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Behavioral Aspects of Arbitrageurs in Timing Games of Bubbles and Crashes

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Abstract

We model a timing game of bubbles and crashes a la Abreu and Brunnermeier (2003), in which arbitrageurs compete with each other to beat the gun in a stock market. However, unlike Abreu and Brunnermeier, instead of assuming sequential awareness, the present paper assumes that with a small probability, each arbitrageur is behavioral and committed to ride the bubble at all times. We show that with incomplete information, even rational arbitrageurs are willing to ride the bubble. In particular, the bubble can persist for a long period as the unique Nash equilibrium outcome.

Keywords: Bubbles and Crashes, Timing Games, Behavioral Arbitrageurs, Positive Feedback Traders, Reputation, Characterization, Uniqueness

JEL Classification Numbers: C720, C730, D820, G140

1. Introduction

This paper demonstrates the theoretical foundation underlying the willingness of rational arbitrageurs to ride the *bubble* in a stock market. We model the stock market as a *timing game* with *incomplete information* among arbitrageurs; the model is inspired by that of Abreu and Brunnermeier (2003).[3] However, our model does not require the assumption of sequential awareness that has played a central role in the above paper. Instead, we assume that each arbitrageur is not necessarily rational, i.e., he is *behavioral* with a small probability, in that he is committed to ride the bubble at all times. With the assumption of incomplete information, the present paper shows that even rational arbitrageurs are willing to ride the bubble for a long period.

The efficient market hypothesis in modern financial theory asserts that by reflecting all relevant information, the stock price is always adjusted to the fundamental value. However, there are considerable evidences that contradict this hypothesis: the stock price sometimes increases beyond the fundamental value and continues to increase until it goes into a free fall. Advocates of behavioral finance, such as Shleifer (2000) and Shiller (2000), argued that the bubble is driven by *positive feedback traders*, who incorrectly believe that the stock price will grow in perpetuity. The efficient market hypothesis, on the other hand, claims that rational arbitrageurs quickly undo this mispricing. The arbitrageurs' selling pressures then dampen the enthusiasm of the positive feedback traders, immediately bursting the bubble.

In contrast to this ideal of rational arbitrageurs, actual professional arbitrageurs who are mostly considered to be rational generally do not think that the best strategy is

[3] See also Abreu and Brunnermeier (2002) and Brunnermeier and Morgan (2006).

to undo mispricing in this manner; instead, they would like to ride the bubble. On the basis of historical experiences, several authors such as Kindleberger (1978) and Soros (1994) even emphasized a self-feeding aspect of bubbles and crashes: speculative price movements involve multiple professional arbitrageurs who continuously drive the stock price up and then sell out at the top to the positive feedback traders.

However, we may disagree with this view, because the arbitrageurs may compete with each other in order to exit the market by selling out, i.e., beat the gun or time the market at the earliest. This phenomenon along with the backward induction method prevents the persistence of a bubble. Hence, in order for this view to be convincing, we need to demonstrate a further theoretical foundation, based on which each rational arbitrageur is willing to terminate this chain reaction of competition and develop a reputation among them to ride the bubble.

On the basis of these arguments, the present paper models the stock market as a timing game, where the stock market operates during the time interval $[0,1]$, and each arbitrageur selects a time to exit the market by selling out his share. As long as no arbitrageur has sold out, the bubble continues to be driven by the positive feedback traders. Once any arbitrageur sells out, the other arbitrageurs vie with each other to follow in the footsteps of this arbitrageur. As soon as their selling pressures exceed a critical amount of shares, the positive feedback traders fail to support the stock price, and then the bubble crashes.

The key assumption of this paper is that every arbitrageur is not necessarily rational. With a small probability, he is behavioral and committed to ride the bubble at all times. We also assume incomplete information, in that whether each arbitrageur is behavioral or rational is unknown to the other arbitrageurs. On witnessing the persistence of a bubble, each rational arbitrageur is increasingly convinced that the other arbitrageurs are behavioral, which incentivizes him to further postpone timing the market. On the basis of this reasoning, each arbitrageur is convinced in the early stage of the timing game that the other arbitrageurs are subject to ride the bubble for a long period, *even if* they are rational.

In this respect, we should mention the relevance to the *reputation* theory in finitely repeated games with incomplete information explored by Kreps, Milgrom, Roberts, and Wilson (1982). With the assumption of incomplete information on whether players are rational or *crazy* enough to have blind faith in implicit collusion, any rational player is willing to mimic the crazy players' collusive behavior. The present paper will apply the basic concept of this theory to the timing games of bubbles and crashes.

With restrictions that the growth rate of a bubble and the probability of each arbitrageur's being behavioral are not very small, it is shown that there exists the *unique* Nash equilibrium. This equilibrium is symmetric and is named the *bubble-crash* equilibrium, where a particular point of critical time $\hat{\tau} \in (0,1)$ exists, such that (i) the bubble never crashes before the critical time $\hat{\tau}$, whereas (ii) after the critical time $\hat{\tau}$, the time at which the bubble crashes is randomly determined according to a constant hazard rate. This hazard rate is very high when the ratio between the total amount of shares that the arbitrageurs possess and the critical amount of share is close to unity; in this case, the bubble can persist for a long period, even if all arbitrageurs are almost certain from the beginning that the bubble will crash just around a particular fixed time.

Without these restrictions, another symmetric Nash equilibrium may exist, which is named the *quick-crash* equilibrium. In this equilibrium, any rational arbitrageur certainly sells out at the initial time, and therefore, the bubble never persists, except for the case in which all arbitrageurs are behavioral. The present paper provides a general characterization of all symmetric Nash equilibria. This characterization implies that whenever the bubble-crash and quick-crash equilibria coexist, there also exists another symmetric Nash equilibrium named the *hybrid* equilibrium. In it, each rational arbitrageur randomizes between the quick-crash equilibrium strategy and a modified version of the bubble-crash equilibrium strategy. In general, there exists *no other* symmetric Nash equilibrium apart from the bubble-crash, quick-crash, and hybrid equilibria.

There exist previous works that identified conditions under which the bubble exists (see the survey by Brunnermeier (2008)). Among these works, the paper by Abreu and Brunnermeier (2003) is particularly relevant to the present paper. This paper modeled the stock market as a timing game similar to ours, and was the first to present a theoretical background that explained that the resilience of the bubble stems from the inability of arbitrageurs to coordinate their selling strategies. Abreu and Brunnermeier assumed that the arbitrageurs become sequentially aware that the bubble has developed. Instead of this sequential awareness, the present paper assumes that each arbitrageur is not necessarily rational and provides an alternative explanation with regard to the inability of arbitrageurs to coordinate their selling strategies, mainly because they use *mixed* strategies.

The rest of the paper is organized as follows. Section 2 defines timing games of bubbles and crashes, and Section 3 investigates the quick-crash equilibrium. In Section 4, we investigate the bubble-crash equilibrium. Section 5 investigates the hybrid equilibria, and Section 6 characterizes all symmetric Nash equilibria. In Section 7, we show a sufficient condition under which the bubble-crash equilibrium is the unique

Nash equilibrium, where we take into account all *asymmetric* Nash equilibria. Section 8 concludes the paper.

2. The Model

This paper considers the trade in a company's shares during the time interval $[0,1]$. The fundamental value of this company is considered to be $y \geq 0$. We assume that the market interest rate is set equal to zero and no dividends are paid. There exist $n \geq 2$ arbitrageurs, each of whom decides the time to sell out his stockholding that is normalized to a single share. Let $N = \{1,...,n\}$ denote the set of arbitrageurs.

Figure 1 illustrates a typical pattern of bubbles and crashes. At the initial time 0, the arbitrageurs recognize that the bubble has occurred, where the stock price is set equal to $1+y$. The bubble persists as long as at most $\tilde{n}-1$ arbitrageurs have sold out their shares, where $\tilde{n}$ is a fixed positive integer and $\tilde{n} < n$. The difference between the stock price and the fundamental value grows exponentially according to a constant rate $\rho > 0$; the stock price per share is considered to be $e^{\rho t} + y$ at any time $t \in [0,1]$.[4]

[Figure 1]

Once any arbitrageur sells out his share, this selling pressure triggers all the other arbitrageurs to sell out immediately, which bursts the bubble because their collective

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[4] Unlike Abreu and Brunnermeier, not the absolute value of the stock price but the distance from the fundamental value grows exponentially.

selling pressure exceeds the critical amount of shares $\tilde{n}$. We assume that even if no arbitrageur sells out at or before the terminal time 1, the bubble crashes just after the terminal time for *exogenous* reasons.[5]

Against the abovementioned background, it is implicit to assume the presence of many positive feedback traders who have psychological biases that lead them to engage in momentum trading. They incorrectly believe that the stock price will grow in perpetuity and attempt to support the high stock price. The moment $\tilde{n}$ or more arbitrageurs sell out their shares, the positive feedback traders fail to support the stock price, which causes it to decline to the fundamental value y. In this respect, like Abreu and Brunnermeier (2002, 2003) and Brunnermeier and Morgan (2006),[6] our dynamic model shares aspects of *coordinated attacks* with the static models of currency attacks in international finance, such as Obstfeld (1996) and Morris and Shin (1998). These models assumed the necessity of speculators' coordination to break a currency peg.[7]

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[5] This paper assumes that short selling is prohibited. Without this assumption, any single arbitrageur can burst the bubble alone; this case is essentially the same as the case of $\tilde{n} = 1$, where the critical amount of shares equals one. Even in this case, we could show that the bubble may survive; however, this possibility is significantly limited.

