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Asset Bubbles, Endogenous Growth, and Financial Frictions

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Abstract

This paper analyzes the existence and the effects of bubbles in an endogenous growth model with financial frictions and heterogeneous investments. Bubbles are likely to emerge when the degree of pledgeability is in the middle range. This suggests that improving the financial market might enhance the possibility of bubbles. We also find that when the degree of pledgeability is relatively low, bubbles boost long-run growth. When it is relatively high, bubbles lower growth. Moreover, we examine the effects of bubbles bursting, and show that the effects depend on the degree of pledgeability, i.e., the quality of the financial system.

Key words: Asset Bubbles, Endogenous Growth, and Financial Frictions

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1 Introduction

Many countries have experienced large movements in asset prices called asset bubbles. The boom and bust of asset bubbles have been associated with significant fluctuations in real economic activity. A notable example is the recent global economic upturn and downturn before and after the financial crisis of 2007. Many economists and policy makers have been anxious to understand why bubbles emerge and how they affect real economies,[1] but it is not yet obvious how bubbles affect economic growth. Moreover, it is still not clear how financial market conditions affect the existence condition of bubbles. In this paper, we examine how the emergence of asset bubbles is related to financial conditions, in other words, whether bubbles are more likely to occur in financially developed economies or financially less-developed ones. Moreover, we investigate the macroeconomic effects of bubbles, in the sense of whether bubbles are growth-enhancing or growth-impairing, and how those effects are related to financial conditions. In the process, we can also analyze how financial conditions determine the effects of bubbles bursting on the growth rate.

It is recognized that emerging market economies often experience bubble-like dynamics. As explored by Caballero (2006) and Caballero and Krishnamurthy (2006), the financial imperfection or less developed financial market is a key element of the existence of bubbles in emerging market economies. However, if financial imperfection is the reason for bubbles, why do less developed countries such as African countries not experience the bubbly economy? We will show that if the financial market is very poor and does not work well, the economic growth rates of less developed countries become too low to support bubbles. On the other hand, when the financial market is working very well, the interest rate becomes high compared to the economic growth rate and bubbles cannot exist. In this sense, there is a non-linear relation between the financial condition of a country and the existence of bubbles in that country. In other words, bubbles may not occur in financially underdeveloped or well-developed economies. They can only occur in financially intermediate-developed ones. In order to capture this intuition, we use an endogenous growth model with heterogeneous investments and financial market imperfection. In our model, some of the entrepreneurs have high productive investments and the others have low productive ones and

[1] See, for example, Akerlof and Shiller (2009).

entrepreneurs can pledge only a fraction of the returns from the investments.

The endogenous growth model with heterogeneous investments is a crucial point for formulating our intuition about the non-linear relationship. For example, recently, Farhi and Tirole (2011) examined the existence of bubbles and they found that bubbles can exist when the pledgeability level is low, although their main focus was the effects of outside liquidity. They have, however, assumed homogeneous investment opportunities. Hence, if the pledgeability is very low, the interest rate becomes very low and the growth rate, which their paper assumes to be zero, becomes relatively high compared to the interest rate. Thus, bubbles can exist even if the condition of the financial market is very poor[2]. On the other hand, if there are heterogeneous investments, the market interest rate may not go down very low even if the financial market is very poor, since the productivity of the low productive investment becomes the lowest bound. Thus, the growth rate becomes very low compared to the interest rate and bubbles cannot exist when the financial market condition is very poor. This result suggests that improving the condition of the financial market might enhance the emergence of bubbles if the initial condition of the financial market is underdeveloped.[3]

Our model also considers the macroeconomic implications of bubbles. We will show that the effect of bubbles on economic growth is dependent upon the financial market condition. We will show that bubbles have both a crowd-out effect and a crowd-in effect on investment and growth rate. Since the existence of bubbles raises the interest rate, it crowds investment out and decreases the economic growth rate. On the other hand, the rise of the interest rate increases the net worth of the entrepreneurs. Their increased net worth crowds in their future investments, that is, the "balance sheet effect" works under the financial imperfection. Our main finding is that the relative impact of these effects depends upon the degree of pledgeability. If the pledgeability level is relatively low, the crowd-in effect dominates the crowd-out effect and the bubbles enhance the economics growth rate. On the other hand, if the pledgeability is relatively high, the crowd-out effect dominates and bubbles decrease the growth rate.

[2]In Caballero (2006) and Caballero and Krishnamurthy (2006), the exogenously given growth rate in emerging countries is assumed to be high. Thus, even their model cannot capture the non-linear relationship.

[3]In this sense, our model is related to Matsuyama (2007, 2008), in which Matsuyama shows that a better credit market might be more prone to financing what he calls bad investments that do not have positive spillover effects on future generations.

This examination also holds an important implication for the effects of bubble bursts. The above result suggests that the effect of bubble bursts is not uniform. It is crucially affected by the financial condition of each country. If the imperfection of the financial market is relatively high (i.e., the pledgeability is relatively low), the bursting of bubbles decreases the growth rate of the country. On the other hand, the bursts may enhance the long-run growth rate if the condition of the financial market is relatively good.

The rest of this paper is organized as follows. In subsection 1.1, we discuss the related works in the literature. In section 2, we present our basic model and describe the economy without bubbles. In section 3, we introduce bubbles to this economy and examine the existence conditions of bubbles. In section 4, we examine the effects of bubbles on economic growth rates and show how the effects are related to financial market conditions. In section 5, we examine the effects of bubbles bursting and in section 6, we conclude our argument.

1.1 Related Work in the Literature

The present paper considers the existence of bubbles in infinitely lived agent model. With regard to the existence of bubbles in infinite horizon economies, it is commonly thought that bubbles cannot arise in deterministic sequential market economies with a finite number of infinitely lived agents (Tirole, 1982). In the Tirole model, the financial market is assumed to be perfect, that is, agents are allowed to borrow and lend freely. Tirole has shown that in such an environment, no equilibrium with bubbles exists. This result is consistent with our result. That is, when the pledgeability is equal to one, which means that the financial market is perfect, bubbles cannot arise even in our setting. We show that bubbles can arise even in infinitely lived agents model if the financial market is imperfect. Of course, the possibility of bubbles in infinite horizon economies with borrowing constraints has been recognized even in previous papers, including Scheinkman and Weiss (1986), Kocherlakota (1992), Santos and Woodford (1997), and Hellwig and Lorenzoni (2009). All of these studies are, however, based on an endowment economy. Our paper's contribution is that we consider a heterogeneous investment model and provide a full characterization on the relation between the existence of bubbles and financial frictions in a production economy.

There are many papers which examined the relation between bubbles and investment. In the literature, however, the crowd-out effect and crowd-in effect are examined separately. The conventional wisdom (Samuelson, 1958; Tirole, 1985) suggests that bubbles crowd investment out and lower output. According to the traditional view, the financial market is perfect and all the savings in the economy flow to investment. In such a situation, once bubbles appear in the economy, they crowd savings away from investment. Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) extend the Samuelson-Tirole model to economies with endogenous growth, and show that bubbles reduce investment and retard long run economic growth.[4][5] Recently, however, some researchers such as Woodford (1990), Caballero and Krishnamurthy (2006), Kiyotaki and Moore (2008), Kocherlakota (2009), Wang and Wen (2009), Martin and Ventura (2011a) developed a model with financial frictions, and showed that bubbles crowd investment in and increase output.[6] In these studies, because of the presence of financial market imperfections, enough resources cannot be transferred to those who have investment from those who do not. As a result, underinvestment occurs. Bubbles help to transfer resources between them.

The novel point of our paper is that we have combined these two effects and shown the degree of financial imperfection is crucial for understanding which of these effects is dominant. In this sense, our work is related to Martin and Ventura (2011b). Martin and Ventura (2011b) also investigated whether bubbles are expansionary or contractionary. There are some significant differences. First, Martin and Ventura (2011b) assume that nobody can borrow or lend through financial markets, because none of the returns from investment can be pledgeable. That is, they consider a situation where financial markets are completely shut down.[7] On the other hand, in our model, the entrepreneurs are allowed to borrow as long as they offer collateral to se-

[4]This crowd-out effect of bubbles has been criticized, because it seems inconsistent with historical evidence that investment and economic growth rate tend to surge when bubbles pop up, and then stagnate when they burst.

[5]Olivier (2000) shows that the conclusions in Saint-Paul (1992), Grossman and Yanagawa (1993), and King and Ferguson (1993) crucially depend on the type of asset being speculated on. Bubbles in equity markets can be growth-enhancing while bubbles in unproductive assets are growth-impairing.

[6]Aoki and Nikolov (2010), Hirano and Yanagawa (2010), Sakuragawa (2010), and Miao and Wang (2011) also show the crowd-in effect of bubbles.

[7]Even in Woodford (1990), none of the returns from investment can be pledgeable. In Kocherlakota (2009), agents can borrow against bubbles in land prices. However, without such bubbles, nobody can borrow and lend.

cure debts, because our main focus is to investigate the relation between the degree of financial imperfection and bubbles. We show that the emergence of bubbles as well as the effects of bubbles is crucially dependent upon the degree of financial imperfection. Second, Martin and Ventura (2011b) use an overlapping generations model, and it is assumed that at each period some fractions of young agents can create new bubbles. This assumption directly produces wealth effects of bubbles (crowd-in effects). They investigated the conditions of new bubble creations for the existence of bubbles. On the other hand, our model does not assume such new bubble creations. Instead, we assume that agents live infinitely, and their type changes stochastically at each period. Agents buy bubbles when they are low productive, and sell them at the time they are high productive, which generates crowd-in effects.[8]

Caballero and Krishnamurthy (2006) developed a theory of stochastic bubbles in emerging markets using an overlapping generations model. In their model, however, the growth rate of a country and the international interest rate are exogenously given. They implicitly assumed that the pledgeability level was low, and that without bubbles the domestic interest rate was lower than the international interest rate. Hence, our argument is a generalization of their argument. The theory by Kiyotaki and Moore (2008) is also relevant. In their theory, since fiat money (bubble) facilitates exchange for its high liquidity, people hold money even though the rate of return on it is low, that is money (bubble) works as a medium of exchange. In our model, however, we focus on the role of bubbles as a store of value.

