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Abstract

In this paper, we present a new logic of indeterminacy of stationary monetary equilibria. We first show that, in a class of dynamic Walrasian market models with fiat money, stationary equilibria are indeterminate; that is, there exists a continuum of stationary equilibria, where the value of money varies across the stationary equilibria. Then we explore the logic of the indeterminacy, and show that it can be applied not only to dynamic Walrasian market models but also to Jean et al. (2010)'s search model, where agents are also allowed for periodic access to Walrasian markets.

Keywords: real indeterminacy, divisible money, dynamic general equilibrium

Journal of Economic Literature Classification Number: D51, E40

1 Introduction

In search models with fiat money, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible fiat money. (See, for example, Green and Zhou (1998), Kamiya and Shimizu (2006), and Matsui and Shimizu (2005).) On the other hand, to the best of our knowledge, indeterminacy of stationary equilibria has never been found in the literature of macroeconomic dynamics.[1] (For a survey, see Benhabib and Farmer (1999).) A question arises as to whether or not there exists indeterminacy of stationary equilibria in dynamic general equilibrium models with fiat money. In this paper, we show that, in a class of dynamic general equilibrium models with fiat money, there exists real indeterminacy of stationary equilibria, i.e., the set of stationary equilibria is a continuum, where the value of money varies across equilibria.

In standard static general equilibrium models, the number of equations is equal to that of variables and thus there typically exist a finite number of equilibria: Walras' Law implies that one market clearing condition is redundant, and homogeneity of degree zero in demand implies that the prices can be normalized such that the sum of prices is one. If the homogeneity of degree zero is violated due to some special structure of a model, there might exist a continuum of equilibria. There are several sources known so

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[1] Gottaridi (1996) and Spear, Srivastava, and Woodford (1990) among others find a continuum of Markov stationary equilibria in OLG models. However, their concept of stationarity is different from ours. Nishimura and Shimomura (2002) show that a dynamic trade model can have a continuum of stationary equilibria.

far to create non-homogeneity. Among them are static Walrasian models with government money taxes and incomplete market models with nominal assets. Lerner (1947) argues that government money taxes can be a reason why fiat money has value, and subsequently Balasko and Shell (1993) show that there exists a continuum of equilibria for each balanced fiscal policy due to the violation of homogeneity. Geanakoplos and Mas-Colell (1989) apply a somewhat similar argument to a two-period incomplete market model with nominal assets, and show that the dimension of the set of equilibria is $S-1$, where S is the number of states. It is worthwhile to note that van der Laan (1982) and Herings, Talman, and Yang (1996) find that the set of equilibria is a one-dimensional manifold linking two trivial equilibria in general equilibrium models with price rigidities, where the homogeneity of demand functions does not hold due to price rigidities.

The above argument by Geanakoplos and Mas-Colell cannot be applied to dynamic Walrasian market models with fiat money. Indeed, in markets with uncertainty, nominal assets are risk sharing devices, and thus can have value. However, in standard dynamic Walrasian market models with fiat money, money has value not as a risk sharing device but as a medium of exchange and a store of value. On the other hand, Balasko and Shell's argument crucially depends on the exogenously given government fiscal policy in a static environment and thus cannot be directly applied to models without tax. In this paper, we present dynamic models, where agents use money as a medium of exchange and a store of value, and the violation of homogeneity is endogenously created.

In a specific money search model, Green and Zhou (1998) show that there is a continuum of stationary equilibria, where money holdings distribution varies across equilibria. Subsequently, Kamiya and Shimizu (2006)(2007) find that, in a quite general framework, there exists a hidden identity in the condition for stationarity of money holdings distributions, and thus there exists indeterminacy of stationary equilibrium money holdings distributions. This argument clearly has nothing to do with homogeneity of demand functions, and our logic of indeterminacy is new in the literature of dynamic monetary models. Therefore there are two sources of indeterminacy in dynamic monetary models: non-homogeneity of demand functions and the hidden identity in stationarity condition. Jean et al. (2010) find indeterminacy in a search model, where agents are also allowed for periodic access to Walrasian markets. We show that their indeterminacy is classified into the first type: indeterminacy caused by non-homogeneity of demand functions.

Below, using a simple environment, we briefly discuss the logic of indeterminacy in our model. We consider a static economy with two agents, one good, and fiat money. The initial endowments are $\omega_1 = \omega_2 = 10$. At the beginning, agents 1 and 2 have \$150 and \$50, respectively. Suppose the government levies tax: both agents must pay \$100. The budget constraints in each period are as follows:

$$Px_1 + 100 = 150 + P \cdot 10,$$

$$Px_2 + 100 = 50 + P \cdot 10,$$

where P is the price of the good. Thus the net consumptions are

$$x_1 - 10 = P^{-1}(150 - 100) = P^{-1} \cdot 50,$$

$$x_2 - 10 = P^{-1}(50 - 100) = -P^{-1} \cdot 50.$$

Consider a price increase. Then agent 1 consumes less amount of good, agent 2 consumes larger amount of good, and the good market clears. Indeed, let $P' > P$, then

$$x_1 - 10 = P^{-1} \cdot 50 > P'^{-1} \cdot 50 = x_1' - \omega_1,$$

$$x_2 - 10 = -P^{-1} \cdot 50 < -P'^{-1} \cdot 50 = x_2' - \omega_2,$$

and

$$x_1' + x_2' - (10 + 10) = P'^{-1}(50 - 50) = 0.$$

Thus there exists real indeterminacy in stationary equilibria: the market clearing condition holds even for P' and the allocation of good changes. The demand is not homogeneous of degree zero, because the amount of money holdings after the deduction of tax are non-zero constants for both agents, and this results in the real indeterminacy. Note that if the utility function $u_i, i = 1, 2$, is strictly concave, the change in P has a real effect on the welfare $u_1(x_1)+u_2(x_2)$. The above arguments crucially depend on the exogenously given tax. In this paper, we show that the need for fiat money in the above arguments can be endogenously created in dynamic general equilibrium models without tax.

