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Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise

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Abstract

For estimating the realized volatility and covariance by using high frequency data, we introduce the Separating Information Maximum Likelihood (SIML) method when there are possibly micro-market noises. The resulting estimator is simple and it has the representation as a specific quadratic form of returns. The SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large under general conditions including *non-Gaussian processes* and *volatility models*. Based on simulations, we find that the SIML estimator has reasonable finite sample properties and thus it would be useful for practice. It is also possible to use the limiting distribution of the SIML estimator for constructing testing procedures and confidence intervals.

Key Words

Realized Volatility, Realized Covariance, Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML).

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1. Introduction

Recently a considerable interest has been paid on the estimation problem of the realized volatility by using high-frequency data in financial econometrics. It may be partly because it is possible now to use a large number of high-frequency data in financial markets including the foreign exchange rates markets and stock markets. Although there were some discussion on the estimation of continuous stochastic processes in the statistical literature, the earlier studies often had ignored the presence of micro-market noises in financial markets when they tried to estimate the underlying stochastic processes. Because there are several reasons why the micro-market noises are important in high-frequency financial data both in economic theory and in statistical measurements, several new statistical estimation methods have been developed. See Anderson, T.G., Bollerslev, T. Diebold,F.K. and Labys, P. (2000), Gloter and Jacod (2001), Ait-Sahalia, Y., P. Mykland and L. Zhang (2005), Hayashi and Yoshida (2005), Zhang, L., P. Mykland and Ait-Sahalia (2005), Hansen P. and A. Lunde (2006), Barndorff-Nielsen, O., P. Hansen, A. Lunde and N. Shepard (2006), Ubukata and Oya (2007) for further discussions on the related topics.

The main purpose of this paper is to develop a new statistical method for estimating the realized volatility and the realized covariance by using high frequency data in the presence of possible micro-market noises. The estimation method we are proposing is called the Separating Information Maximum Likelihood (SIML) estimator, which is regarded as a modification of the standard Maximum Likelihood (ML) method under the Gaussian process. The SIML estimator of the realized volatility and covariance for the underlying continuous (diffusion type) process has the representation as a specific quadratic form of returns. As we shall show in this paper, the SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large and the data frequency interval becomes zero under general conditions including *non-Gaussian processes* and *volatility models*. There has been a theoretical development of the ML estimation of the univariate diffusion process with measurement errors by Gloter and Jacod (2001). Our method can be regarded as a modification or extension of their ML procedure. However, the SIML approach has some different features from the standard ML estimation and it is a a new estimation method. The main motivation of our study is the fact that it is difficult to handle the exact likelihood function and calculate the exact ML estimator of unknown parameters from a large number of data for the underlying continuous stochastic processes with micro-market noises in the multivariate non-Gaussian cases. We denote our estimation method as the Separating Information Maximum Likelihood (SIML) estimator because it gives an interesting extension of the standard ML estimation method. The main merit of the SIML estimation is its simplicity and then it can be practically used for the multivariate (high frequency) financial time series. The SIML estimator has not only desirable asymptotic properties under general conditions including non-Gaussian processes and volatility models, but also it has reasonable finite sample properties.

In Section 2 we introduce the standard model and the SIML estimation of the realized volatility and the realized covariance with micro-market noise. We give the asymptotic properties of the SIML estimator in the standard situation. Then in Section 3 we shall investigate the asymptotic properties of the SIML method in the more general situation. In Section 4 we shall report some finite sample properties of the SIML estimator based on a set of simulations. Then some brief remarks will be given in Section 5. The mathematical derivations of our results will be given in Section 6 and Tables on the simulation results will be in Appendix.

2. The SIML Estimation of Realized Volatility and Covariance with Micro-Market Noise

Let y_{ij} be the $i-$th observation of the $j-$th (log-) price at t_i^n for $j = 1, \cdots, p; 0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1$. We set $\mathbf{y}_i = (y_{i1}, \cdots, y_{ip})$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ be an $n \times p$ matrix of observations. The underlying continuous process $\mathbf{x}_i$ is not necessarily the same as the observed prices and let $\mathbf{v}_i^{'} = (v_{i1}, \cdots, v_{ip})$ be the vector of the micro-market noises. Then we have

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i \tag{2.1}$$

where $\mathcal{E}(\mathbf{v}_i) = \mathbf{0}$ and $\mathcal{E}(\mathbf{v}_i\mathbf{v}_i^{'}) = \mathbf{\Sigma}_v$.

We assume that

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{\Sigma}_x^{1/2}(s)d\mathbf{B}_s \;\; (0 \le t \le 1), \tag{2.2}$$

where $\mathbf{B}_s$ is a $p \times 1$ vector of the standard Brownian motions and we write $\mathbf{\Sigma}_x(s) = (\sigma_{ij}^{(x)}) = \mathbf{\Sigma}_x^{1/2}(s)\mathbf{\Sigma}_x^{1/2}(s)'$. Then the main statistical problem is to estimate the quadratic variations and co-variations

$$\mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_x(s)ds = (\sigma_{ij}^{(x)}) \tag{2.3}$$

of the underlying continuous process $\{\mathbf{x}_t\}$ and also the variance-covariance $\mathbf{\Sigma}_{(v)} = (\sigma_{ij}^{(v)})$ of the noise process from the observed $\mathbf{y}_i$ $(i = 1, \cdots, n)$. Although we assume the Gaussian processes in order to derive the SIML estimation in this section, the asymptotic results do not depend on the Gaussianity of the underlying processes as we shall see in Theorem 1 in Section 2.2 and Theorem 2 in Section 3.

2.1 The Standard Case

We consider the standard situation when $\mathbf{\Sigma}(s) = \mathbf{\Sigma}_x$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independently and normally distributed as $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$. Then given the initial condition $\mathbf{y}_0$ [1], we have

$$\mathbf{Y}_n \sim N_{n\times p}\left(\mathbf{1}_n \otimes \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n\mathbf{C}_n^{'} \otimes h_n\mathbf{\Sigma}_x\right) \;, \tag{2.4}$$

[1] If we regard $\mathbf{y}_0$ as a random vector unconditionally, there is an initial value problem and $\mathcal{E}(\mathbf{y}_1|\mathbf{y}_0) = \mathrm{cov}(\mathbf{y}_1, \mathbf{y}_0)[\mathrm{var}(\mathbf{y}_0)]^{-1}\mathbf{y}_0$. But the coefficients are nearly the identity matrix when the time interval h_n is very short, i.e. the high-frequency financial data.

where $\mathbf{1}_n^{'} = (1, \cdots, 1)$, $h_n = 1/n$ $(= t_i^n - t_{i-1}^n)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} . \tag{2.5}$$

In order to investigate the likelihood function in the standard case, we prepare the next lemma, which may be of independent interest. The proof is given in Section 6.

