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Voluntarily Separable Prisoner's Dilemma with Reference Letters

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Abstract: We consider voluntarily separable repeated Prisoner's Dilemma in which a pair of players meet randomly and repeatedly play Prisoner's Dilemma only by mutual agreement. Fujiwara-Greve and Okuno-Fujiwara (2007) consider the case that once a partnership is dissolved there is no information flow to other partnerships. We consider the case that players can issue a reference letter to the partner if they entered cooperation periods, but the content of a letter is not verifiable. We show that the sheer existence of a letter shortens the trust-building periods of new matches and thus improves efficiency in equilibrium.

Key words: voluntary separation, Prisoner's Dilemma, cooperation, information, random matching.
JLE classification: C 73.

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1 Introduction

Modern economic relationships often take the form that strangers meet randomly to transact or collaborate, and they can continue the relationship by mutual agreement. Internet transactions are prominent examples of this sort. Ordinary repeated-game framework does not apply to such situations because it assumes that players repeat the same game for certain periods without an option to terminate. Fujiwara-Greve and Okuno-Fujiwara (2007) (Greve-Okuno henceforth) formulated a new framework of voluntarily separable repeated game in which one can unilaterally end a partnership and repetition is by mutual agreement only. Focusing on Prisoner's Dilemma as the stage game, they investigated evolutionary stability of strategy distributions.

In Greve-Okuno (2007), it was assumed that actions within a partnership is observable only to the partners, and after a random matching with a new partner they start from a null history. Related literature (Datta, 1996, Ghosh and Ray, 1996, and Kranton, 1996) has the same assumption. Under this no-information-flow assumption, the incentive to exploit the partner and escape is the greatest, hence it is most difficult to sustain cooperation. The literature above are thus mainly concerned with the existence of a cooperative equilibrium and the sufficient conditions for it. Greve-Okuno (2007) showed that there are evolutionary stable distributions consisting of "trust building strategies": for certain periods at the beginning of a new match, players do not cooperate but continue the partnership (these periods are called "trust building periods"), and after that the players cooperate and keep the partnership if and only if the partner also cooperated (these periods are called "cooperation periods"). Moreover, trust building periods are necessary for any symmetric (monomorphic) equilibrium.

In this paper we weaken the no-information-flow assumption and postulate that players can issue a "reference letter" for the partner to bring to the random matching pool. We assume that a reference letter signals only that it exists and not the details of the actions within a partnership.[1] This partial information can still be useful to

[1] This can be justified for example when one can verify a signature but not the actions within a partnership.

infer the reason of a partnership dissolution and thus shorten the trust building periods in a future match. Therefore, an additional information improves efficiency. Such devices are utilized in many parts of the society, and the fundamental idea of this paper should be useful in economic information policies.

2 Model

Consider a large society of a continuum of players with measure 1. The time is discrete. At the end of each period, each player exits from the society for an exogenous reason (which we call a "death") with probability $1-\delta$ $(0<\delta<1)$. If a player dies, a new player enters the society, keeping the population size constant. In each period, players without a partner (including the newly born players) enter the random matching pool and form pairs to play the *Voluntarily Separable Repeated Prisoner's Dilemma with Reference Letters* as follows.

In the matching pool, each player is either with a reference letter (status Y) or without a reference letter (status N). Assume that a newly born player has no reference letter.[2] Randomly matched players observe whether the new partner holds a reference letter or not, and thus they can base their future actions on this observation.

Randomly matched players first play the Prisoner's Dilemma (see Table 1) by choosing action C or D simultaneously. The actions in the Prisoner's Dilemma are observable only by the partners. After that, based on the observation, each player chooses whether to keep the partnership (action k) or end it (action e) simultaneously. If at least one player chooses e, the partnership ends and both players (if they survive) go to the random matching pool in the next period. If both chose action k, unless one of them dies, they play the Prisoner's Dilemma together again in the next period, skipping the matching process. If the partner dies, the surviving player goes to the random matching pool in the next period.

The existence of a reference letter is important when one enters the random

[2] See concluding remark section 5 for a brief discussion of the case when newly born players have reference letters.