[6] Abreu and Brunnermeier (2002, 2003) assumed that whether or not each arbitrageur has sold out is unobservable to the other arbitrageurs, whereas Brunnermeier and Morgan (2006) assumed that it is observable and all arbitrageurs rush to sell out once any arbitrageur times the market. The present paper follows Brunnermeier and Morgan. However, by adding irrelevant complexities, we can apply the basic concept of this paper to the case in which arbitrageurs' transactions are observable, and endogenize the assumption of their rushing in the same manner as Brunnermeier and Morgan.

[7] It might be helpful to think about the game of "musical chairs" as a metaphor for our model: there exist n players and $\tilde{n}$ chairs that are arranged in a circle, where $\tilde{n} < n$, i.e., the number of chairs is less than the number of players. Each player has a whistle in his mouth. Music starts at the initial time 0. While the music is playing, the players walk in unison around

On the basis of the above arguments, we define a *strategy* for each arbitrageur $i \in N$ as a cumulative distribution $q_i : [0,1] \to [0,1]$ that is nondecreasing, right continuous, and satisfies $q_i(\tau_0) = 1$. Following any strategy q_i, arbitrageur i plans to sell out at or before any time $a_i \in [0,1]$ with the probability of $q_i(a_i) \in [0,1]$. Let Q_i denote the set of strategies for arbitrageur i.We will consider $q_i = a_i$ to be a *pure* strategy if

$$q_i(\tau) = 0 \text{ for all } \tau \in [0, a_i), \text{ and } q_i(\tau) = 1 \text{ for all } \tau \in [a_i, \tau_0].$$

Significantly, this paper assumes that any arbitrageur $i \in N$ is not necessarily rational, and therefore, does not necessarily follow any strategy in the set Q_i. Let us fix any arbitrary real number $\varepsilon \in [0,1]$. With regard to the probability of ε, arbitrageur i is behavioral, in that he is committed to ride the bubble, i.e., sell out just after any other arbitrageur first sells out. Hence, any such behavioral arbitrageur does not trigger the burst of the bubble of his own accord. With regard to the remaining probability of $1-\varepsilon$, arbitrageur i is rational, and follows any strategy in Q_i. Whether each arbitrageur is behavioral or rational is independently determined and is not common knowledge among the arbitrageurs. In other words, each arbitrageur does not know whether the other arbitrageurs are rational or behavioral.

Suppose that each arbitrageur $i \in N$ plans to sell out at time $a_i \in [0,1]$. Let us

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the chairs. When a player blows the whistle, the music is immediately shut off, and then every player must race to sit down in one of the chairs. The $\tilde{n}$ players who could sit down are regarded as the winners; each winner obtains $e^{\rho t} + y$ dollars, while any loser obtains y dollars, provided some player blew the whistle and the music was shut off at time $t \in [0,1]$. Even if no player blows the whistle, the music is automatically shut off at the terminal time 1. By assuming that any player who blows the whistle has the advantage of rushing for a seat, we can regard this game as being the same as our model.

arbitrarily set any nonempty subset of arbitrageurs $H \subset N$, and let us suppose that any arbitrageur $i \in H$ is rational, while any arbitrageur $i \in N \setminus H$ is behavioral. Note that any behavioral arbitrageur $i \in N \setminus H$ never sells out at his planned time a_i; he is instead committed to wait for any other arbitrageur to time the market. Let us denote by $\tau \in [0,1]$ the time at which any rational arbitrageur sells out first, which is defined as the earliest time at which the rational arbitrageurs plan to sell out, i.e.,

$$\tau = \min\{a_i \mid i \in H\}.$$

Let $l \in \{1,...,|H|\}$ denote the number of rational arbitrageurs who plans to sell out at this earliest time τ, i.e.,

$$l = |\{i \in H \mid a_i = \tau\}|.$$

If $l > \tilde{n}$, then, with the probability of $\frac{\tilde{n}}{l}$, any rational arbitrageur $i \in H$ who plans to sell out at time τ sells out before the crash of the bubble, and earns $e^{\rho\tau} + y$. With regard to the remaining probability of $1 - \frac{\tilde{n}}{l}$, he fails to sell out before the crash and earns only the fundamental value y.

If $l \le \tilde{n}$, then he certainly sells out before the crash. In this case, $\tilde{n} - l$ further arbitrageurs can sell out before the crash. Hence, even any arbitrageur who either is behavioral or plans to sell out after time τ has the opportunity to sell out before the crash with regard to the positive probability of $\frac{\tilde{n} - l}{n - l}$.[8]

On the basis of these observations, we define the expected earning of any rational

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[8] It is implicit to assume that the behavioral arbitrageurs have the following advantage over the positive feedback traders: the behavioral arbitrageurs can sell out immediately after some rational arbitrageur times the market, while the positive feedback traders cannot sell out until the stock price declines to the fundamental value.

arbitrageur $i \in H$ by

$$v_i(H,a) = \min[1, \frac{K}{l}]e^{\rho\tau} + y \qquad \text{if} \quad a_i = \tau ,$$

and

$$v_i(H,a) = \max[\frac{K-l}{n-l}, 0]e^{\rho\tau} + y \quad \text{if} \quad a_i > \tau .$$

Let $Q = Q_1 \times \cdots \times Q_n$, and let $q = (q_1,...,q_n) \in Q$ denote a strategy profile. The payoff function $u_i : Q \to R$ for each arbitrageur $i \in N$ is defined as follows. For every $q \in Q$, let us specify $u_i(q)$ as being equal to the expected value of $v_i(H,a)$ in terms of (a,H), i.e.,

$$u_i(q) \equiv E[\sum_{H \subset N: i \in N} v_i(H,a)\varepsilon^{n-|H|}(1-\varepsilon)^{|H|-1} \mid q].$$

A strategy profile $q \in Q$ is said to be a Nash equilibrium if

$$u_i(q) \geq u_i(q_i', q_{-i}) \quad \text{for all} \quad i \in N \quad \text{and all} \quad q_i' \in Q_i .$$

A strategy profile $q \in Q$ is said to be symmetric if $q_i = q_1$ for all $i \in N$.

Let us introduce several notations as follows. For each strategy profile $q \in Q$, let us denote the probability that the bubble has crashed at or before any time $t \in [0,1]$ by

$$D(t;q) \equiv 1 - \prod_{i \in N}[\varepsilon + (1-\varepsilon)\{1 - q_i(t)\}].$$

Note that $D(1;q) \equiv 1 - \varepsilon^n < 1$ whenever $\varepsilon > 0$, which implies that the bubble does not necessarily crash during the time interval of $[0,1]$, because it is with the positive probability of $\varepsilon^n > 0$ that all arbitrageurs are behavioral. Let us define the *hazard rate* at which the bubble crashes at any time $t \in [0,1]$ by

$$\theta(t;q) \equiv \frac{\frac{\partial D(t;q)}{\partial t}}{1 - D(t;q)}.$$

Moreover, for each arbitrageur $i \in N$ and each strategy profile $q \in Q$, let us denote the probability that the bubble has crashed at or before any time t, provided arbitrageur

i never bursts the bubble of his own accord, by

$$D_i(t;q_{-i}) = 1 - \prod_{j \in N \setminus \{i\}} [\varepsilon + (1-\varepsilon)\{1 - q_j(t)\}].$$

3. Quick-Crash Equilibrium

We denote by $q^* \equiv (0,...,0)$ the symmetric strategy profile, named the quick-crash strategy profile, according to which any rational arbitrageur plans to sell out at the initial time 0. Hence, according to q^*, the bubble quickly crashes at the initial time 0. Figure 2 illustrates $D(t;q^*)$, which is kept equal to a constant value of $1-\varepsilon^n$.

[Figure 2]

Proposition 1: *The quick-crash strategy profile* q^* *is a Nash equilibrium if and only if*

(1) $$\sum_{\substack{H \subset N \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!}(1-\varepsilon)^{|H|}\varepsilon^{n-1-|H|}\{\min[1,\frac{\tilde{n}}{|H|+1}] - \max[0,\frac{\tilde{n}-|H|}{n-|H|}]\}$$

$$\geq \varepsilon^{n-1}(e^{\rho}-1).$$

Proof: For every $i \in N$,

$$u_i(q^*) = \sum_{\substack{H \subset N \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!}(1-\varepsilon)^{|H|}\varepsilon^{n-1-|H|}\min[1,\frac{\tilde{n}}{|H|+1}] + \varepsilon^{n-1} + y,$$

and for every $a_i \in (0,1]$,

$$u_i(a_i,q^*_{-i}) = \sum_{\substack{H \subset N \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!}(1-\varepsilon)^{|H|}\varepsilon^{n-1-|H|}\max[0,\frac{\tilde{n}-|H|}{n-|H|}]$$

$$+\varepsilon^{n-1}e^{\rho a_i}+y\,.$$

Hence, for every $a_i \in (0,1]$,

$$u_i(q^*)-u_i(a_i,q^*_{-i})$$

$$=\sum_{\substack{H\subset N\\ i\notin H,H\neq\phi}}\frac{(n-1)!}{|H|!(n-1-|H|)!}(1-\varepsilon)^{|H|}\varepsilon^{n-1-|H|}\{\min[1,\frac{\tilde{n}}{|H|+1}]-\max[0,\frac{\tilde{n}-|H|}{n-|H|}\}$$

$$+\varepsilon^{n-1}(1-e^{\rho a_i})\,.$$

From $e^{\rho}\geq e^{\rho a_i}$ for all $a_i \in (0,1]$, and from the above observations, it follows that the inequality of (1) is necessary and sufficient for q^* to be a Nash equilibrium.

Q.E.D.