Lemma 1 : (i) Define an $n \times n$ matrix $\mathbf{A}_n$ by

$$\mathbf{A}_n = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} . \tag{2.6}$$

Then the characteristic roots of $\mathbf{A}_n$ are $\cos \pi(\frac{2k-1}{2n+1})$ $(k = 1, \cdots, n)$ and the associated characteristic vectors are

$$\begin{bmatrix} \cos[\pi(\frac{2k-1}{2n+1})\frac{1}{2}] \\ \cos[\pi(\frac{2k-1}{2n+1})\frac{3}{2}] \\ \vdots \\ \cos[\pi(\frac{2k-1}{2n+1})(n-\frac{1}{2})] \end{bmatrix} \quad (k = 1, \cdots, n). \tag{2.7}$$

(ii) Then we have the spectral decomposition

$$\mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} = \mathbf{P}_n \mathbf{D}_n \mathbf{P}_n^{'} = 2\mathbf{I}_n - 2\mathbf{A}_n \ , \tag{2.8}$$

where $\mathbf{D}_n$ is a diagonal matrix with the k-th elements

$$d_k = 2 \left[1 - \cos(\pi(\frac{2k-1}{2n+1})) \right] \ (k = 1, \cdots, n) \ , \tag{2.9}$$

$$
(2.10) \qquad \mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}
$$

and

$$
(2.11) \qquad \mathbf{P}_n = (p_{jk}) \ , \ p_{jk} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos\left[\pi(\frac{2k-1}{2n+1})(j-\frac{1}{2})\right] \ .
$$

We transform $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$
(2.12) \qquad \mathbf{Z}_n = h_n^{-1/2} \mathbf{P}_n^{'} \mathbf{C}_n^{-1} \left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right)
$$

where

$$
(2.13) \qquad \bar{\mathbf{Y}}_0 = \mathbf{1}_n \otimes \mathbf{y}_0^{'} \ .
$$

We note that given the initial condition of $\mathbf{y}_0$ the above transformation is one-to-one. Then the likelihood function under the Gaussian noises is given by

$$
(2.14) \quad L_n^{*}(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{np} \prod_{k=1}^{n} |a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} e^{\left\{-\frac{1}{2}\mathbf{z}_k^{'} \left(a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x\right)^{-1} \mathbf{z}_k\right\}} ,
$$

where

$$
(2.15) \qquad a_{kn} = 4n \sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right] \ (k = 1, \cdots, n) \ .
$$

Hence the maximum likelihood (ML) estimator can be defined as the solution of minimizing

$$
(2.16) \qquad L_n(\boldsymbol{\theta}) = -\sum_{k=1}^{n} \log |a_{k,n}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} + \frac{1}{2}\sum_{k=1}^{n} \mathbf{z}_k^{'}[a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x]^{-1}\mathbf{z}_k \ .
$$

From this representation we find that the ML estimator of unknown parameters is a rather complicated function of the observations in general. It is mainly because each a_{kn} terms depend on k as well as n. Let denote $a_{k_n,n}$ and then we can evaluate that $a_{k_n,n} \to 0$ as $n \to \infty$ when $k_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ since $\sin x \sim x$ as $x \to 0$. On the other hand, $a_{n+1-l_n,n} = O(n)$ when $l_n = O(n^{\beta})$ $(0 < \beta < 1)$.

When k is small, we expect that $a_{k_n,n}$ is small. Then we may approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$(2.17) \qquad L_{1n}(\boldsymbol{\theta}) = m \log |\boldsymbol{\Sigma}_x| + \sum_{k=1}^{m} \mathbf{z}_k^{'} \boldsymbol{\Sigma}_x^{-1} \mathbf{z}_k \ .$$

It is the standard likelihood function except the fact that we only use the first m terms. Then the SIML estimator of $\hat{\boldsymbol{\Sigma}}_x$ is defined by

$$(2.18) \qquad \hat{\boldsymbol{\Sigma}}_x = \frac{1}{m} \sum_{k=1}^{m} \mathbf{z}_k \mathbf{z}_k^{'} \ .$$

On the other hand, when l is small and $l = n + 1 - l$, we expect that $a_{n+1-l,n}$ is large. Thus we may approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$(2.19) \qquad L_{2n}(\boldsymbol{\theta}) = \sum_{k=n+1-l}^{n} \log |a_{k,n} \boldsymbol{\Sigma}_v| + \sum_{k=n+1-l}^{n} \mathbf{z}_k^{'} [a_{k,n} \boldsymbol{\Sigma}_v]^{-1} \mathbf{z}_k \ .$$

It is also the standard likelihood function approach except the fact that we only use the last l terms. Then the SIML estimator of $\hat{\boldsymbol{\Sigma}}_v$ is defined by

$$(2.20) \qquad \hat{\boldsymbol{\Sigma}}_v = \frac{1}{l} \sum_{k=n+1-l}^{n} a_{k,n}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ .$$

For both $\hat{\boldsymbol{\Sigma}}_v$ and $\hat{\boldsymbol{\Sigma}}_x$, the number of terms m and l should be dependent on n. Then we only need the order requirements that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_v$, respectively.

In the above construction we define the SIML estimator by approximating the exact likelihood function under the Gaussian micro-market noises and the continuous diffusion process with the deterministic covariance. However, we expect that the SIML estimator has some asymptotic robustness. The most important characteristic of the SIML estimator is its simplicity and it has some important aspects for dealing with high-frequency data. It is because the number of observation to use tick data, for instance, becomes enormous from the standard statistical sense. Also it is quite easy to deal with the multivariate high-frequency data in our approach.

Since we have used a linear transformation in (2.12) and using a formula in the appendix, we can rewrite

$$\hat{\mathbf{\Sigma}}_x = \frac{1}{m}(\frac{2n}{n+\frac{1}{2}})\sum_{k=1}^{m}\left[\sum_{s=1}^{n}\mathbf{R}_s \cos\left[\pi(\frac{2k-1}{2n+1})(s-\frac{1}{2})\right]\right]\left[\sum_{t=1}^{n}\mathbf{R}_t^{'} \cos\left[\pi(\frac{2k-1}{2n+1})(t-\frac{1}{2})\right]\right]^{'}$$

$$= \sum_{s,t=1}^{n} c_{s,t}(m,n)\mathbf{R}_s\mathbf{R}_t^{'}$$

$$= \sum_{s=t=1}^{n} c_{s,s}(m,n)\mathbf{R}_s\mathbf{R}_s^{'} + \sum_{l=1}^{n-1} c_{s,t+l}(m,n)\mathbf{R}_s\mathbf{R}_{s+l}^{'} + \sum_{l=1}^{n-1} c_{s+l,t}(m,n)\mathbf{R}_{t+l}\mathbf{R}_t^{'}\ ,$$

where $\mathbf{R}_t = \mathbf{y}_t - \mathbf{y}_{t-1}$ and

$$c_{s,t}(m,n) = \frac{1}{m}(\frac{n}{2n+1})\left[\frac{\sin 2\pi m(\frac{t+s-1}{2n+1})}{\sin(\pi\frac{t+s-1}{2n+1})} + \frac{\sin 2\pi m(\frac{t-s}{2n+1})}{\sin(\pi\frac{t-s}{2n+1})}\right]\ . \tag{2.21}$$

Since we have the representation of the SIML estimator in terms of asset returns (i.e. $\mathbf{y}_t - \mathbf{y}_{t-1} = (y_{tj} - y_{t-1j})$ with the observation interval h_n), we can find the relation between the SIML estimator and other estimation methods. For instance, it may be interesting to see that the SIML estimator is similar but not in the class of the realized kernel estimator which was recently introduced by Barndorff-Nielsen et al. (2006).

2.2 Asymptotic Properties of the SIML estimator in the Standard Case

We can derive the asymptotic properties of the SIML estimator quite easily because it has a simple representation. For the asymptotic theory, we do not necessarily need to assume that the distributions of $\mathbf{x}_i$ $(i = 1, \cdots, n)$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are normal. We first give the asymptotic properties of the SIML estimator of $\mathbf{\Sigma}_x$ when the volatility function is constant.

Theorem 1 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent in (2.1). Suppose that $\mathbf{v}_i$ are mutually independently and distributed with $\mathcal{E}(\mathbf{v}_i) = \mathbf{0}$, $\mathcal{E}(\mathbf{v}_i\mathbf{v}_i^{'}) = \mathbf{\Sigma}_v$ and $\mathcal{E}(\|\mathbf{v}_i\|^4) < \infty$. Also suppose that $\mathbf{x}_i$ is a square integrable martingale with $\mathcal{E}(\mathbf{x}_i - \mathbf{x}_{i-1}) = \mathbf{0}$, $\mathcal{E}\left[n\ (\mathbf{x}_i - \mathbf{x}_{i-1})(\mathbf{x}_i - \mathbf{x}_{i-1})^{'})\right] = \mathbf{\Sigma}_x$ and

$\mathcal{E}(\|\sqrt{n}(\mathbf{x}_i - \mathbf{x}_{i-1})\|^4) < \infty$.