P1 \ P2	C	D
C	c, c	ℓ, g
D	g, ℓ	d, d

Table 1: Prisoner's Dilemma

matching pool. We assume that players choose whether to issue a reference letter to the partner at the time of continuation decision.[3] That is, each player chooses whether to keep the partnership or not and also whether to issue a reference letter (action y) or not (action n) simultaneously. This implies that one decides on the letter issuance based only on the observation of the actions in the Prisoner's Dilemma, without knowing if the partner wants to keep the partnership and/or issue a letter for you. A sequential decision model in which players choose the letter action after the continuation decision can be analyzed similarly but complicates the exposition, hence we do not consider it in this paper. Assume that the reference letter is valid only one period so that the letter status of a player is solely dependent on the most recent choice of the partner.

The payoff in a period is determined only by the action profile in the Prisoner's Dilemma, as in Table 1. Assume that $g > c > d > \ell$ and $2c \geqq g + \ell$. The latter is for simplicity and to make the symmetric action profile (C, C) efficient.

The game continues with probability δ from an individual player's point of view. Thus we focus on the expected total payoff, with δ being the effective discount factor of a player.

We assume that the actions within a partnership are observable perfectly between the partners but not observable by any other player. Therefore newly matched players have no information about the past actions of each other except the letter status. It is then natural to consider the following strategies, which use only the partnership history and the letter status.

Let $t = 1, 2, \dots$ denote the periods in the current partnership. For each t, define

[3]Okuno-Fujiwara et al. (2007) consider a model in which reference letters are generated according to an exogenous rule, based on actions in the Prisoner's Dilemma.

$H_t := \{Y, N\}^2 \times [\{C, D\}^2 \times \{y, n\}^2]^{(t-1)}$ as the set of partnership histories at the beginning of the t-th period of a partnership. If it is a new match ($t = 1$), then only the status of the reference letters is the partnership history.

Definition 1. *A pure strategy s specifies the following $(x_t, y_t, z_t)_{t=1}^{\infty}$.*

For each $t = 1, 2, \ldots,$

(a) $x_t : H_t \to \{C, D\}$ specifies an action in the Prisoner's Dilemma;

(b) $y_t : H_t \times \{C, D\}^2 \to \{k, e\}$ specifies a continuation decision based on the partnership history and the action profile in the current period;

(c) $z_t : H_t \times \{C, D\}^2 \to \{y, n\}$ specifies whether to issue a reference letter to the current partner based on the partnership history and the action profile in the current period.

Since each partnership starts with only the letter status, a player can use the same pure strategy in every match.[4] Let $\boldsymbol{S}$ be the set of pure strategies and $\mathcal{P}(\boldsymbol{S})$ be the set of all probability distributions over $\boldsymbol{S}$. For simplicity, we assume that each player uses only a pure strategy.

Greve-Okuno (2007) showed that if a symmetric strategy distribution is a Nash equilibrium, then it must be a strategy that plays D (but keeps the partnership if (D, D) is observed in the current period) for some initial periods of a partnership, which is called a *trust building* strategy. We also focus on this class of strategies, but since we have additional letter information, we extend the strategy as follows.

Definition 2. *For any $T : \{Y, N\}^2 \to \{0, 1, 2, \ldots\}$,* **letter-based T-trust building strategy** *(written as c_T) specifies the trust building periods based on the letter status combination $q \in \{Y, N\}^2$ at the time of random matching and the T function as follows:*

[4]Kandori (1992) and Ellison (1994) consider strategies that utilize one's own past history in random matching games. It is possible to include such strategies in our analysis, but since the set of players is a continuum, there is no Nash equilibrium that can start the "contagion of defection" to sustain cooperation.

(i) If $t \leqq T(q)$, then play D and choose (k, n) for any observation of the action profile in the Prisoner's Dilemma in the current period;[5]

(ii) If $t > T(q)$, then play C and choose (k, y) if and only if (C, C) is observed in the current period.

3 Payoff Structure and Nash Equilibrium

We consider stability of stationary strategy distributions in the matching pool. Although the strategy distribution in the matching pool may be different from the distribution in the entire society, if the former is stationary, the distribution of various states of matches is also stationary, thanks to the stationary death process. (See Greve-Okuno, 2007.)

