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-Implications for the Central Clearing-

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Abstract

In this paper, we have studied the pricing of a continuously collateralized CDS. We have made use of the "survival measure" to derive the pricing formula in a straightforward way. As a result, we have found that there exists irremovable trace of the counter party as well as the investor in the price of CDS through their default dependence *even under the perfect collateralization*, although the hazard rates of the two parties are totally absent from the pricing formula. As an important implication, we have also studied the situation where the investor enters an offsetting back-to-back trade with another counter party. We have provided simple numerical examples to demonstrate the change of a fair CDS premium according to the strength of default dependence among the relevant names, and then discussed its possible implications for the risk management of the central counter parties.

1 Introduction

The recent financial crisis, exemplified by the collapse of Lehman Brothers, is the major driver of the increased use of collateral agreements based on CSA (credit support annex published by ISDA), which is now almost a market standard among financial institutions [10]. Coupled with the explosion of various basis spreads, such as Libor-OIS and the cross currency swap (CCS) basis, the effects of collateralization on the derivative pricing, particularly from the view point of funding costs, have become important research topics. Johannes & Sundaresan (2007) [11] have first emphasized the cost of collateral using swap rates in U.S. market. In a series of works Fujii & Takahashi (2009, 2010) [5, 6, 7], we have developed a framework of interest-rate modeling in the presence of collateralization and multiple currencies. We have also pointed out the importance of choice of collateral currency and the embedded cheapest-to-deliver option in collateral agreements [8]. Piterbarg (2010) [12] discussed the general issue of option pricing under collateralization.

This financial crisis has also brought about serious research activity on the credit derivatives and counter party default risk, and large amount of research papers have been published since the crisis. The regulators have also been working hard to establish the new rules for the counter party risk management, and also for the migration toward the CCPs (central counter parties) particularly in the CDS (credit default swap) market. The excellent reviews and collections of recent works are available in the books edited by Lipton & Rennie (2011) [4] and Bielecki, Brigo & Patras (2011) [1], for example. However, the effects of collateralization on credit derivatives remain still largely unclear. It seems partly because that the idea of collateral cost appeared only recently, and also because the detailed collateral modeling is very complicated, due to the existence of settlement lag, threshold, and minimum transfer amount, e.t.c..

In this paper, based on the market development toward more stringent collateral management requiring a daily (or even intra-day) margin call, we have studied the pricing of CDS under the assumption of continuous collateralization. This is expected to be particularly relevant for the CCPs dealing with CDS and other credit linked products, for which the assumption of continuous collateralization seems to be a reasonable proxy of the reality. Although we have studied the similar issues for the standard fixed income derivatives in the previous work [9], the result cannot be directly applied to the credit derivatives since the behavior of hazard rates generally violates the so called "no-jump" condition (e.g. Collin-Dufresne, Goldstein & Hugonnier (2004) [2]) if there exists non-trivial default dependence among the relevant parties. In this work, we apply the technique introduced by Schönbucher (2000) [13] and adopted later by [2] in order to eliminate the necessity of this condition.

As a result, we have obtained a simple pricing formula for the collateralized CDS. We will see, under the perfect collateralization, that the CDS price does not depend on the counter party hazard rates at all as expected. However, very interestingly, there remains irremovable trace of the two counter parties through the default dependence. This gives rise to a very difficult question about the appropriate pricing method for CCPs. A CCP acts as a buyer as well as a seller of a given CDS at the same time, by entering a back-to-back trade between the two financial firms. However, the result tells us that the mark-to-market values of the two offsetting CDSs are not equal in general even at the time of inception, if the CCP adopts the same premium rate for the two firms. Although

the detailed work will be left in a separate paper, we think that the result has important implications for the proper operations of CCPs for credit derivatives.

