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Robustness of the Separating Information Maximum Likelihood Estimation of the Realized Volatility with Micro-Market Noise

Naoto Kunitomo
University of Tokyo

Seisho Sato
Institute of Statistical Mathematics

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Naoto Kunitomo †
and
Seisho Sato ‡

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Abstract

For estimating the realized volatility and covariance by using high frequency data, Kunitomo and Sato (2008a,b) have proposed the Separating Information Maximum Likelihood (SIML) method when there are micro-market noises. The SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large under general conditions including *non-Gaussian processes* and *volatility models*. We also show that the SIML estimator has the asymptotic robustness in the sense that it is consistent and it has the asymptotic normality when there are autocorrelations in the market noise terms and there are endogenous correlations between the signal and noise terms.

Key Words

Realized Volatility with Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Endogenous Noise, Autocorrelated Noise, Robustness.

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†Graduate School of Economics, University of Tokyo, Bunkyo-ku, Hongo 7-3-1, Tokyo 113-0033, JAPAN, kunitomo@e.u-tokyo.ac.jp

‡Institute of Statistical Mathematics, Tachikawa-shi, Midori-cho 10-3, Tokyo 190-8562, JAPAN

1. Introduction

Recently a considerable interest has been paid on the estimation problem of the realized volatility by using high-frequency data of financial price processes. Although there were some discussions on the estimation of continuous stochastic processes in the statistical literature, the earlier studies often had ignored the presence of micro-market noises in financial markets when they tried to estimate the underlying stochastic processes. Because there are several convincing reasons why the micro-market noises are important in high-frequency financial data, several new statistical estimation methods have been developed. See Bandorff-Nielsen et al. (2008) for recent discussions on the related topics. In this respect Kunitomo and Sato (2008a, b) have proposed a new statistical method called the Separating Information Maximum Likelihood (SIML) estimator for estimating the realized volatility and the realized covariance by using high frequency data in the presence of possible micro-market noise. The SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the more general case) when the sample size is large and the data frequency interval becomes zero under a set of regularity conditions for the *non-Gaussian* underlying processes and *volatility models*.

In this paper we shall show that the SIML estimator has the desirable asymptotic properties, that is, it is consistent and asymptotically normal even when (i) the noise terms are autocorrelated and (ii) there are endogenous correlations between the market-noise terms and the efficient market price terms. Since these aspects on the signal (i.e. the hidden efficient market price) and noise terms have important roles for the theory and empirical observation on high-frequency data, the SIML estimation is an interesting and useful method. The asymptotic robustness of the SIML method has desirable properties over other estimation methods of unknown parameters from a large number of data for the underlying continuous stochastic processes with micro-market noises in the multivariate non-Gaussian cases. Because the SIML estimation is a simple method, it can be practically used for analyzing the multivariate (high frequency) financial time series.

In Section 2 we introduce the standard SIML method with micro-market noise and discuss the asymptotic properties of the SIML estimator in the general situation. Then in Section 3 we give the asymptotic properties of the SIML estimator when there are autocorrelations in the noise terms, which can be endogenous with the signal terms. In Section 4 we shall report finite sample properties of the SIML estimator based on a set of simulations. Finally, in Section 5 some brief remarks will be given. Some mathematical details and tables based on simulations are given in Appendices.

2. The SIML Estimation of Realized Volatility and Covariance with Micro-Market Noise

2.1 The SIML Method

Let y_{ij} be the $i-$th observation of the $j-$th (log-) price at t_i^n for $j = 1, \cdots, p; 0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1$. We set $\mathbf{y}_i = (y_{i1}, \cdots, y_{ip})'$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ be an $n \times p$ matrix of observations. The underlying continuous process $\mathbf{x}_i$ at t_i^n $(i = 1, \cdots, n)$ is not necessarily the same as the observed prices and let $\mathbf{v}_i^{'} = (v_{i1}, \cdots, v_{ip})$ be the vector of the micro-market noise at t_i^n. We assume

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i \tag{2.1}$$

and

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{C}_x(s) d\mathbf{B}_s \;\; (0 \leq t \leq 1), \tag{2.2}$$

where $\mathcal{E}(\mathbf{v}_i) = \mathbf{0}$, $\mathcal{E}(\mathbf{v}_i \mathbf{v}_i^{'}) = \mathbf{\Sigma}_v$, $\mathbf{B}_s$ is a $q \times 1$ $(q \geq 1)$ vector of the standard Brownian motions, $\mathbf{C}_x(s)$ is a $p \times q$ vector function adapted to the $\sigma-$field $\mathcal{F}(\mathbf{x}_r, \mathbf{B}_r, r \leq s)$, and we write the instantaneous covariance function $\mathbf{\Sigma}_x(s) = (\sigma_{ij}^{(x)}(s)) = \mathbf{C}_x(s)\mathbf{C}_x(s)'$ ($\sigma_{ij}^{(x)}(s)$ is the (i,j)-th element of $\mathbf{\Sigma}_x(s)$). The main statistical problem is to estimate the quadratic variations and co-variations

$$\mathbf{\Sigma}_x = (\sigma_{ij}^{(x)}) = \int_0^1 \mathbf{\Sigma}_x(s) ds \tag{2.3}$$

of the underlying continuous process $\{\mathbf{x}_t\}$ and the covariance $\mathbf{\Sigma}_v = (\sigma_{ij}^{(v)})$ of the noise process from the observed $\mathbf{y}_i$ $(i = 1, \cdots, n)$. We use the notation that $\sigma_{ij}^{(x)}$ and $\sigma_{ij}^{(v)}$ are the (i, j)-th element of $\mathbf{\Sigma}_x(s)$ and $\mathbf{\Sigma}_v$, respectively. In order to derive the estimation method, we consider the standard situation when $\mathbf{x}_t$ $(0 \le t \le 1)$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent with $\mathbf{\Sigma}_x(s) = \mathbf{\Sigma}_x$, and $\mathbf{v}_i$ are independently, identically and normally distributed as $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$. Then given the initial condition $\mathbf{y}_0$, we have

$$\mathbf{Y}_n \sim N_{n\times p}\left(\mathbf{1}_n \cdot \mathbf{y}_0', \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n\mathbf{C}_n' \otimes h_n\mathbf{\Sigma}_x\right) , \tag{2.4}$$

where $\mathbf{1}_n' = (1, \cdots, 1)$, $h_n = 1/n$ $(= t_i^n - t_{i-1}^n)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} . \tag{2.5}$$

We transform $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k'))$ by

$$\mathbf{Z}_n = h_n^{-1/2}\mathbf{P}_n'\mathbf{C}_n^{-1}\left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) \tag{2.6}$$

where

$$\bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0' . \tag{2.7}$$

Then the likelihood function under the Gaussian noise is given by

$$L_n^*(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{np} \prod_{k=1}^{n} |a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} e^{\left\{-\frac{1}{2}\mathbf{z}_k'\left(a_{kn}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x\right)^{-1}\mathbf{z}_k\right\}} , \tag{2.8}$$

where

$$a_{kn} = 4n\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right] \;\; (k = 1, \cdots, n) . \tag{2.9}$$

Because the ML estimator of unknown parameters is a rather complicated function of all observations and each a_{kn} terms depend on k as well as n, one way to have a simple solution of the problem is to approximate the likelihood function in some sense. For this purpose we denote $a_{k_n,n}$ for $a_{k,n}$. When k_n is small, $a_{k_n,n}$ is small and then we approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$L_{1n}(\boldsymbol{\theta}) = m \log |\boldsymbol{\Sigma}_x| + \sum_{k=1}^{m} \mathbf{z}_k^{'} \boldsymbol{\Sigma}_x^{-1} \mathbf{z}_k \ . \tag{2.10}$$

