CIRJE-F-627

Investment Frictions versus Financing Frictions

Takao Kobayashi
University of Tokyo

Risa Sai
Graduate School of Economics, University of Tokyo

July 2009

Investment Frictions versus Financing Frictions

Takao Kobayashi

University of Tokyo

Risa Sai

University of Tokyo

Research Fellow of the Japan Society for the Promotion of Science

July 20, 2009

Abstract

Bertola/Caballero (1994) and Abel/Eberly (1996) extended Jorgenson's classical model of firms' optimal investment. By introducing investment frictions, they were able to capture the role of future anticipations in investment decisions as well as the lumpy and intermittent nature of investment dynamics. We extend Jorgenson's model to the other direction of financing frictions. We construct a model of an equity-only firm, who must pay a linear financing cost for issuing new shares. We show that the firm's optimal investment-financing is a two-trigger policy in which the firm finances investment by issuing new shares (supplementing internal funds) when the shadow price of capital hits the upper trigger value. When the shadow price hits the lower trigger value, she sells a portion of her capital stock and buys back shares (or pays dividends). Values of the shadow price of capital between the two trigger values define a range of "inaction", in which the firm does neither issue nor buy back shares and invests all of her internal funds for expansion.

1 Introduction

Frictions are the primary theme of the theory of the firm and corporate finance. In the past four decades, there has been a flood of theoretical as well as empirical studies on how investment frictions influence the dynamics of corporate investments. In contrast, how financing frictions influence corporate dynamics is a recent topic of research and has been studied mostly via numerical approach which provides realistic but intractable analysis.

Jorgenson's formulation of a dynamic investment problem (Jorgenson (1963)) is where our construction starts. In his classical model the firm can adjust her capital stock without any frictions, and this absence of frictions makes firm's optimal investment policy to be a purely static one in which the marginal product of capital equals the user cost of capital.

The firm's decision becomes a truly dynamic problem, in which anticipations about the future economic environment affect current decisions when frictions prevent instantaneous and costless adjustment of the capital stock. Bertola and Caballero (1994) introduced "irreversibility" of investment in the sense that the firm cannot sell its capital stock. This is equivalent to assuming that the selling price of capital is zero. Abel and Eberly (1996) generalized this model to the case of "costly reversibility"; namely the firm can sell its capital stock but at a price less than the purchase price. They were successful in showing that frictions are the source of nonlinear and intermittent investment dynamics[1].

In this paper, we pursue the other direction of introducing financing frictions to Jorgenson's model. "Irreversibility" in our model means that the firm may return cash to stockholders by paying dividends or buying back shares but cannot obtain additional cash from stockholders by issuing new shares[2]. "Costly reversibility" means that the firm can finance externally but at some cost[3]. Readers will see that our development goes very much

[1] Nonlinear adjustment cost incurred upon changing production capacity is another source of investment frictions. This idea originates in Uzawa's "Penrose curve" (Uzawa (1969)). There is again an ample literature on this topic including Mussa (1977), Abel (1983) and Hayashi (1982). See Abel and Eberly (1994) for a unification of both approaches. In this paper we do not assume the adjustment cost of investment (nor any form of investment frictions) in order to focus solely on the impact of financing frictions.

[2] Neglecting taxation dividends and share repurchase are equivalent in our model. We will use these terms interchangeably, but will mostly use "buying back shares" instead of "dividends" to contrast it to "issuing new shares".

[3] We do not explicitly model asymmmetic information and/or differential taxation to

in parallel with the irreversible investment literature. Yet, by introducing financial frictions our formulation naturally includes financing decisions as well as investment decisions, which is in contrast to the optimal investment literature. In this regard, the benchmark Jorgensonian model may as well be called the Modigliani-Miller model in our context.

We construct a model of an equity-only firm who must pay a linear financing cost for issuing new shares. We show that the firm's optimal investment-financing is a two-trigger policy in which the firm finances its investment by issuing new shares (supplementing internal funds) when the shadow price of capital hits the upper trigger value. When the shadow price hits the lower trigger value, she sells a portion of her capital stock and buys back shares (or pays dividends). Values of the shadow price of capital between the two trigger values define a range of "inaction", in which the firm does neither issue nor buy back shares and invests all of her internal funds for expansion. Analytically we work on the Hamilton-Jacobi-Bellman equation to characterize the optimal investment and financing policy. We will show that financial frictions force the shadow price of capital to satisfy a second-order ordinary linear differential equation, which is dual to the one generated by investment frictions.

Vast amount of researches has been done to study how financial constraints and financial frictions affect corporate behavior. But most of them are empirical and only a handful of papers formulate the firm's investment and financing decisions as a stochastic dynamic optimization problem.

Whited (2006) and Hennessy and Whited (2005, 2007) are most closely related to ours in this regard. Using a discrete-time formulation their interest is to provide a model which produces lumpy and intermittent corporate investments most consistent with reality. Thus their models include such factors as taxation, bankruptcy costs, endogenous defaults, endogenous borrowing rate of interest, and nonlinear equity-issuance costs. However, relying on numerical computation to solve the dynamic optimization problem, the characterization of the optimal investment/financing policy is imperfect and theoretical insights are limited.

As stated in footnote 1 we avoid the introduction of nonlinear adjustment cost of investment. These authors share the same spirit, except that Whited (2006) investigates how lumpy investment behavior depends on nonconvex

endogenize these costs. This is because we want to investigate analytically the impact of these costs on corporate investment and financing decisions in the simplest setting.

physical adjustment cost and the presence of financial constraints.

A latest working paper by Bolton, Chen and Wang (2009) is more in line with ours in that they analytically derive the nature of the optimal policy[4]. Their model includes a wider set of corporate decisions than ours such as cash management and default decisions. On the other hand they work on a revenue function which is proportional to capital, whence nonlinear capital adjustment cost is essential for their formulation. This aspect of their model considerably restricts the generality and depth of their theoretical inquiry.

The paper is organized as follows. Section 2 gives the Jorgenson's model in a format that serves as a benchmark to our considerations. In section 3 we present the analysis of "irreversible investment" and then proceed to the case of "costly reversible investment". We will provide a more straightforward derivation of the optimal policy for the latter case. In section 4 we present a model with "irreversible financing". Section 5 extends the analysis to the case of "costly reversible financing". In section 6 we compute the optimal investment and financing policy numerically by the value-iteration algorithm and show graphically a typical path of the capital stock corresponding to each case. In this paper we assume that the exogenous process driving the market and technological conditions follows a geometric Brownian motion. This technical assumption reduces the problem to one with a single state variable. Section 7 shows how one can extend the model to reflect changing growth-rate, which expands the problem to a two-dimensional state space. Section 8 concludes the paper and proposes additional directions of extending this research.

2 The Classical Jorgensonian Model of Investment

We extend Jorgenson's classical model of investment to a stochastic environment. We also provide an alternative formulation, which is more suited to analyze the influence of financial constraints. *He showed that when a firm can adjust its capital stock without any frictions its optimal investment policy is a static decision in which the construct called the user cost of capital is the key variable affecting investment.*

---

[4] We thank Hayne Leland to teach us of this paper.

Firm's optimization problem

Consider a firm who uses capital stock K_t to produce output. The firm's instantaneous operating cash flow is given by

$$\Pi\left(K_{\,t}, Z_t\right) = K_t^{\alpha} Z_t, \quad 0 < \alpha < 1, \tag{1}$$

where Z_t denotes a random shock which represents the business conditions facing the firm such as strength of demand, costs of inputs and firm's productivity. The specification in (1) can be derived if firm's production and demand functions have constant elasticity. Assume that $\{Z_t\}$ follows a geometric Brownian motion

$$dZ_t = Z_t\left(\mu dt + \sigma dW_t\right), \tag{2}$$

where μ and σ are constants and $\{W_t\}$ is a standard Wiener process.

Capital can be purchased and sold at a constant unit price P[5]. The capital stock depreciates at a constant proportional rate $\delta \geq 0$, so the capital stock evolves according to

$$dK_t = -\delta K_t dt + dG_t, \tag{3}$$

in which $\{G_t\}$ denotes the cumulative gross investment process[6]. At this stage the increments $\{dG_t\}$ is unconstrained in sign; i.e., investments are reversible.

Assume that the firm maximizes the market value of the firm which is defined as the expected present value of net cash flows (free cash flows) discounted at a constant positive rate r[7]. The market value of the firm is

$$V\left(K_0, Z_0\right) \equiv \max_{\{G_t\}} E_0\left[\int_0^{\infty} e^{-rt}\left\{K_t^{\alpha} Z_t dt - P dG_t\right\}\right] \tag{4}$$

[5] We can obviously assume that the price of capital stock also moves randomly. We suppress this generality to focus attention on the fundamental nature of the optimal decisions.

[6] Formally, let $\{\Omega, \mathcal{F}, P\}$ be a filtered probability space supporting a Brownian motion $\{W_t\}$ where $\mathcal{F} = \{\mathcal{F}_t\}$ and $\mathcal{F}_t$ represents the augmented filtration generated by all the information up to time t. We assume that $\{G_t\}$ is adapted to $\mathcal{F}$. All the decision variables in subsequent sections are assumed to be adapted to $\mathcal{F}$.

[7] To keep the generality of the model one can either assume that r is a riskless rate of interest and the expectation is taken under the risk-neutral probability measure or that r is a risk-adjusted rate and the expectation is taken under the natural probability measure.

Since G_t is not differentiable, the second term in (4) is to be interpreted as Itô integral. As usual the subscript of the expectation operator indicates timing of the information with which the conditional expectation is taken.