2 The Model

Consider a discrete-time economy with one homogeneous good and a continuum of entrepreneurs. A typical entrepreneur has the following expected utility,

$$E_0\left[\sum_{t=0}^{\infty}\beta^t \log c_t^i\right], \tag{1}$$

[8]Our paper uses an infinitely lived agens model, while Farhi and Tirole (2011) and Martin and Ventura (2011) are based on an overlapping generations model. The potential benefit of using an infinitely lived agents model is that as Farhi and Tirole point out, it is in principle more suitable for realistic quantitative explorations.

where i is the index for each entrepreneur, and c_t^i is the consumption of him at date t. $\beta \in (0,1)$ is the subjective discount factor, and $E_0[x]$ is the expected value of x conditional on information at date 0.

At each date, each entrepreneur meets high productive investment projects (hereinafter H-projects) with probability p, and low productive investments (L-projects) with probability $1-p$.[9] The investment technologies are as follows:

$$y_{t+1}^i = \alpha_t^i z_t^i, \tag{2}$$

where $z_t^i (\geq 0)$ is the investment level at date t and y_{t+1}^i is the output at date $t+1$. α_t^i is the marginal productivity of investment at date t. $\alpha_t^i = \alpha^H$ if the entrepreneur has H-projects, and $\alpha_t^i = \alpha^L$ if he has L-projects. We assume $\alpha^H > \alpha^L$.[10] The probability p is exogenous, and independent across entrepreneurs and over time. At the beginning of each date t, the entrepreneur knows his own type at date t, whether he has H-projects or L-projects. Assuming that the initial population measure of each type of the entrepreneur is one at date 0, the population measure of each type after date 1 is $2p$ and $2-2p$, respectively. We call the entrepreneurs with H-projects (L-projects) "H-entrepreneurs" ("L-entrepreneurs").

In this economy, we assume that because of frictions in a financial market, the entrepreneur can pledge at most a fraction θ of the future return from his investment to creditors.[11] In such a situation, in order for debt contracts to be credible, debt repayment cannot exceed the pledgeable value. That is, the borrowing constraint becomes:

$$r_t b_t^i \leq \theta \alpha_t^i z_t^i, \tag{3}$$

where r_t and b_t^i are the gross interest rate, and the amount of borrowing

[9] A similar setting is used in Gertler and Kiyotaki (2010), Kiyotaki (1998), Kiyotaki and Moore (2008), and Kocherlakota (2009). In Woodford (1990), the entrepreneurs have investment opportunities in alternating periods.

[10] We can also consider the model where capital goods is produced by the investment technology. For example, let $k_{t+1}^i = \alpha_t^i z_t^i$ be the investment technology, where k is capital goods. Capital fully depreciates in one period. Consumption goods is produced by the following aggregate production function: $Y_t = K_t^\sigma N_t^{1-\sigma} \bar{k}_t^{1-\sigma}$, where K and N are the aggregate capital and labor input, and $\bar{k}$ is per-labor capital of the economy, capturing the externality in order to generate endogenous growth. In this type of the model, we can obtain the same results as this paper.

[11] See Hart and Moore (1994) and Tirole (2006) for the foundations of this setting.

at date t, respectively. The parameter $\theta \in [0,1]$, which is assumed to be exogenous, can be naturally taken to be the degree of imperfection of the financial market.

The entrepreneur's flow of funds constraint is given by

$$c_t^i + z_t^i = y_t^i - r_{t-1} b_{t-1}^i + b_t^i. \tag{4}$$

The left hand side of (4) is expenditure on consumption and investment. The right hand side is financing which comes from the return from investment in the previous period minus debts repayment, and borrowing. We define the net worth of the entrepreneur as $e_t^i \equiv y_t^i - r_{t-1} b_{t-1}^i$.

2.1 Equilibrium

Let us denote the aggregate consumption of H-and L-entrepreneurs at date t as $\sum_{i \in H_t} c_t^i \equiv C_t^H$ and $\sum_{i \in L_t} c_t^i \equiv C_t^L$, where H_t and L_t mean a family of H- and L-entrepreneurs at date t. Similarly, let $\sum_{i \in H_t} z_t^i \equiv Z_t^H$, $\sum_{i \in L_t} z_t^i \equiv Z_t^L$, $\sum_{i \in H_t} b_t^i \equiv B_t^H$, and $\sum_{i \in L_t} b_t^i \equiv B_t^L$ be the aggregate investment and the aggregate borrowing of each type. Then, the market clearing condition for goods, and the market clearing condition for credit are

$$C_t^H + C_t^L + Z_t^H + Z_t^L = Y_t, \tag{5}$$

$$B_t^H + B_t^L = 0, \tag{6}$$

where $\sum_{i \in (H_t \cup L_t)} y_t^i \equiv Y_t$ is the aggregate output at date t.

The competitive equilibrium is defined as a set of prices $\{r_t\}_{t=0}^{\infty}$ and quantities $\left\{c_t^i, b_t^i, z_t^i, y_{t+1}^i, C_t^H, C_t^L, B_t^H, B_t^L, Z_t^H, Z_t^L, Y_t\right\}_{t=0}^{\infty}$, such that (i) the market clearing conditions, (5) and (6), are satisfied, and (ii) each entrepreneur chooses consumption, borrowing, investment, and output to maximize his expected utility (1) under the constraints (2), (3), (4), and the following transversality condition:

$$\lim_{t \to \infty} \beta^t \frac{1}{c_t^i} (z_t^i - b_t^i) = 0. \tag{7}$$

From the maximization problem of the entrepreneur, the borrowing constraint, (3), becomes binding if and only if $\alpha_t^i > r_t$, that is, the rate of return on investment is strictly greater than the interest rate.[12] In equilibrium, since the utility function is log-linear, each entrepreneur consumes a fraction $1-\beta$ of the net worth every period, that is, $c_t^i = (1-\beta)(y_t^i - r_{t-1}b_{t-1}^i)$.[13]

2.2 The Case with $\theta = 1$: Perfect Financial Market

First, let us consider the case with a perfect financial market, that is $\theta = 1$. In this case, if $\alpha^H > r_t$, H-entrepreneur must be willing to borrow an unlimited amount. On the other hand, if $\alpha^L < \alpha^H < r_t$, nobody would borrow. Thus, the equilibrium interest rate must be

$$r_t = \alpha^H.$$

Since each entrepreneur saves a fraction β of the net worth every period, the aggregate saving in the economy is βY_t, which flows to finance H-projects.

Thus, the law of motion of the aggregate output becomes

$$Y_{t+1} = \alpha^H \beta Y_t, \tag{8}$$

and we get the growth rate of the aggregate output as follows,

$$g_t \equiv \frac{Y_{t+1}}{Y_t} = \alpha^H \beta. \tag{9}$$

We observe that the growth rate is independent of wealth distribution. Since we have assumed simple linear production functions, the interest rate is equal to the marginal productivity of H-projects and the steady state growth rate is positive. Moreover, the interest rate is strictly greater than the growth rate of the economy and the transversality condition is satisfied, as explored in the traditional literature.

2.3 The Case with $\theta < 1$: Imperfect Financial Market

Next, we examine the case with an imperfect financial market, that is $\theta < 1$. Even if $\theta < 1$, all of the total saving is used for H-projects and $r_t = \alpha^H$ as long as θ is sufficiently high. Hence, in this section, we focus on the case where the interest rate is strictly lower than α^H and the borrowing constraint

[12] The detail analysis of the maximization problem is given in Appendix 1.

[13] See, for example, chapter 1.7 of Sargent (1988).

is binding for H-enterpreneurs,

$$\alpha^L \leq r_t < \alpha^H.$$

In equilibrium, the interest rate must be at least as high as α^L, since nobody lends to the projects if $r_t < \alpha^L$. We will explore later that which range of θ satisfies the above condition about r_t.

Since each entrepreneur consumes a fraction $1-\beta$ of the net worth, we get the following relation from (4),

$$z_t^i - b_t^i = \beta(y_t^i - r_{t-1}b_{t-1}^i).$$

On the other hand, as long as $r_t < \alpha^H$, the borrowing constraint, (3), is binding and b_t^i satisfies the following relation,

$$b_t^i = \frac{\theta\alpha^H}{r_t} z_t^i.$$

From those two relations, we can derive the following investment function for the H-entrepreneurs at date t,

$$z_t^i = \frac{\beta(y_t^i - r_{t-1}b_{t-1}^i)}{1 - \dfrac{\theta\alpha^H}{r_t}}. \tag{10}$$

This is a popular investment function under financial constraint problems.[14] We see that the investment equals the leverage, $1/\left[1-(\theta\alpha^H/r_t)\right]$, times savings, $\beta(y_t^i - r_{t-1}b_{t-1}^i)$. The leverage increases with θ and is greater than one in equilibrium. This implies that when θ is larger, H-entrepreneurs can finance more investment, z_t^i.