Note that q^* is a Nash equilibrium if $\varepsilon=0$, i.e., it is certain that all arbitrageurs are rational. In this case, the left-hand side of (1) equals zero, while its right-hand side equals $\min[1,\frac{\tilde{n}}{n}]>0$, which automatically implies the inequality of (1). The assumption of $\varepsilon=0$ also implies the uniqueness of the Nash equilibrium; any rational arbitrageur dislikes losing the opportunity to become the single winner of the timing game. This presses him to hasten the time to beat the gun slightly earlier than the others. This aspect of tail-chasing competition eliminates all equilibria other than q^*.

This logic, however, cannot be applied to the case of $\varepsilon>0$. Even if any arbitrageur plans to sell out at the terminal time 1, he still has the opportunity of becoming the single winner with regard to the positive probability of $\varepsilon^n>0$; the assumption of $\varepsilon>0$ may apply the brakes to their tail-chasing competition. The rest of this paper will focus on the case of $\varepsilon>0$.

4. Bubble-Crash Equilibrium

This section specifies a symmetric strategy profile $\tilde{q} = (\tilde{q}_i)_{i \in N} \in Q$, named the bubble-crash strategy profile, according to which *the bubble persists for a long period*. Let us suppose that the growth rate of a bubble $\rho > 0$ and the probability of each arbitrageur being behavioral $\varepsilon \in (0,1)$ satisfies

(2) $$\varepsilon^{n-\tilde{n}} e^{\rho} \geq 1, \text{ i.e., } -1 \leq \frac{(n-\tilde{n})\ln \varepsilon}{\rho} < 0,$$

which implies that ρ and ε are not very small. Let us define a particular time $\tilde{\tau} \in [0,1)$ by

(3) $$\tilde{\tau} \equiv 1 + \frac{(n-\tilde{n})\ln \varepsilon}{\rho},$$

where the inequality of (2) guarantees $\tilde{\tau} \in [0,1)$. We specify the bubble-crash strategy profile $\tilde{q}$ as follows: for every $i \in N$,

$$\tilde{q}_i(a_i) = 0 \quad \text{for all } a_i \in [0, \tilde{\tau}),$$

and

(4) $$\tilde{q}_i(a_i) = \frac{1 - \varepsilon \exp[\frac{\rho(1-a_i)}{n-\tilde{n}}]}{1-\varepsilon} \quad \text{for all } a_i \in [\tilde{\tau}, 1].$$

Note from (3) that $\tilde{q}_i(\tilde{\tau}) = 0$, which implies that $\tilde{q}_i$ is continuous. According to $\tilde{q}$, the bubble *never* crashes before the specified time $\tilde{\tau}$. The following theorem states that $\tilde{q}$ is a Nash equilibrium.

Theorem 2: *With the inequality of (2), the bubble-crash strategy profile $\tilde{q}$ is a Nash equilibrium, and*

(5) $$u_i(\tilde{q}) = \varepsilon^{n-\tilde{n}} e^{\rho} + y.$$

Proof: Since the bubble never crashes before the time $\tilde{\tau}$, it follows from the continuity of $\tilde{q}$ that

$$u_i(a_i, \tilde{q}_{-i}) = e^{\rho a_i} + y \quad \text{for all} \quad a_i \in [0, \tilde{\tau}).$$

Since $\tilde{q}$ is continuous, it follows

$$u_i(\tilde{\tau}, \tilde{q}_{-i}) = e^{\rho \tilde{\tau}} + y > u_i(a_i, \tilde{q}_{-i}) \quad \text{for all} \quad a_i \in [0, \hat{\tau}).$$

For every $a_i \in [\tilde{\tau}, 1]$,

$$u_i(a_i, \tilde{q}_{-i}) = e^{\rho a_i}\{1 - D_i(a_i; \tilde{q}_{-i})\} + \frac{\tilde{n}-1}{n-1}\int_{t=\tilde{\tau}}^{a_i} e^{\rho t} dD_i(t; \tilde{q}_{-i}) + y.$$

The specification of $\tilde{q}$ implies

$$D_i(t; \tilde{q}_{-i}) = 0 \quad \text{for all} \quad t \in [0, \tilde{\tau}),$$

and

(6) $$D_i(t; \tilde{q}_{-i}) = 1 - \varepsilon^{n-1} e^{\frac{n-1}{n-\tilde{n}}\rho(1-t)} \quad \text{for all} \quad t \in [\tilde{\tau}, 1].$$

From (6), the following first-order conditions hold for all $a_i \in [\tilde{\tau}, 1]$:

(7) $$\frac{\partial}{\partial a_i} u_i(a_i, \tilde{q}_{-i}) = \frac{\partial}{\partial a_i}[e^{\rho a_i}\{1 - D_i(a_i; \tilde{q}_{-i})\} + \frac{\tilde{n}-1}{n-1}\int_{t=\tilde{\tau}}^{a_i} e^{\rho t} dD_i(t; \tilde{q}_{-i})]$$

$$= \rho e^{\rho a_i}\{1 - D_i(a_i; \tilde{q}_{-i})\} - (1 - \frac{\tilde{n}-1}{n-1}) e^{\rho a_i} \frac{\partial D_i(a_i; \tilde{q}_{-i})}{\partial a_i}$$

$$= \rho e^{\rho a_i}\{1 - D_i(a_i; \tilde{q}_{-i})\} - \frac{n-\tilde{n}}{n-1} e^{\rho a_i} \frac{\partial D_i(a_i; \tilde{q}_{-i})}{\partial a_i}$$

$$= \rho e^{\rho a_i} \varepsilon^{n-1} e^{\frac{n-1}{n-\tilde{n}}\rho(1-a_i)} - \frac{n-\tilde{n}}{n-1} e^{\rho a_i} \frac{\partial D_i(a_i; \tilde{q}_{-i})}{\partial a_i} = 0,$$

where, from (6), we have derived

$$\frac{\partial D_i(a_i; \tilde{q}_{-i})}{\partial a_i} = \frac{n-1}{n-\tilde{n}} \rho \varepsilon^{n-1} e^{\frac{n-1}{n-\tilde{n}}\rho(1-a_i)},$$

which implies the last equality of (7). Hence, we have proved that

$$u_i(\tilde{\tau}, \tilde{q}_{-i}) = u_i(a_i, \tilde{q}_{-i}) \text{ for all } a_i \in [\tilde{\tau}, 1].$$

From the above arguments, we have proved that $\tilde{q}$ is a Nash equilibrium, i.e.,

$$u_i(\tilde{q}) \geq u_i(q_i, \tilde{q}_{-i}) \text{ for all } q_i \in Q_i.$$

From (3),

$$u_i(\tilde{\tau}, \tilde{q}_{-i}) = e^{\rho\tilde{\tau}} + y = \varepsilon^{n-\tilde{n}} e^{\rho} + y,$$

which along with $u_i(\tilde{q}) \geq u_i(\tilde{\tau}, \tilde{q}_{-i})$ implies the equality of (5). **Q.E.D.**

From the specification of $\tilde{q}$, the probability $D(t;\tilde{q})$ that the bubble has crashed at or before any time $t \in [0,1]$ is given by

$$D(t;\tilde{q}) = 0 \qquad \text{if } 0 \leq t < \tilde{\tau},$$

and

$$D(t;\tilde{q}) = 1 - \varepsilon^n e^{\frac{n}{n-\tilde{n}}\rho(1-t)} \qquad \text{if } \tilde{\tau} < t \leq 1.$$

See Figure 3, which illustrates $D(t;\tilde{q})$.

[Figure 3]

From the specification of $\tilde{q}$, the hazard rate $\theta(t;q)$ at any time $t \in [0,1]$ equals

$$\theta(\tau;\tilde{q}) = 0 \qquad \text{if } 0 \leq t < \tilde{\tau},$$

and

$$\theta(\tau;q^{\hat{\tau}}) = \frac{n\rho}{n-\tilde{n}} \qquad \text{if } \tilde{\tau} < t \leq 1.$$

After the critical time $\tilde{\tau}$, the time at which the bubble crashes is randomly determined according to a *constant* hazard rate $\frac{n\rho}{n-\tilde{n}}$. By slightly postponing the time to beat the gun from time a_i to $a_i + \Delta$, arbitrageur i obtains the gain $\rho e^{\rho a_i}\{1 - D_i(a_i;\tilde{q}_{-i})\}$

from the increase in stock price, whereas he suffers the loss $(1-\frac{\tilde{n}-1}{n-1})e^{\rho a_i}\frac{\partial D_i(a_i;\tilde{q}_{-i})}{\partial a_i}$ from the decrease in winning probability. Since any time choice is the best response in $[\tilde{\tau},1]$, the above gain and loss must be balanced, which implies the equalities of (7), i.e., the first-order condition.

From the specification of $\tilde{q}$, it follows

$$\frac{\frac{\partial \tilde{q}_i(t)}{\partial t}}{1-\tilde{q}_i(t)}=\frac{\frac{\rho}{n-\tilde{n}}\exp[\frac{\rho(1-t)}{n-\tilde{n}}]}{\exp[\frac{\rho(1-t)}{n-\tilde{n}}]-1}.$$

Note $\lim_{t\downarrow\hat{\tau}}\frac{\frac{\partial \tilde{q}_i(t)}{\partial t}}{1-\tilde{q}_i(t)}=0$, which implies that around the critical time $\hat{\tau}$, any rational arbitrageur almost certainly postpones timing the market. Note also $\lim_{t\uparrow 1}\frac{\frac{\partial \tilde{q}_i(t)}{\partial t}}{1-\tilde{q}_i(t)}=+\infty$, which implies that as the terminal time 1 is drawing near, any rational arbitrageur is in a great hurry to time the market. Hence, the reason why even rational arbitrageurs have incentive to ride the bubble is as follows, which bears an analogy to the reputation theory in finitely repeated games:

(i) As the terminal time 1 is drawing near, any arbitrageur is almost convinced that the other arbitrageurs are behavioral.