The valuation (4) reflects the cash flow discount formula. We can transform this to the dividend discount formula as follows. Denote the cumulative dividend process by $\{D_t\}$. Again we have no sign restriction on $\{dD_t\}$ at this stage, so that $dD_t < 0$ means negative dividends; i.e., financing by issuing new shares. At each date we have the budget equation

$$K_t^{\alpha} Z_t dt = dD_t + P dG_t. \tag{5}$$

Using (5) the valuation formula (4) is rewritten as

$$V(K_0, Z_0) = \max_{\{D_t\}} E_0 \left[\int_0^{\infty} e^{-rt} dD_t \right]. \tag{6}$$

In this section we need to use (4) as we will be focusing on the investment constraints such as $dG_t \geq 0$. In section 3 and 4 we will need to work on (6) as we will focus on the financing constraints such as $dD_t \geq 0$. Note that a choice of investment policy $\{G_t\}$ implies a corresponding choice of financing policy $\{dD_t\}$ and *vice versa*[8].

Optimal investment policy under no frictions

Inserting (3) for dG_t into (4) and integrating by parts, one can translate the problem into a maximization problem with respect to $\{K_t\}$:

$$V(K_0, Z_0) = PK_0 + E_0 \left[\int_0^{\infty} e^{-rt} \max_{\{K_t\}} \{K_t^{\alpha} Z_t - (r+\delta) PK_t\} dt \right] \tag{7}$$

This valuation formula shows that the market value of the firm consists of (i) the physical value of the capital stock on hand and (ii) the value created by using the capital stock for business activities. The contribution in the second term during a small time interval $(t, t+\Delta t)$ is the operating cash flow, $(K_t^{\alpha} Z_t) \Delta t$, in excess of Jorgenson's "user cost of capital", $(r+\delta) PK_t \Delta t$. Jorgenson defines the user cost of capital as the opportunity cost of carrying a

[8]Some restriction on the space of $\{G_t\}$ and $\{D_t\}$ (such as restricting $\{G_t\}$ and $\{D_t\}$ as predictable processes) is necessary to guarantee the uniqueness of the process $\{G_t\}$ given the process $\{D_t\}$.

stock of progressively depreciating capital[9]. Alternatively it is the accounting earnings, $(K_t^\alpha Z_t - \delta P K_t)\,\Delta t$, in excess of the required return on capital, $(PK_t)\,r\Delta t$. It is worth noting that this simple "residual income" valuation formula holds only under the assumption of no frictions in investment and financing.

The right hand side of (7) reveals that the firm's maximization problem reduces to a static problem of choosing the capacity K_t at each date which maximizes the periodical residual income. The first-order condition for K_t is

$$\alpha K_t^{\alpha-1} Z_t = (r+\delta)\,P. \tag{8}$$

The firm is best served by the myopic rule of setting the marginal revenue product of capital equal to the user cost of capital at every instance of time. Solving (8) the optimal *frictionless* capital stock is given by the rule

$$K^J(Z_t) = \left(\frac{\alpha}{(r+\delta)\,P}\right)^{1/(1-\alpha)} Z_t^{1/(1-\alpha)} \qquad \forall t. \tag{9}$$

Inserting this result back to (7) and carrying out the stochastic integration, we find that the optimal value of the firm at time t is given by

$$V(K_t, Z_t) = PK_t + AZ_t^{1/(1-\alpha)} \tag{10}$$

where

$$A \equiv \frac{1-\alpha}{r - \frac{1}{1-\alpha}\left[\mu + \frac{\alpha}{2(1-\alpha)}\sigma^2\right]}\left(\frac{\alpha}{(r+\delta)\,P}\right)^{\alpha/(1-\alpha)}. \tag{11}$$

Note that the integral in (7) converges if

$$r > \frac{1}{1-\alpha}\left[\mu + \frac{\alpha}{2(1-\alpha)}\sigma^2\right]. \tag{12}$$

Remark 1 *Check this formula as Bertola's equation (9) has a minus sign.*

Budget equation (5) dictates how the investment is financed. If $K_t^\alpha Z_t \Delta t < \Delta G_t$ the difference is financed by issuing shares, whereas if $K_t^\alpha Z_t \Delta t > \Delta G_t$ the excess cash flow is paid as dividends.

Remark 2 *Optimal investment dictates financing in section 2 and 3. Optimal financing dictates investments in section 4. In paper 2 with banks we return to the former relationship even with financial frictions.*

[9] If the price of capital stock changes over time as Jorgenson assumed, the user cost of capital includes the capital loss (or capital gain) realized on holding the stock.

3 The Model With Investment Frictions

In this section we summarize the analysis provided by Bertola and Caballero (1994) for the case of irreversible investments. We then proceed for the case of costly reversible investments. Both cases allow closed form solutions for the optimal investment policy. We will provide a more straightforward derivation of the optimal policy on costly reversibility than was offered by Abel and Eberly (1996).

Irreversible investments

Irreversible investment can be characterized by the constraint $dG_t \geq 0$. We use the dynamic programming to solve this problem.

The Hamilton-Jacobi-Bellman equation of this problem takes the form

$$rV\left(K_t, Z_t\right) dt = \max_{\{dG_t \geq 0\}} \left\{K_t^{\alpha} Z_t dt - P dG_t + E_t\left[dV\left(K_t, Z_t\right)\right]\right\}, \tag{13}$$

in which $E_t\left[dV\right]$ denotes the infinitessimal generator applied on $V\left(K_t, Z_t\right)$. The left-hand side of equation (13) is the required return on the firm. The right-hand side of (13) is the maximized expected return consisting of net cash flow plus the expected change in the value of firm. Using Itô's lemma we obtain

$$\begin{aligned} rV\left(K_t, Z_t\right) dt &= \max_{\{dG_t \geq 0\}} \left\{\left(V_K - P\right) dG_t + K_t^{\alpha} Z_t dt - \delta K_t V_K dt \right. \\ &\qquad \left. + \mu Z_t V_Z dt + \frac{1}{2}\sigma^2 Z_t^2 V_{ZZ} dt\right\}. \end{aligned} \tag{14}$$

The inequality $dG_t \geq 0$ requires the complementary slackness condition

$$V_K - P \begin{cases} \leq 0 & \forall t; \\ = 0 & \forall t : dG_t > 0 \end{cases} \tag{15}$$

This implies that the shadow price of capital, or the marginal valuation of capital, should never be allowed to exceed its price P.

Adopting this condition (14) reduces to

$$rV\left(K_t, Z_t\right) = K_t^{\alpha} Z_t - \delta K_t V_K + \mu Z_t V_Z + \frac{1}{2}\sigma^2 Z_t^2 V_{ZZ}. \tag{16}$$

Since this equation holds identically along the optimal path, and can be differentiated term-by-term with respect to K_t, taking the partial derivatives of both sides of (16) by K_t yields

$$rV_K = \alpha K^{\alpha-1}Z - \delta V_K - \delta K V_{KK} + \mu Z V_{KZ} + \frac{1}{2}\sigma^2 Z^2 V_{KZZ}. \tag{17}$$

Defining the shadow price of capital by

$$v(K,Z) \equiv V_K(K,Z),$$

(17) is

$$(r+\delta)\,v = \alpha\left(Z/K^{1-\alpha}\right) - \delta K v_K + \mu Z v_Z + \frac{1}{2}\sigma^2 Z^2 v_{ZZ}. \tag{18}$$

It turns out that $v(K,Z)$ depends on (K,Z) through

$$y \equiv \frac{Z^{1/(1-\alpha)}}{K}. \tag{19}$$

Expressing $v(K,Z) \equiv q(y)$ the partial differential equation (18) becomes an ordinary differential equation of the form

$$\frac{\sigma^2}{2(1-\alpha)^2}y^2 q''(y) + \left\{\delta + \frac{\mu}{1-\alpha} + \frac{\alpha\sigma^2}{2(1-\alpha)^2}\right\} y q'(y) - (r+\delta)\,q(y) = -\alpha y^{1-\alpha} \tag{20}$$

In addition to satisfying the differential equation (20), $q(y)$ must satisfy the boundary conditions. Optimal investment is zero when the shadow price of capital, $q(y)$, is less than the price of the capital stock P. Condition (15) requires that the firm should undertake positive gross investment only if $q(y)$ reaches P. The trigger value of y, which we denote by y_G, is given by the smooth-pasting condition

$$q(y_G) = P \tag{21}$$

and the high-contact condition

$$q'(y_G) = 0. \tag{22}$$

Figure 1 illustrates the nature of the shadow price, $q(y)$, corresponding to the optimal solution[10]. The complementary slackness condition (15) requires

[10] It is natural to expect that $q(y)$ increases in y. This is confirmed by the closed-form solution provided below.

that the optimal policy should prevent $q(y)$ from ever exceeding P and should involve no gross investment, $dG_t = 0$, until $q(y)$ reaches the triggered value P. The latter requirement generates the differential equation (20). The smooth-pasting condition guarantees that if the boundary is reached at time t, the shadow price of capital, i.e., the value of additional unit of capital stock, will equal its cost, P. The high-contact condition ensures that the shadow price of capital does not change when investment is non-zero[11].

