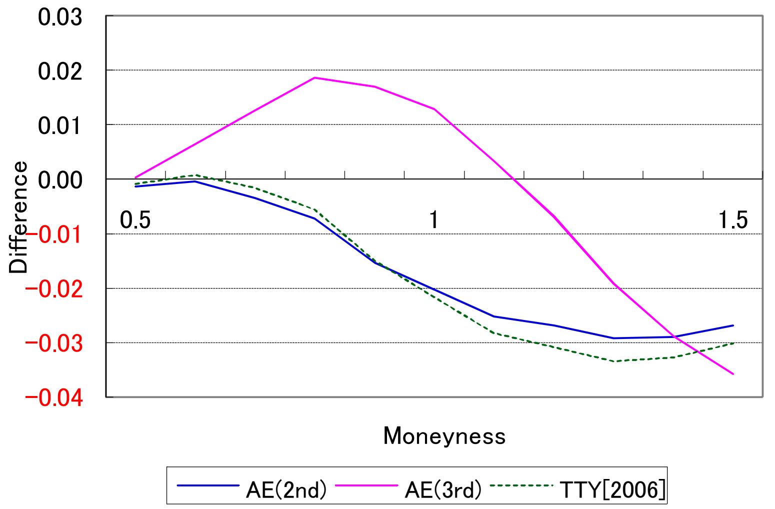
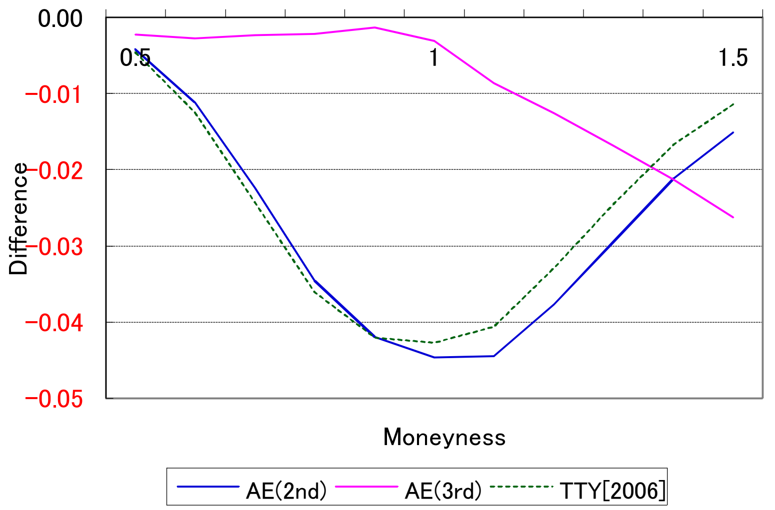


Figure 3:Differences in the **case(iii), C.V., 5y**



Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	39.2223	31.7420	24.6629	18.2340	12.7197	8.3257	5.1064	2.9485	1.6199	0.8579	0.4461
TTY[2006]	39.2221	31.7404	24.6588	18.2285	12.7147	8.3178	5.0953	2.9352	1.6058	0.8456	0.4350
diff.	−0.0002	−0.0016	−0.0041	−0.0055	−0.0050	−0.0079	−0.0111	−0.0133	−0.0141	−0.0123	−0.0111
rel.diff.	0.00%	−0.01%	−0.02%	−0.03%	−0.04%	−0.09%	−0.22%	−0.45%	−0.87%	−1.43%	−2.49%
A.E.(2nd)	39.2220	31.7404	24.6588	18.2284	12.7146	8.3177	5.0952	2.9351	1.6057	0.8456	0.4349
diff.	−0.0003	−0.0016	−0.0041	−0.0056	−0.0051	−0.0080	−0.0112	−0.0134	−0.0142	−0.0123	−0.0112
rel.diff.	0.00%	−0.01%	−0.02%	−0.03%	−0.04%	−0.10%	−0.22%	−0.45%	−0.88%	−1.43%	−2.51%
A.E.(3rd)	39.2225	31.7419	24.6618	18.2337	12.7222	8.3267	5.1038	2.9418	1.6095	0.8462	0.4326
diff.	0.0002	−0.0001	−0.0011	−0.0003	0.0025	0.0010	−0.0026	−0.0067	−0.0104	−0.0117	−0.0135
rel.diff.	0.00%	0.00%	0.00%	0.00%	0.02%	0.01%	−0.05%	−0.23%	−0.64%	−1.36%	−3.03%