(ii) At any time around the critical time $\hat{\tau}$, any arbitrageur is convinced through his rational reasoning that the other arbitrageurs are likely to ride the bubble even if they are rational.

Note that the hazard rate after the critical time $\hat{\tau}$ diverges to infinity as the ratio between the total amount of shares that the arbitrageurs possess and the critical amount of shares $\frac{\tilde{n}}{n}$ approaches to unity. Note also that the critical time $\tilde{\tau}$ depends on, not this ratio, but $(n-\tilde{n})\ln\varepsilon$. This implies that the bubble can persist for a long period even if all arbitrageurs are almost certain that the bubble will crash just around a particular fixed time, i.e., $\tilde{\tau}$.

5. Hybrid Equilibria

This section specifies another symmetric strategy profile named the hybrid strategy profile, according to which any rational arbitrageur sells out at the initial time 0 with a probability that is positive, but less than unity. We arbitrarily set a time $\hat{\tau}\in(0,1)$, where we assume

(8) $$\max[0,1+\frac{(n-\tilde{n})\ln\varepsilon}{\rho}]<\hat{\tau}<1.$$

Let us specify $k\in(0,1)$ by

(9) $$k\equiv\frac{1-\varepsilon\exp[\frac{\rho(1-\hat{\tau})}{n-\tilde{n}}]}{1-\varepsilon},$$

where the inequality of (8) guarantees $k\in(0,1)$. Associated with $\hat{\tau}\in(0,1)$, let us specify the hybrid strategy profile $q^{\hat{\tau}}=(q_i^{\hat{\tau}})_{i\in N}\in Q$ as follows: for every $i\in N$,

$$q_i^{\hat{\tau}}(a_i)=k \quad \text{for all } a_i\in[0,\hat{\tau}),$$

and

$$q_i^{\hat{\tau}}(a_i)=\tilde{q}_i(a_i)=\frac{1-\varepsilon\exp[\frac{\rho(1-a_i)}{n-\tilde{n}}]}{1-\varepsilon} \quad \text{for all } a_i\in[\hat{\tau},1],$$

where specification (9) of k implies $k=q_i^{\hat{\tau}}(\hat{\tau})$, i.e., $q_i^{\hat{\tau}}$ is continuous. According to $q^{\hat{\tau}}$, any rational arbitrageur $i\in N$ plans to sell out at the initial time 0 with the probability of $k > 0$. With regard to the remaining probability of $1 - k > 0$, the rational arbitrageur plans to ride the bubble up to the time $\hat{\tau}$, and he later follows the same strategy as the bubble-crash strategy $\tilde{q}_i$. The following proposition shows the necessary and sufficient condition under which $q^{\hat{\tau}}$ is a Nash equilibrium.

Proposition 3: *The hybrid strategy profile* $q^{\hat{\tau}}$ *is a Nash equilibrium if and only if*

(10) $$\{1-(1-\varepsilon)k\}^{n-1}\{e^{\rho}(\frac{\varepsilon}{1-(1-\varepsilon)k})^{n-\tilde{n}}-1\}$$

$$=\sum_{\substack{H\subset N:\\ i\notin H, H\neq\phi}}[\frac{(n-1)!}{|H|!(n-1-|H|)!}\{(1-\varepsilon)k\}^{|H|}\{1-(1-\varepsilon)k\}^{n-1-|H|}$$

$$\cdot\{\min[1,\frac{\tilde{n}}{|H|+1}]-\max[0,\frac{\tilde{n}-|H|}{n-|H|}]\}] .$$

Proof: See the Appendix.

When the hybrid Nash equilibrium is played, the merit from riding the bubble is severely limited, i.e., $u_i(q^{\hat{\tau}}) = u_i(0, q_{-i}^{\hat{\tau}}) < 1 + y$ must hold, because the time choice of 0 is a best response. The inequality of $\hat{\tau} > \tilde{\tau}$ implies that once the bubble takes off, it tends to grow further than the bubble induced by the bubble-crash equilibrium (See Figure 4, which illustrates $D(t; q^{\hat{\tau}})$).

[Figure 4]

From the specification of $q^{\hat{\tau}}$, it follows that

$$D(t;q^{\hat{\tau}}) = 1 - \varepsilon^n e^{\frac{n}{n-\tilde{n}}\rho(1-\hat{\tau})} \text{ and } \theta(t;q^{\hat{\tau}}) = 0 \qquad \text{if } 0 < t < \hat{\tau},$$

and

$$D(t;q^{\hat{\tau}}) = 1 - \varepsilon^n e^{\frac{n}{n-\tilde{n}}\rho(1-t)} \text{ and } \theta(t;q^{\hat{\tau}}) = \frac{n\rho}{n-\tilde{n}} \qquad \text{if } \hat{\tau} < t \leq 1.$$

Since the first-order condition is the same between the hybrid equilibrium $q^{\hat{\tau}}$ and the bubble-crash equilibrium $\tilde{q}$ at any time $t \in [\hat{\tau},1]$, it follows that the associated hazard rate is the same between $q^{\hat{\tau}}$ and $\tilde{q}$, i.e.,

$$\theta(t;q^{\hat{\tau}}) = \theta(t;\tilde{q}) = \frac{n\rho}{n-\tilde{n}} \text{ for all } t \in (\hat{\tau},1].$$

The following proposition shows that if the quick-crash equilibrium q^* and the bubble-crash equilibrium $\tilde{q}$ *coexist*, there also exists $\hat{\tau} \in (\tilde{\tau},1)$ such that the related hybrid strategy profile $q^{\hat{\tau}}$ is another Nash equilibrium.[9]

Proposition 4: *If the inequalities of (1) and (2) hold without equality, then there exists* $\hat{\tau} \in (\tilde{\tau},1)$ *such that* $q^{\hat{\tau}}$ *is a Nash equilibrium.*

Proof: For every $h \in [0,1]$, let us define

$$B(h) \equiv \{1-(1-\varepsilon)h\}^{n-1}\{e^{\rho}(\frac{\varepsilon}{1-(1-\varepsilon)h})^{n-\tilde{n}} - 1\}$$

$$- \sum_{\substack{H \subset N: \\ i \notin H, H \neq \phi}} [\frac{(n-1)!}{|H|!(n-1-|H|)!}\{(1-\varepsilon)h\}^{|H|}\{1-(1-\varepsilon)h\}^{n-1-|H|}$$

$$\cdot\{\min[1,\frac{\tilde{n}}{|H|+1}] - \max[0,\frac{\tilde{n}-|H|}{n-|H|}]\}].$$

[9] Proposition 4 does not imply the uniqueness of $\hat{\tau} \in (\tilde{\tau},1)$; the hybrid strategy profile $q^{\hat{\tau}}$, which is related to this proposition, is a Nash equilibrium.

Note that $B(h)$ is continuous, $B(0) > 0$, and $B(1) < 0$, which imply that there exists $h \in (0,1)$ such that $B(h) = 0$. This implies the equality of (10). **Q.E.D.**

6. Characterization of Symmetric Nash Equilibria

The following theorem characterizes all symmetric Nash equilibria, which states that there exists *no other* symmetric Nash equilibrium apart from the quick-crash equilibrium q^*, bubble-crash equilibrium $\tilde{q}$, and hybrid equilibrium $q^{\hat{\tau}}$.

Theorem 5: *If any strategy profile $q \in Q$ is a symmetric Nash equilibrium, then either $q = q^*$, $q = \tilde{q}$, or $q = q^{\hat{\tau}}$, where $\hat{\tau}$ satisfies (10).*

Proof: We set any symmetric Nash equilibrium $q \in Q$ arbitrarily, where we assume $q \neq q^*$. We show that $q_1(\tau)$ is continuous in $[0,1]$. Suppose that $q_1(\tau)$ is not continuous in $[0,1]$, i.e., there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. Since $\min[1, \frac{\tilde{n}}{l+1}] - \max[0, \frac{\tilde{n}-l}{n-l}] > 0$ for all $l \in \{0, ..., n-1\}$, it follows from the symmetry of q that by selecting any time that is slightly earlier than time τ', any arbitrageur can drastically increase the probability of his wining the timing game. This implies that no arbitrageur selects time τ', which is a contradiction.

Let

$$\tau^1 = \max\{\tau \in (0,1] : q_1(\tau) = q_1(0)\}.$$

We show that $q_1(\tau)$ is increasing in $[\tau^1,1]$. Suppose that $q_1(\tau)$ is not increasing in $[\tau^1,1]$. From the continuity of q_1 and the definition of τ^1, we can select $\tau',\tau''\in[\tau^1,1]$ such that $\tau'<\tau''$, $q_1(\tau')=q_1(\tau'')$, and the time choice τ' is a best response. Since no arbitrageur selects any time τ in (τ',τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any arbitrageur can increase the winner's gain from $e^{\rho\tau'}+y$ to $e^{\rho\tau''}+y$ without decreasing his wining probability. This is a contradiction.