Similarly, we consider the corresponding terms when $a_{n+1-l_n,n}$ is large and approximate $2 \times L_n(\boldsymbol{\theta})$ by

$$L_{2n}(\boldsymbol{\theta}) = \sum_{k=n+1-l}^{n} \log |a_{kn} \boldsymbol{\Sigma}_v| + \sum_{k=n+1-l}^{n} \mathbf{z}_k^{'} [a_{kn} \boldsymbol{\Sigma}_v]^{-1} \mathbf{z}_k \ . \tag{2.11}$$

Let m and l be dependent on n and we write m_n and l_n formally. (We can take them as integers in an obvious way.) Then we define the SIML estimator of $\hat{\boldsymbol{\Sigma}}_x$ by

$$\hat{\boldsymbol{\Sigma}}_x = \frac{1}{m_n} \sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \tag{2.12}$$

and the SIML estimator of $\hat{\boldsymbol{\Sigma}}_v$ by

$$\hat{\boldsymbol{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ . \tag{2.13}$$

The numbers of terms m_n and l_n we use are dependent on n such that $m_n, l_n \to \infty$ as $n \to \infty$. We impose the order requirements that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_v$, respectively.

2.2 Improving SIML estimation

We consider possible improvements of the original SIML estimation. Without loss of generality, we set $p = q = 1$ and write $\sigma_x^2 = \boldsymbol{\Sigma}_x = \int_0^1 \sigma_x^2(s) ds$. We use the alternative form of the SIML estimator as

$$\hat{\sigma}_{SIML}^2 = \sum_{i,j=1}^{n} c_{ij} (y_i - y_{i-1})(y_j - y_{j-1}) \ , \tag{2.14}$$

where

$$s_{jk} = \cos \left[\frac{2\pi}{2n+1} (j - \frac{1}{2})(k - \frac{1}{2}) \right] \tag{2.15}$$

and

$$\begin{aligned} c_{ij} &= \frac{2}{m}\sum_{k=1}^{m} s_{ik}s_{jk} \ , \\ &= \frac{1}{m}\sum_{k=1}^{m}\left\{\cos\left[\frac{2\pi}{2n+1}(i+j-1(k-\frac{1}{2})\right]+\cos\left[\frac{2\pi}{2n+1}(i-j)(k-\frac{1}{2})\right]\right\} \ . \end{aligned}$$

Then we have investigated the asymptotic (higher order) bias and the alternative forms of the asymptotic variance of the SIML estimator when $\sigma_x(s)$ $(= \mathbf{\Sigma}_x(s))$ is dependent on s and examined the corresponding results of Kunitomo and Sato (2008a). We shall give some detail of derivations of the asymptotic (higher order) bias and the asymptotic variance in Appendix A and here we use their discussion. In order to reduce the possible asymptotic bias in the SIML estimation, we use the relation that for any integer j $(j = 1, \cdots, n)$

(2.16) $$c_{jj} = 1 + \frac{1}{2m}\frac{\sin[\frac{2\pi m}{2n+1}(2j-1)]}{\sin[\frac{2\pi}{2n+1}(2j-1)]}$$

and $|\sin[\frac{2\pi m}{2n+1}(2j-1)]| \le 1$.
Then from the form of (2.14) we notice that for its asymptotic distribution we can control the diagonal quantities $\sqrt{m}(c_{jj}-1)$ $(j = 1, \cdots, n)$ and utilize the condition that $\sqrt{m}\sin[\frac{2\pi}{2n+1}(2j-1)] \to \infty$ as $n \to \infty$. If we take $m = n^{\alpha}$ and $i = i(n) = n^{\gamma}$, it is the same order of $n^{\alpha/2+\gamma-1}$, which implies $\gamma > 1-\alpha/2$. Also it may be desirable to control the off-diagonal quantities $mc_{ij}^2 - 1$ $(i \ne j)$ because we have $n(n-1)/2$ terms. By using some formulas reported in Appendix B, we can evaluate the terms with $i = j$, $i \ne j$ and $k = k'$,$k \ne k'$. Then we can find the condition for $\sqrt{m}\sin[\frac{2\pi}{2n+1}(i+j-1)] \to \infty$ and $\sqrt{m}\sin[\frac{2\pi}{2n+1}(i-j)] \to \infty$ as $n \to \infty$ as a sufficient condition. If we take $m = n^{\alpha}$ and $i+j = (i+j)(n) = n^{\gamma}$, it is the same order of $n^{\alpha/2+\gamma-1}$, which implies $\gamma > 1-\alpha/2$.

One way to define the SIML (or a modified SIML) estimator is to delete some end terms in the original SIML estimation and define

(2.17) $$\hat{\sigma}^2_{MSIML} = \sum_{i,j=1}^{n} c_{ij}^{*}(y_i - y_{i-1})(y_j - y_{j-1}) \ ,$$

where $c_{ij}^{*} = 0$ $(1 \le i+j, n-(i+j) < d\ n^{1-\alpha/2})$, $c_{ij}^{*} = 0$ $(1 \le |i-j, n-(i-j)| < d\ n^{1-\alpha/2})$ with some constant d and $c_{ij}^{*} = c_{ij}$ (otherwise). Because the end effects are not very large, we still have the asymptotic property in the simple form. We summarize the asymptotic distribution of the SIML estimator as the next theorem when $p = 1$. The proof is a combination of the proof of Theorem 1 of Kunitomo and Sato (2008) and the derivations on the asymptotic bias and variance given in Appendix A. (It is straightforward to generalize the result when $p \geq 1$.)

Theorem 1 : We assume that x_i and v_i $(i = 1, \cdots, n)$ follow (2.1)-(2.3) and $\sigma_x^2(s)$ $(= \mathbf{\Sigma}_x(s) \geq 0)$ with $p = q = 1$. Suppose that $r_i = x_i - x_{i-1}$ and v_i are a sequence of martingale differences with $\sup_{1\leq i} \mathcal{E}(\|v_i\|^4) < \infty$, $\sup_{1\leq i} \mathcal{E}\,[\|\sqrt{n}\, r_i\|^4] < \infty$ and σ_x^2 $(= \int_0^1 \sigma_x^2(s)ds = \mathbf{\Sigma}_x)$ is a constant matrix (a.s.).
Then

(2.18)
$$\sqrt{m}\left[\hat{\sigma}^2_{MSIML} - \int_0^1 \sigma_x^2(s)ds\right] \xrightarrow{w} N\left(0, 2\left[\int_0^1 \sigma_x^2(s)ds\right]^2\right) .$$

Since there are n terms with $i = j$ in (2.14) and they are bounded by $m \times n \times (1/n)^2$ at most, it may be better to delete end terms with $i = j$ for removing higher order bias. The choice of α and d in our formulation and the finite sample properties of the possible modifying SIML estimators are currently under investigation.

3. Asymptotic Robustness of the SIML Estimation

3.1 Effects of Autocorrelations of Noise and endogeneity

We shall investigate the effects of the serial correlations of noises on the asymptotic properties of the SIML estimator. Consider the case of $p = q = 1$, $\sigma_x(s) = \mathbf{C}_x(s)$ $((0 \leq s \leq 1))$ and we write

(3.1)
$$r_i = x_i - x_{i-1} = \int_{t_{i-1}}^{t_i} \sigma_x(s)dB_s \ \ (i = 1, \cdots, n)$$

with $0 = t_0 \leq t_1 < \cdots < t_n = 1$ $(i = 1, \cdots, n)$. For the simplicity, we take the equi-distance case as $t_i - t_{i-1} = 1/n$ and the volatility function $\sigma_x(s)$ $(0 \leq s \leq 1)$ is non-stochastic.