2 Fundamental Pricing Formula

2.1 Setup

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where Q is a spot martingale measure, and $\mathbb{F} = \{\mathcal{F}_t \, ; \, t \geq 0\}$ is a sub-σ-algebra of $\mathcal{F}$ satisfying the usual conditions. We denote the set of relevant firms $\mathcal{C} = \{0, 1, 2, ..., n\}$ and introduce a strictly positive random variable τ^i in the probability space as the default time of each party $i \in \mathcal{C}$. We define the default indicator process of each party as $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$ and denote by $\mathbb{H}^i$ the filtration generated by this process. We assume that we are given a background filtration $\mathbb{G}$ containing other information except defaults and write $\mathbb{F} = \mathbb{G} \vee \mathbb{H}^0 \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$. Thus, it is clear that τ^i is an $\mathbb{H}^i$ as well as $\mathbb{F}$ stopping time. We assume the existence of non-negative hazard rate process h^i where

$$M_t^i = H_t^i - \int_0^t h_s^i \mathbf{1}_{\{\tau^i > s\}} ds, \quad t \geq 0 \tag{2.1}$$

is an $(Q, \mathbb{F})$-martingale. We also assume that there is no simultaneous default for simplicity.

For collateralization, we assume the same setup adopted in [9] and repeat it here once again for convenience: *Consider a trade between the party 1 and 2. If the party i ($\in \{1, 2\}$) has a negative mark-to-market value, it has to post the cash collateral* [1] *to the counter party j ($\neq i$), where the coverage ratio of the exposure is denoted by $\delta_t^i \in \mathbb{R}_+$. We assume the margin call and settlement occur instantly. Party j is then a collateral receiver and has to pay collateral rate c_t^i on the posted amount of collateral, which is $\delta_t^i \times (|mark\text{-}to\text{-}market|)$, to the party i. This is done continuously until the end of the contract. Following the market conventions, we set the collateral rate c_t^i as the time-t value of overnight (O/N) rate of the collateral currency used by the party i. It is not equal to the risk-free rate r_t, in general, which is necessary to make the system consistent with the cross currency market* [2]. *We denote the recovery rate of the party i by $R_t^i \in [0, 1]$. We assume that all the processes except default times, such as $\{c^i, r, \delta^i, R^i\}$ are adapted to the background filtration $\mathbb{G}$. As for the details of exposure to the counter party and the recovery scheme, see [9].*

2.2 CDS Pricing

We denote the CDS reference name by party-0, the investor by party-1, and the counter party by party-2, respectively. Let us define $\tau = \tau^0 \wedge \tau^1 \wedge \tau^2$ and its corresponding indicator process, $H_t = \mathbf{1}_{\{\tau \leq t\}}$. We assume that the investor is a protection buyer and party-2 is a seller. Under this setup, the CDS price from the view point of the investor can be written

---

[1] According to the ISDA survey [10], more than 80% of collateral being used is cash. If there is a liquid repo or security-lending market, we may also carry out similar formulation with proper adjustments of its funding cost.

[2] See Ref. [8] for details.

as

$$
\begin{aligned}
S_t \;=\; & \beta_t E^Q \Bigg[\int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau>u\}} \Big(dD_u + q(u,S_u)S_u du \Big) \\
& + \int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau \geq u\}} \Big(Z_u^0 dH_u^0 + Z^1(u,S_{u-})dH_u^1 + Z^2(u,S_{u-})dH_u^2 \Big) \Bigg| \mathcal{F}_t \Bigg] \qquad (2.2)
\end{aligned}
$$

where D denotes the cumulative dividend process representing the premium payment for the CDS, and $\beta_t = \exp\left(\int_0^t r_s ds\right)$ denotes the money-market account with the risk-free interest rate. Other variables are defined as follows (See also [9] for details.):

$$
\begin{aligned}
q(t,v) \;=\; & \delta_t^1 y_t^1 \mathbf{1}_{\{v<0\}} + \delta_t^2 y_t^2 \mathbf{1}_{\{v\geq0\}} \\
Z_t^0 \;=\; & (1-R_t^0) \\
Z^1(t,v) \;=\; & \Big(1 - (1-R_t^1)(1-\delta_t^1)^+ \Big) v \mathbf{1}_{\{v<0\}} + \Big(1 + (1-R_t^1)(\delta_t^2-1)^+ \Big) v \mathbf{1}_{\{v\geq0\}} \\
Z^2(t,v) \;=\; & \Big(1 - (1-R_t^2)(1-\delta_t^2)^+ \Big) v \mathbf{1}_{\{v\geq0\}} + \Big(1 + (1-R_t^2)(\delta_t^1-1)^+ \Big) v \mathbf{1}_{\{v<0\}} \; .
\end{aligned}
$$

Here, $y_t^i = r_t^i - c_t^i$ is the difference of the risk-free and collateral rates relevant for the collateral currency chosen by the party-i at the time t, which represents the instantaneous return of the posted collateral. Thus the term $q(t,v)$ summarizes the return (or cost) of collateral from the view point of the investor. Z^i represents the default payoff when party-i defaults first among the set $\{0,1,2\}$.