By aggregating (10), we get

$$Z_t^H = \frac{\beta E_t^H}{1 - \dfrac{\theta\alpha^H}{r_t}}, \tag{11}$$

where $\sum_{i\in H_t} e_t^i \equiv E_t^H$ is the aggregate net worth of H-entrepreneurs at date t. The movement of the aggregate net worth of H-entrepreneurs evolves

[14] See, for example, Bernanke et al. (1999), Holmstrom and Tirole (1997), and Kiyotaki and Moore (1997).

according to the following equation.

$$E_t^H = p(\alpha^H Z_{t-1}^H - r_{t-1}B_{t-1}^H) + p(\alpha^L Z_{t-1}^L - r_{t-1}B_{t-1}^L) = pY_t. \tag{12}$$

The first term of (12) represents the aggregate net worth of the entrepreneurs who continue to have H-projects from the previous period (we call these H-H entrepreneurs). The second term represents the aggregate net worth of the entrepreneurs who switch from the state with L-projects to the state with H-projects (we call these L-H entrepreneurs). Since every entrepreneur has the same opportunity to invest in H-projects at each period, the aggregate net worth of H-entrepreneurs at date t is a fraction p of the aggregate output at date t. Hence, (11) can be rewritten as

$$Z_t^H = \frac{\beta p Y_t}{1 - \dfrac{\theta\alpha^H}{r_t}}. \tag{13}$$

For L-entrepreneurs, if $r_t = \alpha^L$, lending and borrowing to invest are indifferent. Thus, how much they invest in their own projects is indeterminate at an individual level. However, their aggregate investments' level is determined by the goods market clearing condition:

$$Z_t^H + Z_t^L = \beta Y_t. \tag{14}$$

This implies that the aggregate investments of L-entrepreneurs equal the aggregate saving minus the aggregate investments of H-entrepreneurs. On the other hand, if $r_t > \alpha^L$, Z_t^L must be zero. Thus, the following conditions must be satisfied.

$$Z_t^L(r_t - \alpha^L) = 0,\ Z_t^L \geq 0,\ r_t - \alpha^L \geq 0. \tag{15}$$

The aggregate output is

$$Y_{t+1} = \alpha^H Z_t^H + \alpha^L Z_t^L,$$

and can be rewritten as

$$Y_{t+1} = \beta\alpha^H Y_t - (\alpha^H - \alpha^L) Z_t^L.$$

Hence, the growth rate of Y_t becomes as follows,

$$g_t \equiv \frac{Y_{t+1}}{Y_t} = \beta\alpha^H - (\alpha^H - \alpha^L)\beta l_t, \tag{16}$$

where $l_t \equiv Z_t^L/\beta Y_t$, the ratio of the low productive investment to the total investment. The interpretation of this relation is simple. As long as the amount of L-projects, l_t, is zero, the total savings is allocated to the H-projects, and the growth rate of this economy becomes $\beta\alpha^H$, which is just the same as that under the perfect financial market. If $l_t > 0$, however, the difference in productivity between H-projects and L-projects, $\alpha^H - \alpha^L$, decreases the growth rate and g_t becomes $\beta\alpha^H - (\alpha^H - \alpha^L)\beta l_t$.

Next, we examine the equilibrium level of l_t and how the equilibrium l_t is affected by the degree of financial imperfection, θ. Since $l_t \equiv Z_t^L/\beta Y_t = (\beta Y_t - Z_t^H)/\ \beta Y_t$, we can rewrite l_t and g_t as follows,

$$l_t = 1 - \frac{p}{1 - \dfrac{\theta\alpha^H}{r_t}} = \frac{r_t(1-p) - \theta\alpha^H}{r_t - \theta\alpha^H} = l(r_t, \theta), \tag{17}$$

$$g_t \equiv \frac{Y_{t+1}}{Y_t} = \alpha^H\beta - (\alpha^H - r_t)\beta l(r_t, \theta). \tag{18}$$

From (15), the following relations must be satisfied,

$$(r_t - \alpha^L)\frac{r_t(1-p) - \theta\alpha^H}{r_t - \theta\alpha^H} = 0,\ l_t \geq 0,\ r_t - \alpha^L \geq 0. \tag{19}$$

Those imply that r_t must be α^L or $\theta\alpha^H/(1-p)$. If $\theta\alpha^H/(1-p) \leq \alpha^L$, $r_t = \alpha^L$ since r_t cannot be lower than α^L. On the other hand, if $\theta\alpha^H/(1-p) > \alpha^L$, $r_t = \theta\alpha^H/(1-p)$ since l_t cannot be negative. Hence, we get the following relation.

$$r_t = r(\theta) = \begin{cases} \alpha^L, & \text{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\ \\ \dfrac{\theta\alpha^H}{1-p}, & \text{if } (1-p)\dfrac{\alpha^L}{\alpha^H} \leq \theta < 1-p, \end{cases}$$

and

$$
l_t = l(r(\theta), \theta) = \begin{cases} \dfrac{\alpha^L(1-p) - \theta\alpha^H}{\alpha^L - \theta\alpha^H}, & \text{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\ 0 & \text{if } (1-p)\dfrac{\alpha^L}{\alpha^H} \leq \theta < 1-p. \end{cases}
$$

From those results, we get the following equilibrium growth rate.

$$
g_t = g(\theta) = \begin{cases} \beta\alpha^H - (\alpha^H - \alpha^L)\beta\dfrac{\alpha^L(1-p) - \theta\alpha^H}{\alpha^L - \theta\alpha^H}, & \text{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\ \beta\alpha^H, & \text{if } (1-p)\dfrac{\alpha^L}{\alpha^H} \leq \theta < 1-p, \end{cases} \tag{20}
$$

(20) implies that the growth rate of the economy, g_t, is an increasing function of θ. More savings flow to H-projects from L-projects through the relaxation of the borrowing constraint, which improves the aggregate total factor productivity and enhances growth.[15] Moreover, from the above relation, we can find that if $\theta \geq 1-p$, $r_t = \alpha^H$ and $g_t = \beta\alpha^H$.

In summary, we get the following proposition.

Proposition 1 *When $\theta < 1$ and bubbles do not exist, the equilibrium interest rate, r_t, and the equilibrium growth rate, g_t, are the following increasing functions of θ, respectively.*

$$
r_t = r(\theta) = \begin{cases} \alpha^L, & \textit{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\ \dfrac{\theta\alpha^H}{1-p}, & \textit{if } (1-p)\dfrac{\alpha^L}{\alpha^H} \leq \theta < 1-p, \\ \alpha^H, & \textit{if } 1-p \leq \theta. \end{cases}
$$

[15] The recent macroeconomic literature emphasizes the role of total factor productivity in accounting for business cycles or growth. In our model, the aggregate total factor productivity is endogenously determined depending on saving allocations between H-projects and L-projects, which in turn depends on θ in the steady state.

$$
g_t = g(\theta) = \begin{cases} \beta\alpha^H - (\alpha^H - \alpha^L)\beta \dfrac{\alpha^L(1-p) - \theta\alpha^H}{\alpha^L - \theta\alpha^H}, & \mathit{if}\ 0 \le \theta < (1-p)\dfrac{\alpha^L}{\alpha^H}, \\ \beta\alpha^H, & \mathit{if}\ (1-p)\dfrac{\alpha^L}{\alpha^H} \le \theta < 1-p, \\ \beta\alpha^H, & \mathit{if}\ 1-p \le \theta. \end{cases}
$$

If $0 \le \theta < (1-p)\alpha^L/\alpha^H$, the degree of pledgeability is so small, i.e., financial frictions are so severe, then it is difficult for the financial system to transfer all the savings of the L-entrepreneurs to H-projects. As a result, L-entrepreneurs hold idle savings and end up with investing such idle savings in their own projects with low returns. The more severe financial frictions are, the more L-projects are financed in the economy. However, as financial frictions improve, more savings flow to H-projects. This improvement in savings allocations increases the aggregate total factor productivity, which leads to higher growth. On the other hand, the interest rate is suppressed at a low level of α^L because of the severity of financial frictions.

If $(1-p)\alpha^L/\alpha^H \le \theta < 1-p$, L-entrepreneurs can lend all their savings to H-entrepreneurs, even though the financial market is still imperfect. As a result, only H-entrepreneurs invest. Hence, the economy's growth rate is $\beta\alpha^H$. In this region, together with an improvement in financial frictions, the interest rate rises due to the tightness in the financial market. Note that the borrowing constraint is still binding for H-entrepreneurs in this region, because the interest rate is strictly lower than the rate of return on H-projects.

If $1-p \le \theta$, and the degree of pledgeability is large enough, the interest rate is equal to the rate of return on H-projects. The interest rate becomes equal to the rate of return on H-projects, and H-entrepreneurs does not face credit constraints. The financial system can allocate all the savings in the economy to H-projects, and resource allocation is efficient, even though θ is strictly less than one.[16] In this region, the characteristics of the economy is the same as the one with the perfect financial market.

Figure 1 depicts this situation. On the horizontal axis, we take θ, and on the vertical axis, we take g and r. We show that the relation between g

[16] In $\theta \in (0, 1-p)$, H-entrepreneurs earn $\alpha^H(1-\theta)/\left(1-\theta\alpha^H/r_t\right)$, which is strictly greater than r_t that L-entrepreneurs earn. Thus, income distribution is different between the entrepreneurs. However, in $\theta \in [1-p, 1]$, both entrepreneurs earn the same rate of return, which is α^H. Hence, there is no difference in income distribution.

and θ is nonlinear. As shown below, this nonlinearity plays a crucial role in creating regions where bubbles can arise (bubble regions), or regions where they cannot arise (non-bubble regions).

3 Existence of Asset Bubbles

Now we describe the economy with asset bubbles (we call this a "bubble economy"). We define bubble assets as the assets that produce no real return, i.e., the fundamental value of the assets is zero. Let x_t^i be the level of bubble assets purchased by type i entrepreneur at date t, and let P_t be the per unit price of bubble assets at date t in terms of consumption goods. [17] In the bubble economy, each entrepreneur faces the following three constraints: flow of funds condition, the borrowing constraint, and the short-sale constraint:

$$c_t^{*i} + z_t^{*i} + P_t x_t^i = y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + b_t^{*i} + P_t x_{t-1}^i, \tag{21}$$

$$r_t^* b_t^{*i} \leq \theta(\alpha_t^i z_t^{*i} + P_{t+1} x_t^i), \tag{22}$$

$$x_t^i \geq 0, \tag{23}$$

where * represents the case of bubble economy. Both sides of (21) include bubbles. $P_t x_{t-1}^i$ in the right hand side is the sales of the bubble assets, and $P_t x_t^i$ in the left hand side is the new purchase of them. We define the net worth of the entrepreneur in the bubble economy as $e_t^{*i} \equiv y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + P_t x_{t-1}^i$. (22) is the borrowing constraint under the bubble economy. We assume here that only a fraction θ of the returns from the investment and the bubbles can be pledgeable to creditors. Of course, we can think about a different situation in which the pledgeable fraction on the return on the investment and that of the return on the bubbles are different. Even if we assume that the pledgeable fraction of bubbles' return, say $\theta^x < 1$, is different from that of the investment's return, θ, our results which will be explained below are not affected.[18]

We should add a few remarks about the short-sale constraint (23). As Kocherlakota (1992) has shown, the short-sale constraint is important for

[17] Here we only focus on deterministic bubbles but, even if we allow stochastic bubbles, our qualitative properties are not affected as shown in Appendix 2.