Let $z_{in}^{(1)}$ and $z_{in}^{(2)}$ $(i = 1, \cdots, n)$ be the $k-th$ elements of $\mathbf{Z}_n^{(1)} = h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}(\mathbf{X}_n - \mathbf{Y}_0)$ and $\mathbf{Z}_n^{(2)} = h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}\mathbf{V}_n$, respectively, where $\mathbf{X}_n = (x_i)$ and $\mathbf{V}_n = (v_i)$ are $n \times 1$ vectors with $z_{in} = z_{in}^{(1)} + z_{in}^{(2)}$. Then by following Kunitomo and Sato (2008a), we shall investigate the effects of the (possibly) autocorrelated noise and the endogeity of noise to signal on the asymptotic distribution of $\hat{\sigma}_x^2 - \sigma_x^2$ and $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds$. From *Mathematical Appendix* [1] of Kunitomo and Sato (2008a),

$$(3.2) \qquad \sqrt{m}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[z_{kn}^2 - \sigma_x^2\right]$$

$$= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[z_{kn}^{(1)2} - \sigma_x^2\right] + \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\mathbf{E}[z_{kn}^{(2)2}]$$

$$+ \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}]\right] + 2\frac{1}{\sqrt{m}}\sum_{k=1}^{m}\left[z_{kn}^{(1)}z_{kn}^{(2)}\right] .$$

Then we shall investigate the conditions that three terms except the first one of (3.2) are $o_p(1)$. It is because we could estimate the realized volatility consistently as if there were no noise terms in this situation.

Let $\mathbf{b}_k = \mathbf{e}_k^{'}\mathbf{P}_n^{'}\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0, \cdots, 1, 0, \cdots)$ be an $n \times 1$ vector. We write $z_{kn}^{(2)} = \sum_{j=1}^{n} b_{kj}v_j$ and notice that $\sum_{j=1}^{n} b_{kj}b_{k'j} = \delta(k, k^{'})a_{kn}$. Also we shall use the notations that K_i $(i \geq 1)$ are positive finite constants.
First we impose the condition

$$(\mathbf{I}) \quad \mathbf{E}[v_iv_j] = c_1\rho^{|i-j|} \ (0 \leq \rho < 1) ,$$

where c_1 is a constant.
Then by using the Cauchy-Schwartz inequality

$$(3.3) \qquad \mathbf{E}[z_{kn}^{(2)}]^2 = \mathbf{E}[\sum_{i=1}^{n} b_{ki}v_i \sum_{j=1}^{n} b_{kj}v_j]]$$

$$\leq \sum_{l=0}^{n} c_1(1 + 2l)\rho^l\mathbf{E}[\sum_{i=1}^{n} b_{ki}b_{ki-l}]$$

$$\leq K_1 \times a_{kn} ,$$

provided that $\mathbf{E}[v_i^2]$ are bounded and we define $b_{kj} = 0$ $(j \leq 0)$. Then the second

[1] We have changed their notations slightly in this paper.

term of (3.2) becomes

$$(3.4) \qquad \frac{1}{\sqrt{m}}\sum_{k=1}^{m}\mathbf{E}[z_{kn}^{(2)}]^2 \le K_1\frac{1}{\sqrt{m}}\sum_{k=1}^{m}a_{kn} = O(\frac{m^{5/2}}{n}) \to 0$$

if $0 < \alpha < 0.4$. (See (6.3) of Kunitomo and Sato (2008a).)

For the fourth term of (3.2),

$$(3.5) \qquad \begin{aligned} \mathbf{E}\left[\frac{1}{\sqrt{m}}\sum_{j=1}^{m} z_{kn}^{(1)}z_{kn}^{(2)}\right]^2 &= \frac{1}{m}\sum_{k,k'=1}^{m}\mathbf{E}\left[z_{kn}^{(1)}x_{z'n}^{(1)}z_{kn}^{(2)}z_{k'n}^{(2)}\right] \\ &= \frac{1}{m}\sum_{k,k'=1}^{m}\mathbf{E}\left[2\sum_{j,j'=1}^{n}s_{jk}s_{j'k'}\mathbf{E}(r_jr_{j'})z_{kn}^{(2)}z_{k'n}^{(2)}\right] \\ &= \frac{1}{m}\sum_{k,k'=1}^{m}\mathbf{E}\left[2\sum_{j=1}^{n}s_{jk}s_{jk'}\mathbf{E}(r_j^2)z_{kn}^{(2)}z_{k'n}^{(2)})\right] \\ &\le K_2\left[(\sup_{0\le s\le 1}\sigma_x^2(s))\frac{2}{n}(\frac{n}{2}+\frac{1}{4})\right]\frac{1}{m}\sum_{k,k'=1}^{m}\sqrt{a_{kn}}\sqrt{a_{k'n}} \\ &\le K_3\sum_{k=1}^{m}a_{kn} = O(\frac{m^3}{n}) \end{aligned}$$

by using the relations $\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds \le (1/n)(\sup_{0\le s\le 1}\sigma_x^2(s))$ and $|\sum_{j=1}^{n}s_{jk}s_{jk'}| \le [\sum_{j=1}^{n}s_{jk}^2] = n/2 + 1/4$ for any $k \ge 1$.

Then we need the condition $0 < \alpha < 1/3$. If $\sigma_s = \sigma$ (i.e., the volatility is constant), (3.2) becomes $O(m^2/n)$, which is satisfied if $0 < \alpha < .4$.

For the third term of (3.2), we need to consider the variance of

$$z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}] = \sum_{j,j'=1}^{n} b_{kj}b_{kj'}\left[v_jv_{j'} - \mathbf{E}(v_jv_{j'})\right]$$

and we evaluate the expectation of $\left[z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}]\right]\left[z_{k'n}^{(2)2} - \mathbf{E}[z_{k'n}^{(2)2}]\right]$.

Furthermore we impose the additional condition

$$\textbf{(II)} \quad \mathbf{E}\left[[v_iv_{i'} - \mathbf{E}(v_iv_{i'})[v_{i''}v_{i'''} - \mathbf{E}(v_{i''}v_{i'''})\right] = c_2\rho^{\frac{1}{2}(|i-i'|+|i''-i'''|)} \ (0 \le \rho < 1)\ ,$$

where c_2 is a constant.

The condition (II) is satisfied for the linear processes on $\{v_j\}$ with bounded 4th order moments. The calculations are straightforward, but there are many terms involved. We use the fact that

$$
\begin{aligned}
(3.6) \qquad & \sum_{j,j^{'}=1}^{n} b_{kj}b_{kj^{'}}\rho^{|j-j^{'}|/2} \sum_{j^{''},j^{'''}=1}^{n} b_{k^{'}j^{''}}b_{k^{'}j^{'''}}\rho^{|j^{''}-j^{'''}|/2} \\
& \sim K_4 \left[\sum_{l,l^{'}=1}^{n} \rho^{l+l^{'}} ll^{'}\right] \left[\sum_{j=1}^{n} b_{kj}b_{kj-l}\right] \left[\sum_{j^{''}=1}^{n} b_{k^{'}j^{''}}b_{k^{'}j^{''}-l^{'}}\right] \\
& \sim K_5 \times a_{kn}a_{k^{'}n} \ .
\end{aligned}
$$

By using the condition (II) and (3.6), we obtain

$$
\begin{aligned}
(3.7) \qquad \mathbf{E}\left[\frac{1}{\sqrt{m}}\sum_{j=1}^{m}(z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}])\right]^2 & \leq \frac{1}{m}\sum_{k,k^{'}=1}^{m} a_{kn}a_{k^{'}n} \\
& = O(\frac{1}{m} \times (\frac{m^3}{n})^2) = O(\frac{m^5}{n^2})
\end{aligned}
$$

since $\sum_{k=1}^{m} a_{kn} = O(m^3/n)$.