(i) Let $\mathbf{\Sigma}_x = (\sigma_{ij}^{(x)})$, $\hat{\mathbf{\Sigma}}_x = (\hat{\sigma}_{ij}^{(x)})$ for any i, j $(i, j = 1, \cdots, p)$. We take m $(= m_n)$ as a function of n and $m_n = O(n^{\alpha})$ with $0 < \alpha < \frac{1}{2}$. Then as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} 0 \ . \tag{2.22}$$

Furthermore assume $m_n^5/n^2 \to 0$ as $n \to \infty$. Then

$$\sqrt{m_n}\left[\hat{\sigma}_{ij}^{(x)} - \sigma_{ij}^{(x)}\right] \xrightarrow{w} N\left(0, \sigma_{ii}^{(x)}\sigma_{jj}^{(x)} + \left[\sigma_{ij}^{(x)}\right]^2\right) \ . \tag{2.23}$$

(ii) Let $\mathbf{\Sigma}_v = (\sigma_{ij}^{(v)})$, $\hat{\mathbf{\Sigma}}_v = (\hat{\sigma}_{ij}^{(v)})$ for any i, j $(i, j = 1, \cdots, p)$. We take l $(= l_n)$ as a function of n and $l_n = O(n^{\beta})$ with $0 < \beta < 1$. Then as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_v - \mathbf{\Sigma}_v \xrightarrow{p} 0 \tag{2.24}$$

and

$$\sqrt{l_n}\left[\hat{\sigma}_{ij}^{(v)} - \sigma_{ij}^{(v)}\right] \xrightarrow{w} N\left(0, \sigma_{ii}^{(v)}\sigma_{jj}^{(v)} + \left[\sigma_{ij}^{(v)}\right]^2\right) \ . \tag{2.25}$$

It is obvious that we have the joint normality as the limiting distributions in Theorem 1. One interesting observation is the result that the asymptotic variance of (2.25) does not depend on the fourth order moments under the non-normal disturbances.

There have been testing problems on the realized volatility in the presence of micro-market noise. In the SIML approach the testing procedures and constructing confidence regions can be constructed rather directly by using (2.23) and (2.25) for the covariance of the underlying continuous stochastic process and the covariance of the noises. We can utilize the relation

$$\begin{aligned} \text{(2.26)}\ \hat{\mathbf{\Sigma}}_x &= \frac{1}{n}\sum_{k=1}^{n} \mathbf{z}_k \mathbf{z}_k^{'} \\ &= \left(\frac{m}{n}\right)\frac{1}{m}\sum_{k=1}^{m}\mathbf{z}_k\mathbf{z}_k^{'} + \left(\frac{n-l-m}{n}\right)\frac{1}{n-l-m}\sum_{k=m+1}^{n+1-l}\mathbf{z}_k\mathbf{z}_k^{'} + \left(\frac{l}{n}\right)\frac{1}{l}\sum_{k=n+1-l}^{n}\mathbf{z}_k\mathbf{z}_k^{'} \\ &= \frac{m}{n}\hat{\mathbf{\Sigma}}_x^{(1)} + \frac{n-l-m}{n}\hat{\mathbf{\Sigma}}_x^{(2)} + \frac{l}{n}\hat{\mathbf{\Sigma}}_x^{(2)} , \end{aligned}$$

where $\hat{\mathbf{\Sigma}}_x^{(i)}$ $(i = 1, 2, 3)$ are independent in the standard situation. Since they are asymptotically independent, we can also construct the testing procedure and constructing confidence region on any elements of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ based on them.

One simple testing example is to test the null-hypothesis $H_0 \ : \ \sigma_{ii}^v = 0$ vs. $H_1 \ : \ \sigma_{ii}^v > 0$ for some i, where σ_{ii}^v is the (i,i)-th element of $\mathbf{\Sigma}_v$. For this problem consider

$$T_1 = \sqrt{l_n}\left[\frac{\frac{1}{l_n}\sum_{k=n+1-l_n}^{n} z_{ik}^2}{\frac{1}{n}\sum_{k=1}^{n} z_{ik}^2} - 1\right] , \tag{2.27}$$

where $\mathbf{z}_k = (z_{ik})$ $(i = 1, \cdots, p)$. By using Theorem 1, we find that under H_0

$$T_1 \xrightarrow{w} N(0, 2) \tag{2.28}$$

when $l_n, n \to \infty$ while $l_n/n \to 0$.
Actually the limiting distribution of T_1 is the same under more general conditions as we shall see in Section 3. In this way it is straightforward to construct test statistics and testing procedures in the SIML approach as the standard statistical procedure.

3. Asymptotic Properties of the SIML estimator in the General Case

Since we have introduced the SIML estimator as a modification of the ML estimator in the standard situation, it is important to investigate its properties when the instantaneous volatility function $\mathbf{\Sigma}_x(s)$ of the underlying asset price is not constant over time.

Let the conditional covariance matrix of the (underlying) returns without noise be

$$\mathbf{\Sigma}_i = \mathcal{E}\left[n \ \mathbf{r}_i \mathbf{r}_i^{'} | \mathcal{F}_{n,i-1}\right] , \tag{3.1}$$

where $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ is a sequence of martingale differences and $\mathcal{F}_{n,i-1}$ is the $\sigma-$field generated by $\mathbf{x}_s$ $(s \leq t_{i-1})$ and $\mathbf{v}_s$ $(s \leq t_{i-1})$. In this setting it is natural to impose the condition

$$\text{(3.2)} \qquad \frac{1}{n}\sum_{i=1}^{n}\mathbf{\Sigma}_i \xrightarrow{p} \mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_x(s)ds \ .$$

When the realized volatility and covariance $\mathbf{\Sigma}_x = (\sigma_{ij}^{(x)})$ is a constant (positive definite) matrix, we have the next proposition on the SIML estimator under regularity conditions as a natural extension of Theorem 1.

Theorem 2 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are mutually independent in (2.1), $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ and $\mathbf{v}_i$ are a sequence of martingale differences with (3.1), (3.2), $\sup_{1\le i\le n}\mathcal{E}(\|\mathbf{v}_i\|^4) < \infty$ and $\sup_{1\le i\le n}\mathcal{E}\,[\|\sqrt{n}\,\mathbf{r}_i\|^4] < \infty$. Suppose $\mathbf{\Sigma}_x$ is a constant matrix (a.s.).

(i) As $n \longrightarrow \infty$,

$$\text{(3.3)} \qquad \hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}$$

with $m_n = n^{\alpha}$ $(0 < \alpha < 1/2)$ and

$$\text{(3.4)} \qquad \sqrt{m_n}\left[\hat{\sigma}_{ij}^{(x)} - \sigma_{ij}^{(x)}\right] \xrightarrow{w} N\left(0, \sigma_{ii}^{(x)}\sigma_{jj}^{(x)} + \left[\sigma_{ij}^{(x)}\right]^2\right)$$

with $m_n^5/n^2 \to 0$.

(ii) As $n \longrightarrow \infty$,

$$\text{(3.5)} \qquad \hat{\mathbf{\Sigma}}_v - \mathbf{\Sigma}_v \xrightarrow{p} \mathbf{O}$$

and

$$\text{(3.6)} \qquad \sqrt{l_n}\left[\hat{\sigma}_{ij}^{(v)} - \sigma_{ij}^{(v)}\right] \xrightarrow{w} N\left(0, \sigma_{ii}^{(v)}\sigma_{jj}^{(v)} + \left[\sigma_{ij}^{(v)}\right]^2\right)$$

with $l_n = n^{\beta}$ $(0 < \beta < 1)$.

