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Stationary Monetary Equilibria with Strictly Increasing Value Functions and Non-Discrete Money Holdings Distributions: An Indeterminacy Result

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Abstract

In this paper, we present a search model with divisible money in which there exists a continuum of monetary equilibria with strictly increasing continuous value functions and with non-discrete money holdings distributions.

Keywords: Real Indeterminacy, Random Matching, Money

Journal of Economic Literature Classification Number: C72, C78, D44, D51, D83, E40

1 Introduction

Recently, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible money. (See, for example, Green and Zhou [1] [2], Kamiya and Shimizu [3], Matsui and Shimizu [4], and Zhou [7].) However, all the indeterminacy results found so far are limited to the case that value functions are step functions and money holdings distributions are discrete ones. In this paper, we show that real indeterminacy can occur even in the case of strictly increasing continuous value functions and non-discrete money holdings distributions.

In his introductory paper of symposium volume of *Journal of Economic Theory*, Wallace [5] presents a conjecture that the indeterminacy result is not robust in the following sense:

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> The multiplicity is almost certainly not robust to departing from the assumption that the money is a fiat object. That, is if nominal holdings of the fiat object give utility (can be as paper weights or decoration or burned as fuel), then the kind of multiplicity that has people treating x units of a fiat asset as a new fiat object disappears. ([5] p.225)

Following this conjecture, Wallace and Zhu [6] introduce the concept of commodity-money refinement. To put it shortly, a stationary equilibrium satisfies the commodity-money refinement if it is also a limit of stationary equilibria as the consumption utility of money goes to zero. Wallace and Zhu apply the concept to two specific models. In one model, the commodity-money refinement eliminates stationary equilibria with discrete money holdings distributions, and in the other model the commodity-money refinement eliminates stationary equilibria with value functions that are not strictly increasing, such as step-functions.[1] One might think that Wallace and Zhu's result verifies the above conjecture, for all the previous results of real indeterminacy in money search models are limited to the case of discrete money holdings distributions and step value functions.[2]

The purpose of this note is to present a money search model in which there is a continuum of stationary equilibria with non-discrete money holdings distribution and strictly increasing continuous value functions. Moreover, they satisfy the commodity-money refinement. In Section 2, we present the model, and then in Section 3 we present the indeterminacy result and show that the stationary equilibria indeed satisfies the commodity-money refinement in the sense of Wallace and Zhu.

2 The Model

There is a continuum of agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of perfectly divisible goods. A type $i-1$ agent can produce type i good. (We assume that a type k agent produces type 1 good.) The production technology of each agent is characterized by a fixed cost $\bar{c} > 0$, zero

[1] Note that their results also depend upon their definition of refinement. Zhou [8] defines the commodity-money refinement in another way and shows that there is a continuum of robust stationary equilibria with discrete money holdings distributions and step value functions.

[2] Green and Zhou [2] construct non-stationary equilibria with non-discrete money holdings distributions, but the stationary equilibria in their model have discrete money holdings distributions.

marginal cost, and a capacity constraint $\bar{q} > 0$, where $\bar{c}$ and $\bar{q}$ are common to all agents. More precisely, the cost function of each agent is expressed as:

$$C(q) = \begin{cases} 0, & \text{if } q = 0, \\ \bar{c}, & \text{if } 0 < q \leq \bar{q}, \\ \infty, & \text{if } \bar{q} < q. \end{cases}$$

A type i agent obtains utility only when she consumes type i good and her utility function is expressed by a linear function, $u(q) = aq$, where $a > 0$ is given and q is the amount of good i. Note that a is common to all agents.

Time is discrete. In each period, each agent first chooses either to be a consumer or a seller. Then pairwise random matchings take place. Note that if an agent of type i chooses to be a seller, then she cannot buy type i good even when she meet a type $i-1$ agent. If a type i seller meets a type $i+1$ buyer, then the seller makes a take-it-or-leave-it offer of (q_s, p_s) without knowing the partner's money holding, where q_s is the maximum amount of type $i+1$ good she can sell and p_s is the price of the good. Note that the seller knows the money holdings distribution of the economy. Finally, when (q_s, p_s) is offered, the type i buyer chooses the amount of good $q_b \leq q_s$ he wants to consume.