Thus the third term of (3.2) is negligible if $0 < \alpha < .4$. We summarize the main finding of the asymptotic robustness of the SIML estimator as follows.

Theorem 2 : For (2.1)-(2.3) with $p = q = 1$, define the SIML estimator by (2.14) and (2.17).

(i) Assume Conditions (I) and (II) and set $0 < \alpha < 1/3$. Then the asymptoic distribution of $\sqrt{m_n}\,[\hat{\sigma}_x^2 - \sigma_x^2]$ is asymptotically ($m_n, n \to \infty$) equivalent to that of $(1/\sqrt{m_n})\sum_{k=1}^{m_n}\left[z_{kn}^{(1)2} - \sigma_x^2\right]$ for $\hat{\sigma}_x^2$ is either $\hat{\sigma}_{SIML}^2$ or $\hat{\sigma}_{MSIML}^2$.

(ii) In additon to Conditions of (i), assume the moment conditions of Theorem 1. Then we have (2.18).

In the above discussions we have found that the only term involved in the correlations of noise and signal is the fourth term of (3.2). Thus it is interesting to find the condition that they can be ignored for estimating the realized volatility and covariance. The second line of (3.5) can be written as

$$
(3.8) \qquad \frac{1}{m}\sum_{k,k^{'}=1}^{m} \mathbf{E}\left[z_{kn}^{(1)}z_{k^{'}n}^{(1)}z_{kn}^{(2)}z_{k^{'}n}^{(2)}\right]
$$

$$= \frac{1}{m}\sum_{k,k'=1}^{m}\mathbf{E}\left[(2\sum_{j,j'=1}^{n} s_{jk}s_{j'k'}r_jr_{j'})(\sum_{i=1}^{n} b_{ki}v_i)(\sum_{j=1}^{n} b_{k'j}v_j)\right] .$$

Thus the sufficient condition we need is

$$(\mathbf{I}^{'}) \quad \mathbf{E}[v_iv_j|r_k, k=1,\cdots,n] = c_3\rho_1^{|i-j|} \ (0 \le \rho_1 < 1) \ a.s.,$$

where c_3 is bounded.

We note that c_3 may depend on $\int_0^1 \sigma_s^2 ds$, which is finite (a.s.). In that case we allow that the noises may depend on the volatility structure, but we need the condition that $c_3(\int_0^1 \sigma_s^2 ds)$ should be finite (a.s.) and integrable. By this argument, if both the correlations between signal and noise and the autocorrelations of noise are weak, the SIML estimator is consistent and it has the asymptotic normality. Because this result has an independent interest, We summarize it as follows.

Theorem 3 : Instead of Conditions in Theorem 2, assume Conditions $(\mathrm{I})^{'}$ and (II) and set $0 < \alpha < 1/3$, and relax the independence assumption between the signal and noise terms. Then the results of Theorem 2 hold.

An important feature of our approach is the fact that our arguments go through even when $p \ge 1$. When $p \ge 1$, we take an arbitrary constant vector $\mathbf{c}$ and use $\mathbf{c}^{'}\mathbf{r_i}$ and $\mathbf{c}^{'}\mathbf{v}_i$ $(i = 1, \cdots, n)$. Then we can use the same arguments.

3.2 Autocorrelation of Noise

When there are non-negligible autocorrelations in the noise terms, we want to estimate the dependence structure in the noise terms from the set of observed data. First, we write the (s-th) sample auto-covariance of returns as For this purpose, we can use the sample auto-covariance as

$$(3.9) \qquad \hat{\gamma}_{\Delta y}(s) = \frac{1}{n}\sum_{i=1+l}^{n} \Delta y_i \Delta y_{i-s} \ \ (s = 1, \cdots, q) \ .$$

Because the true (or efficient) price process is a continuous martingale, we find that for $s \ge 1$

$$(3.10) \qquad \hat{\gamma}_{\Delta y}(s) \xrightarrow{p} \gamma_{\Delta y}(s) = -\gamma_v(s-1) + 2\gamma_v(s) - \gamma_v(s+1) \ ,$$

where $\gamma_v(s) = \mathbf{E}[v_i v_{i-s}]$ is the s-th autocovariance.
Hence if we have the condition

$$\gamma_v(s) = O(\rho^s) \ \ , |\rho| < 1 \ ,$$

then

$$\gamma_v(s) = -\left[\sum_{j=1}^{q} j\gamma_{\Delta y}(s+j)\right] + O(\rho^q) \ . \tag{3.11}$$

Alternatively, we can use the SIML estimation on the measurement errors. After some calculations, it is not difficult to show

$$\mathcal{E}\left[\frac{1}{l_n}\sum_{k=n-l_n-1}^{n} a_{k,n}^{-1}[x_{k,n}^{(2)}]^2\right] = \gamma_v(0) + 2\sum_{s=1}^{q}(-1)^s\gamma_v(s) + O(\rho^q) \ . \tag{3.12}$$

Hence there can be several different ways to estimate $\gamma_v(s)$ $(0 \leq s)$ from observations, which may include $\gamma_v(0)$. and the asymptotic normality of the sample auto-covariance under a set of mild conditions. Furthermore, we can extend the arguments to the estimation of auto-correlation in the multivariate cases.

4. Simulations

We have investigated the finite sample distribution of the SIML estimator for the realized variance based on a set of simulations and the number of replications is 1000. As a reasonable setting we have taken $n = 300, 5,000$ and $20,000$, and we have chosen $\alpha = 0.4$ and $\beta = 0.8$ in most cases. In our experiments we have considered the situation that the variance of noises $10^{-2} \sim 10^{-6}$ of the realized variances of the underlying signals [2] .

In our simulation we consider cases when the observations are the sum of signal and micro-market noise when $p = 1$. In our examples the signal is the Brownian motion with the volatility function $\mathbf{\Sigma}_x(s) = \sigma_x^2(s)$ and

$$\sigma_x^2(s) = \sigma(0)^2\left[a_0 + a_1 s + a_2 s^2\right] , \tag{4.1}$$

[2] The simulation procedure is similar to the corresponding one reported by Kunitomo and Sato (2008b).

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. In this case the realized variance $\mathbf{\Sigma}_x = \sigma_x^2$ is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{4.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

In order to investigate the effects of autocorrelations in the market noise terms, first we consider MA(q) model given by

$$v_i = \sum_{j=0}^{q} \theta_j w_{i-j} \ , \tag{4.3}$$

where $\theta_0 = 1$ (for the normalization) and w_j are mutually independent random variables followed by $N(0, \omega^2)$. The bench mark process is $MA(0)$ and we have used $MA(1)$ processes with the coefficient $a = -0.5, 0.5, 0.9, 0.95$ extensively. We have confirmed that our results do not much depend on the MA(q) structure and basically they are the same for the stationary ARMA processes.

As the second problem we have investigated the endogeneity of market noise with the signals. We generate the noise process

$$v_i = \sum_{k=0}^{p} \phi_k \Delta x_{i-k} + \sum_{j=0}^{q} \theta_j w_{i-j} \ , \tag{4.4}$$

where $\Delta x_{i-k} = x_{i-k} - x_{i-k-1}$ are the lagged (stationary) signals. As the preliminary case we have set $\theta_j = 0$ $(j = 2, \cdots, q)$ and $\phi_k = 0$ $(k \neq l, k = 1, \cdots, s)$. By combining the structure of autocorrelations of the noise terms and the endogeneity among the signals and noise terms, there can be many different situations.