When $\mathbf{\Sigma}_x$ is a random matrix, we need the concept of stable convergence, which has been explained by Chapter 3 of Hall and Heyde (1980), and the results of Theorem 2 essentially hold with a careful treatment of weak convergence as we have indicated in the proof given in Section 6. In this situation (3.4) should be replaced by

$$\text{(3.7)} \qquad \sqrt{m_n}\left[\frac{\hat{\sigma}_{ii}^{(x)} - \sigma_{ii}^{(x)}}{\sigma_{ii}^{(x)}}\right] \xrightarrow{w} N(0,2) \quad (i = 1, \cdots, p)$$

as $n \to \infty$.

These results have some important implications in theory as well as in practice. First, the SIML estimator has the asymptotic robustness in the sense that it has the consistency and the asymptotic normality under general conditions. Second, the order of convergences are near to $n^{1/2}$ for the realized volatility and n for the micro-market noise, which could be regarded as the asymptotic bounds. Third, the formulas of the asymptotic variances are so simple that it is very easy to use them for practical applications.

4. Simulations

We have investigated the finite sample distributions of the SIML estimators for the realized variance and the realized covariance based on a set of simulations. The number of replications is 1000. As a reasonable setting we have taken $n = 5000$ and $n = 20000$. We have chosen $\alpha = 0.3$ and $\beta = 0.8$. In our experiments we have considered the situation that the variance of noises $10^{-4}, 10^{-6}$ and 10^{-8} of the realized variances.

In our simulation we consider two cases when the observations are the sum of signal and micro-market noise. In the first example the signal is the Brownian motion with the volatility function

$$\sigma_x(s)^2 = \sigma(0)^2 \left[a_0 + a_1 s + a_2 s^2\right], \tag{4.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. In this case the realized variance is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right] . \tag{4.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

In the second example the volatility follows the stochastic volatility model such that

$$\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n} \sigma_x(t_i^n)^2, \tag{4.3}$$

where $\sigma_x(t_i^n)^2 = e^{h(t_i)}$ $(s = t_i, 0 < t_1^n < \cdots < t_n^n \leq 1)$ and

$$h(t_i^n) = \gamma\, h(t_{i-1}^n) + c\, u(t_i^n)\ . \tag{4.4}$$

In our experiments we have set $\gamma = 0.9, c = 0.2$ and $u(t_i^n)$ are the white noise process followed by $N(0,1)$ as a typical situation.

We summarize our estimation results of the first example in Tables 4.1-4.4 and the second example in Table 4.5, respectively. (See Tables in Appendix.) In each table we have also calculated the value of the historical volatility as HI for comparison. When there are micro-market noise components with the martingale signal part, the value of HI often differs from the true realized volatility of the signal part substantially. However, we have found that it is possible to estimate the realized variance and the noise variance when we have the signal-noise ratio as $10^{-4} \sim 10^{-6}$ at least by the SIML estimation method. Although we have omitted the details of the second example, the estimation results are similar in the stochastic volatility model.

By our simulations we can conclude that we can estimate both the realized volatility of the hidden martingale part and the market noise part reasonably in all cases we have examined by the SIML estimation. When the market noises are extremely small, we have some difficulty to estimate the noise variances, which is a natural phenomenon. In that case, however, we can detect that fact by using the confidence interval constructed by the SIML estimation method.

5. Concluding Remarks

In this paper, we have developed a new estimation method for estimating the realized volatility and the realized covariance by using high-frequency financial data under the presence of noise. The Separating Information Maximum Likelihood (SIML) estimator proposed in this paper can be regarded as a modification of the standard Maximum Likelihood (ML) method. Our SIML estimator has the representation as a specific quadratic form of returns. We have shown that the SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large and the data frequency interval becomes zero under general conditions including non-Gaussian processes and volatility models. The SIML estimator is so simple that it can be practically used not only for the realized volatility but also the realized covariance of the multivariate high frequency financial series.

As an application we are currently investigating a set of high frequency data of Nikkei-225 index and Nikkei-225 Futures, which was the real motivation of our study. The details of our results shall be reported in another occasion.

6 Mathematical Derivations

In this section we give some of the derivations of the results reported in the previous sections.

Proof of Lemma 1 : Let $\mathbf{A}_n = (a_{ij})$ in (2.6) and $\mathbf{x} = (x_j)$ $(i, j = 1, \cdots, n)$ satisfying $\mathbf{A}_n\mathbf{x} = \lambda\mathbf{x}$. Then

$$\text{(6.1)} \qquad \frac{x_1 + x_2}{2} = \lambda x_1 \ ,$$

$$\text{(6.2)} \qquad \frac{x_{t-1} + x_{t+1}}{2} = \lambda x_t \ (t = 2, \cdots, n-1) \ ,$$

$$\text{(6.3)} \qquad \frac{1}{2}x_{n-1} = \lambda x_n \ .$$

Let ξ_i $(i = 1, 2)$ be the solution of (6.2). Since $2\lambda = \xi_1 + \xi_2$ and $\xi_1\xi_2 = 1$, we have the solution

$$\text{(6.4)} \qquad x_t = c_1\xi_1^t + c_2\xi_1^{-t} \ (t = 1, \cdots, n) \ ,$$

where c_i $(i = 1)$ are some constants. Then (6.1) and (6.3) imply

$$\text{(6.5)} \qquad \begin{aligned} 0 &= c_1\xi_1 + c_2\xi_1^{-1} + c_1\xi_1^2 + c_2\xi_1^{-2} - (\xi_1 + \xi_1^{-1})(c_1\xi_1 + c_2\xi_1^{-1}) \\ &= (\xi_1 - 1)(c_1 - c_2\xi_1^{-1}) \ . \end{aligned}$$

Since $c_2 = c_1\xi_1$ and $\xi \neq 1$, we find that $x_t = c_1[\xi_1^t + \xi_1^{-(t-1)}]$ and $\xi_1^{2n+1} = -1$. Then

$$\lambda_k = \cos[\pi \frac{2k-1}{2n+1}] \; (k = 1, \cdots, n) \; . \tag{6.6}$$

By taking $c_1 = (1/2)\xi_1^{-1/2}$, each elements of the characteristic vectors of $\mathbf{A}_n$ with $\cos[\pi(2k-1)/(2n+1)]$ are

$$x_t = \frac{1}{2}\left[\xi_1^{t-1/2} + \xi_1^{-(t-1/2)}\right] = \cos\left[\pi \frac{2k-1}{2n+1}(t - \frac{1}{2})\right] \; . \tag{6.7}$$

Q.E.D.

Lemma 2 : For any integer k

$$\sum_{t=1}^{n}\left[\cos \pi \frac{2k-1}{2n+1}(t-\frac{1}{2})\right]^2 = \frac{n}{2} + \frac{1}{4} \; . \tag{6.8}$$

Proof : We use the relation

$$\begin{aligned}
& \sum_{t=1}^{n}\left(e^{i2\pi[\frac{2k-1}{2n+1}(t-\frac{1}{2})]} + e^{-i2\pi[\frac{2k-1}{2n+1}(t-\frac{1}{2})]}\right) \\
= \; & e^{i2\pi[\frac{2k-1}{2n+1}]\frac{1}{2}} \times \frac{1 - e^{i2\pi[\frac{2k-1}{2n+1}]n}}{1 - e^{i2\pi[\frac{2k-1}{2n+1}]}} + e^{-i2\pi[\frac{2k-1}{2n+1}]\frac{1}{2}} \times \frac{1 - e^{-i2\pi[\frac{2k-1}{2n+1}]n}}{1 - e^{-i2\pi[\frac{2k-1}{2n+1}]}} \\
= \; & \frac{1}{1 - e^{i2\pi[\frac{2k-1}{2n+1}]}}\left(e^{i2\pi[\frac{2k-1}{2n+1}]\frac{1}{2}} - e^{2\pi i[\frac{2k-1}{2n+1}\frac{2n+1}{2}]} - e^{2\pi i[\frac{2k-1}{2n+1}\frac{1}{2}]} + e^{-i2\pi[\frac{2k-1}{2n+1}n]}\right) \\
= \; & \frac{-e^{\pi i[\frac{2k-1}{2n+1}(2n+1)]} + e^{-\pi[\frac{2k-1}{2n+1}(2n+1-2)]}}{1 - e^{i2\pi[\frac{2k-1}{2n+1}]}} \\
= \; & 1 \; .
\end{aligned}$$