The plan of our paper is as follows. Section 2 describes a realistic example of real indeterminacy in a centralized market economy. In section 3, we will present a more general model and discuss the logic of real indeterminacy. We conclude in section 4.

2 An Example

This section describes a realistic example of real indeterminacy in a centralized market economy. We consider an infinite-horizon model with discrete time starting from period 0. There are two types of agents: households and a monopoly firm. The households supply labor and consume two types of goods: an indivisible durable good, e.g., house, car, and refrigerator, and a divisible non-storable good, e.g., food and drink. We assume that each household can consume just one unit of the durable good. In each period, the durable good breaks down with some probability. Then a household can buy a new one. We assume that there is no contingent-claim against the damage, and that the households are subject to cash-in-advance constraints. The monopoly firm produces the indivisible durable goods from labor.

2.1 Households

There is a continuum of homogeneous households in the interval $[0, 1]$. We assume that all households have the same utility function as follows:

$$E_0 \sum_{t=0}^{\infty} \delta^t \left[uI_t + U(X_t) - H_t \right],$$

where $I_t \in \{0, 1\}$ is the amount of indivisible good in period t, $u > 0$ is a given parameter, $X_t \in \mathbb{R}_+$ is the amount of divisible good in period t, $H_t \in \mathbb{R}_+$ is the labor supply in period t, and $\delta \in (0, 1)$ is an exogenously given discount factor. Note that the temporal utility function $uI + U(X) - H$ is quasilinear, i.e., linear in H. We will investigate a general utility function in the next section. We assume that $U(\cdot)$ is increasing, strictly concave, and differentiable, and $\exists X^* \in \mathbb{R}_{++}$ such that $U'(X^*) = 1$ and $U(X^*) > X^*$.

Each household can store divisible money. We assume that there is an upper bound of money holding, denoted by $\bar{m} > 0$.[2] That is, the amount of money a household can hold is at most $\bar{m}$. The total money stock is fixed at $M > 0$. At the beginning of each period, the durable good breaks down with probability $\mu \in (0, 1)$. We assume that the probability is independent across the households. Thus there are two types of households at the beginning of each period: a household with one unit of the durable good and a household without it. Let $V^1(m)$ denote the value of a household with m units of money and one unit of the durable good. Similarly, when the household does not hold the durable good, let its value be $V^0(m)$. At the end of each period, each household receives the same amount of dividend Π from the monopoly firm. Note that Π will be defined in the next subsection. Let P and p be the nominal prices of the divisible good and the durable good, respectively. We assume that the households have a linear technology that converts one unit of labor to one unit of the divisible good. We further assume that captive consumption is possible; that is, a household can consume the divisible good produced by herself.

We first consider the decision problem of a household with one unit of durable good. The Bellman equation is as follows:

$$
\begin{aligned}
V^1(m) = \max_{X,H,s}\; & u + U(X) - H + \delta[\mu V^0(s + \Pi) + (1 - \mu)V^1(s + \Pi)] \\
\text{s.t.}\;\; & PX + s = PH + m, \qquad (1) \\
& P(X - H) \leq m, \\
& X, H \geq 0,\; 0 \leq s \leq \bar{m} - \Pi, \qquad (2)
\end{aligned}
$$

where $s \in \mathbb{R}_+$ is the amount of money left after the trade. Since captive consumption is allowed, the cash-in-advance constraint for the divisible good is $P(X - H) \leq m$.[3] The condition $s \geq 0$ and (1) imply that the cash-in-advance constraint is always satisfied. Since the household receives dividend Π after the trade, the money holding at the beginning of the next period is $s + \Pi$. By the quasilinearity, $X = X^*$ clearly holds at the optimum unless $H \geq 0$ is binding. Then the problem can be rewritten as follows:

$$
V^1(m) = u + U(X^*) - X^* + \frac{m}{P} + \max_s \left\{ -\frac{s}{P} + \delta[\mu V^0(s + \Pi) + (1 - \mu)V^1(s + \Pi)] \mid 0 \leq s \leq \bar{m} - \Pi \right\}. \qquad (3)
$$

[2]The upper bound ensures that labor supply is non-negative. Note that a similar condition can be found in Lagos and Wright (2005). We specify the condition of $\bar{m}$ later in Assumption 1.

[3]We assume that households work at the firm only when $X < H$. If $X \geq H$, they supply labor only for producing their captive consumption. This assumption does not lose generality because receiving P from the firm and producing a divisible good by themselves are the same.