Among many Monte-Carlo simulations, we summarize our main results as Tables of Appendix C. If we knew the fact that the underlying noise process is MA(q) and its distribution is Gaussian, we can use the maximum likelihood (ML) estimation for the model that the observations are the sum of the signal process and the market noise process. By using the standard arguments of statistical asymptotic theory for parametric models, the ML estimation is asymptotically efficient. We have confirmed this fact in Table 11 when the noise process is MA(0). When the noise process is MA(1), however, the ML estimator has lose the asymptotic efficiency while the SIML estimator gives reasonable and stable estimates.

For the standard case of MA(0) and the MA(1) case for noise terms we give the results of the SIML estimation as Tables 1-3. (We note that a stands for the MA(1) coefficient.) The finite sample efficiency of the ML estimator lose its power rather quickly while the SIML estimator has robustness against this type of autocorrelations. (Table 11 shows some results of the ML estimation when we knew that the true process is MA(1).) For the case of the endogenous noise we give Tables 4-6 with and without autocorrelated noise terms. In these tables $l = 0$ stands for the cases of instantaneous endogeneity while $l = 1$ stands for the cases when there are some lagged effects between the signal and noise terms. We also have conducted some extreme experiments such as the cases when $a = 0.95$ and AR(1) with $b = 0.95$. They are summarized in Tables 7-10.

By examining the results of our simulations we can conclude that we can estimate both the realized volatility of the hidden martingale part and the market noise part reasonably in all cases we have examined by the SIML estimation. We also have conducted a number of further simulations and the some details have been given in Kunitomo and Sato (2008b).

5. Conclusions

In this paper, we have shown that the Separating Information Maximum Likelihood (SIML) estimator has the asymptotic robustness in the sense that it is consistent and it has the asymptotic normality under a fairly general conditions even when the standard conditions are not satisfied. They include the cases when the micro-market noises are possibly autocorrelated and they are endogenously correlated with the underlying continuous signal processes. By conducting large number of simulations, we have confirmed that the SIML estimator has reasonable robust properties in finite samples even in these non-standard situations.

As a concluding remark, the SIML estimator is very simple and it can be practically used not only for the realized volatility but also the realized covariance of the multivariate high frequency financial series. Some applications on the analysis of financial futures markets have been reported in Kunitomo and Sato (2008b) for example.

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Appendices

We gather some details in Appendices. In Appendix A we give the mathematical derivation of Theorem 1 and discussion of Theorem 1 of Kunitomo and Sato (2008a). Then in Appendix B we give some formulas used in Appendix A and Section 2. (The derivations are similar to the ones in Kunitomo and Sato (2008a) and they are omitted.) In Appendix C we give some tables.

(I) APPENDIX A : On Derivations of Theorem 1 and Theorem 1 of Kunitomo-Sato (2008a)

From Section 6 of Kunitomo and Sato (2009a), we shall investigate the asymptotic distribution of

$$(A.1)\quad \sqrt{m}\left[\sum_{i,j=1}^{n} c_{ij}r_ir_j - \delta_{ij}\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right] = 2\sqrt{m}\sum_{i>j}c_{ij}r_ir_j + \sqrt{m}\left[\sum_{i=1}^{n}c_{ii}r_i^2 - \int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]\ ,$$

where $\delta_{ij}=1\ (i=j); \delta_{ij}=0\ (i\neq j)$.
The second term is equivalent to

$$(A.2)\quad \sqrt{m}\sum_{i=1}^{n}\left[r_i^2-\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds+(c_{ii}-1)r_i^2\right]$$
$$= \sqrt{\frac{m}{n}}\sqrt{n}\sum_{i=1}^{n}\left[r_i^2-\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]+\sqrt{m}\left[\sum_{i=1}^{n}(c_{ii}-1)(r_i^2-\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds)\right]$$
$$+\sqrt{m}\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]\ .$$

We notice that Jacod-Protter (1998 Annals of Probability) have shown that

$$\sqrt{n}\sum_{i=1}^{n}\left[r_i^2-\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]=O_p(1)\ .$$

By using the fact that $c_{ii}-1\ (i=1,\cdots,n)$ are bounded, we find that as $\sqrt{m}/n\to 0$ the second term of (A.2) is asymptotically equivalent to

$$(A.3)\quad \sqrt{m}\left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]=\sum_{i=1}^{n}\frac{1}{2\sqrt{m}}\left[\frac{\sin[\frac{2\pi m}{2n+1}(2i-1)]}{\sin[\frac{2\pi}{2n+1}(2i-1)]}\right]\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\ .$$

When $\sigma_s = \sigma$ (constant),

$$\begin{aligned}\sqrt{m}\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds &= \sigma^2\frac{\sqrt{m}}{n}\sum_{i=1}^{n}(c_{ii}-1)\\ &= \sigma^2\frac{\sqrt{m}}{n}\left[n+\frac{1}{2}-n\right]\to 0\end{aligned}$$

as $\sqrt{m}/n \to 0$. There are some cases that (A.3) is $o(1)$ because $\int_{t_{i-1}}^{t_i}\sigma_s^2 ds \leq (1/n)\sup_{0\leq s\leq 1 \sigma_s^2}$. Generally, this term may be small because it is bounded and

$$\begin{aligned}|\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds|^2 &= \left[\sum_{i=1}^{n}(c_{ii}-1)^2\right]\left[\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds)^2\right]\\ &\leq \left[\frac{1}{n}\sum_{i=1}^{n}(c_{ii}-1)^2\right]\left[\sup_{0\leq s\leq t}\sigma_x^2(s)\right]^2 = O(\frac{1}{m})\ .\end{aligned}$$

However, we have not proven that it is $o(1)$ under the general situations. The asymptotic variance of the first term of (A.2) is

$$\begin{aligned}&4\sum_{i<j} mc_{ij}^2\mathbf{E}[r_i^2]\mathbf{E}[r_j^2]\\ =\ &2\sum_{i,j=1}^{n} mc_{ij}^2\mathbf{E}[r_i^2]\mathbf{E}[r_j^2] - 2\sum_{i=1}^{n} mc_{ii}^2(\mathbf{E}[r_i^2])^2\\ =\ &2\sum_{i,j=1}^{n}\mathbf{E}[r_i^2]\mathbf{E}[r_j^2] + 2\sum_{i,j=1}^{n}(mc_{ij}^2-1)\mathbf{E}[r_i^2]\mathbf{E}[r_j^2] - 2\sum_{i=1}^{n} mc_{ii}^2(\mathbf{E}[r_i^2])^2\ .\end{aligned}$$

For the third term, we have

$$\sum_{i=1}^{n} mc_{ii}^2(\mathbf{E}[r_i^2])^2 \leq \left[\sup_{0\leq\leq 1}\sigma_x^2(s)\right]^2\frac{m}{n^2}\sum_{i=1}^{n}c_{ii}^2 \to 0$$

as $m/n \to 0$. Then the asymptotic variance of (2.14) becomes

(A.4) V_n

$$\begin{aligned}&= 2\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]^2 + 2\sum_{i,j=1}^{n}(mc_{ij}^2-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\int_{t_{j-1}}^{t_j}\sigma_x^2(s)ds\\ &\to 2\left[\int_0^1\sigma_x^2(s)ds\right]^2 + 2\lim_{n\to\infty}\sum_{i,j=1}^{n}(mc_{ij}^2-1)\left[\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]\left[\int_{t_{j-1}}^{t_j}\sigma_x^2(s)ds\right] = V\ .\end{aligned}$$