Then by using the relation

$$\begin{aligned}
\sum_{t=1}^{n}\left[\cos \pi \frac{2k-1}{2n+1}(t-\frac{1}{2})\right]^2 &= \sum_{t=1}^{n}\left[\frac{1}{2} + \frac{1}{2}\cos \pi \frac{2k-1}{2n+1}2(t-\frac{1}{2})\right] \\
&= \frac{n}{2} + \frac{1}{2}\sum_{t=1}^{n}\cos 2\pi \frac{2k-1}{2n+1}(t-\frac{1}{2}) \; ,
\end{aligned} \tag{6.9}$$

we have (6.8). **Q.E.D.**

A Derivation of (2.21) : We use the relation

$$\sum_{k=1}^{m}\left[\cos \pi \frac{2k-1}{2n+1}(t+s-1))\right] \tag{6.10}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{m}\left(e^{i\pi[\frac{2k-1}{2n+1}(s+t-1)]}+e^{-i\pi[\frac{2k-1}{2n+1}(s+t-1)]}\right)\\
&= \frac{1}{2}\left(e^{i\pi[\frac{s+t-1}{2n+1}]}\times\frac{1-e^{i\pi[\frac{s+t-1}{2n+1}]2m}}{1-e^{i\pi[\frac{s+t-1}{2n+1}]2}}+e^{-i\pi[\frac{s+t-1}{2n+1}]}\times\frac{1-e^{-i\pi[\frac{s+t-1}{2n+1}]2m}}{1-e^{-i\pi[\frac{s+t-1}{2n+1}]2}}\right)\\
&= \frac{1}{2}\frac{1}{1-e^{i\pi[\frac{s+t-1}{2n+1}]2}}\left(e^{i\pi[\frac{s+t-1}{2n+1}]}-e^{\pi i[\frac{s+t-1}{2n+1}(2m+1)}-e^{\pi i[\frac{s+t-1}{2n+1}]}+e^{-i\pi[\frac{s+t-1}{2n+1}(2m-1)]}\right)\\
&= \frac{1}{2}\frac{\sin 2\pi m\frac{s+t-1}{2n+1}}{\sin\pi\frac{s+t-1}{2n+1}}\ .
\end{aligned}
$$

Then we have (2.21).

Proof of Theorem 1 : We first give the proof of Theorem 1 for the asymptotic properties of the SIML estimator of the realized variance. Then we shall apply the method to prove other cases.

(i) For any unit vector $\mathbf{e}_g=(0,\cdots,0,1,0,\cdots,0)^{'}$ $(g=1,\cdots,p)$, we define $\sigma_x^2=\mathbf{e}_g^{'}\mathbf{\Sigma}_x\mathbf{e}_g$, $\hat{\sigma}_x^2=\mathbf{e}_g^{'}\hat{\mathbf{\Sigma}}_x\mathbf{e}_g$, $\sigma_v^2=\mathbf{e}_g^{'}\mathbf{\Sigma}_v\mathbf{e}_g$ and $\hat{\sigma}_v^2=\mathbf{e}_g^{'}\hat{\mathbf{\Sigma}}_v\mathbf{e}_g$. From (2.12) we set $x_{kn}=\mathbf{e}_g^{'}\mathbf{z}_k$ $(k=1,\cdots,n)$ and $x_{kn}=x_{kn}^{(1)}+x_{kn}^{(2)}$, where $x_{kn}^{(1)}$ and $x_{kn}^{(2)}$ correspond to the $(k,g)-$elements of $\mathbf{Z}_n^{(1)}=h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}(\mathbf{X}_n-\mathbf{Y}_0)$ and $\mathbf{Z}_n^{(2)}=h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}\mathbf{V}_n$, respectively. By using Lemma 1, we have $\mathcal{E}[\mathbf{Z}_n^{(1)}\mathbf{e}_g]=\mathbf{0}$, $\mathcal{E}[\mathbf{Z}_n^{(1)}\mathbf{e}_g\mathbf{e}_g^{'}\mathbf{Z}_n^{(1)'}]=(\mathbf{e}_g\mathbf{\Sigma}_x\mathbf{e}_g^{'})\mathbf{I}_n$, $\mathcal{E}[\mathbf{Z}_n^{(2)}\mathbf{e}_g]=\mathbf{0}$ and

$$
(6.11)\qquad \mathcal{E}[\mathbf{Z}_n^{(2)}\mathbf{e}_g\mathbf{e}_g^{'}\mathbf{Z}_n^{(2)'}]=(\mathbf{e}_g\mathbf{\Sigma}_v\mathbf{e}_g^{'})h_n^{-1}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'}=(\mathbf{e}_g\mathbf{\Sigma}_v\mathbf{e}_g^{'})h_n^{-1}\mathbf{D}_n\ .
$$

Then

$$
(6.12)\qquad
\begin{aligned}
\hat{\sigma}_x^2-\sigma_x^2 &= \frac{1}{m}\sum_{k=1}^{m}\left[x_{kn}^2-\sigma_x^2\right]\\
&= \frac{1}{m}\sum_{k=1}^{m}\left[x_{kn}^2-(\sigma_x^2+a_{kn}\sigma_v^2)\right]+\sigma_v^2\left[\frac{1}{m}\sum_{k=1}^{m}a_{kn}\right]\ .
\end{aligned}
$$

By using (2.15),

$$
(6.13)\qquad
\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m}a_{k,n} &= \frac{1}{m}2n\sum_{k=1}^{m}\left[1-\cos(\pi\frac{2k-1}{2n+1})\right]\\
&= 2\frac{n}{m}\left[m-\frac{\sin\pi m\frac{2m}{2n+1}}{\sin\pi\frac{1}{2n+1}}\right]\\
&= O(\frac{m^2}{n})
\end{aligned}
$$

and

$$(6.14)\ \frac{1}{m}\sum_{k=1}^{m} a_{kn}^2 = \frac{1}{m}4n^2\sum_{k=1}^{m}\left[1-2\cos(\pi\frac{2k-1}{2n+1})+\frac{1}{2}(1+\cos(2\pi\frac{2k-1}{2n+1}))\right]$$
$$= \frac{4n^2}{m}\left[\frac{3}{2}m-\frac{\sin\pi m\frac{2m}{2n+1}}{\sin\pi\frac{1}{2n+1}}+\frac{1}{4}\frac{\sin\pi m\frac{4m}{2n+1}}{\sin\pi\frac{2}{2n+1}}\right]$$
$$= O(\frac{m^4}{n^2})$$

as $n\to\infty$. Then (6.13) and (6.14) are $o(1)$ by the condition $\sqrt{m}/n\to 0$ $(n\to\infty)$. Also $\mathcal{E}[x_{kn}^{(2)2}]\le |a_{kn}|\sigma_v^2$, there exists a constant c_1 such that $\mathcal{E}[x_{kn}^{(1)2}]\le c_1$ and

$$\frac{1}{m}\sum_{k=1}^{m}\mathcal{E}\left[x_{kn}^2-(\sigma_x^2+a_{k,n}\sigma_v^2)\right]^2=\frac{1}{m}\sum_{k=1}^{m}\mathcal{E}\left[(x_{kn}^{(1)2}-\sigma_x^2)^2+(x_{kn}^{(2)2}-a_{kn}\sigma_v^2)^2+4x_{kn}^{(1)2}x_{kn}^{(2)2}\right].$$
(6.15)