Next, consider the case of a household who does not hold the durable good. The household chooses whether or not to buy the durable good. Thus the Bellman equation is as follows:

$$V^0(m) = \max_{X,H,I,s} uI + U(X) - H + \delta W(s, I)$$

$$\text{s.t.} \quad PX + pI + s = PH + m, \tag{4}$$

$$pI + \max\{0, P(X - H)\} \leq m. \tag{5}$$

$$X, H \geq 0, \ 0 \leq s \leq \bar{m} - \Pi,$$

where

$$W(s, I) = \begin{cases} \mu V^0(s + \Pi) + (1 - \mu)V^1(s + \Pi) & \text{if } I = 1, \\ V^0(s + \Pi) & \text{if } I = 0. \end{cases} \tag{6}$$

(5) is the cash-in-advance constraint: the total expenditure must be less than or equal to the amount of money.[4] If $X < H$, then the household receives wage income $P(H - X)$. Note that the wage income cannot be used for consuming I in the current period, since $\max\{0, P(X - H)\} = 0$ holds. Unless $H \geq 0$ is binding, we can rewrite V^0 as follows:

$$V^0(m) = U(X^*) - X^* + \frac{m}{P} + \max_{s,I} \left\{ \left(u - \frac{p}{P}\right) I - \frac{s}{P} + \delta W(s, I) \mid 0 \leq s \leq \bar{m} - \Pi, pI \leq m \right\},$$

$$\text{where} \quad W(s, I) = \begin{cases} \mu V^0(s + \Pi) + (1 - \mu)V^1(s + \Pi) & \text{if } I = 1, \\ V^0(s + \Pi) & \text{if } I = 0. \end{cases} \tag{7}$$

Note that the cash-in-advance constraint reduces to $s \geq 0$ in the case of $\max\{0, P(X - H)\} = P(X - H)$, and it reduces to $pI \leq m$ otherwise.

Let the optimal policy functions associated with the Bellman equation be $X^0(m, p, P)$, $X^1(m, p, P)$, $s^0(m, p, P)$, $s^1(m, p, P)$, and $I(m, p, P)$.[5] Note that $I(m, p, P)$, the purchase of the durable good, is defined only when the household does not have it at the beginning of the period.

2.2 Monopoly firm

The monopoly firm has a linear technology that converts one unit of labor to one unit of the durable good. In this economy, the firm has a monopoly power in the indivisible good market, while the divisible good and the labor markets are competitive. Because of the linearity of the technology, the price of labor is equal to that of the divisible good P. Given P, the firm chooses the indivisible good price p so as to maximize the profit. The demand for the indivisible good depends on the policy function $I(m, p, P)$ and a money holdings distribution. Hence, the monopoly firm's profit maximization problem is as follows:

$$\Pi = \max_p py(p) - Py(p),$$

[4] The constraint might seem unrealistic because people often buy a durable good on trust in the real world economy. We use the cash-in-advance constraint only for simplicity. Instead of the cash-in-advance constraint, we can impose a borrowing constraint, i.e., the households can borrow at most some fixed amount of money. Even in this case, a similar indeterminacy result can be derived.

[5] $H^i(m, p, P)$ is derived from $X^i(m, p, P)$, $I(m, p, P)$, and $s^i(m, p, P)$ for $i = 0, 1$.

where the demand for the indivisible good $y(\cdot)$ is derived from the policy function $I(m,p,P)$ and a given money holdings distribution. At the end of the period, the monopoly firm distributes the profit Π equally to all households as dividend.

2.3 Equilibrium

Definition A tuple of value function, (V^0, V^1), policy functions, $X^0(m,p,P)$, $X^1(m,p,P)$, $s^0(m,p,P)$, $s^1(m,p,P)$, and $I(m,p,P)$, a money holdings distribution, and prices, p and P, is called a *stationary monetary equilibrium* if

1. Given p and P, (V^0, V^1) is a solution to the above Bellman equation and the policy functions X^0, X^1, s^0, s^1, and I are maximizers.
2. p is determined by the profit maximization for given P, I, and the money holdings distribution.
3. The money holdings distribution is stationary.
4. The divisible good market clears.
5. The money market clears.

2.4 Real indeterminacy

Before considering equilibrium, Let us define the following assumption to ensure the equilibrium. Define

$$D = \frac{\delta\mu u - (1-\delta)\mu}{\delta\mu[1-\delta(1-\mu)] + (1-\mu)(1-\delta)}.$$

We make the following assumption.

Assumption 1: $1 < D \leq u$ and $M < \bar{m} \leq \left[\frac{(X^*+\mu)M}{D} + (1-\mu)M\right]$.

By the first condition, it will be shown that the utility of the indivisible good is sufficiently high for the household to consume the indivisible good. The second condition says the upper bound of money holdings is not very large. By the condition, it will be shown that the households always consume X^*.

Below, we consider the following candidate for an equilibrium:

A Candidate for an Equilibrium:

a. $X^k(m,p,P) = X^*$ for all feasible (m,p,P), $K = 0,1$.

b. $s^k(m,p,P) = M - \Pi$ for all feasible (m,p,P), $k = 0,1$.

c. $I(m,p,P) = 0$ if $m < p$ and $I(m,p,P) = 1$ otherwise.

d. The support of the money holdings distribution is a singleton $\{M\}$, i.e., almost every households have M at the end of each period.