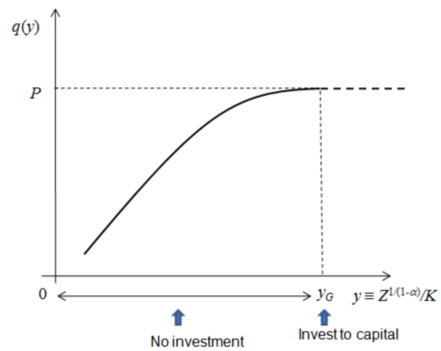

Figure 1. Shadow price of capital (irreversible investment)

The optimal *frictionless* capital stock was given by the function $K^J(Z_t)$, only dependent on the prevailing business condition, Z_t. In contrast, if the investment is irreversible, the optimal capital stock becomes history-dependent. Figure 2 illustrates the firm's optimal investment policy. Using the trigger value, y_G, define function $K^G(Z_t)$ by

$$K^G(Z_t) \equiv \frac{1}{y_G} Z_t^{1/(1-\alpha)} \quad \forall t \tag{23}$$

Denote the currently installed capital stock by K_- and the let Z_- denote the unique solution to $K^G(Z_-) = K_-$. Then the optimal investment is given by the following rule: . If $Z_t > Z_-$ invest immediately so as to obtain $K_t = K^G(Z_t)$; otherwise K_t should be allowed to depreciate. The condition $Z_t > Z_-$ reflects an insufficient capital: The installed capital is too small relative to the firm's anticipation of the current and future business conditions[12]. The firm find herself stuck with excessive stock of capital when $Z_t < Z_-$.

[11] See Dumas (1991) and Abel and Eberly (1996) for a presentation and economic interpretations of the smooth-pasting and high-contact conditions below.

[12] Note that y_G reflects the future anticipations, in particular the volatility parameter σ, as will be shown in (28) and (30).

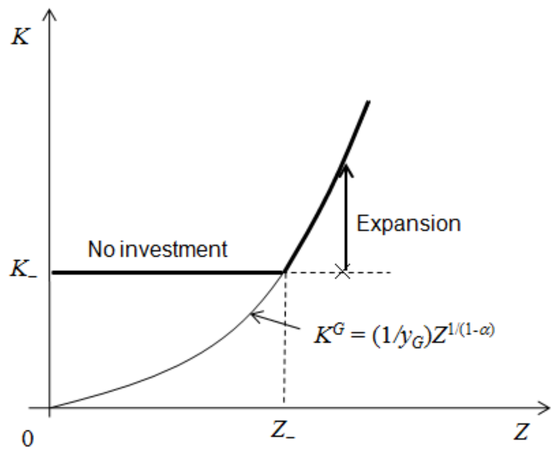

Figure 2. Optimal investment policy (irreversible investment)

The general solution to (20) is given by

$$q\,(y) = \alpha H y^{1-\alpha} + \alpha H C_1 y^{\beta_1} + \alpha H C_2 y^{\beta_2}, \tag{24}$$

where

$$H \equiv \frac{1}{r + \alpha\delta - \mu} \tag{25}$$

and β_1, β_2 are the roots of the characteristic equation

$$\frac{\sigma^2}{2\,(1-\alpha)^2}x\,(x-1) + \left\{\delta + \frac{\mu}{1-\alpha} + \frac{\alpha\sigma^2}{2\,(1-\alpha)^2}\right\}x - (r+\delta) = 0 \tag{26}$$

Since $r + \delta > 0$ this quadratic equation has two roots of opposite sign. Since $Z = 0$ is absorbing for the $\{Z(t)\}$ process, it must be the case that $\lim_{y \to 0} q(y) = 0$, to imply that only the positive root need be considered. Accordingly, we have

$$q\,(y) = \alpha H y^{1-\alpha} + \alpha H C y^{\beta} \qquad \forall y \leq y_G, \tag{27}$$

where and β is the positive root of (26), and B is a constant of integration. Note that $\beta > 1$ can be easily shown from (26) as (12) holds by assumption.

As shown in Appendix A, the solution of (B, y_G) that satisfy (21) and (22) turns out

$$y_G = \left(\frac{cP}{\alpha}\right)^{1/(1-\alpha)} \tag{28}$$

$$C = -\frac{1-\alpha}{\beta}\frac{1}{y_G^{\beta-(1-\alpha)}}, \tag{29}$$

where

$$c \equiv r + \delta + \frac{1}{2(1-\alpha)}\sigma^2\beta. \tag{30}$$

From (23) the desired level of capital is given by

$$K^G(Z_t) = \left(\frac{\alpha}{cP}\right)^{1/(1-\alpha)} Z_t^{1/(1-\alpha)}. \tag{31}$$

Since $c \geq r + \delta$, comparing (9) and (23) we find

$$K^G(Z_t) \leq K^J(Z_t) \qquad \forall Z_t, \tag{32}$$

with equality holding when $\sigma = 0$. The desired capital with irreversible investments is no larger than the optimal capital with reversibility. Intuitively, when capital once acquired cannot be resold again, the firm should be more prudent in investment[13]. If $\sigma = 0$ and the firm can perfectly anticipate the future, irreversibility plays no role and the optimal investment coincides with the *frictionless* case.

The shadow price of capital is given by

$$q(y) = \alpha H y^{1-\alpha} - H\frac{\alpha(1-\alpha)}{\beta}\frac{1}{y_G^{\beta-(1-\alpha)}}y^{\beta} \tag{33}$$

Integrating (33) the value of the firm under the optimal investment policy is given by[14]

$$V(K_t, Z_t) = HK_t^{\alpha}Z_t + H\frac{\alpha(1-\alpha)}{\beta(\beta-1)}\frac{1}{y_G^{\beta-(1-\alpha)}}\left(\frac{Z_t^{1/(1-\alpha)}}{K_t}\right)^{\beta} K_t. \tag{34}$$

[13] Bertola and Caballero (1994) shows that on average the capital stock under investment irreversibility is actually *higher* than the capital stock under *frictionless* investment (see footnote 3).

[14] This equation (34) shows that the value of the firm consists of two terms. The first term is the present value of expected operating cashflow if the firm operates with installed capital without additional investment. The second term is the value of the growth option, i.e. the option to increase the capital stock in the future. Note that both terms are positive. The marginal value of the first term in (33) is positive, but the marginal value of the second term is negative, since addition of capital marginally kills the growth option.

Costly reversible investments

Costless reversible investment and irreversible investment are opposite ends of a spectrum in which there is costly reversibility. Abel and Eberly (1994, 1996) studied investment with costly reversibility by introducing a difference between the purchase price and the sale price of capital.

When investment is costly reversible, firm's optimization problem is redefined as follows:

$$V(K_0, Z_0) \equiv \max_{\{dG_t \geq 0, dC_t \leq 0\}} E_0 \left[\int_0^\infty e^{-rt} \left\{ K_t^\alpha Z_t dt - P_U dG_t - P_L dC_t \right\} \right]$$

subject to

$$dK_t = -\delta K_t dt + dG_t + dC_t.$$

In this formulation $\{G_t\}$ denotes the cumulation of all purchases of capital and $\{C_t\}$ denotes the cumulation of all sales of capital up to time t. The increment dG_t is restricted to be nonnegative, and the increment dC_t is restricted to be nonpositive. The purchase price of capital, P_U, and the sales price of capital, P_L, are assumed to be constants. The wedge between the two prices could arise from transaction costs or from the firm-specific nature of capital. This model includes the Jongenson's case ($P_U = P_L$) and the irreversible investment case ($P_L = 0$).

The Hamilton-Jacobi-Bellman equation becomes

$$\begin{aligned} rV(K_t, Z_t)\, dt &= \max_{\{dG_t \geq 0\}} (V_K - P_U)\, dG_t + \max_{\{dC_t \leq 0\}} (V_K - P_L)\, dC_t \\ &\quad + K_t^\alpha Z_t dt - \delta K_t V_K dt + \mu Z_t V_Z dt + \frac{1}{2}\sigma^2 Z_t^2 V_{ZZ} dt. \end{aligned} \quad (35)$$

The inequality $dG_t \geq 0$ requires the complementary slackness condition

$$V_K - P_U \begin{cases} \leq 0 & \forall t; \\ = 0 & \forall t : dG_t > 0. \end{cases}$$

The inequality $dC_t \leq 0$ requires the complementary slackness condition

$$V_K - P_L \begin{cases} \geq 0 & \forall t; \\ = 0 & \forall t : dC_t > 0. \end{cases}$$

These conditions imply that $\max_{\{dG_t \geq 0\}} (V_K - P_U)\, dG_t = \max_{\{dC_t \leq 0\}} (V_K - P_L)\, dC_t = 0$. Applying these conditions to (35) we find that the shadow price of capital,

$q(y)$, should satisfy the same differential equation (20) that we obtained for the case of irreversible investments. We also have

$$P_L \leq q(y) \leq P_U \tag{36}$$

In this case the optimal investment policy becomes a two-trigger rule: buy capital when $q(y)$ touches the level equal to P_U; sell capital when $q(y)$ touches the level equal to P_L; no investment action when $q(y)$ is between P_U and P_L. The values of y which trigger purchase of capital and sales of capital are denoted by y_G and y_C, respectively. These trigger values are determined by the smooth-pasting condition

$$q(y_C) = P_L \text{ and } q(y_G) = P_U, \tag{37}$$

and the high-contact condition

$$q'(y_C) = 0 \text{ and } q'(y_G) = 0. \tag{38}$$

Figure 3 exhibits the shape of $q(y)$.