Let $m_0 \in [0,1]$ be the measure of agents without money and $f : (0,\infty) \to \mathbb{R}_+$ be a density function of money holdings on $(0,\infty)$. Of course, $1 - m_0 = \int_{(0,\infty)} f d\eta$ must hold. Let $M > 0$ be the nominal stock of fiat money and $\beta \in (0,1)$ be the discount factor.

The conditions for a stationary equilibrium are (i) each agent maximizes the expected value of utility-streams, i.e., the Bellman equation is satisfied, (ii) the money holdings distribution of the economy is stationary, i.e., time-invariant, and (iii) the total amount of money the agents have is equal to M.

We focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the k types are symmetric.

3 The Results

The following theorem is the main result of the present paper.

Theorem 1 Let $d = \frac{a\bar{q}}{\bar{c}}$. Suppose $\frac{3}{2} < d \leq 3$. Then there exists a $\underline{\beta} \in (0,1)$ such that, for any given $\beta \in (\underline{\beta}, 1)$, there exists a continuum of stationary equilibria in which (i) the value functions are continuous, strictly increasing, and concave, and (ii) the money holdings distributions have a full support in some closed interval with a nonempty interior.

Proof:

(I) We focus on the strategy satisfying the following: for some $p > 0$,

- an agent without money always chooses to be a seller and an agent with money holding $\eta > 0$ always chooses to be a buyer,
- a seller always offers $(p, \bar{q})$,
- a buyer with money holding $\eta > 0$ consumes the following amount of her consumption good: there exists a $\underline{p}(\eta) \geq p$ such that, for given (p_s, q_s),

$$q_b(\eta, p_s, q_s) = \begin{cases} \min\{\eta/p_s, q_s\} & \text{if } p_s \leq \underline{p}(\eta), \\ 0 & \text{if } p_s > \underline{p}(\eta), \end{cases} \tag{1}$$

- for some λ and σ, f is expressed by

$$f(\eta) = \begin{cases} 2\lambda\eta + \sigma, & \text{for } \eta \in (0, p\bar{q}], \\ 0, & \text{for } \eta \in (p\bar{q}, \infty]. \end{cases} \tag{2}$$

Note that p, $\underline{p}(\eta)$, λ, and σ will be determined as functions of m_0 later.

(II) Next, we obtain a candidate for a value function $V : \mathbb{R}_+ \to \mathbb{R}$ consistent with the above strategy. From the above strategy, $V(\eta)$ for $\eta \in (0, \infty)$ can be written as a function of $V(0)$ as follows. First, for $\eta \in (0, p\bar{q}]$,

$$V(\eta) = \frac{m_0}{k}\left(a\frac{\eta}{p} + \beta V(0)\right) + \left(1 - \frac{m_0}{k}\right)\beta V(\eta)$$

holds. Thus

$$V(\eta) = A(m_0)\left(a\frac{\eta}{p} + \beta V(0)\right), \quad \text{for } \eta \in (0, p\bar{q}], \tag{3}$$

where $A(m_0) = \frac{m_0}{k-(k-m_0)\beta}$. Note that $A(m_0) < 1$. Similarly, $V(\eta)$ for $\eta \in (p\bar{q}, \infty)$ is written as:

$$V(\eta) = A(m_0)\left(a\bar{q} + \beta V(\eta - p\bar{q})\right), \quad \text{for } \eta \in (p\bar{q}, \infty). \tag{4}$$

Next, since an agent without money always chooses to be a seller, then $V(0)$ is determined by

$$V(0) = \frac{1-m_0}{k}\left[-\bar{c} + \int_{(0,p\bar{q}]} \beta V(\eta)\frac{f(\eta)}{1-m_0}d\eta\right] + \left(1 - \frac{1-m_0}{k}\right)\beta V(0). \tag{5}$$

(III) Below, we focus on equilibria with $V(0) = 0$ and obtain (p, λ, σ) as functions of m_0.[3] First, we decompose $\eta \geq 0$ into an multiple of $p\bar{q}$ and a residual; that is, $\eta = np\bar{q} + \iota$, where n is a nonnegative integer and ι is a nonnegative real number less than $p\bar{q}$. Then, by (3) and (4),

$$V(np\bar{q} + \iota) = \frac{aA(m_0)}{1-\beta A(m_0)}\left\{\bar{q} - (\beta A(m_0))^n \left[\bar{q} - (1-\beta A(m_0))\frac{\iota}{p}\right]\right\} \tag{6}$$

holds. On the other hand, by (2) and (5),

$$(1-m_0)\bar{c} = \frac{a\beta A(m_0)}{p}\int_{(0,p\bar{q}]} \eta f(\eta)d\eta = a\beta A(m_0)\left(\frac{2}{3}\lambda p^2\bar{q}^3 + \frac{1}{2}\sigma p\bar{q}^2\right) \tag{7}$$

holds.