Table 5: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(i), S.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	39.2246	31.7447	24.6653	18.2398	12.7361	8.3510	5.1370	2.9801	1.6482	0.8799	0.4581
TTY[2006]	39.2222	31.7408	24.6596	18.2300	12.7171	8.3209	5.0986	2.9382	1.6082	0.8473	0.4361
diff.	−0.0024	−0.0039	−0.0057	−0.0098	−0.0190	−0.0301	−0.0384	−0.0419	−0.0400	−0.0326	−0.0220
rel.diff.	−0.01%	−0.01%	−0.02%	−0.05%	−0.15%	−0.36%	−0.75%	−1.41%	−2.43%	−3.70%	−4.80%
A.E.(2nd)	39.2240	31.7401	24.6583	18.2281	12.7153	8.3207	5.1012	2.9434	1.6145	0.8531	0.4404
diff.	−0.0006	−0.0046	−0.0070	−0.0117	−0.0208	−0.0303	−0.0358	−0.0367	−0.0337	−0.0268	−0.0177
rel.diff.	0.00%	−0.01%	−0.03%	−0.06%	−0.16%	−0.36%	−0.70%	−1.23%	−2.04%	−3.05%	−3.86%
A.E.(3rd)	39.2240	31.7459	24.6706	18.2499	12.7469	8.3583	5.1376	2.9717	1.6301	0.8551	0.4301
diff.	−0.0006	0.0012	0.0053	0.0101	0.0108	0.0073	0.0006	−0.0084	−0.0181	−0.0248	−0.0280
rel.diff.	0.00%	0.00%	0.02%	0.06%	0.08%	0.09%	0.01%	−0.28%	−1.10%	−2.82%	−6.11%

Table 6: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(ii), S.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	45.4559	36.7930	28.6001	21.1644	14.7889	9.6994	5.9602	3.4417	1.8888	0.9984	0.5144
TTY[2006]	45.4524	36.7826	28.5767	21.1257	14.7371	9.6426	5.9085	3.4049	1.8636	0.9819	0.5054
diff.	−0.0035	−0.0104	−0.0234	−0.0387	−0.0518	−0.0568	−0.0517	−0.0368	−0.0252	−0.0165	−0.0090
rel.diff.	−0.01%	−0.03%	−0.08%	−0.18%	−0.35%	−0.59%	−0.87%	−1.07%	−1.33%	−1.65%	−1.75%
A.E.(2nd)	45.4527	36.7834	28.5781	21.1277	14.7389	9.6425	5.9052	3.3988	1.8563	0.9753	0.5005
diff.	−0.0032	−0.0096	−0.0220	−0.0367	−0.0500	−0.0569	−0.0550	−0.0429	−0.0325	−0.0231	−0.0139
rel.diff.	−0.01%	−0.03%	−0.08%	−0.17%	−0.34%	−0.59%	−0.92%	−1.25%	−1.72%	−2.31%	−2.70%
A.E.(3rd)	45.4553	36.7906	28.5935	21.1551	14.7786	9.6898	5.9512	3.4344	1.8759	0.9776	0.4875
diff.	−0.0006	−0.0024	−0.0066	−0.0093	−0.0103	−0.0096	−0.0090	−0.0073	−0.0129	−0.0208	−0.0269
rel.diff.	0.00%	−0.01%	−0.02%	−0.04%	−0.07%	−0.10%	−0.15%	−0.21%	−0.68%	−2.08%	−5.23%

Table 7: Our approximations for options with maturity five year and their estimates by Monte Carlo Simulations **in the case(iii), S.V.**