Thus we conly need to consider the first term of (6.15). Also we write

$$(6.16)\quad \sqrt{m}\left[\hat{\sigma}_x^2-\sigma_x^2\right]=\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[x_{kn}^2-(\sigma_x^2+a_{kn}\sigma_v^2)\right]+\sigma_v^2\left[\frac{1}{\sqrt{m}}\sum_{k=1}^{m}a_{kn}\right].$$

Under the additional condition that $m^5/n^2\to 0$ as $n\to\infty$ (by using a similar evaluation as (6.13)) we have

$$(6.17)\qquad \frac{1}{\sqrt{m}}\sum_{k=1}^{m}a_{kn}\to 0\ .$$

Then we shall show the consistency and the variance formula in (2.23) under the underlying non-normal distributions. We rewrite

$$\mathcal{E}\left[\frac{1}{m}\sum_{k=1}^{m}(x_{kn}^{(1)2}-\sigma_x^2)\right]^2 = \left[\frac{2n}{2n+1}\right]^2\mathcal{E}\left\{\sum_{i,j=1}^{n}\left[c_{ijm}\,r_{ig}r_{jg}-\delta_{ij}\frac{1}{n}\sigma_x^2\right]\right\}^2$$
$$\sim \mathcal{E}\left\{\sum_{i=j=1}^{n}\left[c_{ijm}r_{ig}^2-\frac{1}{n}\sigma_x^2\right]\right\}^2+\mathcal{E}\left\{\sum_{i\neq j=1}^{n}\left[c_{ijm}r_{ig}r_{jg}\right]\right\}^2,$$

where $\delta_{ij}=1$ $(i=j)$; $\delta_{ij}=0$ $(i\neq j)$, $\mathbf{r}_i=(r_{ij})=\mathbf{x}_i-\mathbf{x}_{i-1}$, $c_{ijm}=(2/m)\sum_{k=1}^{m}s_{ik}s_{jk}$ $(i,j=1,\cdots,n;k=1,\cdots,m)$ and

$$s_{jk}=\cos\left[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})\right]\ .$$

Because $\mathbf{r}_i = (r_{ig})$ $(i = 1, \cdots, n; g = 1, \cdots, p)$ are a sequence of martingale differences,

$$\mathcal{E}\left\{\sum_{i=j=1}^{n}\left[c_{iim}r_{ig}r_{ig} - \frac{1}{n}\sigma_x^2\right]\right\}^2 = \mathcal{E}\left[\sum_{i,j=1}^{n} c_{iim}c_{jjm}(r_{ig}^2 - \frac{1}{n}\sigma_x^2)(r_{jg}^2 - \frac{1}{n}\sigma_x^2)\right]$$
$$= \sum_{i=j=1}^{n} c_{iim}^2 \mathcal{E}\left[r_{ig}^2 - \frac{1}{n}\sigma_x^2\right]^2$$

and

$$\mathcal{E}\left\{\left[\sum_{i\neq j=1}^{n} c_{ijm}r_{ig}r_{jg}\right]^2\right\} = 2\sum_{i\neq j=1}^{n} c_{ijm}^2\mathcal{E}(r_{ig}^2)\mathcal{E}(r_{jg}^2)\ . \tag{6.18}$$

After some calculations, we can evaluate the relations $\sum_{j=1}^{n} s_{jk}s_{jl} = 0$ $(k \neq l)$, $\sum_{j=1}^{n} s_{jk}s_{jl} = n/2 + 1/4$ $(k = l)$ and

$$\sum_{i,j=1}^{n} c_{ijm}^2 = \frac{4}{m}\left[\frac{n}{2} + \frac{1}{4}\right]^2\ . \tag{6.19}$$

Then we have the key relation

$$2\sum_{i,j=1}^{n} c_{ijm}^2\mathcal{E}(r_{ig}^2)\mathcal{E}(r_{jg}^2) \sim 2\times\frac{4}{m}\left[\frac{n}{2}\right]^2\left[\frac{\sigma_x^2}{n}\right]^2 = \frac{2}{m}\left[\sigma_x^2\right]^2\ . \tag{6.20}$$

Hence by using (6.19) and (6.20) we have obtained the consistency and the variance formula (2.23) in the limiting distribution of the SIML estimator.
The remaining step is to apply the martingale central limit theorem (CLT) to the main term of (6.16), that is,

$$\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[x_{kn}^{(1)2} - \sigma_x^2\right]\ . \tag{6.21}$$

For this purpose we decompose

$$\sqrt{m}\left[\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)2} - \sigma_x^2\right] = (\frac{2n}{2n+1})\left[\sum_{i\neq j=1}^{n} c_{ijm}^* r_{ig}r_{jg} + \sum_{i=j=1}^{n} c_{iim}^*(r_{ig}^2 - \sigma_x^2)\right] \tag{6.22}$$

and we consider a sequence of $U_n = \sum_{j=2}^{n}[2\sum_{i=1}^{j-1} c_{ijm}^* r_{ig}]r_{jg}$ is a martingale, where $c_{ijm}^* = \sqrt{m}c_{ijm}$ $(i, j = 1, \cdots, n)$. Then we use the martingale CLT (Theorem 3.5 of Hall and Heyde with $p = 2$) by setting $X_{nj} = (2\sum_{i=1}^{j-1} c_{ijm}^* r_{ig})r_{jg}$ and $Y_{nj} =$

$\mathcal{E}[X_{nj}^2|\mathcal{F}_{nj-1}]$ $(j = 2, \cdots, n)$. We have the condition $\max_{1\le j\le n} Y_{nj} \xrightarrow{p} 0$ because we use the fact that for any $\epsilon > 0$

$$P(\max_{1\le j\le n} Y_{nj} > \epsilon) \le \sum_{j=1}^{n} P(Y_{nj} > \epsilon) \le \left[\frac{1}{\epsilon}\right]^2 \sum_{j=1}^{n} \mathcal{E}[Y_{nj}^2] ,$$

(6.19) and the boundedness of the fourth order moments.

(ii) For the estimation of covariance for any pair of $g, h = 1, \cdots, p$, we define $\sigma_{gh}^{(x)} = \mathbf{e}_g^{'}\mathbf{\Sigma}_x\mathbf{e}_h$, $\hat{\sigma}_{gh}^{(x)} = \mathbf{e}_g^{'}\hat{\mathbf{\Sigma}}_x\mathbf{e}_h$, $\sigma_{gh}^{(v)} = \mathbf{e}_g^{'}\mathbf{\Sigma}_v\mathbf{e}_h$ and $\hat{\sigma}_{gh}^{(v)} = \mathbf{e}_g^{'}\hat{\mathbf{\Sigma}}_v\mathbf{e}_h$. Then we apply the similar arguments of (i). The only difference is to use

$$\mathcal{E}\left\{\left[\sum_{i\ne j=1}^{n} c_{ijm} r_{ig} r_{jh}\right]^2\right\} = \sum_{i\ne j=1}^{n} c_{ijm}^2 \left[\mathcal{E}(r_{ig}^2)\mathcal{E}(r_{jh}^2) + (\mathcal{E}(r_{ig}r_{jh}))^2\right] \tag{6.23}$$

instead of (6.19) and then (6.21) should be modified as

$$\sum_{i,j=1}^{n} c_{ijm}^2 \left[\sigma_{gg}\sigma_{hh} + \sigma_{gh}^2\right] \sim \frac{1}{m}\left[\sigma_{gg}\sigma_{hh} + \sigma_{gh}^2\right] . \tag{6.24}$$