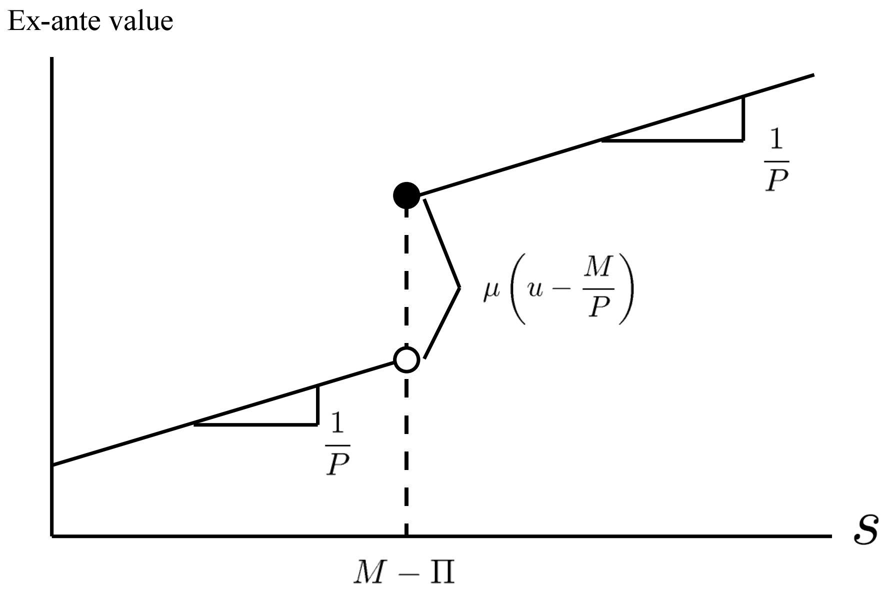

Figure 1: Ex-ante value function

e. $p = M$.

Note that P is not specified in the above. Once P is chosen, then (V^0, V^1) is determined from the above.

Below, we show that there exists a continuum of the above type of equilibria under Assumption 1. That is, for a continuum of P, the above candidate and the corresponding (V^0, V^1) constitute an equilibrium.

Proposition 1: Under Assumption 1, the above candidate for an equilibrium, together with P satisfying $M/P \in [1, D]$, constitutes a stationary monetary equilibrium.

We first discuss the intuition of the proof. Suppose u is large enough for a household to buy the durable good if it is broken. Consider the decision of the monopoly firm. In the candidate, all households have M units of money and they want to buy the durable good if it is broken and $p \leq M$. Thus the monopoly firm chooses the highest price affordable for the households. Namely, $p = M$ holds.

Next, consider the decision of a household. Figure 1 plots $\mu V^0(s+\Pi)+(1-\mu)V^1(s+\Pi)$, the ex-ante value in the next period. The function is linear because the utility function is quasilinear. By (3) and (7), the marginal utility of money is $1/P$ except at $M - \Pi$, and the value has a jump at $M - \Pi$. If $s < M - \Pi$, the household cannot buy a new unit of the durable good when the old one breaks down at the beginning of the next period. On the other hand, if $s \geq M - \Pi$, then the household can afford to buy a new unit.

Consider the following maximization in (3) and (7):

$$\max_s F(s), \quad \text{where } F(s) \equiv -\frac{s}{P} + \delta[\mu V^0(s + \Pi) + (1 - \mu)V^1(s + \Pi)].$$

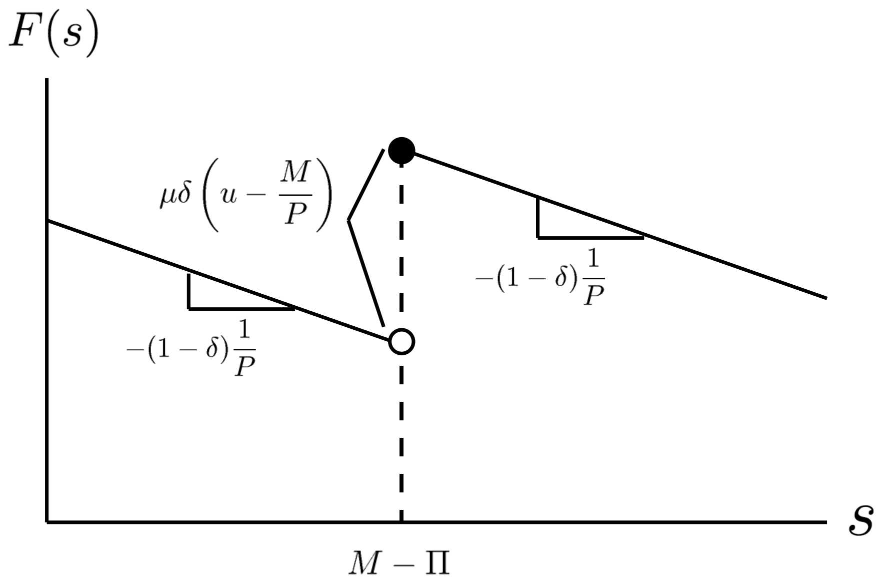

Figure 2: Inside of the maximization of s

A marginal increase in s induces a marginal increase in $\mu V^0(s+\Pi)+(1-\mu)V^1(s+\Pi)$ by $1/P$ except at $M-\Pi$. Thus $F(s)$ decreases by $-(1-\delta)(1/P)$ except at $M-\Pi$, and it only increases at $M-\Pi$. By Figure 2, when $F(0)\leq F(M-\Pi)$, $\arg\max F(s)=M-\Pi$ holds. On the other hand, the monopoly firm has Π units of money right after the trade. Thus the total amount of money the households have is equal to M, i.e., the money market clears. It is worthwhile to note that even if P slightly changes in Figure 2, the demand for money remains the same. That is, there exists a continuum of equilibria.

Finally, we discuss real indeterminacy. If P rises, the price of durable good p relatively decreases. Since the amount of money saved by a household is the same, the budget constraint of a household without the durable good slacks. Thus such a household can decrease her labor supply, and her utility increases. Thus the change in P has a real effect.

We will now present the formal proof.

Proof. Below, we check that the candidate indeed satisfies the equilibrium conditions.