When a strategy $s \in \boldsymbol{S}$ is matched with another strategy $s' \in \boldsymbol{S}$, the *expected length* of the match is denoted as $L(s, s')$ and is computed as follows. Notice that even if s and s' intend to maintain the match, it will only continue with probability δ^2. Suppose that the planned length of the partnership of s and s' is $T(s, s')$ periods, if no death occurs. Then

$$L(s, s') := 1 + \delta^2 + \delta^4 + \cdots + \delta^{2\{T(s,s')-1\}} = \frac{1 - \delta^{2T(s,s')}}{1 - \delta^2}.$$

The *expected total discounted value of the payoff stream of s within the match with s'* is denoted as $V(s, s')$. The *average per period payoff* that s expects to receive within the match with s' is denoted as $v(s, s')$. Clearly,

$$v(s, s') := \frac{V(s, s')}{L(s, s')}, \text{ or } \; V(s, s') = L(s, s')v(s, s').$$

Next we show the structure of the lifetime and average payoff of a player endowed with strategy $s \in \boldsymbol{S}$ in the matching pool, waiting to be matched randomly with a partner. When a strategy distribution in the matching pool is $p \in \mathcal{P}(\boldsymbol{S})$ and is

[5]If players issue a reference letter during the trust building periods, a trust building strategy does not become a best reply to itself, because a strategy that chooses D for one period, receive a reference letter, and then enter the matching pool to exploit the next partner earns a higher total payoff.

stationary, we write the *expected total discounted value of payoff streams* s expects to receive during his lifetime as $V(s;p)$ and the average per period payoff s expects to receive during his lifetime as

$$v(s;p) := \frac{V(s;p)}{L} = (1-\delta)V(s;p),$$

where $L = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta}$ is the expected lifetime of s.

Thanks to the stationary distribution in the matching pool, we can write $V(s;p)$ as a recursive equation. If p has a finite/countable support, then we can write

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') \\ &\quad + [\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\} + \delta^{2\{T(s,s')-1\}}\delta] V(s;p) \Big], \quad (1) \end{aligned}$$

where *supp*(p) is the support of the distribution p, the sum $\delta(1-\delta)\{1+\delta^2+\cdots+\delta^{2\{T(s,s')-2\}}\}$ is the probability that s loses the partner s' before $T(s,s')$, and $\delta^{2\{T(s,s')-1\}}\delta$ is the probability that the match continued until $T(s,s')$ and s survives at the end of $T(s,s')$ and goes back to the matching pool. Thanks to the stationarity of p, the continuation payoff after a match ends for any reason is always $V(s;p)$.

Let $L(s;p) := \sum_{s' \in supp(p)} p(s')L(s,s')$. By computation,

$$\begin{aligned} V(s;p) &= \sum_{s' \in supp(p)} p(s') \Big[V(s,s') + \{1-(1-\delta)L(s,s')\} V(s;p) \Big] \\ &= \sum_{s' \in supp(p)} p(s')V(s,s') + \{1 - \frac{L(s;p)}{L}\} V(s;p). \quad (2) \end{aligned}$$

Hence the average payoff is a nonlinear function of the strategy distribution p:

$$v(s;p) = \frac{V(s;p)}{L} = \sum_{s' \in supp(p)} p(s') \frac{L(s,s')}{L(s;p)} v(s,s'), \quad (3)$$

where the ratio $L(s,s')/L(s;p)$ is the relative length of periods that s expects to spend in a match with s'. This nonlinearity is the important characteristics of the voluntarily separable game and is due to the endogenous duration of partnerships. Note also that, if p is a strategy distribution consisting of a single strategy s', then $v(s;p) = v(s,s')$.

Definition 3. *Given a stationary strategy distribution in the matching pool $p \in \mathcal{P}(\boldsymbol{S})$, $s \in \boldsymbol{S}$ is a best reply against p* if for all $s' \in \boldsymbol{S}$,

$$v(s;p) \geq v(s';p),$$

and is denoted as $s \in BR(p)$.

Definition 4. *A stationary strategy distribution in the matching pool $p \in \mathcal{P}(\boldsymbol{S})$ is a Nash equilibrium* if, for all $s \in \text{supp}(p)$, $s \in BR(p)$.