Below, we obtain (p, λ, σ) as functions of m_0. First, $1 - m_0 = \int_{(0,\infty)} f d\eta$ can be written as follows:

$$1 - m_0 = \int_{(0,p\bar{q}]} f d\eta = \lambda p^2\bar{q}^2 + \sigma p\bar{q}. \tag{8}$$

Since the total amount of money the agents have is equal to M, the following equation must be satisfied:

$$M = \int_{(0,p\bar{q}]} \eta f d\eta = \frac{2}{3}\lambda p^3\bar{q}^3 + \frac{1}{2}\sigma p^2\bar{q}^2. \tag{9}$$

By (7), (8), (9), and $d = \frac{a\bar{q}}{\bar{c}}$, we obtain

$$p = \frac{Ma\beta A(m_0)}{(1-m_0)\bar{c}}, \tag{10}$$

$$\lambda = \frac{3(1-m_0)^3(2-\beta dA(m_0))}{M^2\beta^3 d^3(A(m_0))^3}, \tag{11}$$

$$\sigma = \frac{2(1-m_0)^2(-3+2\beta dA(m_0))}{M\beta^2 d^2(A(m_0))^2}. \tag{12}$$

[3] If $V(0) > 0$, then an agent with a small amount of money does not choose to be a buyer. Indeed, by (3), $\lim_{\eta\downarrow 0} V(\eta) = \beta V(0) < V(0)$ when $V(0) > 0$.

(IV) Next, we check the optimality of the specified strategy.

(i) The optimality of the strategy of an agent with money holding $\eta > 0$:
First, we show that there exists a $\underline{p}(\eta) \geq p$ in (1). If $\eta \in (0, p\bar{q}]$, then by (6),

$$aq + \beta V(\eta - p_s q) = aq\left(1 - \beta A(m_0)\frac{p_s}{p}\right) + a\beta A(m_0)\frac{\eta}{p}$$

holds. Thus if

$$1 - \beta A(m_0)\frac{p_s}{p} \geq 0$$

holds, then she clearly chooses the maximum amount she can buy, and otherwise she chooses $q_b = 0$. Note that $1 - \beta A(m_0) \geq 0$ holds, since $\beta A(m_0) < 1$. Let

$$\underline{p}(\eta) = \frac{p}{\beta A(m_0)}, \quad \text{for } \eta \in (0, p\bar{q}]. \tag{13}$$

Then, (1) is optimal for $\eta \in (0, p\bar{q}]$. Moreover, $\underline{p}(\eta) \geq p$ clearly holds. Similar arguments apply to the case of $\eta \in (p\bar{q}, \infty)$.

Next, we check an incentive for an agent with $\eta > 0$ to become a buyer instead of becoming a seller and offering (p', q'). By (1) and (13), for any $p' > \frac{p}{\beta A(m_0)}$, no buyer accepts such an offer on the equilibrium, and then the value is the same as that of an offer $(p'', 0)$, where $p'' \leq \frac{p}{\beta A(m_0)}$. Therefore, we can restrict our attention to (p', q') such that $p' \in \left[0, \frac{p}{\beta A(m_0)}\right]$ and $q' \in [0, \bar{q}]$. By (1), the value of becoming a seller and offering (p', q') is

$$\frac{1 - m_0}{k}\left[-\bar{c} + \int_{(0,p\bar{q}]} \beta\tilde{V}(\eta, \eta')\frac{f(\eta')}{1 - m_0}d\eta'\right] + \left(1 - \frac{1 - m_0}{k}\right)\beta V(\eta),$$

where

$$\tilde{V}(\eta, \eta') = \begin{cases} V(\eta + \eta'), & \text{if } \eta' \leq p'q', \\ V(\eta + p'q'), & \text{if } \eta' > p'q'. \end{cases} \tag{14}$$