Note that any time choice $\tau\in[\tau^1,1]$ is a best response, because $q_1(\tau)$ is increasing in $[\tau^1,1]$. This implies the following first-order conditions for all $\tau\in[\tau^1,1]$:

$$\frac{\partial u_1(\tau,q_{-1})}{\partial\tau}=\rho e^{\rho\tau}\{1-D_1(\tau;q_{-1})\}-e^{\rho\tau}\frac{\tilde{n}-1}{n-1}\frac{dD_1(\tau;q_{-1})}{d\tau}=0,$$

i.e.,

$$D_1(\tau;q_{-1})=1-Ce^{\frac{-\rho\tau}{1-\lambda}},$$

where C is a positive real number. Since q is symmetric and continuous, it follows that

$$D_1(1;q_{-1})=1-Ce^{\frac{-\rho}{1-\lambda}}=1-\varepsilon^{n-1}, \tag{11}$$

and

$$D_1(\tau^1;q_{-1})=1-Ce^{\frac{-\rho\tau^1}{1-\lambda}}=D_1(0;q_{-1}). \tag{12}$$

From (11), it follows that $C=\varepsilon^{n-1}e^{\frac{\rho}{1-\lambda}}$, and therefore,

$$D_1(\tau;q_{-1})=1-\varepsilon^{n-1}e^{\frac{\rho(1-\tau)}{1-\lambda}}\quad\text{for all }\tau\in[\tau^1,1]. \tag{13}$$

Suppose $q_1(0)=0$. Then, the symmetry of q implies $D_1(0;q_{-1})=0$, which along with (12) and (13) implies

$$\tau^1=1+\frac{(n-\tilde{n})\ln\varepsilon}{\rho}.$$

Hence, it follows from (3) that $\tau^1 = \tilde{\tau}$, and therefore, $q = \tilde{q}$.

Suppose $q_1(0) > 0$. Let $k = q_1(0)$. From (12) and (13),

$$\tau^1 = 1 + \frac{(n-\tilde{n})\ln(\frac{\varepsilon}{1-(1-\varepsilon)k})}{\rho},$$

which along with (9) implies $\tau^1 = \hat{\tau}$, i.e., $q = q^{\hat{\tau}}$. Since $q = q^{\hat{\tau}}$ is a Nash equilibrium, it follows from Proposition 3 that $\tau^1 = \hat{\tau}$ must satisfy (10). **Q.E.D.**

The outline of this proof is as follows. Fix any symmetric Nash equilibrium $q \in Q$ arbitrarily (see Figure 5, where it was supposed that $D(t;q)$ is discontinuous at time $t > 0$). Note that any arbitrageur can drastically increase his wining probability by selling out slightly earlier than time t, i.e., at time $t - \Delta$. This contradicts the Nash equilibrium property. Hence, $D(t;q)$, i.e., q must be continuous.

[Figure 5]

See Figure 6, where it was supposed that $D(t;q)$ is constant in the interval $[\tau', \tau'']$, and the time choice τ' is a best response. Since no arbitrageur sells out in the interval (τ', τ''), any arbitrageur can increase the winner's gain from $e^{\rho\tau'} + y$ to $e^{\rho\tau''} + y$ without decreasing his wining probability. This contradicts the Nash equilibrium property. Hence, it follows that there must exist a time $\hat{\tau}$ such that $D(t;q)$ is constant in $[0, \hat{\tau}]$, whereas $D(t;q)$ is increasing in $[\hat{\tau}, 1]$.

[Figure 6]

During the interval $[\hat{\tau},1]$, the first-order condition must hold, i.e., the hazard rate $\theta(t,q)$ must equal $\frac{n\rho}{n-\tilde{n}}$ (see Figure 7; it must be noted that $\tau^1=\tilde{\tau}$ implies $q=\tilde{q}$; $\tau^1=1$ implies $q=q^*$; and $\tilde{\tau}<\tau^1<1$ implies $q=q^{\hat{\tau}}$, where $\hat{\tau}=\tau^1$).

[Figure 7]

7. Uniqueness

In order for either the quick-crash strategy profile q^* or the hybrid strategy profile $q^{\hat{\tau}}$ to be a Nash equilibrium, the time choice of 0 must be a best response. This along with Theorem 5 implies that whenever the time choice of 0 is a *dominated* strategy, the bubble-crash strategy profile $\tilde{q}$ is the unique symmetric Nash equilibrium. The following theorem states that the uniqueness holds even if we take all asymmetric Nash equilibria into account.

Theorem 6: *The bubble-crash strategy profile* $\tilde{q}$ *is the unique Nash equilibrium if*

(14) $$\varepsilon^{n-1}e^{\rho}>1.$$

Proof: We will show that $\tilde{q}$ is the unique symmetric Nash equilibrium. Note that

$$u_1(0,q'_{-1})\leq 1 \quad \text{and} \quad u_1(1,q'_{-1})>\varepsilon^{n-1}e^{\rho} \quad \text{for all} \quad q'_{-1}\in Q_{-1}.$$

This along with the inequality of (14) implies that the time choice of 0 is dominated by the time choice of 1, i.e.,

$$u_1(1, q'_{-1}) > u_1(0, q'_{-1}) \quad \text{for all} \quad q'_{-1} \in Q_{-1}.$$

Hence, any symmetric Nash equilibrium $q \in Q$ must satisfy $q_1(0) = 0$, which along with Theorem 5 implies $q = \tilde{q}$. Since the inequality of (14) implies the inequality of (2), we have proved that $\tilde{q}$ is the unique symmetric Nash equilibrium.

We will show that $\tilde{q}$ is the unique Nash equilibrium even if all asymmetric Nash equilibria are taken into account. We set any Nash equilibrium $q \in Q$ arbitrarily.

First, we show that $q_i(\tau)$ must be continuous in $[0,1]$ for all $i \in N$. Suppose that $q_i(\tau)$ is not continuous in $[0,1]$. Then, there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_i(\tau) < q_i(\tau')$ for some $i \in N$. Since $\min[1, \frac{\tilde{n}}{l+1}] - \max[0, \frac{\tilde{n}-l}{n-l}] > 0$ for all $l \in \{0,...,n-1\}$, it follows that any other arbitrageur can drastically increase his winning probability by selecting any time that is slightly earlier than time τ'. Hence, any other arbitrageur never selects any time that is either the same as, or slightly later than, the time τ'. This implies that arbitrageur i can increase the winner's gain by postponing timing the market further without decreasing his winning probability. This is a contradiction.

Second, we show that $D(\tau; q)$ must be increasing in $[\tau^1, 1]$, where we denote

$$\tau^1 = \max\{\tau \in (0,1] : q_i(\tau) = q_i(0) \ \textit{for all } i \in N\}.$$

Suppose that $D(\tau; q)$ is not increasing in $[\tau^1, 1]$. Hence, from the continuity of q, we can select $\tau', \tau'' \in (\tau^1, 1]$ such that $\tau' < \tau''$, $D(\tau'; q) = D(\tau''; q)$, and the time choice τ' is a best response for some arbitrageur. Since no arbitrageur selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any arbitrageur can increase the winner's gain from $e^{\rho\tau'} + y$ to $e^{\rho\tau''} + y$ without decreasing his winning probability. This is a contradiction.

Third, we show that q must be symmetric. Suppose that q is asymmetric. Since the inequality of (14) implies that the time choice of 0 is a dominated strategy, it follows $\tau^1 > 0$, and

(15) $$q_i(\tau) = 0 \text{ for all } i \in N \text{ and all } \tau \in [0, \tau^1].$$

Since q is continuous and $D(\tau; q)$ is increasing in $[\tau^1, 1]$, from the supposition $\tau^1 > 0$ and the equality of (15), it follows that there exist $\tau' > 0$, $\tau'' > \tau'$, and $i \in N$ such that

$$q_1(t) = q_j(t) \text{ for all } j \in N \text{ and all } t \in [0, \tau'],$$

(16) $$\frac{\frac{\partial D_i(\tau; q)}{\partial t}}{1 - D_i(\tau; q)} > \min_{h \neq i} \frac{\frac{\partial D_h(\tau; q)}{\partial t}}{1 - D_h(\tau; q)} \text{ for all } t \in (\tau', \tau''),$$

and

(17) $$\frac{\frac{\partial D_i(\tau''; q)}{\partial t}}{1 - D_i(\tau''; q)} = \min_{h \neq i} \frac{\frac{\partial D_h(\tau''; q)}{\partial t}}{1 - D_h(\tau''; q)} > 0,$$

where the last inequality was derived from the increasing property of $D(\tau; q)$ in $[\tau^1, 1]$. Since $D(\tau; q)$ is increasing in $[\tau^1, 1]$, any time choice t in (τ', τ'') must be a best response for any arbitrageur $j \in N$ who satisfies

$$\frac{\frac{\partial D_j(t; q)}{\partial t}}{1 - D_j(t; q)} = \min_{h \neq i} \frac{\frac{\partial D_h(t; q)}{\partial t}}{1 - D_h(t; q)}.$$

Since this equality implies $\frac{\partial q_j(t)}{\partial t} > 0$, it follows from the continuity of q that the following first-order condition holds for arbitrageur j; for every $t \in (\tau', \tau'')$,

$$\frac{\partial u_j(\tau, q_{-j})}{\partial \tau} = \rho e^{\rho\tau}\{1 - D_j(\tau; q_{-j})\} - e^{\rho\tau} \frac{\tilde{n} - 1}{n - 1} \frac{dD_j(\tau; q_{-j})}{d\tau} = 0,$$

i.e.,

$$\rho\frac{n-1}{\tilde{n}-1}=\min_{h\neq i}\frac{\frac{\partial D_h(t;q)}{\partial t}}{1-D_h(t;q)}.$$

Hence, from (16),

$$\rho\frac{n-1}{\tilde{n}-1}<\frac{\frac{\partial D_i(t;q)}{\partial t}}{1-D_i(t;q)},$$

which implies that the first-order condition does not hold for arbitrageur i for every $t\in(\tau',\tau'')$, where

$$\frac{\partial u_i(\tau,q_{-i})}{\partial\tau}=\rho e^{\rho\tau}\{1-D_i(\tau;q_{-i})\}-e^{\rho\tau}\frac{\tilde{n}-1}{n-1}\frac{dD_i(\tau;q_{-i})}{d\tau}<0.$$