Figure 4:Differences in the **case(i), S.V., 5y**

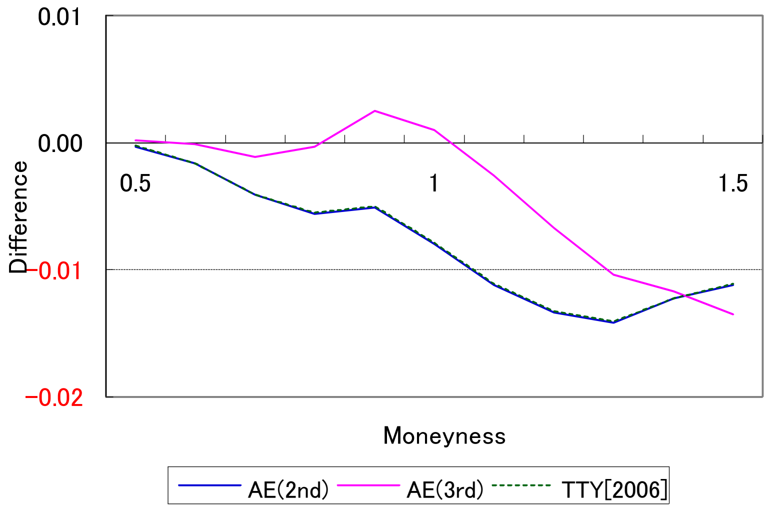


Figure 5:Differences in the **case(ii), S.V., 5y**

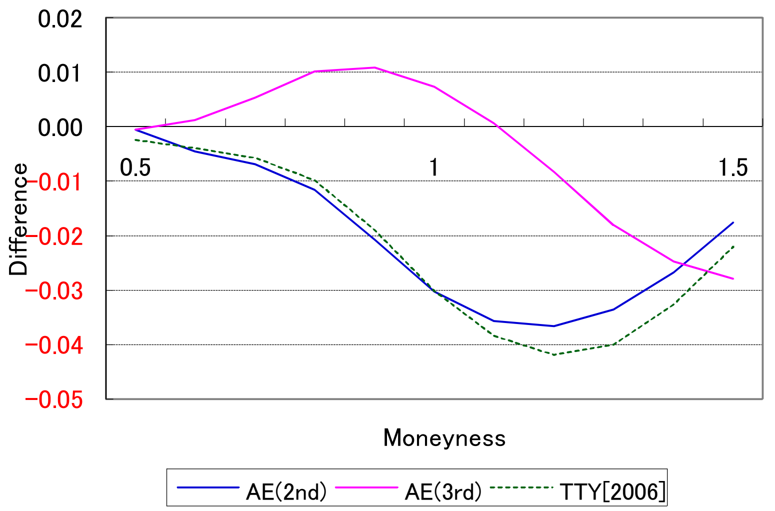
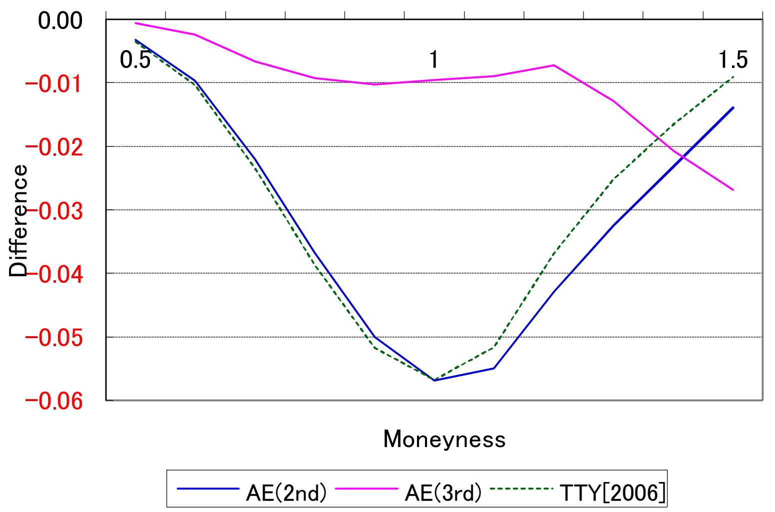