On the other hand, when she becomes a buyer, the value is $V(\eta)$. Thus the difference is

$$\frac{1 - m_0}{k}\left[-\bar{c} + \int_{(0,p\bar{q}]} \beta\left(V(\eta, \eta') - V(\eta)\right)\frac{f(\eta')}{1 - m_0}d\eta'\right] - (1 - \beta)V(\eta). \tag{15}$$

Below, we show

$$V(\eta+\eta')-V(\eta)\le aA(m_0)\frac{\eta'}{p},\quad \text{for } \eta'\in(0,p\bar{q}]. \tag{16}$$

First, there exits a unique nonnegative integer n such that $np\bar{q}\le\eta<(n+1)p\bar{q}$. There are two cases: (a) $\eta+\eta'<(n+1)p\bar{q}$ and (b) $\eta+\eta'\ge(n+1)p\bar{q}$. In case (a), by (6) and $\beta A(m_0)<1$,

$$\begin{aligned}
V(\eta+\eta')-V(\eta)&=\frac{aA(m_0)}{1-\beta A(m_0)}\left\{\bar{q}-(\beta A(m_0))^n\left[\bar{q}-(1-\beta A(m_0))\frac{\eta+\eta'-np\bar{q}}{p}\right]\right\}\\
&\quad-\frac{aA(m_0)}{1-\beta A(m_0)}\left\{\bar{q}-(\beta A(m_0))^n\left[\bar{q}-(1-\beta A(m_0))\frac{\eta-np\bar{q}}{p}\right]\right\}\\
&\le aA(m_0)\left(\beta A(m_0)\right)^n\frac{\eta'}{p}\\
&\le aA(m_0)\frac{\eta'}{p}.
\end{aligned}$$

In case (b), by (6) and $\beta A(m_0)<1$,

$$\begin{aligned}
V(\eta+\eta')-V(\eta)&=\frac{aA(m_0)}{1-\beta A(m_0)}\left\{\bar{q}-(\beta A(m_0))^{n+1}\left[\bar{q}-(1-\beta A(m_0))\frac{\eta+\eta'-(n+1)p\bar{q}}{p}\right]\right\}\\
&\quad-\frac{aA(m_0)}{1-\beta A(m_0)}\left\{\bar{q}-(\beta A(m_0))^n\left[\bar{q}-(1-\beta A(m_0))\frac{\eta-np\bar{q}}{p}\right]\right\}\\
&=aA(m_0)\left(\beta A(m_0)\right)^n\left[(1-\beta A(m_0))\,(n+1)\bar{q}+\beta A(m_0)\frac{\eta'}{p}-(1-\beta A(m_0))\,\frac{\eta}{p}\right]\\
&\le aA(m_0)\left(\beta A(m_0)\right)^n\frac{\eta'}{p}\\
&\le aA(m_0)\frac{\eta'}{p}.
\end{aligned}$$

The fourth line is obtained by $\eta\ge(n+1)p\bar{q}-\eta'$. This completes the proof of (16).

(6), (14), and (16) imply

$$\tilde{V}(\eta,\eta')-V(\eta)\le aA(m_0)\frac{\eta'}{p},\quad \text{for } \eta'\in(0,p\bar{q}].$$

Then, the first term of (15) is less than or equal to

$$\frac{1-m_0}{k}\left[-\bar{c}+\int_{(0,p\bar{q}]}a\beta A(m_0)\frac{\eta'}{p}\frac{f(\eta')}{1-m_0}d\eta'\right].$$

This is equal to zero by the first equality of (7), and thus (15) is non-positive and she becomes a buyer.

(ii) The optimality of the strategy of an agent without money:

By the construction, an agent without money is indifferent between a buyer and a seller. Thus she has an incentive to be a seller. As in the latter part of (i), we restrict our attention to offers (p', q') such that $p' \in \left[0, \frac{p}{\beta A(m_0)}\right]$ and $q' \in [0, \bar{q}]$. By (1) and (6), the value of offering (p', q') is

$$\frac{1-m_0}{k}\left[-\bar{c} + \int_{(0,p\bar{q}]} \beta\tilde{V}(0,\eta')\frac{f(\eta')}{1-m_0}d\eta'\right], \tag{17}$$

where $\tilde{V}$ is defined in (14). If $p'q' \geq p\bar{q}$, $\tilde{V}(0, \eta') = V(\eta')$ for any $\eta' \in (0, p\bar{q}]$. Then, (17) is the same for all $p'q' \geq p\bar{q}$, and therefore the offer $(p, \bar{q})$ is optimal. If $p'q' \leq p\bar{q}$,