Although we can follow the same procedures used in the previous work [9] based on the arguments of Duffie & Huang (1996) [3], we need a careful treatment to avoid the jump in the value process at the time of counter party default [3]. In this work, we apply the measure change technique introduced by Schönbucher [13] and used by Collin-Dufresne et.al. (2004) [2], which leads to the following proposition in a clearcut way.

Proposition 1 *Under the assumptions given in 2.1 and appropriate integrability conditions, the pre-default value V_t corresponding to the CDS contract specified in Eq. (2.2) is given by*

$$
V_t = E^{Q'} \left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \mu(u,V_u) + h_u^0) du \right) \Big(dD_s + Z_s^0 h_s^0 ds \Big) \Bigg| \mathcal{F}_t' \right] \qquad (2.3)
$$

where

$$
\begin{aligned}
\mu(u,v) \;=\; & \Big(y_t^1 \delta_t^1 - (1-R_t^1)(1-\delta_t^1)^+ h_t^1 + (1-R_t^2)(\delta_t^1-1)^+ h_t^2 \Big) \mathbf{1}_{\{v<0\}} \\
& + \Big(y_t^2 \delta_t^2 - (1-R_t^2)(1-\delta_t^2)^+ h_t^2 + (1-R_t^1)(\delta_t^2-1)^+ h_t^1 \Big) \mathbf{1}_{\{v\geq0\}} \qquad (2.4)
\end{aligned}
$$

and it satisfies $S_t = \mathbf{1}_{\{\tau>t\}} V_t$ for all $t \geq 0$. Here, the "survival measure" Q' is defined by

$$
\left. \frac{dQ'}{dQ} \right|_{\mathcal{F}_t} = \prod_{i=0}^2 \mathbf{1}_{\{\tau^i>t\}} \exp\left(\int_0^t \sum_{i=0}^2 h_s^i ds \right) \qquad (2.5)
$$

and the filtration $\mathbb{F}' = (\mathcal{F}_t')_{t\geq0}$ denotes the augmentation of $\mathbb{F}$ under Q'.

---

[3] See the remark just after the proposition 1 in [9].

5

Proof: Using the Doob-Meyer decomposition, one obtains

$$
\begin{aligned}
S_t &= \beta_t E^Q \Bigg[\int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \Big(dD_u + q(u, S_u) S_u du \Big) \\
&\quad + \int_t^T \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \Big(Z_u^0 h_u^0 + Z^1(u, S_u) h_u^1 + Z_u^2(u, S_u) h_u^2 \Big) du \, \Big| \, \mathcal{F}_t \Bigg] .
\end{aligned}
\tag{2.6}
$$

Let us define

$$
\eta_t = \frac{dQ'}{dQ}\Big|_{\mathcal{F}_t} = \mathbf{1}_{\{\tau > t\}} \Lambda_t
\tag{2.7}
$$

where we have used

$$
\Lambda_t = \exp\left(\int_0^t \tilde{h}_s ds \right) .
\tag{2.8}
$$

and $\tilde{h}_s = \sum_{i=0}^2 h_s^i$. Then, we can proceed as

$$
\begin{aligned}
S_t &= \beta_t E^Q \Bigg[\int_{]t,T]} (\beta_u \Lambda_u)^{-1} \eta_u \Big(dD_u + q(u, S_u) S_u du \Big) \\
&\quad + \int_t^T (\beta_u \Lambda_u)^{-1} \eta_u \Big(Z_u^0 h_u^0 + Z^1(u, S_u) h_u^1 + Z^2(u, S_u) h_u^2 \Big) du \, \Big| \, \mathcal{F}_t \Bigg] \\
&= \mathbf{1}_{\{\tau > t\}} E^{Q'} \Bigg[\int_{]t,T]} \frac{\beta_t \Lambda_t}{\beta_u \Lambda_u} \Big\{ dD_u + \Big(q(u, S_u) S_u + Z_u^0 h_u^0 + Z^1(u, S_u) h_u^1 + Z^2(u, S_u) h_u^2 \Big) du \Big\} \, \Big| \, \mathcal{F}'_t \Bigg] .
\end{aligned}
$$