4 Efficiency Improvement by Reference Letters

In Greve-Okuno (2007), the information regarding past partnerships was completely lacking for newly formed matches. Thus they focused on strategies that starts from the null history for each new match and derived equilibria. They showed that in any symmetric Nash equilibrium, the players must build trust, i.e., play D for some initial periods of any new match. This is to impose cost on newly formed matches to make a long-term cooperative relationship valuable, since the players cannot distinguish the past of others.

By contrast, in our model, players can choose different continuation strategies depending on the letter status of the new partner. Therefore we investigate whether we can reduce the trust-building periods to zero for (Y, Y)-pair of players. If that holds in an equilibrium, the sheer existence of reference letter improves the efficiency of the society, although the contents of the letter is irrelevant.

Below we focus on a particular T-function such that $T(Y, Y) = 0$ and $T(q) = T$ for any $q \neq (Y, Y)$ for some integer $T \geqq 1$ and find a sufficient condition (depending on T) for a symmetric strategy distribution consisting of c_T-strategy to be a Nash equilibrium.

4.1 Distribution in the Matching Pool

In this subsection we derive a unique stationary distribution of Y and N status players in the matching pool, which is consistent with the stationary c_T-strategy

	δ	$1-\delta$
δ	keep	Y, N
$1-\delta$	N, Y	N, N

Table 2: Patterns and probabilities of players entering the matching pool from a YY-pair or a pair after trust building

distribution. Let the measure of players in the matching pool be 1 and the fraction of Y-status players be α. This α must satisfy a certain equality under the stationary strategy distribution of c_T.

By the random matching process, the fraction of pairs of players such that both have reference letters, (called a *YY-pair*) is $\frac{1}{2}\alpha^2$. The fraction of pairs in which exactly one player has a reference letter (called a *NY-pair*) is $\alpha(1-\alpha)$, since there are two ways that a pair is of this form. The fraction of pairs in which none has a reference letter (called a *NN-pair*) is $\frac{1}{2}(1-\alpha)^2$. The probability that these pairs continue for t periods is the probability that both players survive (with probability δ^2) for t consecutive periods. Thus, the fraction of YY-pairs that continued for t periods is $\frac{1}{2}\alpha^2\delta^{2t}$, that of NY-pairs that continued for t periods is $\alpha(1-\alpha)\delta^{2t}$, and that of NN-pairs that continued for t periods is $\frac{1}{2}(1-\alpha)^2\delta^{2t}$.

At a point of time, these pairs with $t = 1, 2, 3, \ldots$ periods of duration co-exist in the society. The total fraction of YY-pairs is $\frac{1}{2}\alpha^2\{1+\delta^2+\cdots\}$. Among these, those who go back to the matching pool are the ones that the partner died, since c_T-strategy prescribes that matched players issue reference letters to each other in every period of the partnership. The probability of exactly one player dies in a partnership is $\delta(1-\delta)$ and there are two cases. (See Table 2.) Therefore, the fraction of Y-status players entering the matching pool from YY-pairs is

$$Y(Y,Y) := 2\delta(1-\delta)\frac{1}{2}\alpha^2\{1+\delta^2+\cdots\} = \frac{\delta(1-\delta)\alpha^2}{1-\delta^2}.$$

Dead players in YY-pairs will generate newly born players with N status in the matching pool. The probability that exactly one player dies is $2\delta(1-\delta)$ and the probability that both die is $(1-\delta)^2$. Hence the fraction of N-status players entering

during TB	δ	$1-\delta$
δ	keep	N, N
$1-\delta$	N, N	N, N

during CP	δ	$1-\delta$
δ	keep	Y, N
$1-\delta$	N, Y	N, N

Table 3: Patterns and probabilities of players entering the matching pool from NY or NN-pairs

the matching pool from YY-pairs is

$$N(Y,Y) := \{2\delta(1-\delta)+2(1-\delta)^2\}\frac{1}{2}\alpha^2\{1+\delta^2+\cdots\} = \frac{(1-\delta)\alpha^2}{1-\delta^2}.$$