[18] As will be explained below, the crucial point for our results is that the previous return from bubbles increases the net worth for a borrower.

the existence of bubbles in deterministic economies with a finite number of infinitely lived agents. Without the constraint, bubbles always represent an arbitrage opportunity for an infinitely lived agent; he can gain by permanently reducing his holdings of the asset. However, it is well known that in such economies, equilibria can only exist if agents are constrained not to engage in Ponzi schemes. Kocherlakota (1992) has demonstrated that the short-sale constraint is one of no-Ponzi-game conditions and hence, it can support bubbles by eliminating the agent's ability to permanently reduce his holdings of the asset.[19]

In order that bubble assets be held in equilibrium, the rate of return on bubbles has to be equal to the interest rate:

$$r_t^* = \frac{P_{t+1}}{P_t}. \tag{24}$$

Each entrepreneur chooses the levels of consumption, investment, output, borrowing, and bubble assets $\left\{c_t^{*i}, z_t^{*i}, y_{t+1}^{*i}, b_t^{*i}, x_t^i\right\}$, to maximize the expected utility (1) subject to (21), (22), and (23), given the interest rate and the current and future price of bubbles, r_t^*, P_t, and P_{t+1}. Moreover, on the optimal feasible plan, the following transversality condition must be satisfied:[20]

$$\liminf_{t\to\infty} \beta^t \frac{1}{c_t^{*i}} P_t x_t^i = 0. \tag{25}$$

In equilibrium, as before, the entrepreneur consumes a fraction $1-\beta$ of the net worth every period.[21] As in the previous section, we focus on the case where the equilibrium interest rate is strictly lower than the productivity of the H-projects, that is, L-entrepreneurs purchase bubbles and the borrowing constraint on H-entrepreneurs is binding. The investment function of the entrepreneurs who have H-projects at date t is

$$z_t^* = \frac{\beta e_t^{*i}}{1 - \dfrac{\theta \alpha^H}{r_t^*}}.$$

[19] See Kocherlakota (1992) for details.

[20] See Kocherlakota (1992) for the transversality condition in economies with the short sale constraint.

[21] The detail analysis of the maximization problem for the entrepreneur is given in Appendix.

where $e_t^{*i} = -r_{t-1}^* b_{t-1}^{*i} + P_t x_{t-1}^i$ for L-H entrepreneurs, and $e_t^{*i} = y_t^{*i} - r_{t-1}^* b_{t-1}^{*i}$ for H-H entrepreneurs. For L-H entrepreneurs, since they purchased bubbles in the previous period, they are able to sell bubbles at the time they encounter H-projects. As a result, their net worth increases (compared to the bubbleless case) and boosts their investments, that is, the "balance sheet effect" works.[22] Moreover, the expansion level of the investment is more than the direct increase of net worth because of the leverage effect. H-H entrepreneurs, however, are not able to take advantage of this merit, because they did not buy bubbles in the previous period.

Next, we describe the aggregate economy. Since both H-and L-entrepreneurs consume a fraction $1-\beta$ of their net worth, the goods market clearing condition can be written as

$$Z_t^{*H} + P_t X = \beta(Y_t^* + P_t X), \tag{26}$$

where $Y_t^* + P_t X$ and $\beta(Y_t^* + P_t X)$ are the aggregate wealth (total asset) and the aggregate saving in the bubble economy. X is the aggregate quantity of bubbles, which is exogenously fixed. We see that some of the aggregate saving flow to bubble assets as well as H-projects, which can be the source for raising the interest rate in the financial market. The aggregate demand for bubbles, $P_t X$, is equal to the aggregate saving minus the aggregate investment of H-entrepreneurs, $\beta(Y_t^* + P_t X) - Z_t^{*H}$.

The aggregate investment function becomes,

$$Z_t^{*H} = \frac{\beta E_t^{*H}}{1 - \dfrac{\theta\alpha^H}{r_t^*}},$$

where $E_t^{*H} = p(Y_t^* - r_{t-1}^* B_{t-1}^{*H}) + p(P_t X - r_{t-1}^* B_{t-1}^{*L}) = p\,(Y_t^* + P_t X)$. The first term is the aggregate net worth of H-H entrepreneurs at date t and the second term represents the one of L-H entrepreneurs at date t. Hence, it can be written as

$$Z_t^{*H} = \frac{\beta p\,(Y_t^* + P_t X)}{1 - \dfrac{\theta\alpha^H}{r_t^*}}. \tag{27}$$

[22] In Kiyotaki and Moore (1997), the rise in land price increases the entrepreneurs' net worth, which results in balance sheet effects, thereby increasing investment. In this paper, bubbles play a similar role as the land in Kiyotaki and Moore's paper.

The aggregate wealth under the bubble economy can be written as,

$$Y_{t+1}^* + P_{t+1}X = \alpha^H Z_t^{*H} + r_t^* P_t X. \tag{28}$$

In order to characterize the economic growth rate, let us define as follows,

$$k_t = \frac{P_t X}{\beta(Y_t^* + P_t X)},$$

$$g_t^k = \frac{Y_{t+1}^* + P_{t+1}X}{Y_t^* + P_t X},$$

where k_t is the relative size of bubbles and g_t^k is the growth rate of the aggregate wealth, $Y_t^* + P_t X$. From (27) and these definitions, (28) can be written as

$$g_t^k = \alpha^H \frac{\beta p}{1 - \dfrac{\theta \alpha^H}{r_t^*}} + r_t^* \beta k_t. \tag{29}$$

Moreover, from (26), we get

$$\frac{\beta p}{1 - \dfrac{\theta \alpha^H}{r_t^*}} + \beta k_t = \beta. \tag{30}$$

From (29) and (30), the growth rate of the aggregate wealth, g_t^k, becomes

$$g_t^k = \alpha^H \beta (1 - k_t) + r_t^* \beta k_t = \alpha^H \beta - (\alpha^H - r_t^*)\beta k_t.$$

Furthermore,

$$k_t = 1 - \frac{p}{1 - \dfrac{\theta \alpha^H}{r_t^*}} = \frac{r_t^*(1-p) - \theta \alpha^H}{r_t^* - \theta \alpha^H} = l(r_t^*, \theta). \tag{31}$$

This means that given r_t and θ, the relative size of bubbles, k_t, is just equal to the relative size of L-projects, $l_t = l(r_t, \theta)$ under the bubbleless economy. Hence, we can derive the following growth rate function,

$$g_t^k(r_t^*,\theta) = \beta\alpha^H - (\alpha^H - r_t^*)\beta l(r_t^*,\theta).$$

This means the growth rate function in the bubble economy is just same as the bubbleless economy. However, the growth rate becomes different since the equilibrium interest rate is different.

Next we examine the determination process of r_t^*. From the definition of k_t

$$\frac{k_{t+1}}{k_t} = \frac{r_t}{g_t^k} = \frac{r_t}{\alpha^H\beta - (\alpha^H - r_t)\beta k_t}. \tag{32}$$

In order to satisfy the transversality condition (25),

$$\frac{k_{t+1}}{k_t} = \frac{r_t^*}{g_t^k} \leq 1 \tag{33}$$

should be satisfied. Moreover, if $k_{t+1}/k_t < 1$, the economy converges to the asymptotically bubbleless economy. Hence, in this paper, we focus on the case where $k_{t+1}/k_t = 1$, that is, the share of the bubble assets is constant under the steady state. Hence, at each period, the following condition must be satisfied:

$$r^*(\theta) = \alpha^H\beta - (\alpha^H - r^*(\theta))\beta l(r^*(\theta),\theta). \tag{34}$$

From (31) and $k_{t+1}/k_t = 1$, we get the equilibrium interest rate and the relative size of bubbles as follows,

$$\begin{aligned} r_t^* &= r^*(\theta) = \alpha^H \frac{(1-\beta)\theta + p\beta}{1-\beta+p\beta}, \\ k_t &= l(r^*(\theta),\theta) = \frac{\beta(1-p)-\theta}{\beta(1-\theta)}. \end{aligned}$$

Furthermore, since the growth rate of the total output, $g_t^* \equiv Y_{t+1}^*/Y_t^*$ is just equal to g_t^k from $k_{t+1}/k_t = 1$, we get

$$g_t^* = g_t^k = r^*(\theta) = \beta\alpha^H - (\alpha^H - r^*(\theta))\beta l(r^*(\theta),\theta) = \alpha^H \frac{(1-\beta)\theta + p\beta}{1-\beta+p\beta}. \tag{35}$$

Obviously, the equilibrium growth rate is an increasing function of θ. An increase of θ decreases the relative size of bubbles, k_t, and raises the growth rate.

3.1 Existence Condition of Bubbles

In this subsection, we examine the existence condition of bubbles. For the existence of bubbles, the following two conditions must be satisfied. First, the equilibrium interest rate must not be lower than α^L at each period. Second, k_t must be non-negative. In other words,

$$\begin{aligned} \alpha^L &\leq r^*(\theta) = \alpha^H \frac{(1-\beta)\theta + p\beta}{1-\beta+p\beta} \leq \alpha^H, \\ k_t &= l(r^*(\theta), \theta) = \frac{\beta(1-p) - \theta}{\beta(1-\theta)} > 0. \end{aligned}$$

From these conditions, we get the following proposition(Hereafter, proofs of all Propositions are in Appendix 1).

Proposition 2 *Bubbles can exist as long as θ satisfies the following condition,*

$$\underline{\theta} \equiv Max\left[\frac{\alpha^L - \beta\left[\alpha^L + (\alpha^H - \alpha^L)p\right]}{\alpha^H(1-\beta)}, 0\right] \leq \theta < \overline{\theta} \equiv \beta(1-p).$$

From this proposition, we can understand that bubbles tend to exist when the degree of financial imperfection, θ, is in the middle range. In other words, improving the condition of the financial market might enhance the existence of bubbles when the initial condition of θ is low.[23] This result is in sharp contrast with the results in the previous literature such as Farhi and Tirole (2010), in which bubbles are more likely to emerge when the finacial market is more imperfect (when the pledgeability is more limited).

Figure 2 is a typical case representing the relation between θ and bubble regions.[24] It is shown that if the degree of financial frictions is sufficiently

[23] Researchers such as Kaminsky and Reinhart (1999) and Allen and Gale (1999) point out that financial liberalization causes bubbles. An interpretation of this effect based on our model is as follows. For instance, before financial liberalization, the economy is in non-bubble regions. After the liberalization, θ increases, and the borrowing constraint is relaxed, so that the economy enters bubble regions. Consequently, we might think of the increase in θ as a measure of financial liberalization.