This inequality implies that arbitrageur i prefers time τ' rather than any time in $(\tau',\tau''+\varepsilon)$, and therefore,

$$\frac{\partial D_i(\tau;q)}{\partial t}=0 \quad \text{for all} \quad \tau\in(\tau',\tau''+\varepsilon),$$

where ε was positive but close to zero. This is a contradiction, because the inequality of (17) implied $\frac{\partial D_i(\tau'';q)}{\partial t}>0$. Hence, we have proved that any Nash equilibrium q must be symmetric. **Q.E.D.**

The brief sketch of this proof is as follows. We can prove the continuity of q and the increasing property of q in a manner similar to the proof of Theorem 5. In order to show that any Nash equilibrium q must be symmetric, we have used the inequality of (14) as follows. The first-order condition implies

$$\text{(18)} \qquad \frac{\frac{\partial D_i(t;q)}{\partial t}}{1-D_i(t;q)}=\rho\frac{n-1}{\tilde{n}-1} \quad \text{for all} \quad i\in N \quad \text{and all} \quad t\in[\tau^1,1].$$

The inequality of (14) implies

$$\text{(19)} \qquad q_i(\tau^1)=0 \quad \text{for all} \quad i\in N,$$

which along with (18) implies that $q_i = q_1$ for all $i \in N$, i.e., q is symmetric. Without the inequality of (14), however, we may not be able to show this symmetry; we may not be able to exclude the possibility that any asymmetric Nash equilibrium q exists, such that

$$q_i(0) > 0 \text{ and } q_i(0) \neq q_j(0) \text{ for some } i \in N \text{ and some } j \in N \setminus \{i\}.$$

8. Conclusion

This paper modeled the stock market as a timing game with incomplete information, where it was assumed that each arbitrageur is not necessarily rational, and is committed to ride the bubble with a small but positive probability. We showed a sufficient condition under which there exists the unique Nash equilibrium, where this equilibrium induces the bubble to persist for a long period even if all arbitrageurs are rational. We also characterized all symmetric Nash equilibria in general.

It is important to generalize our model in several directions. For instance, the present paper assumed that each arbitrageur has a single share in common. If we drop this assumption and permit heterogeneity among arbitrageurs in terms of their shareholdings, it might be conjectured that larger shareholders are more likely to insist on riding the bubble than smaller ones. This type of interesting but careful analysis is, however, beyond the purpose of this paper, and is regarded as a pending problem for possible future researches.

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Appendix: Proof of Proposition 3

From the specification of $q^{\hat{\tau}}$, it follows that $u_i(0, q_{-i}^{\hat{\tau}})$ equals

$$\text{(A-1)} \qquad \sum_{\substack{H \subset N: \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!} \{(1-\varepsilon)k\}^{|H|} \{1-(1-\varepsilon)k\}^{n-1-|H|} \min[1, \frac{\tilde{n}}{|H|+1}]$$

$$+\{1-(1-\varepsilon)k\}^{n-1} + y,$$

and for every $a_i \in (0,1]$, $u_i(a_i, q_{-i}^{\hat{\tau}})$ equals

$$\text{(A-2)} \qquad \sum_{\substack{H \subset N: \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!} \{(1-\varepsilon)k\}^{|H|} \{1-(1-\varepsilon)k\}^{n-1-|H|} \max[0, \frac{\tilde{n}-|H|}{n-|H|}]$$

$$+\{1-(1-\varepsilon)k\}^{n-1} e^{\rho a_i} + y \qquad \text{if } a_i \in (0, \hat{\tau}),$$

and

$$\text{(A-3)} \qquad \sum_{\substack{H \subset N: \\ i \notin H, H \neq \phi}} \frac{(n-1)!}{|H|!(n-1-|H|)!} \{(1-\varepsilon)k\}^{|H|} \{1-(1-\varepsilon)k\}^{n-1-|H|} \max[0, \frac{\tilde{n}-|H|}{n-|H|}]$$

$$+e^{\rho a_i}\{1-D_i(a_i; q_{-i}^k)\}$$

$$+\frac{\tilde{n}-1}{n-1} \int_{t=0}^{a_i} e^{\rho t} dD_i(t; q_{-i}^k) + y \qquad \text{if } a_i \in [\hat{\tau}, 1].$$

Moreover, from the specification of $q^{\hat{\tau}}$,

$$D_i(t; q_{-i}^{\hat{\tau}}) = 1 - \{\varepsilon + (1-\varepsilon)(1-k)\}^{n-1} \quad \text{for all } t \in [0, \hat{\tau}), \text{ and}$$

$$\text{(A-4)} \qquad D_i(t; q_{-i}^{\hat{\tau}}) = 1 - \varepsilon^{n-1} e^{\frac{n-1}{n-\tilde{n}}\rho(1-t)} \quad \text{for all } t \in [\hat{\tau}, 1].$$

Since $\tilde{q}$ is continuous,

$$u_i(\hat{\tau}, q_{-i}^{\hat{\tau}}) - u_i(a_i, q_{-i}^{\hat{\tau}}) = \{1-(1-\varepsilon)k\}^{n-1}(e^{\rho\hat{\tau}} - e^{\rho a_i}) > 0 \quad \text{for all } a_i \in (0, \hat{\tau}).$$

From (A-1) and (A-2), for every $i \in N$ and every $a_i \in (0, \hat{\tau})$,

$$u_i(0,q_{-i}^{\hat{\tau}})-u_i(a_i,q_{-i}^{\hat{\tau}})$$

$$=\sum_{\substack{H\subset N:\\ i\notin H,H\neq\phi}}\frac{(n-1)!}{|H|!(n-1-|H|)!}\{(1-\varepsilon)k\}^{|H|}\{1-(1-\varepsilon)k\}^{n-1-|H|}\{\min[1,\frac{\tilde{n}}{|H|+1}]-\max[0,\frac{\tilde{n}-|H|}{n-|H|}]\}$$

$$+\{1-(1-\varepsilon)k\}^{n-1}(1-e^{\rho a_i})>0\,.$$

From the equalities of (9) and (10), it follows that for every $i\in N$,

$$u_i(0,q_{-i}^{\hat{\tau}})-u_i(\hat{\tau},q_{-i}^{\hat{\tau}})$$

$$=\sum_{\substack{H\subset N:\\ i\notin H,H\neq\phi}}\frac{(n-1)!}{|H|!(n-1-|H|)!}\{(1-\varepsilon)k\}^{|H|}\{1-(1-\varepsilon)k\}^{n-1-|H|}\{\min[1,\frac{\tilde{n}}{|H|+1}]-\max[0,\frac{\tilde{n}-|H|}{n-|H|}]\}$$

$$+\{1-(1-\varepsilon)k\}^{n-1}(1-e^{\rho\hat{\tau}})=0\,,$$

where the last equality was derived from (9) and (10). From (A-3), the following first-order condition holds for every $a_i\in[\hat{\tau},1]$;

(A-5)
$$\frac{\partial}{\partial a_i}u_i(a_i,q_{-i}^{\hat{\tau}})=\frac{\partial}{\partial a_i}[e^{\rho a_i}\{1-D_i(a_i;q_{-i}^{\hat{\tau}})\}+\frac{\tilde{n}-1}{n-1}\int_{t=0}^{a_i}e^{\rho t}dD_i(t;q_{-i}^{\hat{\tau}})]$$

$$=\rho e^{\rho a_i}\{1-D_i(a_i;q_{-i}^{\hat{\tau}})\}-\frac{n-\tilde{n}}{n-1}e^{\rho a_i}\frac{\partial D_i(a_i;q_{-i}^{\hat{\tau}})}{\partial a_i}$$

$$=\rho e^{\rho a_i}\varepsilon^{n-1}e^{\frac{n-1}{n-\tilde{n}}\rho(1-t)}-\frac{n-\tilde{n}}{n-1}e^{\rho a_i}\frac{\partial D_i(a_i;q_{-i}^{\hat{\tau}})}{\partial a_i}=0\,,$$

where, from (A-4), we have derived

$$\frac{\partial D_i(a_i;q_{-i}^{\hat{\tau}})}{\partial a_i}=\frac{n-1}{n-\tilde{n}}\rho\varepsilon^{n-1}e^{\frac{n-1}{n-\tilde{n}}\rho(1-t)}\,,$$

which implies the last equality of (A-5). Hence,

$$u_i(\hat{\tau},q_{-i}^{\hat{\tau}})=u_i(a_i,q_{-i}^{\hat{\tau}})\quad\text{for all}\quad a_i\in[\hat{\tau},1]\,,$$

and therefore, we have proved that

$$u_i(q^{\hat{\tau}})\geq u_i(q_i,q_{-i}^{\hat{\tau}})\quad\text{for all}\quad q_i\in Q_i\quad\text{and all}\quad i\in N\,.$$