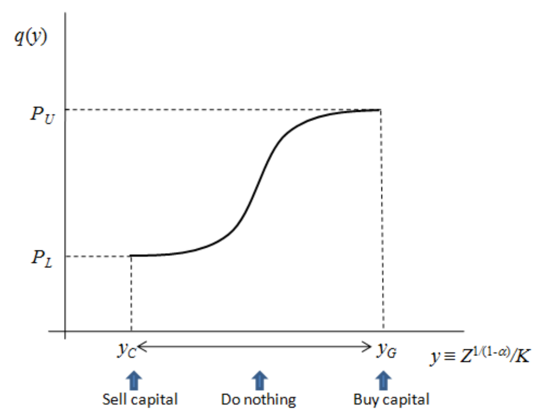

Figure 3. Shadow price of capital (costly reversibility)

Using the trigger values y_G and y_C define functions $K^G(Z_t)$ and $K^C(Z_t)$ by

$$K^G(Z_t) \equiv \frac{1}{y_G} Z_t^{1/(1-\alpha)}, \ K^C(Z_t) \equiv \frac{1}{yC} Z_t^{1/(1-\alpha)} \quad \forall t \tag{39}$$

Figure 4 illustrates the optimal investment policy. Let Z_-^G denote the unique solution to $K^G\left(Z_-^G\right) = K_-$, in which K_- is the current capital stock. Similarly let Z_-^C denote the unique solution to $K^C\left(Z_-^C\right) = K_-$. The optimal investment is given by the following rule: (i) If $Z_t > Z_-^G$ invest immediately so as to obtain $K_t = K^G(Z_t)$; (ii) If $Z_t < Z_-^C$ sell capital immediately so as to obtain $K_t = K^C(Z_t)$; (iii) otherwise no investment action and let capital depreciate.

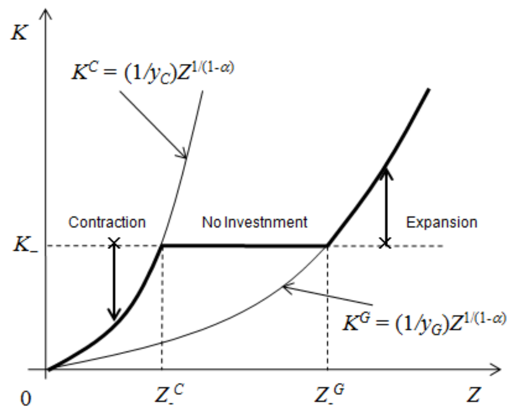

Figure 4. Optimal investment policy (costly reversibility)

We summarize the closed-form solution below[15]. Let $\beta_1 > 1$ and $\beta_2 < 0$ denote the two roots of the characteristic equation (26). Define functions $\theta(x)$ and $\phi(x)$ by

$$\theta(x) \equiv \frac{x^{\beta_1} - x^{1-\alpha}}{x^{\beta_1} - x^{\beta_2}}, \tag{40}$$

$$\phi(x) \equiv H\left\{1 - \frac{(1-\alpha)}{\beta_1}[1-\theta(x)] - \frac{(1-\alpha)}{\beta_2}\theta(x)\right\}. \tag{41}$$

Then y_G and y_C are obtained by

$$\alpha y_G^{1-\alpha} = \frac{P_U}{\phi(G^{-1})}, \quad \alpha y_C^{1-\alpha} = \frac{P_L}{\phi(G)}, \tag{42}$$

where $G \equiv y_G/y_C$ is the solution to

$$G^{1-\alpha}\frac{\phi(G^{-1})}{\phi(G)} = \frac{P_U}{P_L}. \tag{43}$$

Abel and Eberly (1996) showed that even a tiny wedge between the purchase price and the sale price of capital produces a substantial range of inaction.

[15] See Apendix B for an alternative and more straightforward derivation of the solution.

The shadow price of capital is given by

$$q(y) = \alpha H y^{1-\alpha} - H\frac{\alpha(1-\alpha)}{\beta_1}[1-\theta(G)]\, y_C^{(1-\alpha)-\beta_1} y^{\beta_1} - H\frac{\alpha(1-\alpha)}{\beta_2}\theta(G)\, y_C^{(1-\alpha)-\beta_2} y^{\beta_2}. \tag{44}$$

Integrating (44) the value of the firm under the optimal investment policy is given by[16]

$$\begin{aligned} V(K_t, Z_t) \quad = \quad & HK_t^\alpha Z_t + H\frac{\alpha(1-\alpha)}{\beta_1(\beta_1-1)}[1-\theta(G)]\frac{1}{y_C^{\beta_1-(1-\alpha)}}\left(\frac{Z_t^{1/(1-\alpha)}}{K_t}\right)^{\beta_1} K_t \\ & -H\frac{\alpha(1-\alpha)}{\beta_2(1-\beta_2)}\theta(G)\, y_C^{(1-\alpha)-\beta_2}\left(\frac{Z_t^{1/(1-\alpha)}}{K_t}\right)^{\beta_2} K_t. \end{aligned} \tag{45}$$

4 The Model With No External Financing

We now extend the foregoing analysis to the firm's optimal investment subject to financial frictions. To contrast the impact of financial frictions upon firm's investment to that of investment frictions we assume *frinctionless* investment, i.e., the firm can purchase or sell the capital stock at a constant price, P.

Assume that the firm cannot finance externally. She has the choice between earnings retention and dividends. All retained earnings are used to purchase capital stock. Thus investment decision is equivalent to dividend decision.

The decision problem is formulated as follows:

$$V(K_0, Z_0) \equiv \max_{\{dD_t \geq 0\}} E_0\left[\int_0^\infty e^{-rt} dD_t\right] \tag{46}$$

subject to (3) and

$$K_t^\alpha Z_t dt = dD_t + PdG_t, \tag{47}$$

[16] As in (34) the first term is the value of maintaining the current operation and the second term is the value of the growth option. The third term is the value of the abandonment option, i.e. the option to sell the capital stock in the future. Note that all terms are positive. In contrast, the marginal value of the second term in (44) is negative and the marginal value of the third term is positive. To verify these signs note that $0 < \theta(G) < 1$.

where $\{D_t\}$ denotes the cumulation of all dividends up to time t, which is restricted to have nonnegative increments ($dD_t \geq 0$). If we remove the nonnegativity constraint on $\{dD_t\}$, we return to the Jorgenson's case. In this regard the Jorgenson's case is more properly called the Modigliani-Miller case. "Irreversibility" in financing means that the firm may return cash to stockholders by paying dividends or buying back shares but cannot obtain additional cash from stockholders by issuing new shares. This financial constraint generates an optimal dynamic dividend policy which is nonlinear and path-dependent.

Solving the problem

The Hamilton-Jacobi-Bellman equation for this optimization problem is

$$rV\left(K_t, Z_t\right) dt = \max_{\{dD_t \geq 0\}} \left\{ dD_t + E_t\left[dV\left(K_t, Z_t\right)\right]\right\}. \tag{48}$$

Substituting (47) into (3) we can cancel the term dG_t and get

$$dK_t = -\delta K_t dt + \frac{1}{P}\left(K_t^{\alpha} Z_t dt - dD_t\right).$$

Using Itô's lemma, we obtain

$$\begin{aligned} rV\left(K_t, Z_t\right) dt \quad = \quad & \max_{\{dD_t \geq 0\}} \left\{ \left(1 - \frac{V_K}{P}\right) dD_t \right. \\ & \left. + V_K\left(-\delta K_t dt + \frac{1}{P} K_t^{\alpha} Z_t dt\right) + \mu Z_t V_Z dt + \frac{1}{2}\sigma^2 Z_t^2 V_{ZZ} dt \right\}. \end{aligned}$$

The inequality $dD_t \geq 0$ requires the complementary slackness condition

$$1 - \frac{V_K}{P} \begin{cases} \leq 0 & \forall t; \\ = 0 & \forall t : dD_t \geq 0 \end{cases} \tag{49}$$

It enforces the shadow price of installed capital never to fall below the price of capital P.

Adopting this condition the Hamilton-Jacobi-Bellman equation reduces to

$$rV\left(K, Z\right) = V_K\left(-\delta K + \frac{1}{P} K^{\alpha} Z\right) + \mu Z V_Z + \frac{1}{2}\sigma^2 Z^2 V_{ZZ} \tag{50}$$

Defining $v(K,Z) \equiv V_K(K,Z)$, $y \equiv Z^{1/(1-\alpha)}/K$, and $q(y) \equiv v(K,Z)$ as before, we can derive the following equation for $q(y)$ (see Appendix C for derivation).

$$\frac{\sigma^2}{2(1-\alpha)^2} y^2 q''(y) + \left\{ \delta + \frac{\mu}{1-\alpha} + \frac{\alpha\sigma^2}{2(1-\alpha)^2} - \frac{1}{P} y^{1-\alpha} \right\} y q'(y) - \left(r + \delta - \frac{\alpha}{P} y^{1-\alpha} \right) q(y) = 0. \tag{51}$$

This second-order linear differential equation is very similar to (20), except that it is a homogeneous equation. On the other hand, the power function $y^{1-\alpha}$ which was in the non-homogeneous term is now in the coefficient of $q'(y)$ and $q(y)$. This prevents a closed-form expression for $q(y)$.

The nature of optimal investment and dividend policy

The firm cannot raise additional capital from her stockholders. This financial constraint motivates the firm to store capital by restricting dividends. The complementary slackness condition (49) prevents $q(y)$ from falling below P. Figure 5 illustrates the shape of $q(y)$. The optimal dividend policy directs to pay dividends only when the shadow price drops to P. Otherwise, the firm should grow by investing all of the earned income. Distributing dividends by selling one unit of capital stock (or by distributing earnings in an amount equal to buying one unit of capital stock) should be weighted against losing the marginal future income. If $q(y) > P$, the stockholders' opportunity cost of losing marginal future income outweighs the benefit of receiving dividends. The firm should pay dividends only when $q(y) = P$.