Figure 6:Differences in the **case(iii), S.V., 5y**



Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	30.9214	25.5431	20.7434	16.6063	13.1440	10.3135	8.0420	6.2410	4.8291	3.7301	2.8791
TTY[2006]	30.8968	25.5038	20.6883	16.5398	13.0693	10.2350	7.9619	6.1640	4.7567	3.6633	2.8180
diff.	−0.0246	−0.0393	−0.0551	−0.0665	−0.0747	−0.0785	−0.0801	−0.0770	−0.0724	−0.0668	−0.0611
rel.diff.	−0.08%	−0.15%	−0.27%	−0.40%	−0.57%	−0.76%	−1.00%	−1.23%	−1.50%	−1.79%	−2.12%
A.E.(2nd)	30.8960	25.5021	20.6857	16.5365	13.0657	10.2315	7.9588	6.1615	4.7550	3.6625	2.8180
diff.	−0.0254	−0.0410	−0.0577	−0.0698	−0.0783	−0.0820	−0.0832	−0.0795	−0.0741	−0.0676	−0.0611
rel.diff.	−0.08%	−0.16%	−0.28%	−0.42%	−0.60%	−0.80%	−1.03%	−1.27%	−1.53%	−1.81%	−2.12%
A.E.(3rd)	30.9144	25.5317	20.7246	16.5808	13.1131	10.2781	8.0019	6.1987	4.7841	3.6820	2.8273
diff.	−0.0070	−0.0114	−0.0188	−0.0255	−0.0309	−0.0354	−0.0401	−0.0423	−0.0450	−0.0481	−0.0518
rel.diff.	−0.02%	−0.04%	−0.09%	−0.15%	−0.24%	−0.34%	−0.50%	−0.68%	−0.93%	−1.29%	−1.80%

Table 8: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations **in the case(i), C.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	30.9365	25.5789	20.8081	16.7109	13.2956	10.5138	8.2886	6.5336	5.1613	4.0972	3.2731
TTY[2006]	30.8991	25.5085	20.6956	16.5494	13.0807	10.2474	7.9745	6.1764	4.7683	3.6740	2.8276
diff.	−0.0374	−0.0704	−0.1125	−0.1615	−0.2149	−0.2664	−0.3141	−0.3572	−0.3930	−0.4232	−0.4455
rel.diff.	−0.12%	−0.28%	−0.54%	−0.97%	−1.62%	−2.53%	−3.79%	−5.47%	−7.61%	−10.33%	−13.61%
A.E.(2nd)	30.8861	25.4910	20.6804	16.5429	13.0881	10.2698	8.0105	6.2228	4.8215	3.7303	2.8844
diff.	−0.0504	−0.0879	−0.1277	−0.1680	−0.2075	−0.2440	−0.2781	−0.3108	−0.3398	−0.3669	−0.3887
rel.diff.	−0.16%	−0.34%	−0.61%	−1.01%	−1.56%	−2.32%	−3.36%	−4.76%	−6.58%	−8.95%	−11.88%
A.E.(3rd)	30.9586	25.6099	20.8380	16.7255	13.2805	10.4574	8.1809	6.3659	4.9293	3.7974	2.9072
diff.	0.0221	0.0310	0.0299	0.0146	−0.0151	−0.0564	−0.1077	−0.1677	−0.2320	−0.2998	−0.3659
rel.diff.	0.07%	0.12%	0.14%	0.09%	−0.11%	−0.54%	−1.30%	−2.57%	−4.49%	−7.32%	−11.18%

Table 9: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations in the case(ii), C.V.

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	41.8053	34.6344	28.2061	22.6409	17.9642	14.1288	11.0424	8.5945	6.6712	5.1712	4.0076
TTY[2006]	41.4950	34.2558	27.7925	22.2245	17.5663	13.7614	10.7092	8.2944	6.4035	4.9339	3.7972
diff.	−0.3103	−0.3786	−0.4136	−0.4164	−0.3979	−0.3674	−0.3332	−0.3001	−0.2677	−0.2373	−0.2104
rel.diff.	−0.74%	−1.09%	−1.47%	−1.84%	−2.21%	−2.60%	−3.02%	−3.49%	−4.01%	−4.59%	−5.25%
A.E.(2nd)	41.5101	34.2753	27.8058	22.2233	17.5468	13.7222	10.6531	8.2261	6.3284	4.8571	3.7235
diff.	−0.2952	−0.3591	−0.4003	−0.4176	−0.4174	−0.4066	−0.3893	−0.3684	−0.3428	−0.3141	−0.2841
rel.diff.	−0.71%	−1.04%	−1.42%	−1.84%	−2.32%	−2.88%	−3.53%	−4.29%	−5.14%	−6.07%	−7.09%
A.E.(3rd)	41.6234	34.4631	28.0597	22.5205	17.8613	14.0294	10.9312	8.4578	6.5007	4.9610	3.7540
diff.	−0.1819	−0.1713	−0.1464	−0.1204	−0.1029	−0.0994	−0.1112	−0.1367	−0.1705	−0.2102	−0.2536
rel.diff.	−0.44%	−0.49%	−0.52%	−0.53%	−0.57%	−0.70%	−1.01%	−1.59%	−2.56%	−4.06%	−6.33%

Table 10: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations in the case(iii), C.V.