Thus, on the set $\{\tau > t\}$, we can write

$$
V_t = E^{Q'} \Bigg[\int_{]t,T]} \frac{\beta_t \Lambda_t}{\beta_u \Lambda_u} \Big\{ dD_u + \Big(q(u, V_u) V_u + Z_u^0 h_u^0 + Z^1(u, V_u) h_u^1 + Z^2(u, V_u) h_u^2 \Big) du \Big\} \, \Big| \, \mathcal{F}'_t \Bigg] .
$$

Simple algebra gives us

$$
V_t = E^{Q'} \Bigg[\int_{]t,T]} e^{-\int_t^s (r_u + \tilde{h}_u) du} \Big\{ dD_s + Z_s^0 h_s^0 ds + \Big(\tilde{h}_s - h_s^0 + \mu(s, V_s) \Big) V_s ds \Big\} \, \Big| \, \mathcal{F}'_t \Bigg] .
$$

Integrating the linear terms gives us the desired result. ∎

3 Financial Implications

Following the approximation method in [9], we can derive price adjustments in various situations. However, in the reminder of the paper, let us concentrate on the simplest but important situation where the CDS is perfectly collateralized by the domestic currency, which should be also the most relevant case for the operations of CCPs. In this case, we have $(\delta^1 = \delta^2 = 1)$ and $(y^1 = y^2 = r - c)$. Thus, using the result of Proposition 1, the pre-default value of the CDS is given by

$$
V_t = E^{Q'} \Bigg[\int_{]t,T]} \exp\left(-\int_t^s (c_u + h_u^0) du \right) \Big(dD_s + Z_s^0 h_s^0 ds \Big) \, \Big| \, \mathcal{F}'_t \Bigg]
\tag{3.1}
$$

6

and one can easily confirm that the hazard rates of the investor 1 as well as the counter party 2 are absent from the pricing formula. Naively, it looks as if we succeed to recover the risk-free situation by the stringent collateral management. However, we will just see that it is quite misleading and dangerous to treat the result of Eq. (3.1) as the usual risk-free pricing formula.

The key point resides in the new measure Q' and the filtration $\mathbb{F}'$. As was emphasized in the works [13, 2], the transformation in Eq. (2.5) puts zero weight on the events where the parties $\{0, 1, 2\}$ default. It can be easily checked as follows: By construction, we know that

$$M_t = H_t - \int_0^t (1 - H_s)\tilde{h}_s ds \tag{3.2}$$

is a $(Q, \mathbb{F})$-martingale. Then, Maruyama-Girsanov's theorem implies that

$$M_t' = M_t - \int_0^t \frac{d\langle M, \eta \rangle_s}{\eta_{s-}} \tag{3.3}$$

should be a $(Q', \mathbb{F}')$-martingale, where $\langle \cdot, \cdot \rangle$ denotes the (conditional or predictable) quadratic covariation. Now, one can easily check that

$$
\begin{aligned}
M_t' &= M_t + \int_0^t (1 - H_s)\tilde{h}_s ds \\
&= H_t
\end{aligned}
\tag{3.4}
$$

and thus $H_t = \mathbf{1}_{\{\tau \le t\}}$ itself becomes a $(Q', \mathbb{F}')$-martingale. In other words, under the new measure, the parties $\{0, 1, 2\}$ do not default almost surely. Since the density process in (2.7) has finite variation, there is no change of dynamics in the continuous part.