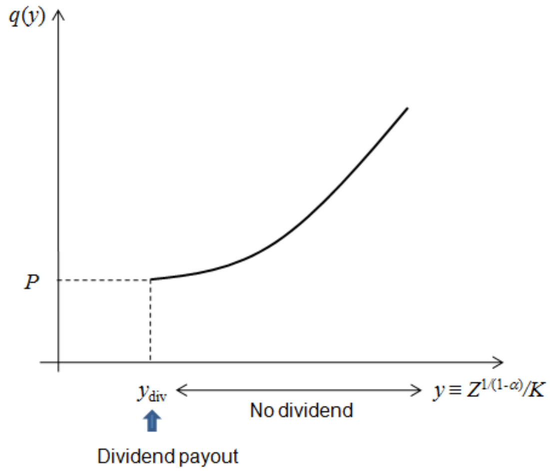

Figure 5. Shadow price of capital (no external financing)

Let y_{div} denote the value of y which triggers dividend payments. At this boundary we have the smooth-pasting condition

$$q\left(y_{\text{div}}\right) = P, \tag{52}$$

and the high-contact condition

$$q'\left(y_{\text{div}}\right) = 0. \tag{53}$$

Figure 6 illustrates the firm's optimal dividend and investment policy. Using the trigger value y_{div}, define $K^{\text{div}}\left(Z_t\right)$ by

$$K^{\text{div}}\left(Z_t\right) \equiv \frac{1}{y_{\text{div}}} Z_t^{1/(1-\alpha)} \quad \forall t \tag{54}$$

For the current installed capital stock, K_{-}, let Z_{-} denote the unique solution to $K^{\text{div}}\left(Z_{-}\right) = K_{-}$. Then the optimal decision rule is the following: If $Z_t > Z_{-}$, retain all her earnings and invest in the capital stock; if $Z_t \leq Z_{-}$, sell capital stock and distribute the proceeds as dividends until $K_t = K^{\text{div}}\left(Z_t\right)$ is realized.

Note that in the case of $Z_t > Z_{-}$ the new level of capital stock may be above or below $K^{\text{div}}\left(Z_t\right)$, depending on the amount of the current income. It is when the firm has an excessive capital stock (relative to the current and anticipated future business conditions) that the firm should distribute returns to her stockholders.

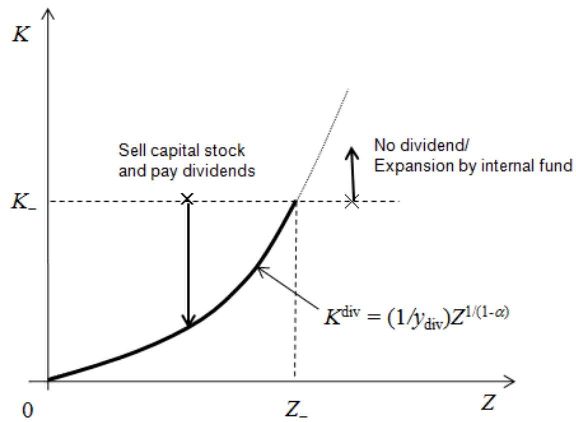

Figure 6. Optimal investment policy (no external financing)

Since no closed-form solution is available we will show how the optimal policy operates by solving the stochastic dynamic programming problem numerically in section 6.

5 The Model With Costly Equity Finance

In this section we assume that the firm can raise additional capital by issuing new shares. This assumption relaxes the assumption of irreversible financing. As illustrated in Introduction we assume that a financing cost of ϕ per dollar is incurred whenever the firm finances by issuing shares.

The firm's optimization problem is given by

$$V(K_0, Z_0) \equiv \max_{\{dD_t \geq 0, dE_t \leq 0\}} E_0 \left[\int_0^\infty e^{-rt} (dD_t + dE_t) \right]$$

subject to (3) and

$$K_t^\alpha Z_t dt = dD_t + (1-\phi) dE_t + P dG_t, \tag{55}$$

where $\{E_t\}$ denotes the cumulation of all equity finance up to time t, which is restricted to have nonpositive increments ($dE_t \leq 0$).

The Hamilton-Jacobi-Bellman equation for this problem is

$$\begin{aligned} rV(K_t, Z_t)\, dt &= \max_{\{dD_t \geq 0\}} \left(1 - \frac{V_K}{P}\right) dD_t + \max_{\{dE_t \leq 0\}} \left(1 - (1-\phi)\frac{V_K}{P}\right) dE_t \\ &\quad + V_K \left(-\delta K_t dt + \frac{1}{P} K_t^\alpha Z_t dt\right) + \mu Z_t V_Z dt + \frac{1}{2}\sigma^2 Z_t^2 V_{ZZ} dt. \end{aligned}$$

The inequality $dD_t \geq 0$ requires (49). The inequality $dE_t \leq 0$ requires

$$1 - (1-\phi)\frac{V_K}{P}\begin{cases} \geq 0 & \forall t; \\ = 0 & \forall t : dE_t < 0. \end{cases} \tag{56}$$

Adopting these conditions the Hamilton-Jacobi-Bellman equation reduces again to the differential equation (51) for $q(y)$. We also obtain the inequality

$$P \leq q(y) \leq \frac{P}{1-\phi}. \tag{57}$$

The nature of optimal investment and financing policy

When the firm can raise capital by issuing new shares with issuing cost $\phi > 0$, the optimal policy becomes a two-trigger rule: pay dividends (or repurchase shares) when $q(y)$ touches the level equal to P; issue new shares when $q(y)$ touches the level equal to $P/(1-\phi)$; no financing action when $q(y)$ is between P and $P/(1-\phi)$. Denote the trigger values by y_{div} for dividend payments, and by y_{issue} for issuing shares. These trigger values are determined by the boundary conditions (52), (53), and

$$q(y_{\text{issue}}) = \frac{P}{1-\phi}, \quad q'(y_{\text{issue}}) = 0, \tag{58}$$

as exhibited in Figure 7.

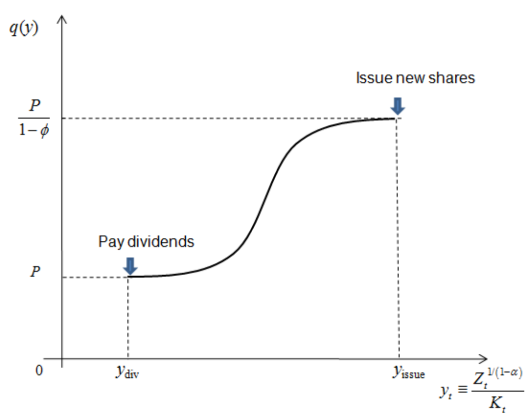

Figure 7. Shadow price of capital (costly equity finance)

As in the previous section, if $q(y) > P$ then the stockholders' opportunity cost of losing marginal future income outweights the benefit of receiving dividends. If $q(y) < P/(1-\phi)$, then increasing one unit of capital stock by issuing new shares will hurt the shareholders since the marginal contribution to the firm value is lower than the shareholders' cash contributions per unit of capital. Thus the firm should remain inactive in cashing-in or cashing-out the stockholders if the shadow price of capital is in $(P, P/(1-\phi))$.

Figure 8 illustrates the firm's optimal financing and investment policy. Together with $K^{\text{div}}(Z_t)$ given by (54) define $K^{\text{issue}}(Z_t)$ by

$$K^{\text{issue}}(Z_t) \equiv \frac{1}{y_{\text{issue}}} Z_t^{1/(1-\alpha)} \quad \forall t \tag{59}$$

Let Z_-^{issue} denote the unique solution to $K^{\text{issue}}\left(Z_-^{\text{issue}}\right) = K_-$ and let Z_-^{div} denote the unique solution to $K^{\text{div}}\left(Z_-^{\text{div}}\right) = K_-$, in which K_- is the current capital stock. The optimal decisions are given by the following rule: (i) If $Z_t > Z_-^{\text{issue}}$ issue new shares to finance investment immediately so as to obtain $K_t = K^{\text{issue}}(Z_t)$; (ii) If $Z_t < Z_-^{\text{div}}$ sell capital immediately so as to obtain $K_t = K^{\text{div}}(Z_t)$ and distribute the sales proceeds as dividends ; (iii) if $Z_-^{\text{div}} \leq Z \leq Z_-^{\text{issue}}$ there should be no external finance nor dividend payouts and the firm should invest all of the current income for capital expansion. Note that case (i) may involve no equity finance if the current income is sufficient to realize $K_t = K^{\text{issue}}(Z_t)$.

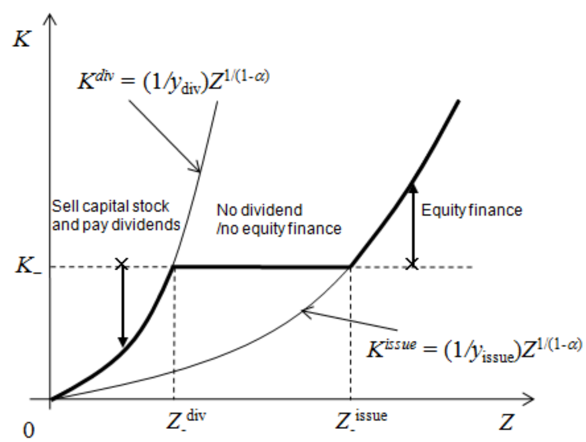

Figure 8. Optimal financing policy (costly equity finance)

When the financing cost $\phi = 0$, we have the Jorgenson/Modigliani-Miller case. In this case (57) requires $q(y) = P$, $\forall y$, i.e., the shadow price of capital always equals the price of capital stock. It is worth noting that if we substitute $q'(y) \equiv q''(y) \equiv 0$ into (51) we obtain

$$r + \delta = \frac{\alpha}{P} y^{1-\alpha}, \tag{60}$$

which coincides with (8)[17]. The two curves in Figure 8 collapses to a single curve, and it coincides with the Jorgenson's desired capital $K^J(Z_t)$. The capital should adjust immediately to this level independent of history. The amount of dividends or equity finance is determined by such requirement. That is, financing decisions are completely subjected to the investment decisions if financing is frictionless.