[24] Even though the growth rate is strictly greater than the interest rate, bubbles cannot arise in the economy unless people expect to be able to pass bubbles on to other people.

large or small, bubbles can not exist. This suggests that in financially underdeveloped or well-developed economies, bubbles can not emerge. They can only arise in financially intermediate-developed ones.[25] An intuitive reason for this result is as follows. If θ is low, H-entrepreneurs cannot borrow sufficiently and the growth rate must be low even with bubbles. On the other hand, the interest rate cannot be lower than α^L, since there is an opportunity to invest in L-projects even if θ is low. Hence, under a very low θ level, the interest rate becomes higher than the growth rate and bubbles cannot exist. Since we assume heterogeneous investment opportunities, the interest rate has the lowest bound and we can obtain the result that is different from the previous literature.[26]

Moreover, we can use the structure of the bubbleless economy to characterize the existence condition. The existence condition of the bubble is that the growth rate is not lower than the interest rate under the bubbleless economy. This condition is consistent with the existence condition in the previous literature such as Tirole(1985).

Proposition 3 *The necessary condition for the existence of bubble is that the equilibrium growth rate is not lower than the equilibrium interest rate under the bubbleless economy.*

This expectation is the sufficient condition for the existence of bubbles. Here, we assume that the condition is satisfied when bubbles appear.

[25] Readers may wonder why the phenomenon which looks like bubbles occurs repeatedly in the real world where the financial system is continually developing over time, even though our model suggests that bubbles do not appear in high θ regions. We propose one interpretation from our model. In the paper, we assume a common θ on both high and low investment. However, we can put different θ on those projects. In such a case, the important factor for the existence of bubbles is θ^H, which is placed on high-profit investments, not on low-profit investments. Taking this into account, consider the situation where the existing projects with α^L disappear, and new investment opportunities with higher profitability than the existing α^H appear in the economy. In such a situation, the θ that is placed on those new projects is important for the existence of bubbles. If the θ is low, the economy will enter bubble regions again even if it was in non-bubble regions with high θ before. In the real world, this process might repeat itself.

[26] Martin and Ventura (2010) assume two types of investment opportunities but bubbles may be able to exist in an economy without credit markets. The crucial difference is that their paper allows new bubble creations at each period.

4 Asset Bubbles and Economic Growth

In this section, we examine how bubbles affect the economic growth rate. We will show here that the effect of bubbles on the growth rate is dependent on the financial market condition, θ, even if the existence condition of bubbles is satisfied. We can derive that there is a threshold level of θ, $\theta^* = \beta(1-p)\alpha^L/\alpha^H$.

Proposition 4 *Let us define $\theta^* \equiv \beta(1-p)\alpha^L/\alpha^H$. If $\underline{\theta} \leq \theta \leq \theta^*$, the growth rate under the bubble economy is higher than that under the bubbleless economy at each period. If $\theta^* < \theta < \overline{\theta}$, the growth rate under the bubble economy is lower than that under the bubbleless economy at each period.*

Proposition 4 implies that in the economies within the bubble regions and with relatively low θ, bubbles enhance growth while in the economies with relatively high θ, they impede it. Here we explain an intuitive reason of this result. From (20) and (35), we get that

$$g_t = \beta\alpha^H - (\alpha^H - \alpha^L)\beta l(\alpha^L, \theta)$$

$$g_t^* = \beta\alpha^H - (\alpha^H - r^*(\theta))\beta l(r^*(\theta), \theta)$$

The difference between the two growth rates arises mainly from the difference in the interest rates. When bubbles appear in the economy, the interest rate rises, which produces two competing effects. One is a crowd-out effect. That is, H-entrepreneurs are forced to cut back on their investments because they experience tight borrowing constraints. This reduces the growth rate of the aggregate net worth of H-entrepreneurs, which in turn crowds investment out. The other is a crowd-in effect. Due to the hike in the interest rate, the interest income for L-entrepreneurs, which is the returns from purchasing bubbles, rises. This increases the growth rate of the aggregate net worth of H-entrepreneurs, which in turn crowds investment in. More precisely, the difference between the growth rates can be written as follows.

$$g_t^* - g_t = (r^*(\theta) - \alpha^L)\beta l(\alpha^L, \theta) - (\alpha^H - r^*(\theta))\beta(l(r^*(\theta), \theta) - l(\alpha^L, \theta)). \quad (36)$$

In this formulation, the first term of the right hand side represents the crowd-in effect and the second term of the right hand side represents the crowd-out effect. Since $r^*(\theta) \geq \alpha^L$, the first term is (weakly) positive. This term captures the effect that an increase in the interest rate raises the income of the entrepreneurs who invested in the bubbles and enhances the economic growth rate. More precisely, if there is no bubble, some L-entrepreneurs have to invest in L-projects, $l(\alpha^L, \theta)$, as long as the borrowing constraint binds the H-entrepreneurs. If they have a chance to invest in bubble assets instead of L-projects, they can earn $r^*(\theta)l(\alpha^L, \theta)$ instead of $\alpha^L l(\alpha^L, \theta)$. This increased earning contributes to enhancing the H-investment at the time they become H-entrepreneurs in the future. Thus, this income effect increases the growth rate by the increase of H-investments.

The second term represents the crowd-out effect. As you can see from (17), $l(r^*, \theta)$ is an increasing function of r^*. A rise in the interest rate tightens the borrowing constraint of the H-entrepreneuers and increases the investments in the L-projects or bubbles. Hence, $l(r^*(\theta), \theta) - l(\alpha^L, \theta)$ is positive. Under the bubble economy, if θ is low, $\beta l(\alpha^L, \theta)$ is high and the crowd-in effect is high. On the other hand, if θ is high, $\beta l(\alpha^L, \theta)$ becomes low and the crowd-in effect is dominated by the crowd-out effect. Thus, in the economies with relatively low θ, the crowd-in effect dominates the crowd-out effect, but, in the economies with relatively high θ, the crowd-out effect dominates the crowd-in effect.[27]

Here, we will add a few remarks on the effect of bubbles on aggregate productivity. In the bubble economy, L-entrepreneurs stop investing and only H-entrepreneurs invest, i.e., bubbles improve efficiency in production by eliminating low-productive investments. Thus, the total factor productivity increases together with the emergence of bubbles. It moves procyclically with economic growth if θ is relatively low. This implies that bubble bursts results in productive inefficiency.

5 Effects of the Bubbles Bursting

In this section, we examine the effects of bubble bursts. In this perfect foresight model, an unexpected shock may generate a burst of bubbles.[28]

[27] In our model, the presence of L-projects plays a crucial role in showing that bubbles crowd investment in and enhance growth. Without them, in the bubbleless economy, the interest rate adjusts such that all the savings in the economy flow to H-projects. In such a situation, once bubble assets appear in the economy, they crowd savings away from H-projects, thereby lowering the growth.

[28] When we assume stochastic bubbles, bubbles can burst even without exogenous shocks. The effects of bubbles bursting are examined in Appendix 2 and we can show that qual-

Let us suppose that there is an unexpected shock at $t = s$ that decreases the productivity from α^H to $\alpha^S < \alpha^H$. First, we examine the case where this shock is permanent (or at least this shock is expected to be permanent at $t = s$.). As we have shown in the previous section, $\beta\alpha^H \geq \alpha^L$ is a necessary condition for the existence of bubbles. Hence, if α^S is strictly smaller than α^L/β, bubbles must burst for any θ. Even if $\alpha^S \geq \alpha^L/\beta$, bubbles may burst if θ is relatively low. Since $\underline{\theta}$ is a decreasing function of α^H, bubbles must collapse in the countries whose pledgeability is lower than $\underline{\theta}(\alpha^S)$. This result shows that even if the shock is common, the effect of the shock differs from country to country and in particular, the effect on the stock price in a country is crucially affected by the financial market conditions of this country.

Next, we examine how the growth rate in each country is affected by the unexpected shock at $t = s$. After the collapse of bubbles, the growth rate is determined by the mechanism explained in Section 2. Hence, the growth rate after the shock becomes,

$$g_t = g(\theta) = \begin{cases} \beta\alpha^S - (\alpha^S - \alpha^L)\beta\dfrac{\alpha^L(1-p) - \theta\alpha^S}{\alpha^L - \theta\alpha^S} & \text{if } 0 \leq \theta < (1-p)\dfrac{\alpha^L}{\alpha^S}, \\ \\ \beta\alpha^S, & \text{if } (1-p)\dfrac{\alpha^L}{\alpha^S} \leq \theta \leq 1. \end{cases}$$

Since the productivity is lower than before, the growth rate becomes lower than that of the bubble periods. Although the growth rate under the bubble economy is not lower than α_L, the growth rates must be lower than α_L after the burst when $\beta\alpha^S < \alpha_L$. Furthermore, the variance in growth rates among countries becomes higher after the burst. The reason is as follows. The countries whose θ is $\underline{\theta} \leq \theta \leq \theta^*$ experienced relatively high growth rate due to the existence of bubbles. This means that after the bubbles burst, those countries experience decreased growth rate due to the two reasons, the decreased productivity and the burst bubbles. Hence, those countries experience relatively very low growth rate after the bubbles burst. On the other hand, the countries whose θ is $\theta^* < \theta < \overline{\theta}$ suffer decreased growth rate due to the decrease in productivity, but this effect must be offset by the burst's positive effect on the growth rate, since the existence of bubbles decreased the growth rates of those countries. In summary, the low (high) θ

itative properties of bubbles bursting are almost the same even if we assume stochastic bubbles.

countries experience relatively lower (higher) growth rates; thus the variance in growth rate becomes higher even though the average growth rate must be lower than before the burst. This result may be consistent with an empirical observation. Figure 3 shows the growth rates of Asian countries before and after the financial crisis. The figure shows that the variance in the growth rates becomes higher after the crisis. Although, actual growth rates will be affected by many factors, our result is not inconsistent with this interesting observation.