The second term is bounded because

$$\frac{1}{n^2}\sum_{i,j=1}^{n}|mc_{ij}^2-1| \leq 1+\frac{m}{n^2}\sum_{i,j=1}^{n}c_{ij}^2$$

and it is likely to be small, but we have not proven that it is $o(1)$. Hence presently the statement of Theorem 1 of Kunitomo and Sato (2008a) should be slightly modified as

(A.5) $$\sqrt{m}\left[\hat{\sigma}_x^2 - \sigma_x^2 - \sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right] \xrightarrow{w} N\left[0,V\right] \quad .$$

When the volatility function is constant ($\sigma_x(s)=\sigma$),

$$\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds = \sigma^2\frac{1}{n} \ ,$$

the second term vanishes because

$$\frac{1}{n^2}\sum_{i,j=1}^{n}(mc_{ij}^2-1) = \frac{1}{n^2}\left[n^2+n+\frac{1}{4}-n^2\right] \to 0$$

and then

(A.6) $$V = 2\left[\int_0^1 \sigma_x^2(s)ds\right]^2 \ .$$

There are some other cases that second term of V_n is $o(1)$ because we have (A.4).

(II) APPENDIX B : Some Formulas

In our derivations we have used elementary relations of trigonometric functions and their sums. From (2.16), we have

(B.7) $$\sum_{j=1}^{n} c_{jj} = n + \frac{1}{2} \ .$$

From (2.15), we rewrite

$$\begin{aligned}
c_{ij}^2 &= (\frac{2}{m})^2 \sum_{k,k'=1}^{m} s_{ik}s_{jk}s_{ik'}s_{jk'} \\
&= (\frac{1}{m})^2 \sum_{k,k'=1}^{m} \left[\cos[\frac{2\pi}{2n+1}(i+j-1)(k-\frac{1}{2})] + \cos[\frac{2\pi}{2n+1}(i-j)(k-\frac{1}{2})]\right] \\
&\qquad \times \left[\cos[\frac{2\pi}{2n+1}(i+j-1(k'-\frac{1}{2})] + \cos[\frac{2\pi}{2n+1}(i-j)(k'-\frac{1}{2})]\right] \\
&= (\frac{1}{m})^2 \sum_{k,k'=1}^{m} \left\{\frac{1}{2}\cos[\frac{2\pi}{2n+1}(i+j-1)(k+k'-1) + \frac{1}{2}\cos[\frac{2\pi}{2n+1}(i+j-1)(k-k')\right. \\
&\qquad +2\left[\frac{1}{2}\cos[\frac{2\pi}{2n+1}((i+j-1)(k-\frac{1}{2}) + (k'-\frac{1}{2})(i-j))]\right.
\end{aligned}$$

$$+\frac{1}{2}\cos[\frac{2\pi}{2n+1}((i+j-1)(k-\frac{1}{2})-(k^{'}-\frac{1}{2})(i-j))]\Big]$$
$$+\frac{1}{2}\cos[\frac{2\pi}{2n+1}(i-j)(k+k^{'}-1)+\frac{1}{2}\cos[\frac{2\pi}{2n+1}(i-j)(k-k^{'})\Big\} .$$

and hence

$$\begin{aligned} c_{jj}^2 &= 1+\frac{2}{m}\sum_{k=1}^{m}\cos[\frac{2\pi}{2n+1}(k-\frac{1}{2})(2j-1)] \\ &+\frac{1}{2m^2}\sum_{k,k^{'}=1}^{m}\Big[\cos[\frac{2\pi}{2n+1}(2j-1)(k+k^{'}-1)]+\cos[\frac{2\pi}{2n+1}(2j-1)(k-k^{'})]\Big] . \end{aligned}$$

By using the relation $\sum_{j=1}^{n}\cos[\frac{2\pi}{2n+1}(2k-1)(2j-1)]=\frac{1}{2}$ for $k\geq 1$ and after some calculations, it is straightforward to show

(B.8) $$\frac{1}{n}\sum_{i=1}^{n}(c_{ii}^2-1)\to 0 ,$$

(B.9) $$\frac{1}{n}\sum_{i=1}^{n}(c_{ii}-1)^2\to 0 ,$$

and

(B.10) $$m\sum_{i,j=1}^{n}c_{ij}^2=(n+\frac{1}{2})^2$$

by using the relation $\sum_{j=1}^{n}s_{jk}s_{jk^{'}}=\delta_{kk^{'}}(\frac{n}{2}+\frac{1}{4})$.

(III) APPENDIX C : TABLES

In Tables 1-11 σ_x^2 and σ_v^2 are the true parameter variances and we give their corresponding estimates when we have the constant volatility model. Mean and SD in Tables are the sample mean and the standard deviation of the SIML estimator (or the ML estimator) in simulations. H-vol stands for the historical volatility.

Table 1 : Estimation of Realized Volatility (standard case, $v_t \sim i.i.d.Normal$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.05E-04	2.18E-06	1.40E-03	2.00E-04	3.82E-07	3.19E-04	2.01E-04	1.84E-07	2.01E-04
SD	9.61E-05	3.22E-07	1.35E-04	9.16E-05	5.61E-08	2.67E-05	9.29E-05	2.69E-08	1.61E-05
MSE	9.27E-09	1.36E-13		8.40E-09	3.61E-14		8.63E-09	3.39E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.07E-04	2.01E-06	2.02E-02	2.01E-04	2.10E-07	2.20E-03	2.00E-04	1.23E-08	2.20E-04
SD	5.37E-05	9.44E-08	4.93E-04	5.12E-05	9.68E-09	5.22E-05	5.27E-05	5.68E-10	4.35E-06
MSE	2.92E-09	8.99E-15		2.62E-09	2.03E-16		2.78E-09	1.06E-16	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.05E-04	2.00E-06	8.02E-02	2.00E-04	2.03E-07	8.20E-03	2.00E-04	4.54E-09	2.80E-04
SD	4.10E-05	5.47E-08	9.82E-04	3.98E-05	5.43E-09	9.91E-05	3.90E-05	1.19E-10	2.87E-06
MSE	1.70E-09	3.00E-15		1.59E-09	3.59E-17		1.52E-09	6.46E-18	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Table 2 : Estimation of Realized Volatility (MA(1) noise, $a = 0.5$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.03E-04	3.53E-06	1.87E-03	2.03E-04	5.18E-07	3.68E-04	1.99E-04	1.86E-07	2.01E-04
SD	9.62E-05	5.12E-07	1.99E-04	9.69E-05	7.45E-08	3.21E-05	9.35E-05	2.69E-08	1.64E-05
MSE	9.27E-09	2.61E-12		9.40E-09	1.07E-13		8.74E-09	3.44E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.04E-04	3.52E-06	2.82E-02	2.00E-04	3.62E-07	3.00E-03	2.00E-04	1.38E-08	2.28E-04
SD	5.27E-05	1.63E-07	7.61E-04	5.12E-05	1.69E-08	7.94E-05	5.09E-05	6.45E-10	4.57E-06
MSE	2.79E-09	2.35E-12		2.62E-09	2.65E-14		2.59E-09	1.39E-16	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.05E-04	3.56E-06	1.12E-01	1.98E-04	3.57E-07	1.14E-02	2.00E-04	6.09E-09	3.12E-04
SD	3.95E-05	9.59E-08	1.53E-03	4.00E-05	9.58E-09	1.54E-04	3.92E-05	1.62E-10	3.30E-06
MSE	1.58E-09	2.43E-12		1.61E-09	2.48E-14		1.54E-09	1.67E-17	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process:

$$y_t = x_t + \sqrt{\sigma_v^2/(1+a^2)}v_t$$
$$x_t = x_{t-1} + \sqrt{\sigma_x^2/n}u_t$$
$$v_t = w_t - aw_{t-1}$$
$$u_t \sim i.i.d.N(0,1), w_t \sim i.i.d.N(0,1)$$