Next, consider players going to the matching pool from NY-pairs or NN-pairs. The fraction of these types of pairs is $\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2$. From these types of pairs, during the trust building periods ($t \leqq T$), two N-status players are generated in the matching pool regardless of the reason of dissolution, and no Y-status player is generated. Let $N(q;TB)$ be the fraction of N-status players entering the matching pool from $q \neq (Y,Y)$-pairs during their trust-building periods. It is expressed as

$$\begin{aligned} N(q;TB) &:= 2(1-\delta^2)\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{1+\delta^2+\cdots+\delta^{2(T-1)}\} \\ &= \frac{(1-\delta^2)(1-\alpha^2)(1-\delta^{2T})}{1-\delta^2}. \end{aligned}$$

After the trust building periods, the fraction of Y-status players and N-status players entering the matching pool are the same as those from YY-pairs. (See Table 3.) Let $Y(q;CP)$ be the fraction of Y-players entering the matching pool from cooperation periods for $q \neq (Y,Y)$. It is expressed as

$$\begin{aligned} Y(q;CP) &:= 2\delta(1-\delta)\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{\delta^{2T}+\delta^{2(T+1)}+\cdots\} \\ &= \frac{\delta(1-\delta)(1-\alpha^2)\delta^{2T}}{1-\delta^2}, \end{aligned}$$

The fraction of newly born players (with N-status) from $q \neq (Y,Y)$ pairs during the cooperation periods is

$$\begin{aligned} N(q;CP) &:= \{2\delta(1-\delta)+2(1-\delta)^2\}\{\alpha(1-\alpha)+\frac{1}{2}(1-\alpha)^2\}\{\delta^{2T}+\delta^{2(T+1)}+\cdots\} \\ &= \frac{(1-\delta)(1-\alpha^2)\delta^{2T}}{1-\delta^2}. \end{aligned}$$

In total, the overall fraction of Y-status players entering the matching pool is

$$Y(Y,Y) + Y(q;CP) = \frac{\delta\{\delta^{2T} + \alpha^2(1-\delta^{2T})\}}{1+\delta} =: f(\alpha).$$

If there exists $\alpha \in (0,1)$ such that $f(\alpha) = \alpha$, that is the stationary fraction. Since f is a continuous, monotone-increasing function of α and $0 < f(0) = \frac{\delta^{2T+1}}{1+\delta} < f(1) = \frac{\delta}{1+\delta} < 1$, by the Mean-Value Theorem, such α uniquely exists. (It is easy to check that this α also satisfies $N(Y,Y) + N(q;TB) + N(q;CP) = 1 - \alpha$.) In summary we have the following lemma.

LEMMA 1. *For any $\delta \in (0,1)$ and any T, there exists a unique $\alpha \in (0,1)$ which makes the stationary strategy distribution of c_T-strategies consistent with the flow of players over time.*

In the following we assume this α as the fraction of Y-status players in the matching pool.

4.2 Expected Payoff

Let us write the total expected payoff of a player who is just randomly matched with another player as $V(Y,Y)$ and $V(q)$ ($q \neq (Y,Y)$), depending on the letter status combination of the newly formed pair. These values must satisfy the following system of simultaneous equations. First, if they start as a YY-pair, the player gets a reference letter in any period and thus the recursive equation is as follows.

$$\begin{aligned} V(Y,Y) &= \frac{c}{1-\delta^2} + \delta(1-\delta)(1+\delta^2+\cdots)[(1-\alpha)V(q) + \alpha V(Y,Y)] \\ &= \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha)V(q) + \alpha V(Y,Y)]. \qquad (4) \end{aligned}$$

Second, consider the case when the player started in an NY or NN-pair. During the trust building periods, if the partner dies the player goes back to the matching pool without the reference letter and thus $V(q)$ is the continuation payoff. After the trust building periods, the continuation payoff is the same $(1-\alpha)V(q) + \alpha V(Y,Y)$

as that of a player starting from a YY-pair. Hence,

$$\begin{aligned}
V(q) &= \frac{1-\delta^{2T}}{1-\delta^2}d \\
&\quad +\delta(1-\delta)(1+\delta^2+\cdots+\delta^{2(T-1)})V(q) \\
&\quad +\frac{\delta^{2T}}{1-\delta^2}c \\
&\quad +\delta(1-\delta)(\delta^{2T}+\delta^{2(T+1)}+\cdots)[(1-\alpha)V(q)+\alpha V(Y,Y)] \\
&= \frac{1-\delta^{2T}}{1-\delta^2}d+\frac{\delta(1-\delta)(1-\delta^{2T})}{1-\delta^2}V(q) \\
&\quad +\frac{\delta^{2T}}{1-\delta^2}c+\frac{\delta(1-\delta)\delta^{2T}}{1-\delta^2}[(1-\alpha)V(q)+\alpha V(Y,Y)]. \qquad (5)
\end{aligned}$$