1. We first show that the constraint $H\geq 0$ is not binding. By (1) and (4),

$$H\geq X^*-\frac{m}{P}+\frac{M-\Pi}{P}$$

holds for all households, since X^* is chosen if m is not very large and some $X>X^*$ is chosen otherwise. Thus, by $\Pi=\mu(M-P)$, a sufficient condition for $H\geq 0$ is

$$P(X^*+\mu)+(1-\mu)M\geq\bar{m}.$$

By $\frac{M}{P}\leq D$, a lower bound of P is $\frac{M}{D}$. By Assumption 1, the above inequality holds for

$P = \frac{M}{D}$, and thus holds for all $P \geq \frac{M}{D}$. Thus the constraint $H \geq 0$ is not binding and $X^i(m,p,P) = X^*$ holds for the candidate.

Then we show that $I(m,p,P)$ in the candidate is indeed optimal. Consider (7), a household's problem when she has no durable good. Clearly, $V^1(m) \geq V^0(m)$ holds for all m. Then

$$\left(u - \frac{M}{P}\right) + \delta W(s,1) - \delta W(s,0) \geq u - \frac{M}{P} \geq u - D \geq 0,$$

holds, where the second and the third inequalities follow from the premise of the proposition and Assumption 1, respectively. Thus the household chooses $I = 1$, if $m \geq p$.

Second, we show that $s^0(m,p,P)$ and $s^1(m,p,P)$ in the candidate are optimal. First, consider the case of $m \geq p$. Then $I(m,p,P) = 1$ in (7). By the quasilinearity, the maximization problem is independent of m. Let

$$F(s) = -\frac{s}{P} + \delta[\mu V^0(s+\Pi) + (1-\mu)V^1(s+\Pi)].$$

For $s \neq M - \Pi$, $\partial F(s)/\partial s = -(1-\delta)(1/P)$ holds, and thus $F(s)$ is decreasing for $s \neq M - \Pi$. By (3) and (7), the following equation holds in both cases with and without the durable good:

$$F(M-\Pi) - \lim_{s \nearrow (M-\Pi)} F(s) = \mu\delta[V^0(M) - \lim_{m \nearrow M} V^0(m)] = \mu\delta\left(u - \frac{M}{P}\right).$$

Since $F(s)$ is strictly decreasing except at $M - \Pi$, the only candidates for the maximal value of $F(s)$ are $F(0)$ and $F(M-\Pi)$. Below, we show that $F(M-\Pi) \geq F(0)$. By (3), (7), and $\Pi = (M-P)\mu$,

$$\begin{aligned}
& F(M-\Pi) \geq F(0) \\
\Leftrightarrow\ & -\frac{M-\Pi}{P} + \delta[\mu V^0(M) + (1-\mu)V^1(M)] \geq \delta[\mu V^0(\Pi) + (1-\mu)V^1(\Pi)] \\
\Leftrightarrow\ & \delta\mu[V^0(M) - V^0(\Pi)] + \delta(1-\mu)[V^1(M) - V^1(\Pi)] \geq \frac{M-\Pi}{P} \\
\Leftrightarrow\ & \delta\mu\left\{u - \frac{\Pi}{P} + \delta(1-\mu)\left[V^1(M) - V^0(M)\right]\right\} + \delta(1-\mu)\frac{M-\Pi}{P} \geq \frac{M-\Pi}{P} \\
\Leftrightarrow\ & \delta\mu\left[u - \frac{\Pi}{P} + \delta(1-\mu)\frac{M}{P}\right] \geq [1-\delta(1-\mu)]\frac{M-\Pi}{P} \qquad (8) \\
\Leftrightarrow\ & \delta\mu u - (1-\delta)\mu \geq \{\delta\mu[1-\delta(1-\mu)] + (1-\mu)(1-\delta)\}\frac{M}{P}.
\end{aligned}$$

The last inequality clearly follows from $M/P \in [1, D]$.

Second, consider the case of $m < p$. If a household has the durable good, the maximization problem is similar to the above, and the same arguments apply. If a household does not have the durable good, she maximizes

$$\tilde{F}(s) = -\frac{s}{P} + \delta V^0(s+\Pi).$$

As in the above, it suffices to show that $\tilde{F}(M-\Pi) \geq \tilde{F}(0)$. By (3) and (7),

$$
\begin{aligned}
& \tilde{F}(M-\Pi) \geq \tilde{F}(0) \\
\Leftrightarrow & -\frac{M-\Pi}{P} + \delta V^0(M) \geq \delta V^0(\Pi) \\
\Leftrightarrow & \ \delta\left[u - \frac{\Pi}{P} + \delta(1-\mu)[V^1(M) - V^0(M)]\right] \geq \frac{M-\Pi}{P} \\
\Leftrightarrow & \ \delta\left[u - \frac{\Pi}{P} + \delta(1-\mu)\frac{M}{P}\right] \geq \frac{M-\Pi}{P}
\end{aligned}
$$

holds. Assumption 1 implies (8), and by $\frac{1-\delta(1-\mu)}{\mu} > 1$ the last inequality in the above follows. Thus $\tilde{F}(M-\Pi) \geq \tilde{F}(0)$ holds.

2. A household without the durable good buys a new unit of the durable good if and only if $p \leq M$. Since $M \geq P$ follows from the premise of the proposition, the monopoly firm chooses $p = M$.

3. Since all households carry over $M - \Pi$ units of money and receive dividend Π, all households always have M units of money at the beginning of each period. Thus the candidate money holdings distribution is stationary.