Figure 1

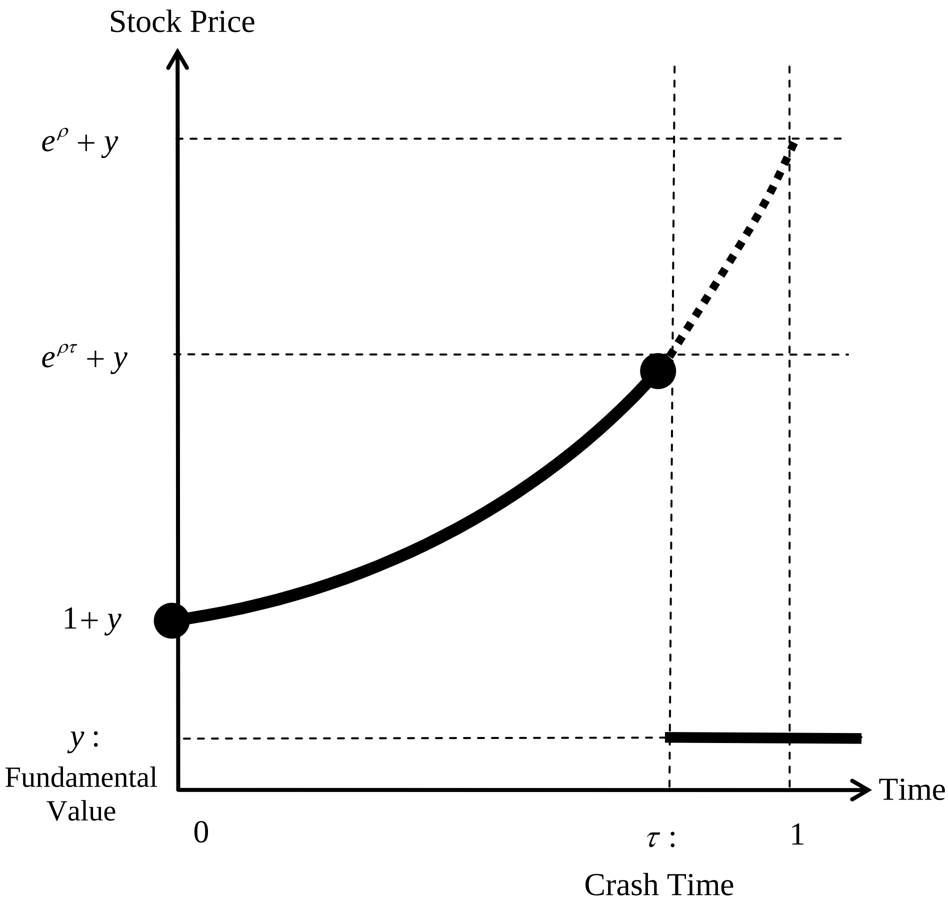

Stock Price
$e^{\rho}+y$
$e^{\rho\tau}+y$
$1+y$
$y:$
Fundamental
Value
Time
0
$\tau:$
1
Crash Time

Figure 2

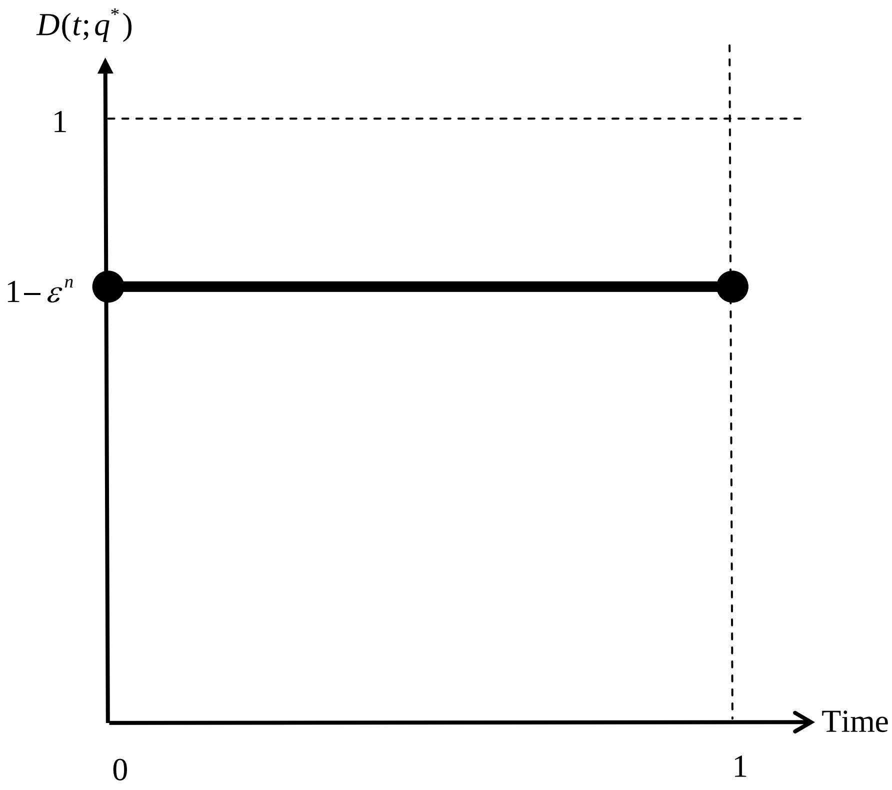


Figure 3

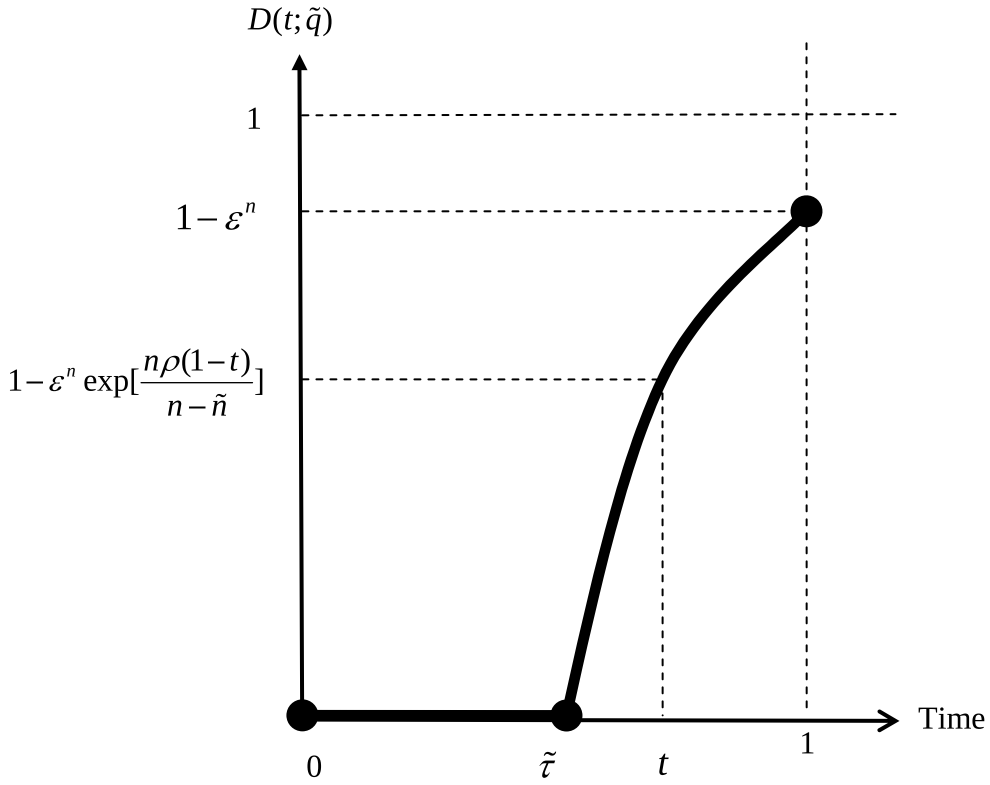

$D(t;\tilde{q})$
1
$1-\varepsilon^n$
$1-\varepsilon^n \exp[\frac{n\rho(1-t)}{n-\tilde{n}}]$
Time
0
$\tilde{\tau}$
t
1

Figure 4

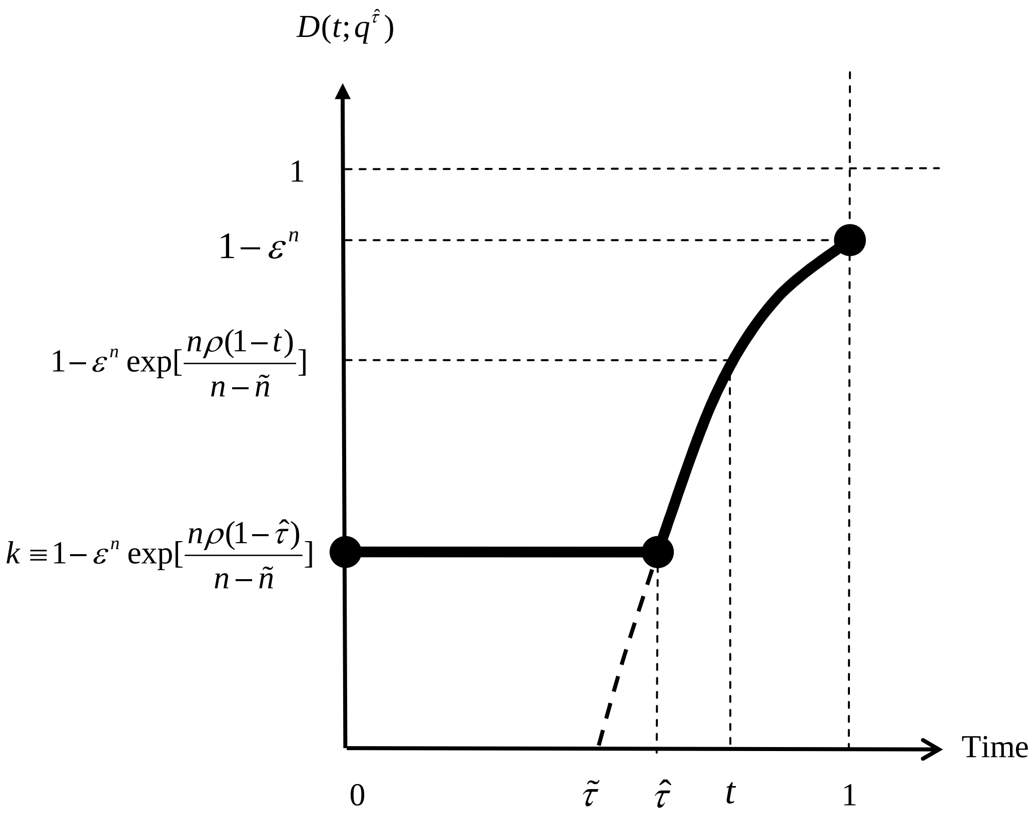

$D(t;q^{\hat{\tau}})$
1
$1-\varepsilon^n$
$1-\varepsilon^n \exp[\frac{n\rho(1-t)}{n-\tilde{n}}]$
$k \equiv 1-\varepsilon^n \exp[\frac{n\rho(1-\hat{\tau})}{n-\tilde{n}}]$
Time
0
$\tilde{\tau}$
$\hat{\tau}$
t
1