Solving (4) and (5) simultaneously, we explicitly obtain

$$V(Y,Y) = \frac{c(1+\delta^{2T+1})+(1-\alpha)\delta(1-\delta^{2T})d}{(1-\delta)\{1+\delta-\alpha\delta(1-\delta^{2T})\}} \qquad (6)$$

$$V(q) = \frac{\delta^{2T}(1+\delta)c+\{1+(1-\alpha)\delta\}(1-\delta^{2T})d}{(1-\delta)\{1+\delta-\alpha\delta(1-\delta^{2T})\}} \qquad (7)$$

It is straightforward to show that $V(Y,Y) > V(q)$. Also, notice that a newly born player has N status and thus each player's lifetime expected payoff is $V(q)$.

4.3 Nash Equilibrium

Let us find a condition of T that the symmetric strategy distribution of c_T constitutes a Nash equilibrium. By the dynamic programming, it is sufficient to show that no strategy that differ from c_T in one step obtains a strictly better payoff.

First, consider one-step deviations during the trust building periods. If a player started in a non-YY-pair, the partners are supposed to play (D,D) for the first T periods. During this time no one-step deviation would earn a higher payoff than c_T's.

Second, consider one-step deviations to play D in the Prisoner's Dilemma, when a player started in a YY-pair or after trust building periods, so that the partner is expected to play C. Such one-step deviation earns g in this period and receive no reference letter so that the continuation payoff is $V(q)$. Thus the total payoff is $g+\delta V(q)$. By contrast, the payoff of c_T-strategy is

$$\frac{c}{1-\delta^2}+\frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha)V(q)+\alpha V(Y,Y)].$$

To explain, the first term $\frac{c}{1-\delta^2}$ is the total payoff within the current partnership which continues with probability δ^2. If the partner dies, there are two cases of continuation payoffs. One possibility is that you meet an N-player in the matching pool, with probability $(1-\alpha)$, in which case the continuation payoff is $V(q)$. The other possibility is that you meet a Y-player with probability α, in which case the continuation payoff is $V(Y,Y)$ since you have a reference letter. Therefore, the condition that the one-step deviations do not earn higher payoff is

$$g + \delta V(q) \leqq \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2}[(1-\alpha)V(q) + \alpha V(Y,Y)]. \tag{8}$$

Third, one-step deviations to choose e when it is supposed to choose k has the continuation payoff of $V(q)$ during the trust building periods and $(1-\alpha)V(q) + \alpha V(Y,Y)$ during the cooperation periods. During the trust building periods, c_T strategy's continuation payoff is at least (when there are $T-1$ periods of trust building remains)

$$\begin{aligned} V(q;T-1) \quad &:= \quad \frac{1-\delta^{2(T-1)}}{1-\delta^2}d + \frac{\delta(1-\delta)(1-\delta^{2(T-1)})}{1-\delta^2}V(q) \\ &\quad + \frac{\delta^{2(T-1)}}{1-\delta^2}c + \frac{\delta(1-\delta)\delta^{2(T-1)}}{1-\delta^2}[(1-\alpha)V(q) + \alpha V(Y,Y)] \\ &> \quad V(q). \end{aligned}$$

During the cooperation periods, the continuation payoff of c_T-strategy is $V(Y,Y)$, which is clearly greater than $(1-\alpha)V(q) + \alpha V(Y,Y)$.

Finally, consider one-step deviations to choose a different letter decision than c_T prescribes. This deviation does not change the player's payoff and thus there is no incentive to do so.