Next we examine the case where the unexpected shock is temporary and it is expected to be so after the shock. In this case, bubbles might exist even after the shock since all agents can expect that this shock is temporary. In order to sustain the bubble path after the shock, however, the price of bubbles, P_s, must drop according to the shock. The reason is as follows. Let us suppose the shock is temporary and that the productivity recovers to α^H after $t = s + 1$. Under the shock, from $t = s + 1$, the growth rate of each country can recover to $g_t^*(\theta)$ but Y_t must be lower since Y_{s+1} is decreased by the shock. Hence, in order to sustain the bubble path, the price of bubbles must decrease at $t = s$. This result suggests that the decrease in asset prices does not directly mean that the bubbles burst. It might be the adjustment process of bubbles. Even after the drop in asset prices, bubbles can exist even under the perfect foresight economy as long as there is an unexpected shock.

It is not necessary, however, that people continue to choose the bubble path even after the unexpected shock. People may choose the bubbleless path after the shock. Hence, bubbles may burst if agents revise their expectation as a result of the shock and expect that the value of the bubble is zero even if the productivity shock is temporary and α^H recovers to the original level at $t = s + 1$. Next, we examine how the bubble bursts affect the economic growth rates in this case. Since the bubbles have burst at $t = s$, the growth rate follows (20) from $t = s + 1$. This implies that the difference between the growth rates before and after the bubbles burst can be characterized by the difference between the growth rates of the bubble economy and the bubbleless economy. Hence, if $\underline{\theta} \leq \theta \leq \theta^*$, the growth rate becomes lower after the bubbles burst but if $\theta^* < \theta < \overline{\theta}$, the growth rate becomes higher (except $t = s$) after the bubbles burst.[29] This result suggests that the effect of

[29] In the standard real business cycle models, a temporary productivity shock has only temporal effects on output. However, in our model, even a small temporary shock on

bubble bursts is not uniform. It is crucially affected by the financial condition of each country. If the imperfection of the financial market is relatively high, the bursting of bubbles decreases the growth rate of the country but the bursts may enhance the long run growth rate if the condition of the financial market is relatively good. This point is shown in Figure 4.[30] In other words, the bubbles bursting explores the "true" economic condition of each country. This result also means that the variance in growth rates among countries becomes higher and, once again, this result is consistent with the observation in Figure 3.

6 Conclusion

In this paper, we assumed the imperfection of the financial markets and examined the effects of bubbles under the imperfect financial market condition. We explored how the existence condition of bubbles is related to the condition of the financial market and how the middle range of pledgeability allows for the existence of bubbles. This suggests that improving the condition of the financial market might enhance the possibility of bubbles if the initial condition of the financial market is underdeveloped.[31] Moreover, the effects of bubbles on the economic growth rates are also related to the financial market's condition. If the pledgeability is relatively low, bubbles increase the growth rate; but bubbles decrease the growth rate if the pledgeability is relatively high. This result has an important implication for the effects of bubble bursts. The bursting of the bubbles decreases the growth rate when

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productivity of the entrepreneurs' investment has permanent effects on the aggregate productivity and the long run growth rate.

[30] If $\underline{\theta} \leq \theta \leq \theta^*$, the growth rate at $t = s$ might be higher than $t = s - 1$ if the temporary shock is not so large since the growth rate is enhanced by the burst of bubbles even at $t = s$.

[31] In our model, the volatility of the growth rate is high when θ is in the middle range, because bubbles can occur. This may be consistent with the empirical evidences such as Easterly et al. (2000), and Kunieda (2008), which shows that macroeconomic volatility is high when financial development is an intermediated level. As theoretical papers, there are Aghion et al. (1999), and Matsuyama (2007, 2008), in which they show that macroeconomic volatility is high when the financial market is intermediatedly developed. However, the source of high volatility is different between these papers and ours. In our paper, it is from the appearance of bubbles, while in their papers, it comes from the interest rate or quality of investments.

the condition of the financial market is not so good, but the bursts may enhance the growth rate when the financial market's condition is relatively good. These imply that the bubbles bursting explores the "true" economic condition of each country. In order to sustain high long-run growth rates, realizing the high quality of the financial system is important.

Our model could be extended in several directions. One direction would be to endogenize the pledgeability. In this model, we assume that the level of the pledgeability is exogenously given. It would be interesting to examine how the pledgeability is affected by legal systems or behaviors of financial sectors and how these factors affect the bubble regions. Another direction would be to extend our model into a two-countries model with different pledgeability levels, and investigate how globalization such as capital account liberalization affects the emergence of bubbles in each country. Finally, we have not analyzed the welfare implications of bubbles, policy-oriented issues such as government's intervention after bubble bursts, or the role of financial market regulations on the emergence of bubbles. These would be promising areas for future research.[32]

[32] Lorenzoni (2008) presents an interesting framework to study policies in the presence of pecuniary externality which comes from amplification in asset prices. Analyzing bubbles within Lorenzoni's framework will be interesting research for understanding regulations which prevent bubbles or government's intervention after bursts of bubbles.

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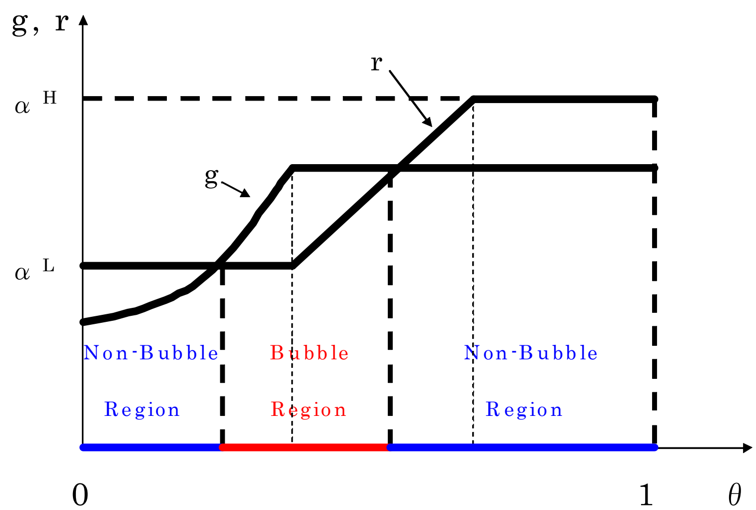


Figure 1: Bubble region and θ

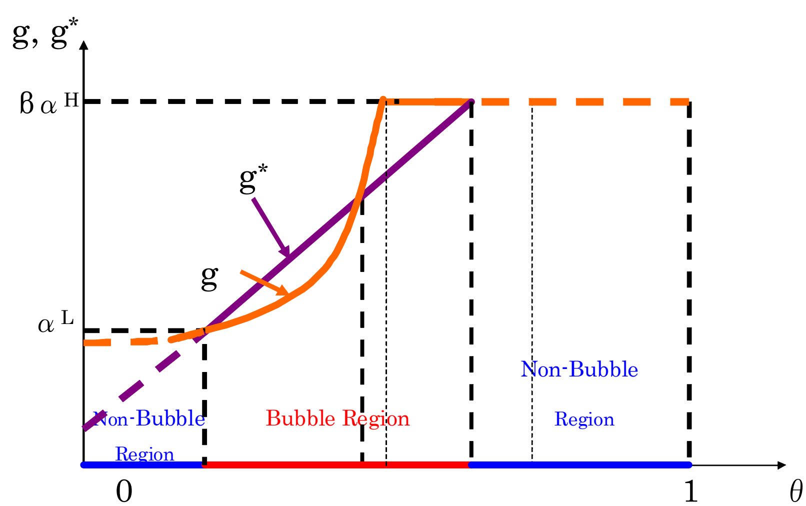
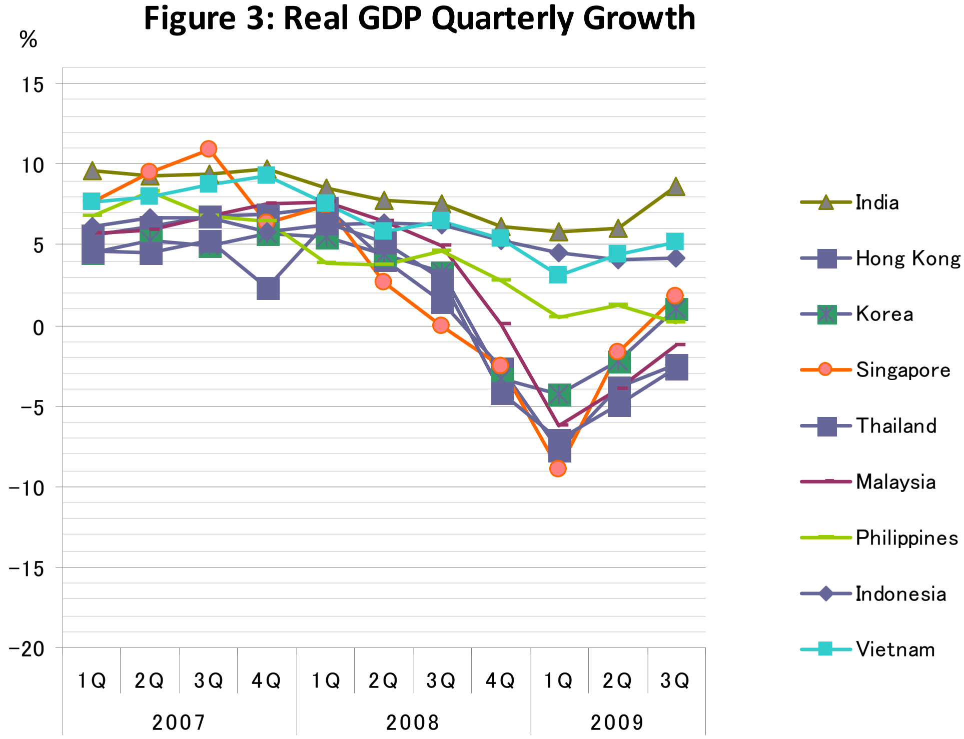