Figure 7:Differences in the **case(i), C.V., 10y**

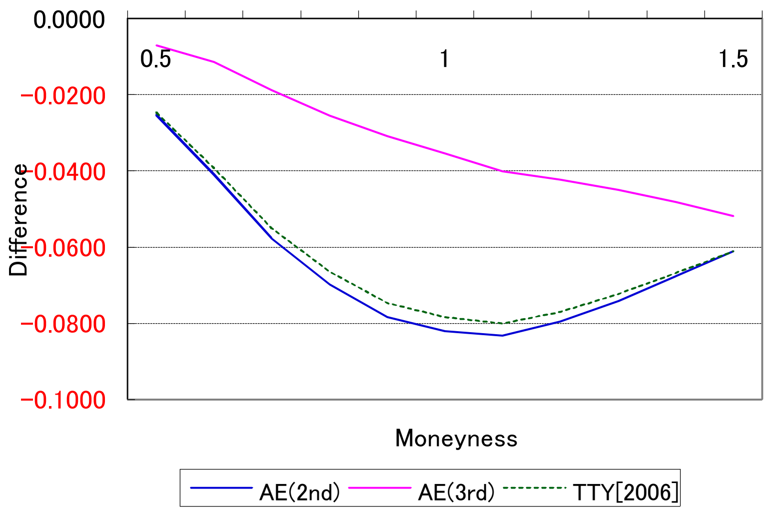


Figure 8:Differences in the **case(ii), C.V., 10y**

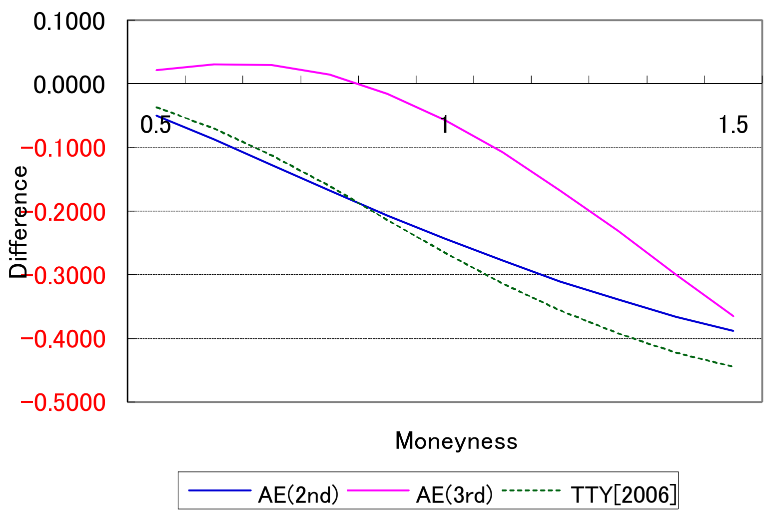
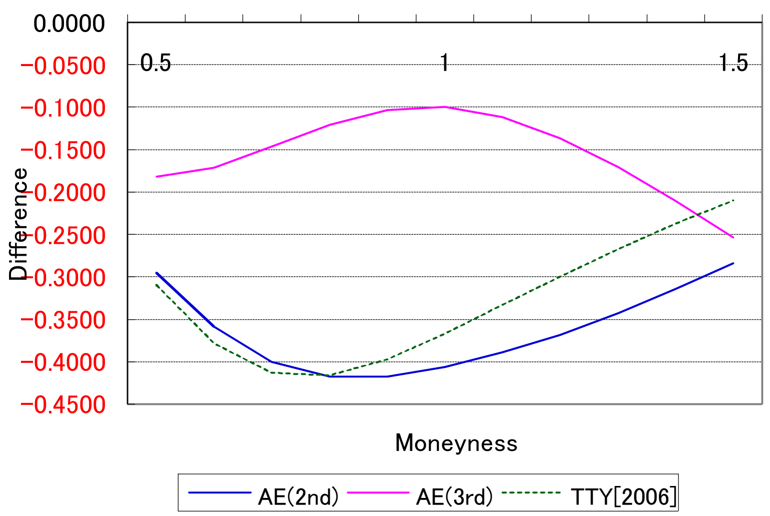