4. $H = X^* + (M-\Pi)/P = X^* + \frac{(1-\mu)M+\mu P}{P}$ holds if a household does not have the durable good, and $H = X^* - \Pi/P = X^* - \frac{\mu(M-P)}{P}$ holds otherwise. On the other hand, the monopoly firm inputs labor μ. Then the aggregate supply of the divisible good is

$$\mu\left[X^* + \frac{(1-\mu)M+\mu P}{P}\right] + (1-\mu)\left[X^* - \frac{\mu(M-P)}{P}\right] - \mu = X^*.$$

Thus the aggregate supply is equal to the aggregate demand X^*.

5. The money demand is $s + \Pi = M - \Pi + \Pi = M$ and is clearly equal to the exogenously given money supply. Hence, the money market clears. □

Corollary 1 The indeterminacy is real.

Proof. For a household without the durable good, $H = X^* + (M-\Pi)/P$ holds. On the other hand, for a household with the durable good, $H = X^* - \Pi/P$ holds. Thus the labor supply H depends on P. Thus the distribution of utilities varies across P. □

By the quasilinearlity of the utility, it can be shown that the average utility is constant in equilibria. Indeed, if the durable good breaks down, then the utility is equal to

$$u + U(X^*) - \left(X^* + \frac{M-\Pi}{P}\right) = u + U(X^*) - X^* - (1-\mu)\frac{M}{P} - \mu,$$

and otherwise it is equal to

$$u + U(X^*) - \left(X^* - \frac{\Pi}{P}\right) = u + U(X^*) - X^* + \mu\frac{M}{P} - \mu.$$

Hence, the Benthamian welfare, which is clearly equal to the expected value of the above, is as follows:

$$\sum_{t=0}^{\infty}\delta^t\left\{\mu\left[u + U(X^*) - X^* - (1-\mu)\frac{M}{P} - \mu\right] + (1-\mu)\left[u + U(X^*) - X^* + \mu\frac{M}{P} - \mu\right]\right\}$$
$$= \frac{1}{1-\delta}\left[u + U(X^*) - X^* - \mu\right].$$

Clearly, it does not depend on P.

In the following section, we show that not only the distribution of utilities but also the expected value vary across P unless the utility function is quasilinear.

3 Discussion

In this section, we present the conditions for real indeterminacy and show how these conditions lead to the indeterminacy. First, we present a general model that captures the key features of the indeterminacy. We also derive that the welfare changes according to fluctuation in equilibrium prices. Lastly, we end this section with a discussion on the relationship between our result and the existing money search literature on real indeterminacy.

3.1 A General Model

There are two key features in the model in the previous section:

1. Each household optimally saves a constant amount of money in each period no matter what the equilibrium prices are.

2. There is heterogeneity of budget constraints; that is, in each period there exist at least two types of households with different budget constraints.

First, the utility of the durable good is sufficiently high for households to buy it, and the amount of money they save is always equal to the price of the good, i.e., $p = M$. Second, only a fraction μ of the households lose their durable good, and their budget, putting aside the money for the durable good, is less than that of the households who do not loose it. Thus the budget constraints are heterogeneous among households.

Below, we present a general model that captures the above key features, and show that they are sufficient for indeterminacy of stationary equilibria. In the economy, there are N households indexed by i and L divisible goods indexed by j.[6] Each household has a temporal utility function $U : \mathbb{R}_+^L \to \mathbb{R}_+$, and obtains a fixed amount of endowment

[6] Even in the case of a continuum of households, the same result can be obtained.

$\omega^i \in \mathbb{R}^L_{++}$ in each period. There is fiat money with the supply being $M > 0$. We assume that each household has an incentive to save $p > 0$ units of fiat money in each period. An example would be a non-stochastic version of the model in the previous section: an indivisible good is produced by a monopoly firm, each household wants to have one unit of the indivisible good of which price is p, and the consumption is subject to the cash-in-advance constraint. Another example would be a dynamic version of Balasko and Shell (1993)'s model: in each period, the government levies p units of money from each household, and redistributes it to the households as subsidy, where the amounts of subsidy varies across the households. We assume that the divisible goods are traded in Walrasian markets, and that p is determined outside the Walrasian markets. For example, in the case of the indivisible good, it is determined by the monopoly firm, and in the case of tax-subsidy, it is determined by the government. Note that there is no stochastic shock in the economy, and thus the model in the previous section is not a special case of the general model. The case of a general model with stochastic shock will be discussed later.

Let $\delta \in (0, 1)$ be a discount factor. Then household i maximizes the discounted sum of a utility stream as follows:

$$\max \sum_{t=0}^{\infty} \delta^t \left(U(X_t) + u\right)$$

$$\text{s.t.} \quad P_t \cdot (X_t - \omega^i) + m^i_{t+1} + p \leq W^i_t, \quad m^i_{t+1} \geq p,$$

where $X_t = (X_{1t}, \ldots, X_{Lt})$ is the consumption vector, $P_t = (P_{1t}, \ldots, P_{Lt})$ denotes the price vector of the divisible goods at period t, and m^i_{t+1} is the amount of savings. In the case of indivisible good, u is the utility of the indivisible good, and in the case of tax-subsidy, it is zero. W^i_t is the amount of money holdings at the beginning of period t. In the case of indivisible good,

$$W^i_t = m^i_t + \pi^i_t, \tag{9}$$

where π^i_t is the dividend from the firm, and in the case of tax-subsidy,

$$W^i_t = m^i_t + s^i_t, \tag{10}$$

where s^i_t is the subsidy from the government.