Let us consider the financial implications of this fact. By our construction of filtration, $(Q, \mathbb{F})$ hazard rate process of party i can be written in the following form in general:

$$h_t^i = \sum_{\{D \in \Pi; \ i \notin D\}} \prod_{j \in D} \mathbf{1}_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C} \backslash D} \mathbf{1}_{\{\tau^k > t\}} h_D^i(t) \tag{3.5}$$

where Π denotes the set of all the subgroups of $\mathcal{C} = \{0, 1, 2, \cdots, n\}$ including also the empty set. Let us define $\mathbb{H}^D = \prod_{j \in D} \vee \mathbb{H}^k$, then $h_D^i(t)$ is adapted to the filtration $\mathbb{G} \vee \mathbb{H}^D$, and hence includes the information of default times of the parties included in the set D although they are not explicitly shown in the formula. Importantly, it is not the hazard rate (or default intensity) under the new measure $(Q', \mathbb{F}')$.

In the new measure, the parties included in $\mathcal{S} = \{0, 1, 2\}$ are almost surely alive for all $t \ge 0$, and hence h_t^i given in Eq. (3.5) is equivalent to $h_t'^i$ given below:

$$h_t'^i = \sum_{\{D \in \Pi'; \ i \notin D\}} \prod_{j \in D} \mathbf{1}_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C}' \backslash D} \mathbf{1}_{\{\tau^k > t\}} h_D^i(t) \tag{3.6}$$

where $\mathcal{C}' = \mathcal{C} \backslash \mathcal{S}$, and Π' denotes the set of all the subgroups of $\mathcal{C}'$ and the empty set. Thus, the pre-default value of the CDS can also be written as

$$V_t = E^{Q'} \left[\int_{]t,T]} \exp\left(-\int_t^s (c_u + h_u'^0) du \right) \left(dD_s + Z_s^0 h_s'^0 ds \right) \bigg| \mathcal{F}_t' \right]. \tag{3.7}$$

Now, let us consider the difference from the situation where the same CDS is being traded between the two completely default-free parties. In this case, the pre-default value of the CDS is given by

$$V_t^{\mathrm{rf}} = E^Q \left[\int_{]t,T]} \exp\left(-\int_t^s (c_u + h_u^0) du \right) \left(dD_s + Z_s^0 h_s^0 ds \right) \middle| \mathcal{F}_t \right] . \qquad (3.8)$$

Here, we have assumed that the collateralization is still being carried out. In the reminder of the paper, we study the important difference

$$V_t^{\mathrm{rf}} - V_t \qquad (3.9)$$

in details.

Here, let us briefly discuss the cause of the above difference intuitively. As one can see from (3.5), the hazard rate in the full filtration $\mathbb{F}$ jumps at the default of any firm if there is non-zero default dependence with the reference firm [4]. Suppose the firm A has positive dependence to the reference name. Then, the value of CDS protection bought from the default-free firm jumps upward when A defaults and the V_0^{rf} contains the discounted contribution from this scenario. Now, suppose the situation where we bought a protection from the firm A. In this case, the contribution from the future scenarios where A defaults before the contract expires is smaller than that in the previous situation. This is because the collateral posted by the firm A cannot cover the upward jump of CDS at the time of default. The similar effect also arises from the investor himself. As a result, if there is positive (or contagious) default dependence among the participants and the reference name, the CDS value is expected to be smaller than V^{rf}. Although it is difficult to calculate this difference in generic setup, we will provide the explicit formula in the two simplified but important situations in the following sections.

4   Special Cases

We first list up the two special cases which provide us better understanding. Especially, the second example can be applied to the back-to-back trade, which is the most relevant situation for the CCPs.

4.1   3-party Case

The simplest situation to calculate the collateralized CDS is the case where there are only three relevant names, $\mathcal{C} = \{0, 1, 2\}$, which are the reference entity, the investor and the counter party, respectively. In this case, since $\mathcal{S} = \mathcal{C}$, the set Π' contains only the empty set $\{\emptyset\}$. Thus, under the survival measure $(Q', \mathbb{F}')$, we have

$$h_t'^i = h_{\{\emptyset\}}^i(t) \qquad (4.1)$$

and particularly,

$$h_t'^0 = h_{\{\emptyset\}}^0(t) . \qquad (4.2)$$

Since we know that $h_{\{\emptyset\}}^i$ is adapted to the background filtration $\mathbb{G}$, the evaluation of CDS value is quite straightforward in this case.

---

[4]See the discussion in Sec. 5.