Figure 2: Bubbles and Economic Growth

Figure 3: Real GDP Quarterly Growth



Thomson Reuter, Datastream

* year-over-year basis

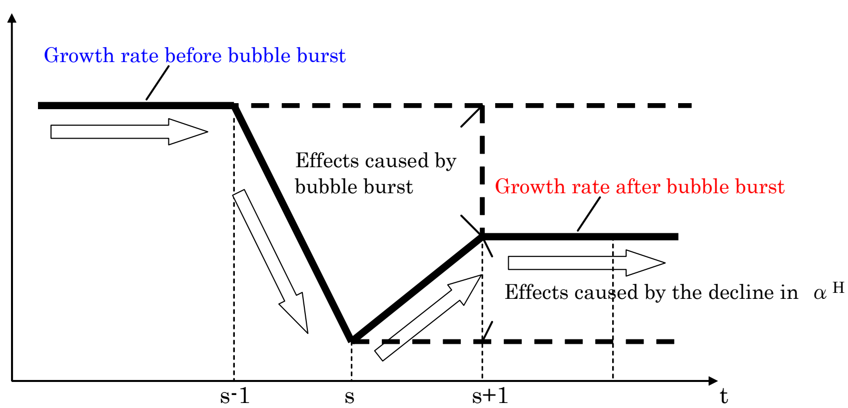


Figure 4-1: The effect of bubbles' bursting in relatively low θ

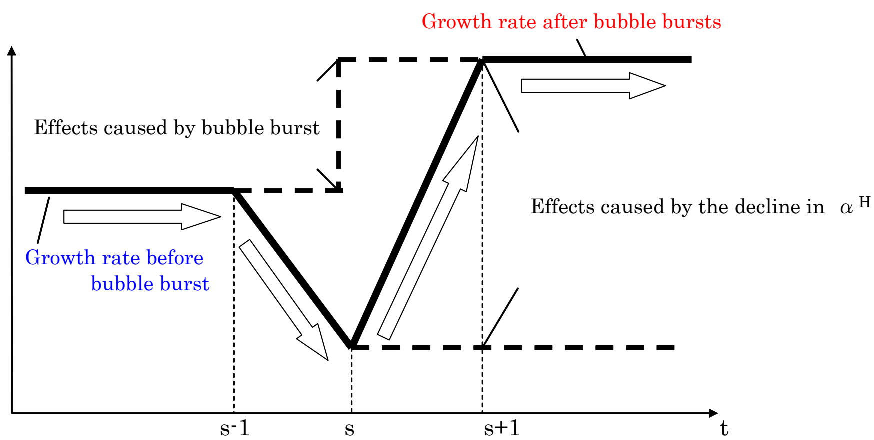


Figure 4-2: The effects of bubbles' bursting in relatively high θ

Appendix 1

Maximization Problem for the entrepreneur in the bubbleless economy

We derive the first order conditions of this problem by solving the Lagrangian ($\mathcal{L}$)

$$\mathcal{L}_0^i = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log(\alpha_{t-1}^i z_{t-1}^i - r_{t-1} b_{t-1}^i + b_t^i - z_t^i) + \lambda_t^i (\theta \alpha_t^i z_t^i - r_t b_t^i) + \phi_t^i z_t^i \right],$$

where λ_t^i, ϕ_t^i are Lagrange multipliers on the borrowing constraint and non-negative constraint ($z_t^i \geq 0$), respectively.

First order conditions:

$$\frac{\partial \mathcal{L}_t^i}{\partial z_t^i} = -\frac{1}{c_t^i} + \lambda_t^i \theta \alpha_t^i + \phi_t^i + E_t \left[\beta \frac{\alpha_t^i}{c_{t+1}^i} \right] = 0,$$

$$\frac{\partial \mathcal{L}_t^i}{\partial b_t^i} = \frac{1}{c_t^i} - \lambda_t^i r_t - E_t \left[\beta \frac{r_t}{c_{t+1}^i} \right] = 0.$$

Complementary slackness condition:

$$\begin{aligned} \lambda_t^i &\geq 0, \ r_t b_t^i \leq \theta \alpha_t^i z_t^i, \ \lambda_t^i (\theta \alpha_t^i z_t^i - r_t b_t^i) = 0, \\ \phi_t^i &\geq 0, \ z_t^i \geq 0, \ \phi_t^i z_t^i = 0. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\text{If } \alpha_t^i > r_t, \quad \lambda_t^i > 0 \text{ and } \phi_t^i = 0, \\ &\text{If } \alpha_t^i = r_t, \quad \lambda_t^i = 0 \text{ and } \phi_t^i = 0, \\ &\text{If } \alpha_t^i < r_t, \quad \lambda_t^i = 0 \text{ and } \phi_t^i > 0. \end{aligned}$$

Maximization Problem for the entrepreneur in the bubble economy

We derive the first order conditions of this problem by solving the Lagrangian ($\mathcal{L}$)

$$\mathcal{L}_0^{*i} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log(\alpha_{t-1}^i z_{t-1}^{*i} - r_{t-1}^* b_{t-1}^{*i} + b_t^{*i} - z_t^{*i}) + \lambda_t^{*i}(\theta \alpha_t^i z_t^{*i} - r_t^* b_t^{*i}) + \mu_t^{*i} P_t x_t^i + \phi_t^{*i} z_t^{*i} \right],$$

where λ_t^{*i}, μ_t^{*i}, ϕ_t^{*i} are Lagrange multipliers on the borrowing constraint, the short sale constraint, and non-negative constraint ($z_t^{*i} \geq 0$), respectively.

First order conditions:

$$\frac{\partial \mathcal{L}_t^{*i}}{\partial z_t^{*i}} = -\frac{1}{c_t^{*i}} + \lambda_t^{*i} \theta \alpha_t^i + \phi_t^{*i} + E_t \left[\beta \frac{\alpha_t^i}{c_{t+1}^{*i}} \right] = 0,$$

$$\frac{\partial \mathcal{L}_t^{*i}}{\partial b_t^{*i}} = \frac{1}{c_t^{*i}} - \lambda_t^{*i} r_t - E_t \left[\beta \frac{r_t^{*i}}{c_{t+1}^{*i}} \right] = 0,$$

$$\frac{\partial \mathcal{L}_t^{*i}}{\partial P_t x_t^i} = -\frac{1}{c_t^{*i}} + \lambda_t^{*i} \theta \frac{P_{t+1}}{P_t} + \mu_t^{*i} + E_t \left[\beta \frac{1}{c_{t+1}^{*i}} \frac{P_{t+1}}{P_t} \right] = 0.$$

Complementary slackness condition:

$$\begin{aligned}
\lambda_t^{*i} &\geq 0,\ r_t^* b_t^{*i} \leq \theta(\alpha_t^i z_t^{*i} + P_{t+1} x_t^i),\ \lambda_t^{*i} \left[\theta(\alpha_t^i z_t^{*i} + P_{t+1} x_t^i) - r_t^* b_t^{*i} \right] = 0, \\
\mu_t^{*i} &\geq 0,\ x_t^i \geq 0,\ \mu_t^{*i} x_t^i = 0, \\
\phi_t^{*i} &\geq 0,\ z_t^{*i} \geq 0,\ \phi_t^{*i} z_t^{*i} = 0.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\text{If } \alpha_t^i > r_t^{*i},\quad \lambda_t^{*i} > 0,\ \mu_t^{*i} > 0 \text{ and } \phi_t^{*i} = 0, \\
&\text{If } \alpha_t^i = r_t^{*i},\quad \lambda_t^{*i} = 0,\ \mu_t^{*i} = 0 \text{ and } \phi_t^{*i} = 0, \\
&\text{If } \alpha_t^i < r_t^{*i},\quad \lambda_t^{*i} = 0,\ \mu_t^{*i} = 0 \text{ and } \phi_t^{*i} > 0.
\end{aligned}$$

Proof of Proposition 2

If $\beta\alpha^H < \alpha^L$, the growth rate of this economy cannot be equal to the interest rate and bubbles cannot exist. This situation is the case where $> \beta(1-p)$ and θ which satisfies the above condition does not exist. Thus, we focus on the cases where $\beta\alpha^H \geq \alpha^L$. When $\frac{\alpha^L-\beta\{\alpha^L+(\alpha^H-\alpha^L)p\}}{\alpha^H(1-\beta)} > 0$, $r^*(\underline{\theta}) = \alpha^L$ and $l(r^*(\underline{\theta}), \underline{\theta}) > 0$. Since $r^*(\theta)$ is a strictly increasing function of θ, $r^*(\theta) < \alpha^L$ and bubbles cannot exist if $\theta < \underline{\theta} = \frac{\alpha^L-\beta\{\alpha^L+(\alpha^H-\alpha^L)p\}}{\alpha^H(1-\beta)}$. On the other hand, $r^*(\theta) \geq \alpha^L$ if $\theta \geq \underline{\theta} = \frac{\alpha^L-\beta\{\alpha^L+(\alpha^H-\alpha^L)p\}}{\alpha^H(1-\beta)}$. When $\frac{\alpha^L-\beta\{\alpha^L+(\alpha^H-\alpha^L)p\}}{\alpha^H(1-\beta)} \leq 0$, $\underline{\theta} = 0$ and $r^*(0) \geq \alpha^L$. But θ cannot be negative. Hence, we do not have to consider the case of $\theta < \underline{\theta}$ and $r^*(0) \geq \alpha^L$ as long as $\theta \geq \underline{\theta} = 0$. However, $l(r^*(\theta), \theta)$ is a decreasing function of θ and $l(r^*(\theta), \theta)$ becomes zero when $\theta = \beta(1-p) = \overline{\theta}$. Therefore, bubbles can exist as long as $\underline{\theta} \leq \theta < \overline{\theta}$.[33]

Proof of Proposition 3

If $\beta\alpha^H < \alpha^L$, bubbles cannot exist as explained in the proposition 1, and the growth rate under the bubbleless economy is lower than the interest rate under the bubbleless economy for any θ. Next we check the case where $\beta\alpha^H \geq \alpha^L$. From the definition of $\underline{\theta}$, the following relation is satisfied.

$$\beta\alpha^H - (\alpha^H - \alpha^L)\beta l^*(\alpha^L, \underline{\theta}) = \alpha^L.$$

This relation means that at $\underline{\theta}$, the growth rate under the bubbleless economy (the left hand side) is equal to the interest rate under the bubbleless economy (the right hand side). When θ is a little higher than $\underline{\theta}$ but smaller than $(1-p)\alpha^L/\alpha^H$, the growth rate under the bubbleless economy becomes higher than α^L but the interest rate is still α^L. Thus, the growth rate is higher than the interest rate under the bubbleless economy. If θ becomes higher than $(1-p)\alpha^L/\alpha^H$, the interest rate becomes $\theta\alpha^H/(1-p)$ which is higher than α^L and the growth rate becomes $\beta\alpha^H$. Hence, the growth rate becomes lower than the interest rate when θ becomes higher than $\overline{\theta} \equiv \beta(1-p)$. In summary, the growth rate is higher than the interest rate under the bubbleless economy when $\underline{\theta} \leq \theta \leq \overline{\theta}$, and $\underline{\theta} \leq \theta \leq \overline{\theta}$ is exactly the necessary condition of existence of bubbles.