$$\begin{aligned}
\int_{(0,p\bar{q}]} \tilde{V}(0,\eta')f(\eta')d\eta' &= \int_{(0,p'q']} V(\eta')f(\eta')d\eta' + \int_{(p'q',p\bar{q}]} V(p\bar{q})f(\eta')d\eta' \\
&\leq \int_{(0,p'q']} V(\eta')f(\eta')d\eta' + \int_{(p'q',p\bar{q}]} V(\eta')f(\eta')d\eta' \\
&= \int_{(0,p\bar{q}]} \tilde{V}(0,p\bar{q})f(\eta')d\eta',
\end{aligned}$$

where the inequality is obtained by (6) and (14). Then, the offer $(p, \bar{q})$ is optimal. This completes the proof of (IV).

(V) Finally, we check $f(\eta) \geq 0$ for all $\eta \in (0, p\bar{q}]$. Since f is linear, it suffices to show $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$. By (10), (11), and (12),

$$f(0) = \sigma = \frac{2(1-m_0)^2(-3+2\beta dA(m_0))}{M\beta^2 d^2(A(m_0))^2}$$

and

$$f(p\bar{q}) = 2\lambda p\bar{q} + \sigma = \frac{2(1-m_0)^2(3-\beta dA(m_0))}{M\beta^2 d^2(A(m_0))^2}$$

hold. A sufficient condition for $f(0) \geq 0$ and $f(p\bar{q}) \geq 0$ is clearly

$$\frac{3}{2} \leq \beta dA(m_0) \leq 3.$$

By the assumption $d \leq 3$, $\beta dA(m_0) \leq 3$ is always satisfied. It is easily verified that $\frac{3}{2} \leq \beta dA(m_0)$ is equivalent to

$$m_0 \geq \frac{3k(1-\beta)}{\beta(2d-3)}. \tag{18}$$

Setting $\underline{\beta} = \frac{3k}{3(k-1)+2d}$, we can show that for any $\beta \in (\underline{\beta}, 1)$ there exists a continuum of m_0 satisfying (18) and $m_0 \in (0,1)$. Indeed, $\underline{\beta} < 1$ follows from the assumption $d > \frac{3}{2}$, and $1 > \frac{3k(1-\beta)}{\beta(2d-3)}$ follows from $\beta > \underline{\beta}$. Note that the stationarity of money holdings distribution clearly holds. This concludes the proof. ∎

We show that any equilibrium in Theorem 1 satisfies the commodity-money refinement defined by Wallace and Zhu [6]. Wallace and Zhu assume that agents receive utility $\varepsilon\gamma(\eta)$ by consuming η amount of money, where $\varepsilon \geq 0$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, bounded, strictly increasing, and concave, and satisfies $\gamma(0) = 0$ and $\gamma'(0)$ finite. A stationary equilibrium in the case of fiat money (i.e., $\varepsilon = 0$) satisfies the commodity-money refinement if and only if it is a limit of some sequence of stationary equilibria as $\varepsilon \downarrow 0$, i.e., a limit of commodity money equilibria.

By (6), in any stationary equilibrium described in Theorem 1, the value function is positively linear with respect to money holding $\eta \in [0, p\bar{q}]$. This implies that, even in the case of commodity money, if ε is sufficiently small, then agents never consume the money. Therefore, the set of equilibria in the case of small $\varepsilon > 0$ is the same as that in the case of $\varepsilon = 0$.

Corollary 1 Any equilibrium in Theorem 1 satisfies the commodity-money refinement. This implies that there is a continuum of stationary equilibria that satisfies the commodity-money refinement.

We define the welfare as the average of values. Then by (6), (7), (8), and (9), we obtain

$$\begin{aligned} W &= m_0 V(0) + \int_{(0,p\bar{q}]} V(\eta) f(\eta) d\eta \\ &= \frac{\bar{c}}{\beta}(1 - m_0). \end{aligned}$$

In other words, the smaller m_0 is, the higher the welfare level is, as long as V is an equilibrium value function. Then $m_0 = \frac{3k(1-\beta)}{\beta(2d-3)}$ yields the highest welfare level among this class of equilibria. This implies the above indeterminacy result is real; there is a continuum of Pareto-rankable stationary equilibria.

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