4.2 4-party Case

Now, let us add one more party and consider the 4-party case, $\mathcal{C} = \{0, 1, 2, 3\}$. We first consider the trade of a CDS between the investor (party-1) and the counter party (party-2). The reference entity is party-0. In this case, we have $\mathcal{C}' = \{3\}$. Therefore, the relevant processes under the survival measure are

$$h_t'^0 = \mathbf{1}_{\{\tau^3 > t\}} h_{\{\emptyset\}}^0(t) + \mathbf{1}_{\{\tau^3 \leq t\}} h_{\{3\}}^0(t, \tau^3) \tag{4.3}$$

$$h_t'^3 = \mathbf{1}_{\{\tau^3 > t\}} h_{\{\emptyset\}}^3(t) . \tag{4.4}$$

Here, we have made the dependence on the default time in $h_{\{3\}}^0$ explicitly. Since the feedback effect only appears through the party 3 in the survival measure, we can proceed in similar fashion done in Ref. [2].

In this case, the perfectly collateralized CDS pre-default value turns out to be

$$
\begin{aligned}
V_t &= \mathbf{1}_{\{\tau^3 \leq t\}} E^{Q'} \left[\int_{]t,T]} e^{-\int_t^s \left(c_u + h_{\{3\}}^0(u, \tau^3) \right) du} \left(dD_s + Z_s^0 h_{\{3\}}^0(s, \tau^3) ds \right) \middle| \mathcal{F}_t' \right] \\
&+ \mathbf{1}_{\{\tau^3 > t\}} \left\{ E^{Q'} \left[\int_{]t,T]} e^{-\int_t^s \left(c_u + h_{\{\emptyset\}}^0(u) \right) du} \left(e^{-\int_t^s h_{\{\emptyset\}}^3(u) du} \right) \left(dD_s + Z_s^0 h_{\{\emptyset\}}^0(s) ds \right) \middle| \mathcal{F}_t' \right] \right. \\
&+ E^{Q'} \left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h_{\{\emptyset\}}^0(u) + \int_v^s h_{\{3\}}^0(u,v) \right) du} \left[e^{-\int_t^v h_{\{\emptyset\}}^3(x) dx} h_{\{\emptyset\}}^3(v) \right] dv \right) dD_s \middle| \mathcal{F}_t' \right] \\
&+ E^{Q'} \left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h_{\{\emptyset\}}^0(u) + \int_v^s h_{\{3\}}^0(u,v) \right) du} \left[e^{-\int_t^v h_{\{\emptyset\}}^3(x) dx} h_{\{\emptyset\}}^3(v) \right] h_{\{3\}}^0(s, v) dv \right) Z_s^0 ds \middle| \mathcal{F}_t' \right] \right\} .
\end{aligned}
\tag{4.5}
$$

Here, the important point is not the possible contagious effect from the party-3, but rather the *lack* of the contagious effects from the other names included in the set $\mathcal{S}$, which are included in the marginal default intensity of the reference entity.

Now, because of the symmetry, if the investor enters a back-to-back trade with the counter party 3, the pre-default value of this offsetting contract is given as follows:

$$
\begin{aligned}
V_t^{B2B} &= -\mathbf{1}_{\{\tau^2 \leq t\}} E^{Q''} \left[\int_{]t,T]} e^{-\int_t^s \left(c_u + h_{\{2\}}^0(u, \tau^2) \right) du} \left(dD_s + Z_s^0 h_{\{2\}}^0(s, \tau^2) ds \right) \middle| \mathcal{F}_t'' \right] \\
&- \mathbf{1}_{\{\tau^2 > t\}} \left\{ E^{Q''} \left[\int_{]t,T]} e^{-\int_t^s \left(c_u + h_{\{\emptyset\}}^0(u) \right) du} \left(e^{-\int_t^s h_{\{\emptyset\}}^2(u) du} \right) \left(dD_s + Z_s^0 h_{\{\emptyset\}}^0(s) ds \right) \middle| \mathcal{F}_t'' \right] \right. \\
&+ E^{Q''} \left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h_{\{\emptyset\}}^0(u) + \int_v^s h_{\{2\}}^0(u,v) \right) du} \left[e^{-\int_t^v h_{\{\emptyset\}}^2(x) dx} h_{\{\emptyset\}}^2(v) \right] dv \right) dD_s \middle| \mathcal{F}_t'' \right] \\
&+ E^{Q''} \left[\int_{]t,T]} e^{-\int_t^s c_u du} \left(\int_t^s e^{-\left(\int_t^v h_{\{\emptyset\}}^0(u) + \int_v^s h_{\{2\}}^0(u,v) \right) du} \left[e^{-\int_t^v h_{\{\emptyset\}}^2(x) dx} h_{\{\emptyset\}}^2(v) \right] h_{\{2\}}^0(s, v) dv \right) Z_s^0 ds \middle| \mathcal{F}_t'' \right] \right\} .
\end{aligned}
\tag{4.6}
$$