Our model clearly has the above key features unless $W^i, i = 1, \ldots, N$, are the same across the households. We first show that $m^i_{t+1} \geq p$ must be binding in stationary equilibria under standard assumptions; that is, all households save the same amount of money p. The intuition is as follows. In our model, money is needed to pay p. Therefore, if a household saves an amount of money larger than p in stationary equilibria, she will not use $m - p$ in the future. We assume U is strictly increasing, strictly concave, and differentiable on $\mathbb{R}^L$. Let λ_t and ϕ_t be the Lagrange multipliers of the first and second inequalities, respectively. The first order condition is as follows:

$$
\begin{aligned}
&\frac{\partial U}{\partial X_{t\ell}} - \lambda_t P_{t\ell} = 0, \quad \forall \ell = 1, \ldots, L, && (11)\\
&-\lambda_t + \phi_t + \delta\lambda_{t+1} = 0, && (12)\\
&\lambda_t [W_t^i - P_t \cdot X_t - m_{t+1}^i - p] = 0, && (13)\\
&\phi_t (m_{t+1}^i - p) = 0, && (14)\\
&\lambda_t \geq 0, \quad \phi_t \geq 0.
\end{aligned}
$$

Below, we investigate the case of stationary equilibria, and thus the subscript t is omitted. Suppose $m > p$. Then $\phi = 0$ follows from (14). Thus, by (12), $\lambda = 0$ holds. Then, by (11), $\partial U/\partial X_\ell = 0$. This contradicts the assumption that U is strictly increasing. Therefore, $m = p$ holds at stationary equilibria.

The following equations follows from the above first order condition:

$$
\begin{aligned}
&\frac{\partial U}{\partial X_\ell} = \frac{P_\ell}{P_1}\frac{\partial U}{\partial X_1}, \quad \forall \ell = 2, \ldots, L, && (15)\\
&P \cdot X = W^i - m - p, && (16)\\
&m = p. && (17)
\end{aligned}
$$

As shown in the above, the money demand $m(P)$ is always equal to p. For a given P, the demand for divisible goods $X^i(P) = (X_1^i(P), \ldots, X_L^i(P))$ is uniquely determined by equations (15) and (16) and the strict concavity of U. Now, we are ready to show that the equilibrium is indeterminate. The market clearing condition of the divisible goods and money are as follows:

$$
\begin{aligned}
&\sum_{i=1}^N X_\ell^i(P) + Z_\ell(P) = \sum_{i=1}^N \omega_\ell^i, \qquad \forall \ell = 1, \ldots, L, && (18)\\
&\sum_{i=1}^N m(P) = M, && (19)
\end{aligned}
$$

where in the case of production economies $Z(P) = (Z_1(P), \ldots, Z_L(P))$ is an input vector for producing N units of the indivisible good. In the case without production, $Z = (0, \ldots, 0)$. The system of equations contains $L + 1$ equations and L unknown variables $(P_1, \ldots, P_L)$. By Walras' law, one equation is redundant. Moreover, as we have shown, $m(P) = p = \frac{M}{N}$ holds for all P, and thus equation (19) is independent from P. Then, there exist $L - 1$ linearly independent equations and L unknowns, and thus the system has one degree of freedom and the equilibrium price vector P is indeterminate.

It is worthwhile to compare our model with the standard dynamic general equilibrium models with cash-in-advance constraints. In such models, the demand for goods $X(P)$ is homogeneous of degree zero and the demand for money $m(P)$ is homogeneous of degree one. By Walras' law, the number of independent equations is equal to that of unknowns. Unlike such models, (19) is not an equation but an identity in our case and thus the number of equations is one less than that of unknowns. Moreover, a proportional change in P affects the equilibrium consumptions, since $X(P)$ is not homogeneous of degree zero. Indeed, since p is a constant in the budget constraint, then $X(P)$ is not homogeneous of degree zero in P.

Even in the case of an economy with stochastic shocks, the equilibrium conditions can be expressed by equations similar to (18) and (19). If $m(P) = p$ holds for all P, then equation (19) is independent from P. In the case of indivisible good, the monopoly firm chooses $p = M$, and thus (19) always holds. In the case of tax-subsidy, (19) holds if the budget of the government is balanced. Thus even in the case of stochastic shocks, the equilibrium price vector P is indeterminate.

Next, we show that the indeterminacy is real; that is, the real allocation changes according to P. This follows from the second key feature, i.e., the heterogeneity of the budget constraints. In stationary equilibria, the budget constraint of household i is

$$P \cdot (X^i - \omega^i) + m^i + p = W^i.$$

By the heterogeneity of W^i, the change in P results in varying effects for the households.

Finally, we show that the social welfare changes according to P. In the durable good model, the social welfare is unchanged because of the quasilinear utility function. In this section, we assume that the utility function is strictly concave. Then the welfare changes according to price fluctuation. For simplicity, we assume no production, $L = 1$, and the initial endowments are the same, i.e., $\omega^i = \omega$ for all $i = 1, \dots, N$.

By the budget constraint, the consumption of household i is

$$X^i = \frac{W^i - 2p}{P} + \omega.$$

Without loss of generality, we assume that $W^1 \leq W^2 \leq \cdots \leq W^N$ and $W^1 < W^N$. Since $\sum_{i=1}^N X^i = N\omega$, there exists some integer $1 \leq n < N$ such that $X^i \leq \omega$ and $W^i - 2p \leq 0$ for $i \leq n$ and $X^i > \omega$ and $W^i - 2p > 0$ otherwise. As we have shown, P is indeterminate. Below, we show that a change in P has asymmetric effects on X^i. This effects together with strict concavity of utility function make the welfare depends on P. Indeed, an increase in P reduces the consumption of household i if $X^i > \omega$, and increases it otherwise.