(iii) For σ_v^2, we have

$$\begin{aligned} \hat{\sigma}_v^2 - \sigma_v^2 &= \frac{1}{l_n}\sum_{k=n+1-l}^{n} a_{k,n}^{-1}\left[x_{kn}^2 - a_{k,n}\sigma_v^2\right] \\ &= \frac{1}{l_n}\sum_{k=n+1-l}^{n} a_{k,n}^{-1}\left[x_{kn}^2 - (\sigma_x^2 + a_{kn}\sigma_v^2)\right] + \sigma_x^2\left[\frac{1}{l}\sum_{k=n+1-l}^{n} a_{k,n}^{-1}\right] . \end{aligned} \tag{6.25}$$

We rewrite

$$a_{k,n} = 2n\left[1 + \cos\pi[\frac{2l_n}{2n+1}]\right] = 4n\cos^2\frac{\pi}{2}[\frac{2l_n}{2n+1}] ,$$

where $l_n = n + 1 - k_n$. Because we take $l_n = o(n)$, there esists n_0 such that for $n \ge n_0$ we find that $|a_{kn}^{-1}| = O(n^{-1})$. Hence

$$\frac{1}{l}\sum_{k=n+1-l}^{n} a_{kn}^{-1} \to 0 . \tag{6.26}$$

By applying the CLT to

$$\frac{1}{\sqrt{l_n}}\sum_{k=n+1-l}^{n}\left[a_{kn}^{-1}x_{kn}^2 - \sigma_v^2\right] , \tag{6.27}$$

which is the main term of $\sqrt{l_n}\,[\hat{\sigma}_v^2 - \sigma_v^2]$. We notice that $\mathcal{E}[a_{kn}^{-1}(x_{kn}^{(2)2} - a_{kn}\sigma_v^2)]^2$ are bounded for $k = n+1-l_n$ and n. Then by using the similar arguments as (i) and evaluating the asymptotic variance, we have (2.25). For the estimation of covariance of the noise term, we can use the same arguments as (ii) and it is omitted. **Q.E.D.**

Proof of Theorem 2 : Because the basic method is similar to the proof of Theorem 1, we illustrate the proof of Theorem 2 by using the simple case when $p = 1$. We decompose $z_k = x_{kn}^{(1)} + x_{kn}^{(2)}$ $(k = 1, \cdots, m)$ such that

$$x_{kn}^{(1)} = \sqrt{\frac{4n}{2n+1}} \sum_{j=1}^{n} s_{jk} r_j \tag{6.28}$$

and

$$x_{kn}^{(2)} = \sum_{j=1}^{n} b_{jk} v_j \ , \tag{6.29}$$

where

$$\mathbf{B}_n = (b_{jk}) = h_n^{-1/2} \mathbf{P}_n^{'} \mathbf{C}_n^{-1} \ .$$

Then

$$\frac{1}{m}\sum_{k=1}^{m} z_k^2 = \frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)2} + \frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(2)2} + 2\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)} x_{kn}^{(2)} \ . \tag{6.30}$$

First we can evaluate

$$\mathcal{E}\left[\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(2)2}\right] = \frac{1}{m}\sum_{k=1}^{m} \mathcal{E}\left[\sum_{j=1}^{n} b_{jk}^2 v_j^2\right] = \sigma_v^2 \frac{1}{m}\sum_{k=1}^{m} a_{km} \longrightarrow 0$$

as $n \to \infty$ if we have $m_n = O(n^{\alpha})$ $(0 < \alpha < 1/2)$. Then

$$\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(2)2} \xrightarrow{p} 0 \ . \tag{6.31}$$

Also we use the fact that

$$\left[\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)} x_{kn}^{(2)}\right]^2 \leq \left[\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)2}\right]\left[\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(2)2}\right] \xrightarrow{p} 0$$

if $(1/m)\sum_{k=1}^{m} x_{kn}^{(1)2}$ converges to a constant because of (6.30). Then the important step is to show

$$\frac{1}{m}\sum_{k=1}^{m} x_{kn}^{(1)2} - \frac{1}{n}\sum_{j=1}^{n}\sigma_j^2 = \left[\frac{4n}{2n+1}\right]\frac{1}{m}\sum_{k=1}^{m}\left[\sum_{i,j=1}^{n} s_{ik}s_{jk}r_i r_j\right] - \frac{1}{n}\sum_{j=1}^{n}\sigma_j^2 \tag{6.32}$$

$$= \sum_{i,j=1}^{n}\left[(\frac{2n}{2n+1})(\frac{2}{m}\sum_{k=1}^{m}s_{ik}s_{jk})r_ir_j\right]-\frac{1}{n}\sum_{j=1}^{n}\sigma_j^2$$
$$\xrightarrow{p} 0$$

as $n \to \infty$ in the same way as the proof of Theorem 1. Because

$$\sum_{k=1}^{m}s_{ik}s_{jk}=\frac{1}{2}\sum_{k=1}^{m}\left\{\cos\left[\frac{2\pi}{2n+1}(i+j-1)(k-\frac{1}{2})\right]+\cos\left[\frac{2\pi}{2n+1}(i-j)(k-\frac{1}{2})\right]\right\},$$

for large m we have the relation $(2/m)\sum_{k=1}^{m}s_{ik}s_{jk}=\delta_{ij}+o(1)$ and for any $i \neq j$ and large m

$$\frac{4}{m}\sum_{k,l=1}^{m}s_{ik}s_{jk}s_{il}s_{jl}=\delta_{kl}+o(\frac{1}{\sqrt{m}})\ . \tag{6.33}$$

Then we can evaluate the asymptotic variance of

$$\sqrt{m}\left[\frac{1}{m}\sum_{k=1}^{m}x_{kn}^{(1)2}-\frac{1}{n}\sum_{j=1}^{n}\sigma_j^2\right]=\sqrt{m}\sum_{i,j=1}^{n}\left[(\frac{2n}{2n+1})(\frac{2}{m}\sum_{k=1}^{m}s_{ik}s_{jk})r_ir_j-\delta_{ij}\frac{1}{n}\sum_{j=1}^{n}\sigma_j^2\right] \tag{6.34}$$

and we decompose its terms as (6.22) in the proof of Theorem 1. By using the fact that the order of $(1/n)^2\sum_{i=j=1}^{n}\sum_{k,l=1}^{m}(4/m)s_{ik}s_{jk}s_{il}s_{jl}=o(1)$, and r_i $(i=1,\cdots,n)$ are a sequence of martingale differences with $\mathcal{E}(r_i^2|\mathcal{F}_{n,i-1})=\sigma_i^2$ and the bounded fourth-order moments, we have

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\sum_{k,l=1}^{m}\frac{4}{m}s_{ik}s_{jk}s_{il}s_{jl}\sigma_i^2\sigma_j^2-\left[\frac{1}{n}\sum_{j=1}^{n}\sigma_j^2\right]^2\longrightarrow 0 \tag{6.35}$$

as $n \to \infty$.

By using the martingale CLT with (3.1)-(3.2) (Theorem 3.5 of Hall and Heyde with $p=2$), the asymptotic normality of $\hat{\sigma}_x^2$ (or the stable convergence in the general case when $\mathbf{\Sigma}_x$ is a random matrix) follows. The variance of the limiting distribution is given by (3.4) in this case.