9

Here, Q'' is defined according to the new survival set $\mathcal{S}^{B2B} = \{0, 1, 3\}$ by

$$\left.\frac{dQ''}{dQ}\right|_{\mathcal{F}_t} = \prod_{i \in \mathcal{S}^{B2B}} \mathbf{1}_{\{\tau^i > t\}} \exp\left(\int_0^t \sum_{i \in \mathcal{S}^{B2B}} h_s^i ds\right) \tag{4.7}$$

and the filtration $\mathbb{F}'' = (\mathcal{F}_t'')_{t \geq 0}$ denotes the augmentation of $\mathbb{F}$ under Q''.

Now let us consider $V_0 + V_0^{B2B}$. It is easy to check that it is not zero in general and does depend on the default intensities of party-2 and -3, and also their contagious effects to the reference entity. Suppose that the investor is a CCP just entered into the back-to-back trade with the party-2 and -3 who have the same marginal default intensities. Even under the perfect collateralization, if the CCP applies the same CDS price (or premium) to the two parties, it has, in general, the mark-to-market loss or profit even at the inception of the contract. For example, consider the case where the protection seller (party-2) has very high default dependence with the reference entity, while the buyer (party-3) of the protection from the CCP has smaller one. In this case, the h^0 in $(Q', \mathbb{F}')$ should be smaller than that of $(Q'', \mathbb{F}'')$. If the CCP uses the same CDS premium to the two parties, CCP should properly recognizes the loss, which stems from the difference of the contagion size from the default of the two parties. Since the party 2 has high default dependence, the short protection position of the CCP with the party 3 suffers bigger loss at the time of the default of the party 2.

5 Examples using a Copula

In order to separate the marginal intensity and default dependence, we will adopt here the copula framework. After explaining the general setup, we will apply Clayton copula to demonstrate quantitative impact.

5.1 Framework

Suppose that we are given a non-negative process λ^i adapted to the background filtration $\mathbb{G}$ for each party $i \in \mathcal{C} = \{0, 1, 2, \cdots, n\}$. Suppose also that there exists a uniformly distributed random variable $U^i \in [0, 1]$. We assume that, under $(Q, \mathcal{F}_0)$, the $(n+1)$-dimensional random vector

$$\vec{U} = (U^0, U^1, \cdots, U^n) \tag{5.1}$$

is distributed according to the $(n+1)$-dimensional copula

$$C(\vec{u}) . \tag{5.2}$$

We further assume that $\vec{U}$ is independent from $\mathcal{G}_\infty$ and also that copula function C is $(n+1)$-times continuously differentiable.

Now, let us define the default time τ^i as

$$\tau^i = \inf\left\{t; \ e^{-\int_0^t \lambda_s^i ds} \leq U^i\right\} . \tag{5.3}$$