Define the variable y_J by

$$y_J \equiv \left(\frac{(r+\delta) P}{\alpha} \right)^{1/(1-\alpha)} \tag{61}$$

Comparing (61) with (28) we find that

$$y_G \geq y_J \tag{62}$$

holds. If investment is irreversible, the firm has strong disincentive to over-invest. This is manifested in (62), or equivalently $K^G(Z_t) \leq K^J(Z_t)$ for all Z_t (see (32)).

We can extend this observation. If accessibility to the equity market for external financing is absent, the firm has strong incentive to accumulate capital more than is necessary for producing goods to fulfill the current demand. This intuition suggests the following inequality

$$y_{\text{div}} \leq y_J, \tag{63}$$

or equivalently, $K^{\text{div}}(Z_t) \geq K^J(Z_t)$ .There is an alternative root to prove (62), by which we can also prove the inequality (63).

Recall the differential equation (20) for $q(y)$. As explained above the first term and the second term of the left-hand-side are both zero at $y = y_J$. In contrast the first term is negative ($q''(y_G) < 0$ as one can see from Figure 1) and the second term is zero at $y = y_G$. Observe that $q(y_J) = P$ (Jorgensonian shadow price) and $q(y_G) = P$ (the smooth-pasting condition), implying that

[17] Naturally the same result obtains when we substitute $q'(y) \equiv q''(y) \equiv 0$ into (20).

the third term is identical at y_J and y_G. Using these properties one can easily show (63) by contradiction.

Similarly on the differential equation (51) we have $q''(y_{\mathrm{div}}) > 0$ (see Figure 5), $q'(y_{\mathrm{div}}) = 0$, and $q(y_{\mathrm{div}}) = P$. From this we can prove the inequality (63) by contradiction.

Thus the inequality

$$K^G(Z_t) \le K^J(Z_t) \le K^{\mathrm{div}}(Z_t) \tag{64}$$

follows.

6 Numerical Solution of the Problem

The models of Section 3 have closed-form solutions, but the models of section 4 and section 5 do not. Hence we solved all the stochastic dynamic programming problems numerically. Namely, we transformed the problem to a discrete-time, discrete-state Markov decision problem and used the method of value iteration to obtain the optimal solution[18].

We set the risk-free interest $r = 5\%$, the depreciation rate $\delta = 10\%, \alpha = 0.7, \sigma = 2\%$ and $\mu = \sigma^2/2$. For the model of section 5 we set the equity issuance cost at $\phi = 1\%$. The algorithm produces the value function $V(K, Z)$ and the values of all decision variables at each node of (K, Z). We let $\ln(Z)$ to have 800 points of support in $[-100\sigma, 100\sigma]$. The capital stock K lies in the set $\left[\bar{K}, \bar{K}(1-\delta\Delta t), \bar{K}(1-\delta\Delta t)^2, ..., \bar{K}(1-\delta\Delta t)^{500}\right]$, where we set $\Delta t = 0.2$ and the maximum allowable capital stock, $\bar{K}$, is determined by

$$\frac{\partial \Pi\left(K, \bar{Z}\right)}{\partial K} - \delta = 0, \tag{65}$$

in which $\bar{Z}$ is the maximum grid value of Z. We used Gauss-Hermite quadature to evaluate the conditional expectation of next period's firm value in implementing the value iteration.

[18] We double-checked our result by comparing with the solution that we obtained by numerically solving the second-order linear differential equations of (20) or (51).

Irreversible Investment and Costly Reversible Investment

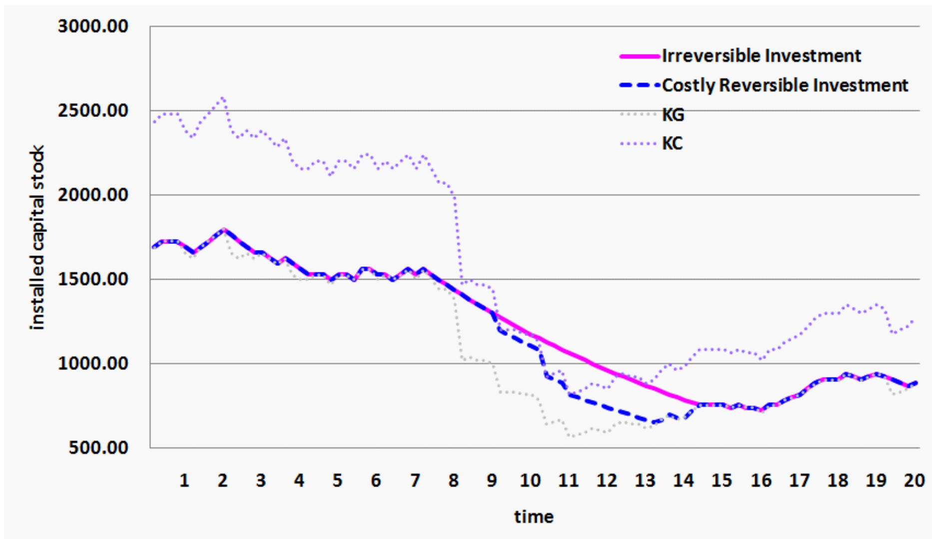

Figure 9. Typical investment path (irreversible investment and costly reversibility)

Figure 9 exhibits a typical path of optimal capital stock when investment is costly reversible. The lower dotted line shows the path of $K^G(Z_t)$, the level of capital that triggers buying additional capital, while the upper dotted line shows the path of $K^C(Z_t)$, the level of capital that triggers selling capital. If the installed capital stock is strictly inside these trigger bounds, the firm takes no investment action and the capital stock is left to depreciate. Note that these trigger bounds fluctuate according to the movement of Z_t. The bold dotted line shows the optimal path of capital stock.

It is interesting and very much worth a formal proof that the level of buy trigger is independent of the selling price of capital stock[19]. This further implies that trigger level of buying capital, $K^G(Z_t)$, remains the same when we move to the case of irreversible investment. We superimpose the optimal solution for the irreversibility case on the same figure. The straight line shows the optimal path of capital stock when investment is irreversible. The path of installed capital stock is higher for this case than when the case of costly

[19] We believe that this is a fairly general property of optimization problems that involve two-sided trigger. We found no reference to this property in Abel and Eberly (1994, 1996) and in related literature. The proof is left to the readers.

reversibility on occasions when costly reversibility enforces the firm to sell capital.

No equity finance and costly equity finance

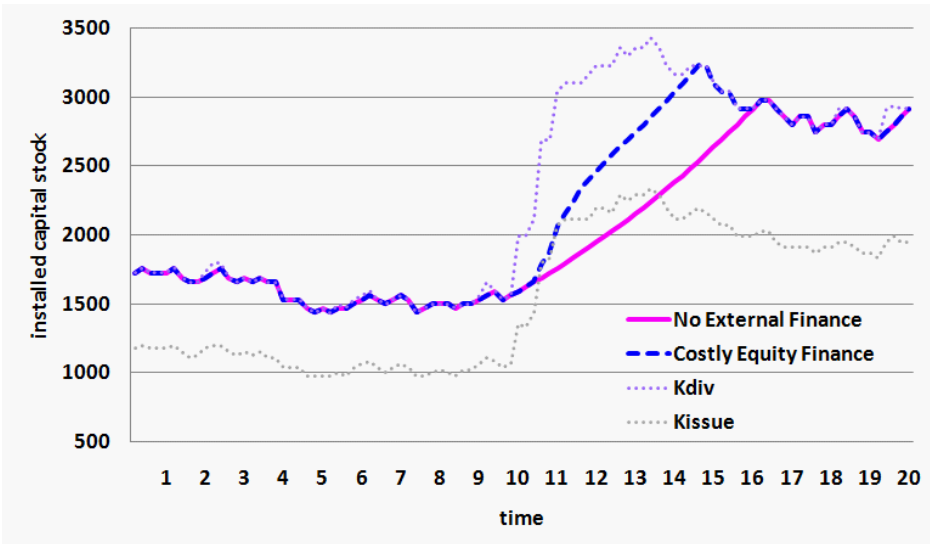

Figure 10. Typical investment path (no equity finance and costly equity finance)

Figure 10 exhibits a typical path of capital stock when additional equity finance is available but costly. Again the optimal strategy involves a trigger rule, but in this case an action means cash transactions between the firm and the stockholders. The lower dotted line shows the path of $K^{\text{issue}}(Z_t)$, the level of capital that triggers issuing new shares, while the upper dotted line shows the path of $K^{\text{div}}(Z_t)$, the level of capital that triggers dividend payouts, or repurchase of shares. If the installed capital stock is strictly inside these trigger bounds, the firm takes no financing action, which means that the firm uses all internal funds for buying capital stock. The bold dotted line shows the optimal path of capital stock.