We define the social welfare as follows:

$$\text{Welfare} = \frac{1}{1-\delta} \sum_{i=1}^N U(X^i) = \frac{1}{1-\delta} \sum_{i=1}^N U\left(\frac{W^i - 2p}{P} + \omega\right).$$

Roughly speaking, an increase in P reduces the inequality of consumptions, and the welfare increases because of the strict concavity of U. Formally,

$$\begin{aligned}
\frac{\partial \text{ Welfare}}{\partial P} &= \frac{1}{1-\delta}\sum_{i=1}^{N}\left[-\frac{W^i-2p}{P^2}U'\left(\frac{W^i-2p}{P}+\omega\right)\right] \\
&= \frac{1}{P^2(1-\delta)}\sum_{i=1}^{n}\left[(2p-W^i)U'\left(X^i\right)\right] - \frac{1}{P^2(1-\delta)}\sum_{i=n+1}^{N}\left[(W^i-2p)U'\left(X^i\right)\right] \\
&> \frac{1}{P^2(1-\delta)}\sum_{i=1}^{n}\left[(2p-W^i)U'\left(X^n\right)\right] - \frac{1}{P^2(1-\delta)}\sum_{i=n+1}^{N}\left[(W^i-2p)U'\left(X^n\right)\right] \\
&= \frac{U'(X^n)}{P(1-\delta)}\left[\sum_{i=1}^{n}\left(\frac{2p-W^i}{P}\right)-\sum_{i=n+1}^{N}\left(\frac{W^i-2p}{P}\right)\right] \\
&= \frac{U'(X^n)}{P(1-\delta)}\sum_{i=1}^{N}(\omega-X^i)=0.
\end{aligned}$$

We can conclude that the welfare is a strictly increasing function of P.

Finally, note that there is room for policy intervention. In search theory models, Kamiya and Shimizu (2007) show that some government policies with unbalanced budgets can lead the economy to an efficient allocation. In this model, similar government intervention can lead the economy to an efficient equilibrium.

3.2 Related Literature

Our work is highly motivated by Jean et al. (2010). Their model is a modified version of Lagos and Wright (2005)'s model, where the market is divided into two submarkets: a decentralized market and a centralized market. In the former market, households are randomly matched pairwise to trade an indivisible good. More precisely, when a single-coincidence-of-wants occurs between a randomly matched pair, the seller posts a take-it-or-leave-it price offer. In the latter market, a divisible good is traded in a Walrasian market. Under a condition that utility from consuming the indivisible good is sufficiently high, they show the existence of a continuum of stationary equilibria, where, in the centralized market, each household saves money in order to buy the indivisible good when she meets a seller in the decentralized market. In equilibria, the amount of saving is always equal to the posted price in the decentralized market. Their model clearly satisfies the key features above. Indeed, each household optimally saves a constant amount of money in the centralized market and there are three types of households with different budget constraints at the beginning of the centralized market: a household who was a buyer, a household who was a seller, and a household who did not trade in the decentralized market.

We would like to emphasize two main findings of our paper. First, the type of real indeterminacy in Jean et al. arises not only in Lagos-Wright type models with an indivisible good but also in more general setups. Apart from the Lagos-Wright framework, we showed real indeterminacy of stationary equilibria in a dynamic general equilibrium model without random matching between buyers and sellers. Second, the type of real indeterminacy shown in Jean et al. is different from what causes real indeterminacy in Green and Zhou (1998) and Kamiya and Shimizu (2006)(2007). The reason for the existence of real indeterminacy in their models is due to indeterminacy of stationary money holdings distribution. This happens because there is an identity in the system of equations of the stationarity of money holdings. The identity reduces the number of linearly independent equations in the system, and this makes the stationary money holdings distribution indeterminate. In contrast, our model has the unique stationary money holdings distribution under some assumption on preferences. We want to emphasize that indeterminacy in our model is due to the fact that the prices of the indivisible good and the divisible goods are determined somewhat separately. Then the equilibrium price of the indivisible good remains fixed while the price of the divisible good fluctuates. This fluctuation in the equilibrium price is the source of real indeterminacy.

This paper can be thought of as an extended version of Balasko and Shell (1993)'s static model in a dynamic setup. In Balasko and Shell, tax is charged at the end of the period and hence all households must prepare the amount needed to pay the tax. Since the tax is a given fixed price and is chosen independently from the household's demand, it follows that the demand for money remains the same despite the small fluctuations in the price of money. Although in our example model we are allowing for the possibility that a household may not consume the indivisible good, it is not optimal to do so given certain conditions on the utility function. Therefore, the model is as if all households are required to consume the good. Their model and our model are the same in a way that all households choose to prepare a constant amount of money at the end of the period as long as the price of money remains in some interval. Since a change in the price of money alters the consumption of divisible goods, an equilibrium utility value from consuming divisible goods changes as the price of money changes. Hence, the welfare differs according to the equilibrium prices that result in real indeterminacy of stationary equilibria.

4 Conclusion

In this paper, we considered a dynamic Walrasian model with divisible fiat money. We found the real indeterminacy of stationary equilibria where the value of money varies across stationary equilibria. When it is optimal to bring over a constant amount of money into the next period and the budget constraints are heterogeneous among agents, then the demand function of the divisible goods violates homogeneity of degree zero. On the other hand, there always exists at least one degree of freedom in characterizing an equilibrium price vector for the divisible good. Thus real indeterminacy of stationary equilibria exists. We showed the above type of real indeterminacy in a realistic model with a durable good and a quasilinear utility. We also considered a model with a general utility function.

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