Figure 9:Differences in the **case(iii), C.V., 10y**



Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	31.1372	25.7907	20.9299	16.6437	12.9837	9.9555	7.5175	5.6026	4.1336	3.0277	2.2065
TTY[2006]	31.1163	25.7563	20.8795	16.5827	12.9168	9.8839	7.4446	5.5314	4.0633	2.9575	2.1371
diff.	−0.0209	−0.0344	−0.0504	−0.0610	−0.0669	−0.0716	−0.0729	−0.0712	−0.0703	−0.0702	−0.0694
rel.diff.	−0.07%	−0.13%	−0.24%	−0.37%	−0.52%	−0.72%	−0.97%	−1.27%	−1.70%	−2.32%	−3.14%
A.E.(2nd)	31.1156	25.7549	20.8773	16.5797	12.9132	9.8800	7.4408	5.5282	4.0610	2.9561	2.1369
diff.	−0.0216	−0.0358	−0.0526	−0.0640	−0.0705	−0.0755	−0.0767	−0.0744	−0.0726	−0.0716	−0.0696
rel.diff.	−0.07%	−0.14%	−0.25%	−0.38%	−0.54%	−0.76%	−1.02%	−1.33%	−1.76%	−2.36%	−3.15%
A.E.(3rd)	31.1312	25.7816	20.9153	16.6256	12.9624	9.9286	7.4860	5.5682	4.0947	2.9823	2.1542
diff.	−0.0060	−0.0091	−0.0146	−0.0181	−0.0213	−0.0269	−0.0315	−0.0344	−0.0389	−0.0454	−0.0523
rel.diff.	−0.02%	−0.04%	−0.07%	−0.11%	−0.16%	−0.27%	−0.42%	−0.61%	−0.94%	−1.50%	−2.37%

Table 11: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations **in the case(i), S.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	31.1565	25.8286	20.9966	16.7468	13.1320	10.1523	7.7630	5.8951	4.4698	3.4024	2.6131
TTY[2006]	31.1183	25.7601	20.8858	16.5917	12.9282	9.8971	7.4587	5.5456	4.0769	2.9698	2.1480
diff.	−0.0382	−0.0685	−0.1108	−0.1551	−0.2038	−0.2552	−0.3043	−0.3495	−0.3929	−0.4326	−0.4651
rel.diff.	−0.12%	−0.27%	−0.53%	−0.93%	−1.55%	−2.51%	−3.92%	−5.93%	−8.79%	−12.71%	−17.80%
A.E.(2nd)	31.1089	25.7451	20.8675	16.5757	12.9217	9.9063	7.4867	5.5919	4.1378	3.0400	2.2215
diff.	−0.0476	−0.0835	−0.1291	−0.1711	−0.2103	−0.2460	−0.2763	−0.3032	−0.3320	−0.3624	−0.3916
rel.diff.	−0.15%	−0.32%	−0.61%	−1.02%	−1.60%	−2.42%	−3.56%	−5.14%	−7.43%	−10.65%	−14.99%
A.E.(3rd)	31.1682	25.8457	21.0160	16.7583	13.1217	10.1067	7.6744	5.7578	4.2751	3.1428	2.2844
diff.	0.0117	0.0171	0.0194	0.0115	−0.0103	−0.0456	−0.0886	−0.1373	−0.1947	−0.2596	−0.3287
rel.diff.	0.04%	0.07%	0.09%	0.07%	−0.08%	−0.45%	−1.14%	−2.33%	−4.36%	−7.63%	−12.58%

Table 12: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations **in the case(ii), S.V.**