Figure 5

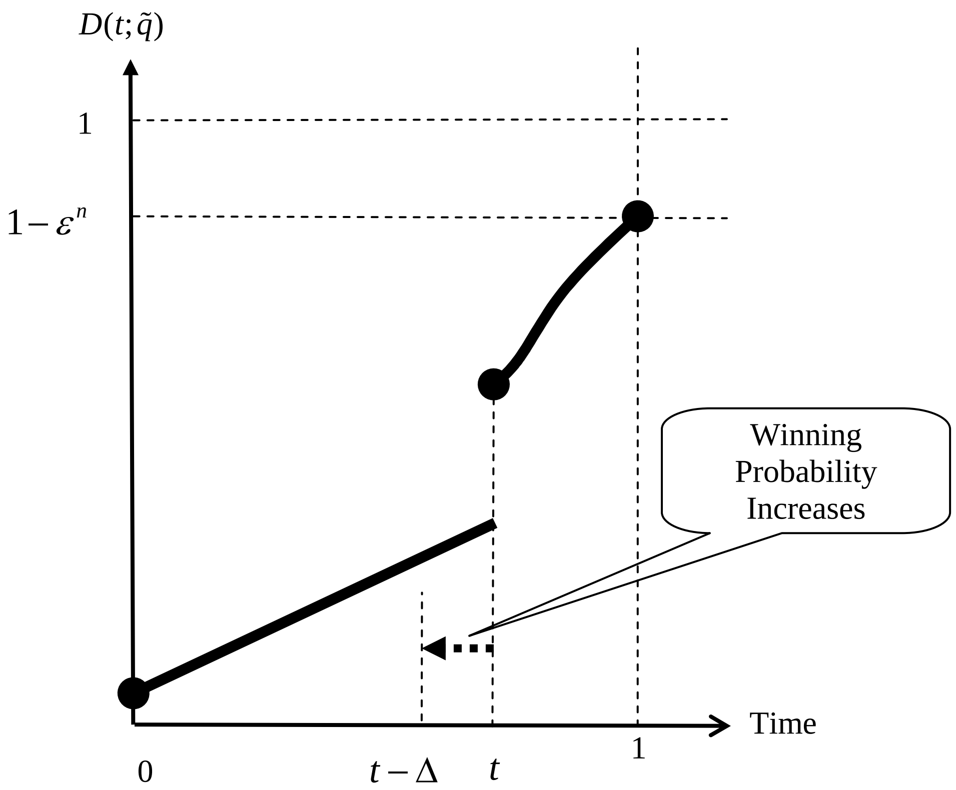

$D(t;\tilde{q})$
1
$1-\varepsilon^n$
Winning Probability Increases
Time
0
$t-\Delta$
t
1

Figure 6

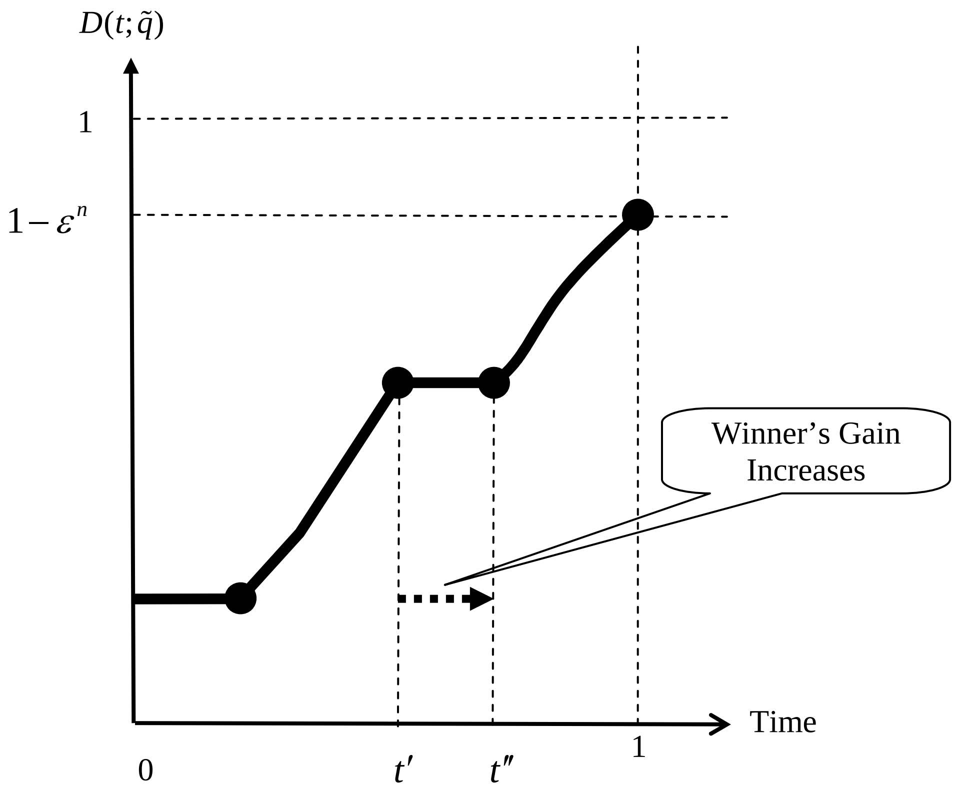
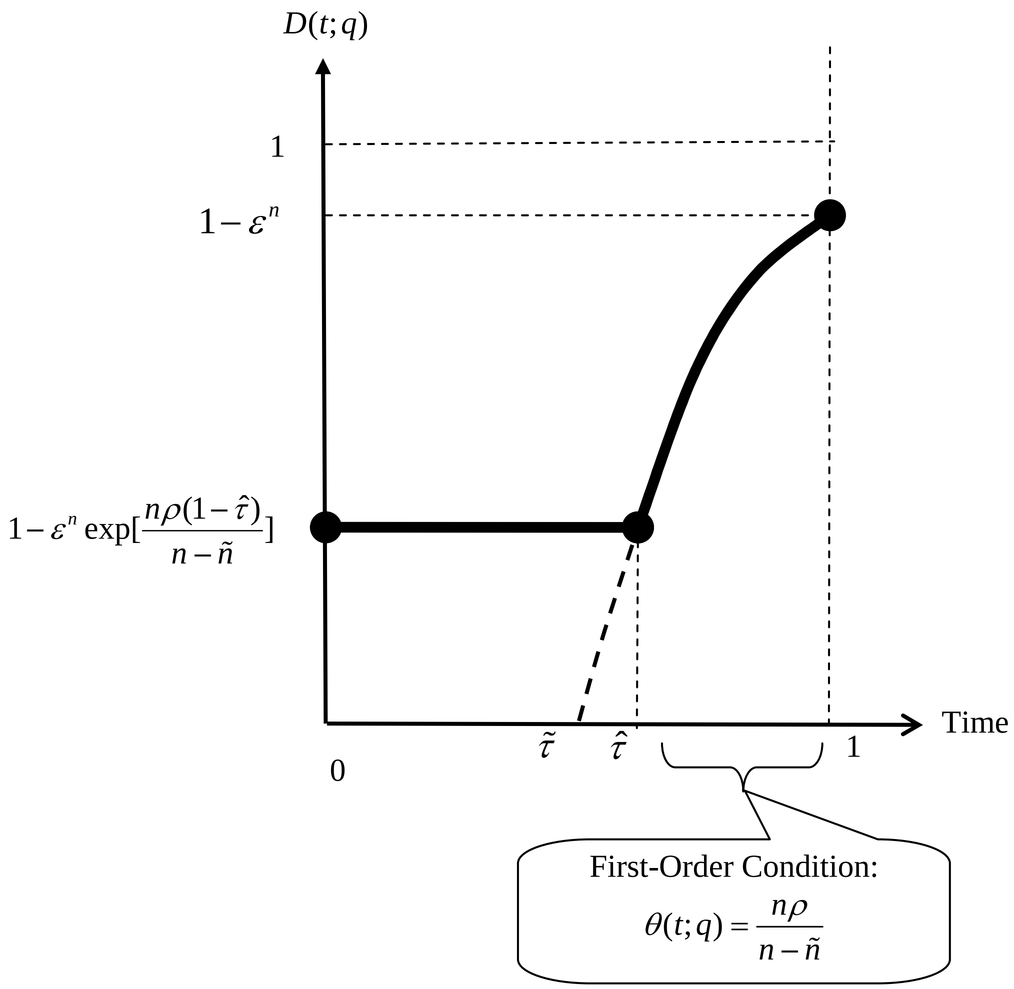


Figure 7



$D(t;q)$
1
$1-\varepsilon^n$
$1-\varepsilon^n \exp[\frac{n\rho(1-\hat{\tau})}{n-\tilde{n}}]$
0
$\tilde{\tau}$
$\hat{\tau}$
1
Time
First-Order Condition:
$\theta(t;q)=\frac{n\rho}{n-\tilde{n}}$