Table 3 : Estimation of Realized Volatility (MA(1) noise, $a = -0.5$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.07E-04	8.40E-07	9.17E-04	2.02E-04	2.47E-07	2.71E-04	2.02E-04	1.82E-07	2.00E-04
SD	9.59E-05	1.25E-07	8.22E-05	9.65E-05	3.65E-08	2.26E-05	9.40E-05	2.66E-08	1.57E-05
MSE	9.24E-09	1.36E-12		9.32E-09	3.53E-15		8.83E-09	3.33E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.05E-04	4.96E-07	1.22E-02	2.02E-04	5.89E-08	1.40E-03	2.00E-04	1.08E-08	2.12E-04
SD	5.21E-05	2.34E-08	2.75E-04	5.30E-05	2.81E-09	3.10E-05	5.29E-05	4.88E-10	4.35E-06
MSE	2.75E-09	2.26E-12		2.81E-09	1.99E-14		2.79E-09	7.71E-17	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.06E-04	4.52E-07	4.82E-02	2.01E-04	4.75E-08	5.00E-03	2.01E-04	2.99E-09	2.48E-04
SD	4.01E-05	1.23E-08	5.47E-04	3.84E-05	1.26E-09	5.58E-05	4.01E-05	8.07E-11	2.45E-06
MSE	1.64E-09	2.40E-12		1.48E-09	2.33E-14		1.61E-09	9.88E-19	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process: same as Table 2.

Table 4 : Estimation of Realized Volatility (Endogenous noise, $\rho = 0.5, l = 0$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.02E-04	1.76E-06	1.14E-03	1.98E-04	4.64E-07	3.68E-04	1.97E-04	2.02E-07	2.11E-04
SD	9.54E-05	2.54E-07	1.07E-04	9.31E-05	6.62E-08	3.14E-05	9.44E-05	2.98E-08	1.71E-05
MSE	9.11E-09	1.23E-13		8.67E-09	7.40E-14		8.93E-09	4.10E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.02E-04	1.15E-06	1.16E-02	2.01E-04	1.55E-07	1.65E-03	1.99E-04	1.58E-08	2.55E-04
SD	5.27E-05	5.42E-08	2.82E-04	5.27E-05	7.22E-09	3.77E-05	5.13E-05	7.48E-10	5.19E-06
MSE	2.78E-09	7.20E-13		2.78E-09	2.08E-15		2.64E-09	1.90E-16	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.03E-04	1.07E-06	4.30E-02	2.00E-04	1.25E-07	5.09E-03	2.00E-04	5.78E-09	3.29E-04
SD	3.99E-05	2.87E-08	5.21E-04	3.91E-05	3.37E-09	6.12E-05	3.96E-05	1.57E-10	3.50E-06
MSE	1.60E-09	8.60E-13		1.53E-09	5.66E-15		1.57E-09	1.43E-17	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process:

$$y_t = x_t + \sqrt{\sigma_v^2} v_t$$
$$x_t = x_{t-1} + \sqrt{\sigma_x^2/n} u_t$$
$$v_t = (1-\rho)w_t + \rho u_{t-l}$$
$$u_t \sim i.i.d.N(0,1), w_t \sim i.i.d.N(0,1)$$

Table 5 : Estimation of Realized Volatility (MA(1) and Endogenous noise, $a = 0.5, \rho = 0.5, l = 0$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.01E-04	2.55E-06	1.35E-03	2.04E-04	5.68E-07	3.81E-04	1.99E-04	2.05E-07	2.10E-04
SD	9.55E-05	3.79E-07	1.44E-04	9.55E-05	8.31E-08	3.50E-05	9.29E-05	3.02E-08	1.73E-05
MSE	9.11E-09	4.48E-13		9.14E-09	1.42E-13		8.63E-09	4.23E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.01E-04	1.95E-06	1.55E-02	2.01E-04	2.44E-07	2.00E-03	2.01E-04	1.78E-08	2.54E-04
SD	5.21E-05	9.14E-08	4.20E-04	5.14E-05	1.13E-08	5.24E-05	5.18E-05	8.43E-10	5.34E-06
MSE	2.72E-09	1.09E-14		2.64E-09	2.03E-15		2.68E-09	2.51E-16	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.00E-04	1.87E-06	5.87E-02	2.00E-04	2.10E-07	6.60E-03	1.99E-04	7.25E-09	3.36E-04
SD	4.03E-05	5.12E-08	8.14E-04	3.92E-05	5.60E-09	8.82E-05	3.87E-05	1.96E-10	3.63E-06
MSE	1.63E-09	1.98E-14		1.54E-09	1.22E-16		1.50E-09	2.76E-17	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process:

$$y_t = x_t + \sqrt{\sigma_v^2/(1+a^2)}v_t$$

$$x_t = x_{t-1} + \sqrt{\sigma_x^2/n}u_t$$

$$v_t = \epsilon_t - a\epsilon_{t-1}$$

$$\epsilon_t = (1-\rho)w_t + \rho u_{t-l}$$

$$u_t \sim i.i.d.N(0,1), w_t \sim i.i.d.N(0,1)$$

Table 6 : Estimation of Realized Volatility (MA(1) and Endogenous noise, $a = 0.8, \rho = 0.9, l = 1$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	1.98E-04	1.71E-06	7.86E-04	2.00E-04	1.88E-08	6.91E-05	2.01E-04	1.39E-07	1.74E-04
SD	9.23E-05	2.47E-07	9.13E-05	9.30E-05	2.94E-09	7.78E-06	9.42E-05	2.02E-08	1.39E-05
MSE	8.53E-09	1.43E-13		8.65E-09	3.28E-14		8.88E-09	1.92E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.00E-04	2.81E-06	2.10E-02	2.01E-04	2.13E-07	1.51E-03	2.00E-04	2.12E-09	1.11E-04
SD	5.31E-05	1.33E-07	6.02E-04	5.16E-05	1.02E-08	4.47E-05	5.15E-05	1.02E-10	2.51E-06
MSE	2.82E-09	6.67E-13		2.66E-09	2.61E-16		2.65E-09	2.51E-20	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.01E-04	3.01E-06	9.06E-02	2.01E-04	2.65E-07	7.70E-03	2.00E-04	7.30E-11	7.13E-05
SD	4.01E-05	8.08E-08	1.28E-03	4.05E-05	6.97E-09	1.08E-04	3.96E-05	1.96E-12	9.98E-07
MSE	1.61E-09	1.04E-12		1.64E-09	4.28E-15		1.56E-09	3.71E-18	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process: same as Table 5.

Table 7 : Estimation of Realized Volatility (MA(1) noise, $a = 0.95$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	1.99E-04	1.71E-06	7.86E-04	1.99E-04	1.90E-08	6.89E-05	2.02E-04	1.39E-07	1.74E-04
SD	9.40E-05	2.53E-07	9.35E-05	9.22E-05	2.96E-09	7.87E-06	9.67E-05	2.05E-08	1.44E-05
MSE	8.83E-09	1.49E-13		8.51E-09	3.28E-14		9.35E-09	1.91E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.02E-04	2.81E-06	2.10E-02	1.99E-04	2.12E-07	1.51E-03	1.99E-04	2.12E-09	1.11E-04
SD	5.25E-05	1.30E-07	5.94E-04	5.17E-05	9.97E-09	4.37E-05	5.08E-05	9.98E-11	2.45E-06
MSE	2.76E-09	6.71E-13		2.67E-09	2.52E-16		2.58E-09	2.50E-20	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.03E-04	3.02E-06	9.07E-02	1.99E-04	2.65E-07	7.69E-03	2.01E-04	7.30E-11	7.13E-05
SD	3.99E-05	8.17E-08	1.26E-03	3.90E-05	7.18E-09	1.10E-04	3.95E-05	1.98E-12	1.01E-06
MSE	1.60E-09	1.04E-12		1.52E-09	4.27E-15		1.56E-09	3.71E-18	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process: same as Table 2.