[33] At $\theta = \left\{\alpha^L - \beta\left[\alpha^L + (\alpha^H - \alpha^L)p\right]\right\} / \left[\alpha^H(1-\beta)\right]$, the interest rate, the rate of return on L-projects and bubbles are the same. Thus, L-entrepreneurs invest in their own L-projects as well as buy bubbles and lend to H-entrepreneurs.

Proof of Proposition 4

From (18) and (35), $g_t = g(\theta) = \beta\alpha^H - (\alpha^H - \alpha^L)\beta\frac{\alpha^L(1-p)-\theta\alpha^H}{\alpha^L-\theta\alpha^H}$, and $g_t^* = r^*(\theta) = \alpha^H\frac{(1-\beta)\theta+p\beta}{1-\beta+p\beta}$. We derive θ which satisfies $g_t = g_t^*$, that is

$$\begin{aligned}\theta^2 - \frac{-p\beta\alpha^H + \alpha^L(1-\beta) + \beta\alpha^L(1-\beta+p\beta)}{\alpha^H(1-\beta)}\theta \\ + \frac{-p\beta\alpha^L + \left[\alpha^L + (\alpha^H - \alpha^L)p\right]\beta\frac{\alpha^L}{\alpha^H}(1-\beta+p\beta)}{\alpha^H(1-\beta)} = 0.\end{aligned} \tag{A1}$$

By solving the above quadratic function (A1), we can derive that

$$\underline{\theta} = \frac{\left\{\alpha^L - \beta\left[\alpha^L + (\alpha^H - \alpha^L)p\right]\right\}}{\alpha^H(1-\beta)},$$

$$\theta^* = \frac{\alpha^L}{\alpha^H}\beta(1-p).$$

Furthermore, from the quadratic function (A1), we can derive that $g_t < g_t^*$ if $\theta < \theta^*$, and $g_t > g_t^*$ if $\theta > \theta^*$.

Appendix 2: Stochastic Bubbles

This appendix presents stochastic bubbles version of the basic model of Section 3. Following Weil (1987), we assume that bubble price becomes zero (bubble bursts) with probability $1-\pi$ at date t conditional on positive bubble price at date $t-1$, and once they burst, they never arise again. This implies that bubbles continue with probability $\pi(<1)$ and their prices are positive until they switch to being equal to zero forever. Let P_t be the per unit price of bubble assets at date t in terms of consumption goods when bubbles do not collapse at date t.

The maximization problem for an entrepreneur is as follows:

$$\underset{\left\{c_t^{*i},\, z_t^{*i},\, b_t^{*i},\, x_t^i\right\}_{t=0}^{\infty}}{Max} \qquad E_0\left[\textstyle\sum_{t=0}^{\infty}\beta^t \log c_t^{*i}\right]$$

$$c_t^{*i} + z_t^{*i} + P_t x_t^i = y_t^{*i} - r_{t-1}^* b_{t-1}^{*i} + b_t^{*i} + P_t x_{t-1}^i,$$

$$r_t^* b_t^{*i} \leq \theta(\alpha_t^i z_t^{*i} + E_t\left[\tilde{P}_{t+1} x_t^i\right]),$$

$$x_t^i \geq 0,$$

$$\lim_{t\to\infty}\inf \beta^t \frac{1}{c_t^{*i}} P_t x_t^i = 0,$$

where $\tilde{P}_{t+1}$ is a random variable, because bubbles collapse stochastically. As before, we assume that only a fraction θ of the returns from the investment and the expected return on bubbles can be pledgeable to the creditors.

H-entrepreneurs act much like they do in the deterministic case. On the other hand, for L-entrepreneurs, their portfolio problem is more complicated than in the deterministic case. Since bubble assets deliver no return with probability $1-\pi$, hence L-entrepreneurs may want to hedge themselves by investing in their L-projects.

For date t L-entrepreneurs, from the first order conditions,

$$\frac{1}{c_t^{*i}} = \pi\beta\frac{r_t^*}{c_{t+1}^{*i,\pi}} + (1-\pi)\beta\frac{r_t^*}{c_{t+1}^{*i,1-\pi}}, \tag{A2}$$

$$\frac{1}{c_t^{*i}} = \pi\beta\frac{1}{c_{t+1}^{*i,\pi}}\frac{P_{t+1}}{P_t}, \tag{A3}$$

where $c_{t+1}^{i,\pi}$ and $c_{t+1}^{i,1-\pi}$ are the consumption level at date $t+1$ when bubbles continue and collapse at date $t+1$, respectively.

From (A2) and (A3),

$$\pi(\frac{P_{t+1}}{P_t} - r_t^*)c_{t+1}^{*i,1-\pi} = (1-\pi)r_t^* c_{t+1}^{*i,\pi}. \tag{A4}$$

The aggregate counterpart to (A4) is

$$\pi(\frac{P_{t+1}}{P_t} - r_t^*)C_{t+1}^{*1-\pi} = (1-\pi)r_t^* C_{t+1}^{*\pi}, \tag{A5}$$

where $C_{t+1}^{*\pi} = (1-\beta)(\alpha^L Z_t^{*L} - r_t^* B_t^{*L} + P_{t+1}X)$ and $C_{t+1}^{*1-\pi} = (1-\beta)(\alpha^L Z_t^{*L} - r_t^* B_t^{*L})$. Note that if $r_t^* = \alpha^L$, $Z_t^{*L} \geq 0$ and if $r_t^* > \alpha^L$, $Z_t^{*L} = 0$.

Rearranging (A5) by using the aggregate flow of funds constraint of date t L-entrepreneurs, $P_t X + Z_t^{*L} + B_t^{*H} = \beta E_t^{*L}$,

$$k_t = \begin{cases} \frac{(\pi \frac{P_{t+1}}{P_t} - \alpha^L)(1-p)}{(\frac{P_{t+1}}{P_t} - \alpha^L)} & \text{if } r_t^* = \alpha^L \\ \frac{(\pi \frac{P_{t+1}}{P_t} - r_t^*)(1-p)}{(\frac{P_{t+1}}{P_t} - r_t^*)} & \text{if } r_t^* > \alpha^L \end{cases}. \tag{A6}$$

Evolution of the aggregate wealth's growth rate follows

$$g_t^k = \begin{cases} \alpha^H \frac{\beta p}{1-\frac{\theta \alpha^H}{\alpha^L}} + \alpha^L \left[\beta - \frac{\beta p}{1-\frac{\theta \alpha^H}{\alpha^L}} \right] + (\frac{P_{t+1}}{P_t} - \alpha^L)\beta k_t & \text{if } r_t^* = \alpha^L \\ \alpha^H \frac{\beta p}{1-\frac{\theta \alpha^H}{r_t^*}} + \frac{P_{t+1}}{P_t} \left[\beta - \frac{\beta p}{1-\frac{\theta \alpha^H}{r_t^*}} \right] & \text{if } r_t^* > \alpha^L \end{cases}. \tag{A7}$$

When $r_t^* > \alpha^L$, the interest rate is determined by equation (30), given k_t :

$$\frac{p}{1 - \frac{\theta \alpha^H}{r_t^*}} + k_t = 1. \tag{A8}$$

Existence of Stochastic Bubbles

Here following Farhi and Tirole (2010), we restrict our attention to an equilibria where the variables $(g_t^k, g_t^*, r_t^*, \frac{P_{t+1}}{P_t}, k_t)$ are constant over time, which Farhi and Tirole call a conditional bubbly steady state. In this equilibria, wealth's growth rate, output's growth rate, the interest rate, the rate of return on bubbles, and bubble share are constant until the bubbles crash.

In such a conditional bubbly steady state,

$$g^k = g^* = \frac{P_{t+1}}{P_t}. \tag{A9}$$

From (A6)-(A9), we can obtain the conditional bubbly steady state. In order that the conditional bubbly steady state can exist, the following conditions must be satisfied:

$$k > 0, \tag{A10}$$

$$\frac{Z_t^{*L}}{\beta(Y_t^* + P_t X)} > 0 \quad \text{if } r_t^* = \alpha^L. \tag{A11}$$

From (A10) and (A11), we can obtain the existence condition of the conditional bubbly steady state.

Proposition 5 *The existence condition of the conditional bubbly steady state becomes*

$$Max\left[\frac{\alpha^L - \pi\left[\alpha^L + (\alpha^H - \alpha^L)p\right]\beta}{\alpha^H(1-\pi\beta)}, 0\right] < \theta < \pi\beta(1-p).$$

Compared to the existence condition of deterministic bubbles (Proposition 2), the condition becomes tightened.

Here we should note two things. The first one is that if

$$\frac{\alpha^L - \pi\left[\alpha^L + (\alpha^H - \alpha^L)p\right]\beta}{\alpha^H(1-\pi\beta)} < 0,$$

then, even if $\theta = 0$, the conditional bubbly steady state can exist.

The second one is that if

$$\frac{\alpha^H}{\alpha^L} < \frac{1}{\pi\beta},$$

then, the conditional bubbly steady state can not exist.

Moreover, within the bubble regions, growth effects of stochastic bubbles become different depending upon θ.

Proposition 6 *If* $Max\left[\frac{\alpha^L - \pi\left[\alpha^L + (\alpha^H - \alpha^L)p\right]\beta}{\alpha^H(1-\pi\beta)}, 0\right] < \theta \leq \theta_1$, *the growth rate under the stochastic bubble economy is higher than that under the bubbleless economy. If* $\theta_1 < \theta < \pi\beta(1-p)$, *the growth rate under the stochastic bubble economy is lower than that under the bubbleless economy.* θ_1 *is the greater value of the following quadratic equation:* $\alpha^H \frac{\beta[1-\pi(1-p)]+(1-\beta)\theta}{1-\pi\beta(1-p)} = \alpha^H \frac{\alpha^L\beta p}{\alpha^L - \theta\alpha^H} + \alpha^L(\beta - \frac{\alpha^L\beta p}{\alpha^L - \theta\alpha^H})$.