Moneyness(K/F)	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Monte Carlo	42.0912	34.9785	28.4891	22.7375	17.8014	13.6935	10.3726	7.7525	5.7414	4.2195	3.0853
TTY[2006]	41.7893	34.5938	28.0480	22.2813	17.3615	13.2910	10.0164	7.4473	5.4749	3.9882	2.8846
diff.	−0.3019	−0.3847	−0.4411	−0.4562	−0.4399	−0.4025	−0.3562	−0.3052	−0.2665	−0.2313	−0.2007
rel.diff.	−0.72%	−1.10%	−1.55%	−2.01%	−2.47%	−2.94%	−3.43%	−3.94%	−4.64%	−5.48%	−6.50%
A.E.(2nd)	41.8000	34.6103	28.0667	22.2945	17.3603	13.2680	9.9685	7.3763	5.3866	3.8904	2.7852
diff.	−0.2912	−0.3682	−0.4224	−0.4430	−0.4411	−0.4255	−0.4041	−0.3762	−0.3548	−0.3291	−0.3001
rel.diff.	−0.69%	−1.05%	−1.48%	−1.95%	−2.48%	−3.11%	−3.90%	−4.85%	−6.18%	−7.80%	−9.73%
A.E.(3rd)	41.8928	34.7739	28.3035	22.5895	17.6870	13.5986	10.2796	7.6504	5.6105	4.0535	2.8787
diff.	−0.1984	−0.2046	−0.1856	−0.1480	−0.1144	−0.0949	−0.0930	−0.1021	−0.1309	−0.1660	−0.2066
rel.diff.	−0.47%	−0.58%	−0.65%	−0.65%	−0.64%	−0.69%	−0.90%	−1.32%	−2.28%	−3.93%	−6.70%

Table 13: Our approximations for options with maturity ten year and their estimates by Monte Carlo Simulations **in the case(iii), S.V.**

Figure 10:Differences in the **case(i), S.V., 10y**

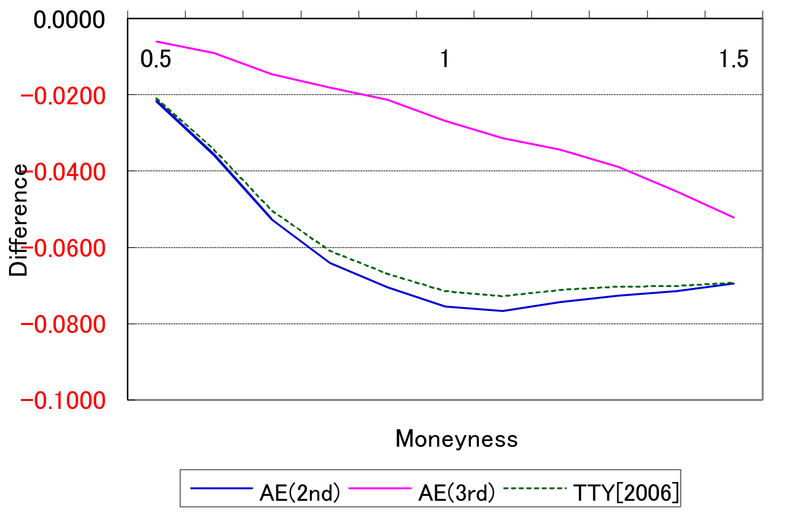


Figure 11:Differences in the **case(ii), S.V., 10y**

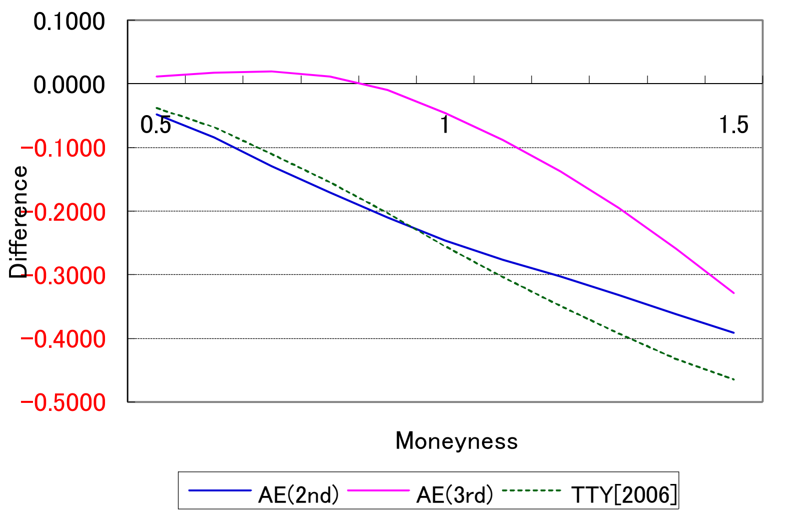


Figure 12:Differences in the **case(iii), S.V., 10y**