Table 8 : Estimation of Realized Volatility (MA(1) and Endogenous noise, $a = 0.9, \rho = 0.9, l = 1$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.02E-04	2.49E-06	1.66E-03	1.99E-04	2.64E-07	3.46E-04	2.02E-04	1.63E-07	2.01E-04
SD	9.64E-05	3.67E-07	1.62E-04	9.41E-05	4.74E-08	3.21E-05	9.59E-05	2.42E-08	1.69E-05
MSE	9.29E-09	3.75E-13		8.86E-09	6.40E-15		9.21E-09	2.66E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.00E-04	2.91E-06	2.48E-02	1.98E-04	2.38E-07	2.65E-03	1.99E-04	4.36E-09	2.24E-04
SD	5.14E-05	1.35E-07	6.68E-04	5.20E-05	1.10E-08	6.51E-05	5.30E-05	2.28E-10	4.78E-06
MSE	2.64E-09	8.45E-13		2.71E-09	1.53E-15		2.81E-09	5.64E-18	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	1.99E-04	3.07E-06	9.84E-02	2.00E-04	2.74E-07	1.00E-02	2.00E-04	7.45E-10	2.98E-04
SD	3.88E-05	8.40E-08	1.36E-03	3.91E-05	7.39E-09	1.29E-04	3.93E-05	2.59E-11	3.42E-06
MSE	1.50E-09	1.14E-12		1.53E-09	5.60E-15		1.54E-09	1.58E-18	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process: same as Table 5.

Table 9 : Estimation of Realized Volatility (AR(1) noise, $b = 0.95$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.24E-04	2.39E-07	2.59E-04	2.04E-04	1.89E-07	2.06E-04	2.00E-04	1.82E-07	1.99E-04
SD	1.05E-04	3.43E-08	2.11E-05	9.72E-05	2.80E-08	1.72E-05	9.65E-05	2.63E-08	1.63E-05
MSE	1.17E-08	3.10E-12		9.47E-09	9.17E-16		9.31E-09	3.29E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.43E-04	6.31E-08	1.20E-03	2.04E-04	1.56E-08	3.00E-04	1.99E-04	1.03E-08	2.01E-04
SD	6.34E-05	2.89E-09	2.39E-05	5.22E-05	7.30E-10	6.00E-06	5.12E-05	4.91E-10	4.10E-06
MSE	5.89E-09	3.75E-12		2.74E-09	3.40E-14		2.62E-09	6.96E-17	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.35E-04	5.46E-08	4.20E-03	2.04E-04	7.75E-09	6.00E-04	2.01E-04	2.59E-09	2.04E-04
SD	4.62E-05	1.44E-09	4.22E-05	4.08E-05	2.10E-10	6.05E-06	3.97E-05	6.97E-11	2.03E-06
MSE	3.35E-09	3.78E-12		1.68E-09	3.70E-14		1.58E-09	3.56E-19	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process:

$$y_t = x_t + \sqrt{\sigma_v^2(1-a^2)}v_t$$
$$x_t = x_{t-1} + \sqrt{\sigma_x^2/n}u_t$$
$$v_t = bv_{t-1} + w_t$$
$$u_t \sim i.i.d.N(0,1), w_t \sim i.i.d.N(0,1)$$

Table 10 : Estimation of Realized Volatility (MA(1) and Endogenous noise, $a = 0.9, \rho = 0.9, l = 0$)

n=300	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.01E-04	4.46E-06	2.14E-03	2.00E-04	8.75E-07	4.91E-04	1.99E-04	2.24E-07	2.15E-04
SD	9.44E-05	6.43E-07	2.43E-04	9.61E-05	1.28E-07	4.94E-05	9.48E-05	3.31E-08	1.76E-05
MSE	8.90E-09	6.48E-12		9.24E-09	4.72E-13		8.99E-09	5.03E-14	
AVAR	8.17E-09	8.34E-14		8.17E-09	8.34E-16		8.17E-09	8.34E-20	
n=5000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.00E-04	3.53E-06	2.66E-02	2.00E-04	4.37E-07	3.26E-03	1.99E-04	2.43E-08	2.84E-04
SD	5.20E-05	1.67E-07	7.58E-04	5.28E-05	2.08E-08	9.32E-05	5.01E-05	1.15E-09	6.17E-06
MSE	2.70E-09	2.38E-12		2.79E-09	5.65E-14		2.51E-09	4.98E-16	
AVAR	2.65E-09	8.79E-15		2.65E-09	8.79E-17		2.65E-09	8.79E-21	
n=20000	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol	σ_x^2	σ_v^2	H-vol
true-val	2.00E-04	2.00E-06		2.00E-04	2.00E-07		2.00E-04	2.00E-09	
mean	2.00E-04	3.40E-06	1.02E-01	2.01E-04	3.80E-07	1.12E-02	1.99E-04	1.13E-08	4.18E-04
SD	3.92E-05	9.07E-08	1.44E-03	3.91E-05	1.03E-08	1.59E-04	3.85E-05	3.03E-10	4.93E-06
MSE	1.54E-09	1.96E-12		1.53E-09	3.25E-14		1.49E-09	8.62E-17	
AVAR	1.52E-09	2.90E-15		1.52E-09	2.90E-17		1.52E-09	2.90E-21	

Data generating process: same as Table 5.

Table 11 : Comparing with ML (MA(1) noise, $a = 0.5, \alpha = 0.45$)

n=300	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML
true-val	5.00E-05	5.00E-07	5.00E-05	5.00E-05	5.00E-09	5.00E-05	5.00E-05	0	5.00E-05
mean	5.34E-05	2.10E-07	1.11E-04	4.91E-05	4.73E-08	5.10E-05	4.97E-05	4.54E-08	4.95E-05
SD	2.11E-05	3.05E-08	3.31E-05	1.96E-05	7.02E-09	7.23E-06	1.94E-05	6.83E-09	7.18E-06
n=5000	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML
true-val	5.00E-05	5.00E-07	5.00E-05	5.00E-05	5.00E-09	5.00E-05	5.00E-05	0	5.00E-05
mean	5.24E-05	1.24E-07	1.98E-04	5.00E-05	3.78E-09	6.66E-05	4.96E-05	2.57E-09	4.99E-05
SD	1.09E-05	5.74E-09	4.21E-05	1.05E-05	1.81E-10	2.78E-06	9.98E-06	1.19E-10	1.69E-06
n=20000	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML	σ_x^2	σ_v^2	ML
true-val	5.00E-05	5.00E-07	5.00E-05	5.00E-05	5.00E-09	5.00E-05	5.00E-05	0	5.00E-05
mean	5.20E-05	1.13E-07	2.25E-04	4.99E-05	1.76E-09	9.69E-05	4.96E-05	6.35E-10	5.00E-05
SD	8.00E-06	3.02E-09	3.81E-05	7.55E-06	4.74E-11	2.85E-06	7.53E-06	1.76E-11	8.43E-07

Data generating process: same as Table 5.