The proof of the asymptotic properties on $\mathbf{\Sigma}_v$ is similar to that of Theorem 1 by utilizing the above arguments and we have omitted the details. **Q.E.D.**

References

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APPENDIX : TABLES

Table 4.1 : Estimation of realized volatility :
Case I ($a_0 = 1, a_1 = a_2 = 0$)

5000	*sgx*	*sgv*	HI	*sgx*	*sgv*	HI
True	2.000E-04	2.000E-06		2.000E-04	2.000E-07	
Mean	2.06E-04	2.03E-06	2.02E-02	2.04E-04	2.26E-07	2.20E-03
SD	8.62E-05	4.27E-08	4.76E-04	8.27E-05	4.69E-09	5.08E-05
True	2.000E-04	2.000E-08		2.000E-04	2.000E-09	
Mean	2.01E-04	4.65E-08	4.00E-04	2.00E-04	2.85E-08	2.20E-04
SD	8.22E-05	1.14E-09	8.30E-06	8.19E-05	7.84E-10	4.37E-06
20000	*sgx*	*sgv*	HI	*sgv*	*sgx*	HI
True	2.000E-04	2.000E-06		2.000E-04	2.000E-07	
Mean	2.01E-04	2.01E-06	8.02E-02	2.01E-04	2.08E-07	8.20E-03
SD	6.49E-05	2.12E-08	9.74E-04	6.51E-05	2.33E-09	1.01E-04
True	2.000E-04	2.000E-08		2.000E-4	2.000E-09	
Mean	1.98E-04	2.84E-08	1.00E-03	1.99E-04	1.04E-08	2.80E-04
SD	6.57E-05	3.28E-10	1.16E-05	6.14E-05	1.53E-10	2.82E-06

Note : In Table 4.1, *sgx* and *sgv* correspond to the estimates for the variances $\boldsymbol{\Sigma}_x$ (4.2) and $\boldsymbol{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 4.2 : Estimation of realized volatility :

Case II ($a_0 = 1, a_1 = 1, a_2 = 1$)

5000	sgx	sgv	HI	sgx	sgv	HI
True	3.667E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.66E-04	2.05E-06	2.04E-02	3.63E-04	2.49E-07	2.37E-03
SD	1.62E-04	4.33E-08	4.80E-04	1.55E-04	5.25E-09	5.39E-05
True	2.000E-04	2.000E-08		2.000E-04	2.000E-09	
Mean	3.57E-04	6.86E-08	5.66E-04	3.57E-04	5.05E-08	3.87E-04
SD	1.51E-04	1.82E-09	1.18E-05	1.54E-04	1.47E-09	8.02E-06
20000	sgx	sgv	HI	sgv	sgx	HI
True	3.667E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.62E-04	2.02E-06	8.04E-02	3.63E-04	2.15E-07	8.36E-03
SD	1.21E-04	2.13E-08	9.76E-04	1.24E-04	2.43E-09	1.02E-04
True	3.667E-04	2.000E-06		3.667E-04	2.000E-07	
Mean	3.58E-04	3.54E-08	1.17E-03	3.59E-04	1.74E-08	4.47E-04
SD	1.23E-04	4.39E-10	1.31E-05	1.16E-04	2.75E-10	4.59E-06

Note : In Table 4.2, sgx and sgv correspond to the estimates for the variances $\mathbf{\Sigma}_x$ in (4.2) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 4.3 : Estimation of realized volatility :

Case III ($a_0 = 1, a_1 = -1, a_2 = 1$)

5000	*sgx*	*sgv*	HI	*sgx*	*sgv*	HI
True	1.667E-04	2.000E-06		1.677E-04	2.000E-07	
Mean	1.72E-04	2.02E-06	2.02E-02	1.70E-04	2.22E-07	2.17E-03
SD	7.24E-05	4.26E-08	4.76E-04	6.93E-05	4.59E-09	5.02E-05
True	1.667E-04	2.000E-08		1.667E-04	2.000E-09	
Mean	1.67E-04	4.21E-08	3.67E-04	1.67E-04	2.41E-10	3.74E-06
SD	6.85E-05	1.02E-09	7.70E-06	6.90E-05	6.62E-10	3.74E-06
20000	*sgx*	*sgv*	HI	*sgv*	*sgx*	HI
True	1.667E-04	2.000E-06		1.667E-04	2.000E-07	
Mean	1.68E-04	2.01E-06	8.02E-02	1.67E-04	2.-7E-07	8.16E-03
SD	5.47E-05	2.12E-08	9.73E-04	5.47E-05	2.31E-09	1.00E-04
True	1.667E-04	2.000E-8		1.667E-04	2.000E-09	
Mean	1.65E-05	2.70E-08	9.67E-04	1.65E-04	8.99E-09	2.47E-04
SD	5.48E-05	3.07E-10	1.13E-05	5.11E-05	1.30E-10	2.49E-06

Note : In Table 4.3, *sgx* and *sgv* correspond to the estimates for the variances $\mathbf{\Sigma}_x$ in (4.2) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 4.4 : Estimation of realized volatility :
Case IV ($a_0 = 3, a_1 = -3, a_2 = 1$)

5000	sgx	sgv	HI	sgx	sgv	HI
True	1.833E-04	1.000E-06		1.833E-04	1.000E-07	
Mean	1.90E-04	1.03E-06	1.02E-02	1.91E-04	1.24E-07	1.18E-03
SD	8.13E-05	2.17E-08	2.40E-04	8.16E-05	2.61E-09	2.67E-05
True	1.833E-04	1.000E-08		1.833E-04	1.000E-09	
Mean	1.90E-04	3.43E-08	2.83E-04	1.88E-04	2.53E-08	1,93E-04
SD	8.24E-05	9.09E-10	5.78E-06	8.10E-05	7.41E-10	4.00E-06
20000	sgx	sgv	HI	sgv	sgx	HI
True	1.833E-04	1.000E-06		1.833E-04	1.000E-07	
Mean	1.85E-04	1.01E-06	4.02E-02	1.86E-04	1.08E-07	4.18E-03
SD	6.32E-05	1.06E-08	4.87E-04	6.30E-05	1.21E-09	5.11E-05
True	1.833E-04	1.000E-08		1.833E-04	1.000E-09	
Mean	1.84E-04	1.77E-08	5.84E-04	1.85E-05	8.69E-09	2.23E-04
SD	6.36E-05	2.22E-10	6.58E-06	6.02E-05	1.42E-10	2.34E-06

Note : In Table 4.4, sgx and sgv correspond to the estimates for the variances $\mathbf{\Sigma}_x$ in (4.4) and $\mathbf{\Sigma}_v$, respectively. Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.

Table 4.5 : Estimation of realized volatility :

Case V (Stochastic Volatility)

5000	*sgx*	*sgv*	HI	*sgx*	*sgv*	HI
True	2.303E-04	5.000E-07	2.303E-04	2.303E-04	5.000E-08	
Mean	2.302E-04	5.125E-07	5.237E-03	2.308E-04	6.196E-08	7.319E-04
SD	1.034E-04	2.366E-08	1.304E-04	1.042E-04	3.347E-09	3.348E-05
True	2.303E-04	5.000E-09	2.303e-04	2.303E-04	5.000E-10	
Mean	2.335E-04	1.690E-08	2.822E-04	2.274E-04	1.39E-08	2.371E-04
SD	8.24E-05	9.09E-10	5.78E-06	1.081E-04	2.061E-09	2.970E-05
20000	*sgx*	*sgv*	HI	*sgv*	*sgx*	HI
True	2.303E-04	5.000E-07	2.303E-04	2.303E-04	5.000E-08	
Mean	2.373E-04	5.034E-07	2.024E-02	2.302E-04	5.299E-08	2.232E-03
SD	7.911E-05	1.366E-08	2.564E-04	7.868E-05	1.432E-09	3.054E-05
True	2.303E-04	5.000E-09	2.303E-04	2.303E-04	5.000E-10	
Mean	2.310E-04	7.948E-09	4.314E-04	2.338E-04	3.440E-09	2.516E-04
SD	8.066E-05	3.145E-10	1.564E-05	7.939E-05	2.458E-10	1.567E-05

Note : In Table 4.5, *sgx* and *sgv* correspond to the estimates for the variances $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ when we have the stochastic volatility model of (4.3) and (4.4). Mean and SD are the sample mean and the standard deviation of the SIML estimator in the simulation. HI stands for the historical volatility.