Again we find that the level capital that triggers dividend payouts is independent of the equity financing cost, ϕ. This in turn implies that the level of capital that triggers dividend payouts, $K^{\text{div}}(Z_t)$, remains the same when we move to the case of no equity finance. We superimpose this case of no equity finance on the same figure. In this case we have only the dividend-trigger as the firm cannot issue shares. The straight line shows the optimal path of capital stock when equity finance is unavailable. The path of installed capital stock is lower when equity finance is unavailable than when it is available on occasions when the growing demand enforces the firm to issue shares for further accumulation of capital.

7 A Model With Changing Growth Rate

We have worked with the assumption that the exogenous process Z_t describing the business conditions fluctuates with constant expected rate of growth and constant volatility. In this section we propose an alternative model of Z_t which may make our model more realistic.

Assume that $\{Z_t\}$ follows the process

$$dZ_t = \mu_t Z_t dt, \tag{66}$$

in which μ_t follows an Ornstein-Uhlenbeck mean-reverting process

$$d\mu_t = (\theta - \zeta\mu_t)\,dt + \sigma_\mu dW_t. \tag{67}$$

where θ, ζ and σ_μ are constants and $\{W_t\}$ is a standard Brownian motion. The value of (θ/ζ) is the long-run rate of growth and the parameter ζ indicates the speed of adjustment in the mean-reversion.We only consider the case with investment irreversibility ($dG_t \geq 0$), i.e., the first model of Section 3. The extension of the following analysis to the other three cases is obvious.

Besides K_t and Z_t, we have an additional state variable μ_t in the set of state variables, so that the value function is given by $V\left(K_t, Z_t, \mu_t\right)$. Applying Itô's lemma the Hamilton-Jacobi-Bellman equation (13) becomes

$$\begin{aligned} rV\left(K_t, Z_t, \mu_t\right)dt &= \max_{\{dG_t \geq 0\}} \left(V_K - P\right)dG_t + K_t^\alpha Z_t dt - \delta K_t V_K dt + \mu_t Z_t V_Z dt \\ &\quad + \left(\theta - \zeta\mu_t\right)V_\mu dt + \frac{1}{2}\sigma_\mu^2 V_{\mu\mu} dt. \end{aligned} \tag{68}$$

Defining $y \equiv Z^{\frac{1}{1-\alpha}}/K$ as before and $\chi\left(y,\mu\right)\left(y,\mu\right) \equiv V_K\left(K, Z, \mu\right)$, we can repeat the same process as in section 3 and obtain the linear partial differential equation

$$\frac{\sigma_\mu^2}{2}\chi_{\mu\mu} + \left(\theta - \zeta\mu\right)\chi_\mu + \left(\delta + \frac{\mu}{1-\alpha}\right)y\chi_y - \left(r+\delta\right)\chi = -\alpha y^{1-\alpha}. \tag{69}$$

The complementary slackness gives the inequality constraint $\chi(y,\mu) \leq P$ for all (y,μ) and $dG_t > 0$ only when $\chi(y,\mu) = P$.

One can show that the solution to (69) is given in an additive form

$$\chi(y,\mu) = q(y) + \psi(\mu) \tag{70}$$

where $q(y)$ satisfies

$$\left(\delta + \frac{\mu}{1-\alpha}\right) yq'(y) - (r+\delta)\, q(y) = -\alpha y^{1-\alpha} \tag{71}$$

and $\psi(\mu)$ satisfies

$$\frac{\sigma_\mu^2}{2}\psi''(\mu) + (\theta - \zeta\mu)\,\psi'(\mu) - (r+\delta)\,\psi(\mu) = 0. \tag{72}$$

These two ordinary differential equations together with the smooth-pasting and high-contact conditions leads to the optimal solution of the problem. The challenge of solving this free-boundary problem is left to the readers.

Figure 11 illustrates firm's optimal investment policy.

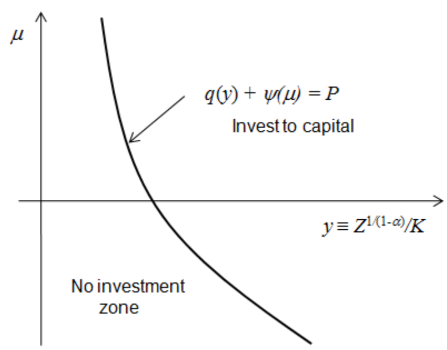

Figure 11. Optimal investment policy (a model with changing growth rate)

The curve exhibited on this state space of (y,μ) gives the trigger boundary, which is the locus of points for which the equality $q(y) + \psi(\mu) = P$ holds. The optimality condition requires that the firm should maintain the shadow price of capital within a range not greater than P. If the current state is to the south-west of this curve, the firm should not invest in the capital stock and distribute all of the current income as dividends. If the state touches the curve, the firm should invest immediately.

The desired level of capital stock is given by $K_t^G(Z_t,\mu_t) = \eta(\mu_t)\, Z_t^{1/(1-\alpha)}$, in which $\eta(\mu_t)$ is some increasing function of μ_t. Let K_- denote the install capital stock and let $Z_{-} \equiv \left(K_{-}/\phi\left(\mu_t\right)\right)^{1-\alpha}$. Then $Z_t > Z_{-}$ reflects an insufficient capital: The installed capital is too small relative to the firm's anticipation of the current and future business conditions. The firm should invest immediately to obtain the desired level of capital stock. On the other hand, when $Z_t < Z_{-}$ the firm finds herself stuck with excessive stock of capital and hence distribute all of the current income to stockholders. It conforms to intuition that the no-investment zone for Z_t gets smaller as the anticipated short-run growth rate μ_t increases.

8 Conclusion

In this paper we have shown that the theory of investment under irreversibility and costly reversibility can be naturally extended to construct a dynamic theory of firm's financing decisions. "Irreversibility" in our model meant that the firm may buy back shares but cannot issue new shares. "Costly reversibility" meant that the firm can issue shares at some cost. We have shown that financial frictions force the shadow price of capital to satisfy a second-order ordinary linear differential equation, which is "dual" to the one generated by investment frictions. Although the dual differential equation does not have a closed-form solution, we could characterize the nature of the optimal financing policy analytically. We also provided some results which compare the impact of financing frictions on investment behavior to that of investment frictions.

We assumed no investment frictions to contrast the role of financing frictions to investment frictions. But, since the firm can buy or sell her capital stock without any frictions in our model, the capital stock got the additional role of providing the vehicle of corporate savings. An obvious extension is to include transactions with banks, i.e., bank savings and borrowings at different rates of interest[20]. A subsequent paper will show how one can extend the present analysis to include these additional spectrum of financing.

To Include investment frictions into the model and investigate the interaction between the two would be another topic of interest. Adding fixed

[20] If we allow that the firm can save or borrow at the same rate of interest (i.e. "reversible" financing in regard to bank transactions), the frictions on equity financing plays no essential role and we go back to the Modigliani-Miller paradigm in which financing decisions are irrelevant. Thus we need to model differential rates of interest between saving and borrowing.

and/or nonlinear adjustment cost in financing and investment may bring the prediction of the model closer to reality in corporate investment and financing dynamics.

Appendix A. Derivation of the solution for the case of irreversible investment

In this appendix we show that the parameters C and y_G in (27) should be given by (28), (29), and (30) to satisfy the boundary conditions (21) and (22).

Using (21) and (22), we get

$$\alpha H y_G^{1-\alpha} + \alpha H C y_G^{\beta} = P \tag{A1}$$

and

$$\alpha (1-\alpha) H y_G^{-\alpha} + \alpha H C \beta y_G^{\beta-1} = 0. \tag{A2}$$

Solving these two equations we find

$$y_G = \left\{ \frac{\beta}{\beta - (1-\alpha)} \frac{P}{\alpha H} \right\}^{1/(1-\alpha)} \tag{A3}$$

and (29). We further show that (A3) has a simpler expression of (28). Let β be a solution to $a'x^2 + b'x + c' = 0$ for an arbitrary pair (a', b', c'). Rewriting this equation as

$$(x - (1-\alpha)) \left(a'x - \frac{c'}{1-\alpha} \right) = - \left((1-\alpha) a' + b' + \frac{c'}{1-\alpha} \right) x$$

gives the expression

$$\frac{x - (1-\alpha)}{x} = - \frac{(1-\alpha) a' + b' + \frac{c'}{1-\alpha}}{a'x - \frac{c'}{1-\alpha}}.$$

In our case, β is the positive root of the characteristic equation (26) and thus

$$\begin{aligned} \frac{\beta - (1-\alpha)}{\beta} &= - \frac{(1-\alpha) \frac{\sigma^2}{2(1-\alpha)^2} + \left[\delta + \frac{1}{1-\alpha} \left(\mu - \frac{\sigma^2}{2} \right) \right] - \frac{1}{1-\alpha} (r + \delta)}{\frac{\sigma^2}{2(1-\alpha)^2} \beta + \frac{1}{1-\alpha} (r+\delta)} \\ &= \frac{1}{\left(r + \delta + \frac{1}{2(1-\alpha)} \sigma^2 \beta \right) H} \end{aligned}$$

Insert this relationship into (A3) to get

$$\begin{aligned} y_G &= \left\{ \left(r + \delta + \frac{1}{2(1-\alpha)} \sigma^2 \beta \right) \frac{PH}{\alpha H} \right\}^{1/(1-\alpha)} \\ &= \left(\frac{cP}{\alpha} \right)^{1/(1-\alpha)}, \end{aligned}$$

where c is given by (30).

Appendix B. Derivation of the solution for the case of costly reversible investment

We provide an alternative and more straightforward derivation of the optimal solution for the case of costly reversible investment, which supplements Abel and Eberly (1996).