In summary, the sufficient condition for the symmetric strategy distribution of c_T-strategy to be a Nash equilibrium is (8), or in terms of the average payoff,

$$g + \frac{\delta}{1-\delta}v(q) \leqq \frac{c}{1-\delta^2} + \frac{\delta}{1-\delta^2}[(1-\alpha)v(q) + \alpha v(Y,Y)]. \tag{9}$$

Now we compare T that satisfies (9) with the minimum trust building periods of $\underline{\tau}(\delta)$ for the model without reference letters (Greve-Okuno, 2007). With no information flow, the continuation payoff right after a partnership dissolution is the same regardless of the cause of the dissolution, which is also the total expected payoff at the time of entering the matching pool, i.e., one's lifetime payoff. Let this be V^{NL} and the average lifetime payoff of the no-information-flow model be v^{NL}.[6] The sufficient condition to prevent deviation to play D after trust building periods, under no information flow, is

$$\begin{aligned} & g + \delta V^{NL} \leqq \frac{c}{1-\delta^2} + \frac{\delta(1-\delta)}{1-\delta^2} V^{NL} \\ \iff \quad & g + \frac{\delta}{1-\delta} v^{NL} \leqq \frac{c}{1-\delta^2} + \frac{\delta}{1-\delta^2} v^{NL}. \end{aligned} \tag{10}$$

Let v^{BR} be the upper bound to v^{NL} that satisfies (10), i.e., v^{BR} satisfies (10) with equality. Greve-Okuno (2007) shows that for any $\delta \in (\underline{\delta}, 1)$ (for some $\underline{\delta} > 0$ defined in Greve-Okuno, 2007), there exists the minimum trust building periods $\underline{\tau}(\delta) < \infty$ that warrants $v^{NL}(c_{\underline{\tau}(\delta)}) \leqq v^{BR}$. Since $v(q) < (1-\alpha)v(q) + \alpha v(Y, Y)$, (9) is also satisfied by $\underline{\tau}(\delta)$-trust building strategy with reference letters. Thus we have the following existence result.

PROPOSITION 1. *For any $\delta \in (\underline{\delta}, 1)$ (where $\underline{\delta} > 0$ is defined in Greve-Okuno, 2007), there exists the minimum $T = \underline{\tau}(\delta)$ such that the letter-based T-trust building strategy is a Nash equilibrium.*[7]

Moreover, $v(q) < (1-\alpha)v(q) + \alpha v(Y, Y)$ implies that the upper bound of $v(q)$ that warrants (9) is strictly greater than v^{BR}. This implies that the sheer existence of reference letter improved the efficiency.

PROPOSITION 2. *The highest equilibrium average payoff under the letter-based trust building strategy is strictly greater than that of the trust building strategy without reference letters.*

5 Concluding Remarks

We analyzed the role of reference letters in voluntarily separable repeated Prisoner's Dilemma. During the cooperation periods, if both partners cooperated they issue

[6] NL stands for No Letter.

[7] Clearly, letter-based trust-building strategies with longer trust building periods for non-YY-pairs are also equilibria.

reference letters to each other, and, in the matching pool, those with reference letters can start cooperation right away. We derived a necessary and sufficient condition for such strategy combination to be a Nash equilibrium and showed that this equilibrium has average payoff greater than any symmetric equilibrium in the no information flow model. Therefore the sheer existence of reference letters, which only signals the cause of partnership dissolution, improves the efficiency.

In this paper we assumed that newly born players do not have a reference letter, but an analogous result obtains when newly born players are assumed to have reference letters. Moreover, if newly born players have reference letters, on the equilibrium path, all players entering the matching pool have the reference letters. This strengthens the incentive to cooperate, because one can start in a YY-pair always in the matching pool, unless one deviates. Hence efficiency is even more improved. In reality, diplomas, transcripts, and recommendation letters from college or high school can be interpreted as a reference letter. If these are considered to be credible evidence of the quality of a new employee by the society, initial trust building periods can be significantly shortened. By contrast, if the society do not respect these, not only the payoff of new employees but also the one for the society reduces.

We focused on strategies such that if a player with a reference letter met a player without a reference letter, they start with trust building. It is possible that in that case one wants to "skip" this match and wait for a match with a player with a reference letter so that they can start the cooperation periods right away. If a player chooses to stay in the matching pool for more than the transition time, the model can include the unemployment. By the logic of the efficiency wage theory,[8] the possibility of unemployment serves as a disciplining device. This is an interesting future research direction.

[8] See Shapiro-Stiglitz (1984) and Okuno-Fujiwara (1987).

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