We know that the general solution to the second-order linear differential equation (20) is given by (24) for $y_C \leq y \leq y_G$, where H is defined by (25) and β_1, β_2 are the roots of the characteristic equation (26) with $\beta_1 > 1$ and $\beta_2 < 0$.

The constants (C_1, C_2, y_C, y_G) in $q(y)$ are determined by the four boundary conditions (37) and (38):

$$y_C^{1-\alpha} + C_1 y_C^{\beta_1} + C_2 y_C^{\beta_2} = \frac{P_L}{\alpha H} \tag{B1}$$

$$y_G^{1-\alpha} + C_1 y_G^{\beta_1} + C_2 y_G^{\beta_2} = \frac{P_U}{\alpha H} \tag{B2}$$

$$(1-\alpha) y_C^{1-\alpha} + C_1 \beta_1 y_C^{\beta_1} + C_2 \beta_2 y_C^{\beta_2} = 0 \tag{B3}$$

$$(1-\alpha) y_G^{1-\alpha} + C_1 \beta_1 y_G^{\beta_1} + C_2 \beta_2 y_G^{\beta_2} = 0. \tag{B4}$$

As the first step, using (B3) and (B4) we express (C_1, C_2) as functions of (y_C, y_G). Defining

$$G \equiv \frac{y_G}{y_C}, \tag{B5}$$

we obtain

$$\begin{aligned} C_1 &= -\frac{(1-\alpha)}{\beta_1} \frac{G^{1-\alpha} - G^{\beta_2}}{G^{\beta_1} - G^{\beta_2}} y_C^{(1-\alpha)-\beta_1}, \\ C_2 &= -\frac{(1-\alpha)}{\beta_2} \frac{G^{\beta_1} - G^{1-\alpha}}{G^{\beta_1} - G^{\beta_2}} y_C^{(1-\alpha)-\beta_2}. \end{aligned}$$

Define function $\theta(x)$ by

$$\theta(x) \equiv \frac{x^{\beta_1} - x^{1-\alpha}}{x^{\beta_1} - x^{\beta_2}}, \tag{B6}$$

which implies

$$1 - \theta(x) = \frac{x^{1-\alpha} - x^{\beta_2}}{x^{\beta_1} - x^{\beta_2}}. \tag{B7}$$

Using $\theta(x)$ we can rewrite C_1 and C_2 as

$$C_1 = -\frac{(1-\alpha)}{\beta_1}\left[1 - \theta(G)\right] y_C^{(1-\alpha)-\beta_1} \tag{B8}$$

$$C_2 = -\frac{(1-\alpha)}{\beta_2}\theta(G)\, y_C^{(1-\alpha)-\beta_2}. \tag{B9}$$

The conditions (B1) and (B2) then determines y_C and y_G. Substituting (B8) and (B9), the left-hand-side of (B1) is

$$y_C^{1-\alpha}\left\{1 - \frac{(1-\alpha)}{\beta_1}\left[1-\theta(G)\right] - \frac{(1-\alpha)}{\beta_2}\theta(G)\right\} \equiv y_C^{1-\alpha}\frac{\phi(G)}{H}$$

if we define

$$\phi(x) \equiv H\left\{1 - \frac{(1-\alpha)}{\beta_1}\left[1-\theta(x)\right] - \frac{(1-\alpha)}{\beta_2}\theta(x)\right\}. \tag{B10}$$

Thus (B1) reduces to

$$\alpha y_C^{1-\alpha}\phi(G) = P_L. \tag{B11}$$

The left-hand-side of (B2) is

$$y_C^{1-\alpha}G^{1-\alpha}\left\{1 - \frac{(1-\alpha)}{\beta_1}\left[1-\theta(G)\right]G^{\beta_1-(1-\alpha)} - \frac{(1-\alpha)}{\beta_2}\theta(G)\, G^{\beta_2-(1-\alpha)}\right\}$$

Using the identities

$$\begin{aligned}\left[1-\theta(x)\right]x^{\beta_1-(1-\alpha)} &\equiv 1 - \theta\left(x^{-1}\right),\\ \theta(x)\, x^{\beta_2-(1-\alpha)} &\equiv \theta\left(x^{-1}\right),\end{aligned}$$

it is rewritten as

$$y_C^{1-\alpha}G^{1-\alpha}\left\{1 - \frac{(1-\alpha)}{\beta_1}\left[1-\theta\left(G^{-1}\right)\right] - \frac{(1-\alpha)}{\beta_2}\theta\left(G^{-1}\right)\right\}.$$

Hence the condition (B2) reduces to

$$\alpha y_G^{1-\alpha}\phi\left(G^{-1}\right) = P_U. \tag{B12}$$

Dividing both sides of (B11) and (B12) generates an equation for G :

$$G^{1-\alpha}\frac{\phi\left(G^{-1}\right)}{\phi\left(G\right)} = \frac{P_U}{P_L}. \tag{B13}$$

Substituting (B8) and (B9) back into (24), we obtain the expression (44) for the shadow price of capital.

Appendix C. Derivation of the differential equation (51)

Taking the partial derivatives of both sides of (50) in K, we get

$$\begin{aligned} rV_K &= \left(\frac{\alpha}{P}K^{\alpha-1}Z-\delta\right)V_K+\left(\frac{1}{P}K^{\alpha}Z-\delta K\right)V_{KK} \\ &\quad +\mu Z V_{KZ}+\frac{1}{2}\sigma^2 Z^2 V_{KZZ}. \end{aligned} \tag{C1}$$

Using $v\left(K,Z\right)\equiv V_K\left(K,Z\right)$ this can be rewritten as

$$\begin{aligned} 0 &= -\left(r+\delta-\frac{\alpha}{P}K^{\alpha-1}Z\right)v+\left(\frac{1}{P}K^{\alpha}Z-\delta K\right)v_K \\ &\quad +\mu Z v_Z+\frac{1}{2}\sigma^2 Z^2 v_{ZZ}. \end{aligned} \tag{C2}$$

Defining

$$y\equiv\frac{Z^{1/(1-\alpha)}}{K},$$

and letting $v\left(K,Z\right)\equiv q\left(y\right)$, the partial derivatives appearing in (C2) are:

$$\begin{aligned} v_K\left(K,Z\right) &= q'\left(y\right)\frac{\partial y}{\partial K}=-yq'\left(y\right)K^{-1} \\ v_Z\left(K,Z\right) &= q'\left(y\right)\frac{\partial y}{\partial Z}=\frac{1}{1-\alpha}q'\left(y\right)y^{\alpha}K^{\alpha-1} \\ v_{ZZ}\left(K,Z\right) &= \frac{1}{1-\alpha}K^{\alpha-1}\left\{q''\left(y\right)y^{\alpha}\frac{\partial y}{\partial Z}+\alpha q'\left(y\right)y^{\alpha-1}\frac{\partial y}{\partial Z}\right\} \\ &= \frac{1}{\left(1-\alpha\right)^2}K^{2\alpha-2}\left\{q''\left(y\right)y^{2\alpha}+\alpha q'\left(y\right)y^{2\alpha-1}\right\}. \end{aligned}$$

If we substitute these equalities into (C2) we obtain

$$\begin{aligned} 0 &= -\left(r+\delta-\frac{\alpha}{P}y^{1-\alpha}\right)q\left(y\right)-\left(\frac{1}{P}y^{1-\alpha}-\delta\right)yq'\left(y\right)+\frac{\mu}{1-\alpha}yq'\left(y\right) \\ &\quad +\frac{\sigma^{2}}{2\left(1-\alpha\right)^{2}}\left\{y^{2}q''\left(y\right)+\alpha yq'\left(y\right)\right\}, \end{aligned}$$

which is rewritten as (50).

References

[1] Abel, Andrew B., 1983. Optimal Investment under Uncertainty. *American Economic Review* 73, 228-33

[2] Abel, Andrew B. and Janice C. Eberly, 1994. A Unified Model of Investment under Uncertainty. *American Economic Review* 84, 1369-1384.

[3] Abel, Andrew B. and Janice C. Eberly, 1996. Optimal Investment with Costly Reversibility. *Review of Economic Studies* 63, 581-593.

[4] Bertola, Guiseppe and Ricardo J. Caballero, 1990. Irreversibility and Aggregate Investment. *Review of Economic Studies* 61, 223-246.

[5] Bolton, Patrick., Hui Chen, and Neng Wang, 2009. A Unified Theory of Tobin's q, Corporate Investment, Financing, and Risk Management. unpublished manuscript, March.

[6] **Dumas, S.,** 1991. Super Contact and Related Optimality Conditions, *Journal of Economic Dynamics and Control*, 15, 675-685.

[7] Hayashi, Fumio, 1982. Tobin's Marginal q and Average q: A Neoclassical Interpretation. *Econometrica* 50, 213-224.

[8] Hennessy, Christopher A. and Toni M. Whited, 2007. How Costly Is External Financing? Evidence from a Structural Estimation. *Journal of Finance* 62, 1705-1745.

[9] Hennessy, Christopher A. and Toni M. Whited, 2007. Debt Dynamics. *Journal of Finance* 60, 1129-1165.

[10] Jorgenson, Dale, 1963. Capital Theory and Investment Behavior. *American Economic Review* 53, 247-259.

[11] Mussa, Michael 1977. External and Internal Adjustment Costs and the Theory of Aggregate and Firm Investment. *Econometrica* 44, 163-78.

[12] Uzawa, Hirofumi, 1969. Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth. *Journal of Political Economy* 77, 628-652.

[13] Whited, Toni M., 2006. External finance constraints and the intertemporal pattern of intermittent investment. *Journal of Financial Economics* 81, 467-502.