Then, given the information $\mathcal{G}_\infty$, one obtains the joint default distribution as

$$Q\left(\tau^0 > T^0, \tau^1 > T^1, \cdots, \tau^n > T^n \middle| \mathcal{G}_\infty\right) = C(\vec{\gamma}(\vec{T})) \tag{5.4}$$

where we have used the notation of

$$C\big(\vec{\gamma}(\vec{T})\big) = C\big(\gamma^0(T^0), \cdots, \gamma^n(T^n)\big) \tag{5.5}$$

and

$$\gamma^i(T) = \exp\left(-\int_0^T \lambda_s^i ds\right) . \tag{5.6}$$

Following the well known procedures [5], one obtains

$$
\begin{aligned}
Q^i(t,T) &= Q\big(\mathbf{1}_{\{\tau^i > T^i\}}\big| \mathcal{F}_t\big) \\
&= \sum_{\{D \in \Pi; \, i \notin D\}} \prod_{j \in D} \mathbf{1}_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C}\backslash D} \mathbf{1}_{\{\tau^k > t\}} \left. \frac{E^Q\left[\vec{\partial}_D C\big(\gamma^i(T^i), \vec{\gamma}^{\mathcal{C}\backslash\{D,i\}}(t), \vec{\gamma}^D(\vec{t^D})\big)\big| \mathcal{G}_t\right]}{\vec{\partial}_D C\big(\vec{\gamma}^{\mathcal{C}\backslash D}(t), \vec{\gamma}^D(\vec{t^D})\big)}\right|_{\vec{t^D} = \vec{\tau}^D} .
\end{aligned}
\tag{5.7}
$$

Here, we have defined Π as the set containing all the subgroups of $\mathcal{C}$ with the empty set, and

$$\vec{\partial}_D = \prod_{i \in D} \frac{\partial}{\partial u^i} . \tag{5.8}$$

$\vec{\gamma}^D(\vec{\tau^D})$ is the set of $\gamma^i(\tau^i)$ for all $i \in D$, and similarly for $\vec{\gamma}^{\mathcal{C}\backslash D}$. In the expression of Eq. (5.7), we have not properly ordered the arguments of the copula function just for simplicity, which should be understood in the appropriate way.

$(Q, \mathbb{F})$ hazard rate of party-i is calculated as

$$
\begin{aligned}
h_t^i &= \left. -\frac{\partial}{\partial T} \ln Q^i(t,T)\right|_{T=t} \\
&= \lambda_t^i \gamma^i(t) \sum_{\{D \in \Pi; \, i \notin D\}} \prod_{j \in D} \mathbf{1}_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C}\backslash D} \mathbf{1}_{\{\tau^k > t\}} \frac{\partial_i \vec{\partial}_D C\big(\vec{\gamma}^{\mathcal{C}\backslash D}(t), \vec{\gamma}^D(\vec{\tau}^D)\big)}{\vec{\partial}_D C\big(\vec{\gamma}^{\mathcal{C}\backslash D}(t), \vec{\gamma}^D(\vec{\tau}^D)\big)} .
\end{aligned}
\tag{5.9}
$$

Hence, as we have done in Eq (3.6) for the survival set $\mathcal{S} = \{0, 1, 2\}$, h_t^i can be replaced as follows under the new measure $(Q', \mathbb{F}')$:

$$h_t'^i = \lambda_t^i \gamma^i(t) \sum_{\{D \in \Pi'; \, i \notin D\}} \prod_{j \in D} \mathbf{1}_{\{\tau^j \le t\}} \prod_{k \in \mathcal{C}'\backslash D} \mathbf{1}_{\{\tau^k > t\}} \frac{\partial_i \vec{\partial}_D C\big(\vec{\gamma}^{\mathcal{C}\backslash D}(t), \vec{\gamma}^D(\vec{\tau}^D)\big)}{\vec{\partial}_D C\big(\vec{\gamma}^{\mathcal{C}\backslash D}(t), \vec{\gamma}^D(\vec{\tau}^D)\big)} \tag{5.10}$$

where $\mathcal{C}' = \mathcal{C}\backslash\mathcal{S}$ and Π' is the set of all the subgroups of $\mathcal{C}'$ with an additional empty set. One can observe that both h^i and h'^i are equal to the marginal intensity λ^i if there is no default dependence (or in the case of the product copula).

Now, let us apply the copula framework to the two special cases given in Sec. 4. (3-party Case): In the 3-party case with $\mathcal{C} = \{0, 1, 2\}$, we have

$$h_t'^0 = \lambda_t^0 \gamma^0(t) \frac{\partial_0 C\big(\vec{\gamma}(t)\big)}{C\big(\vec{\gamma}(t)\big)} \tag{5.11}$$

---

[5]See, the works[14, 15], for example.

where $\vec{\gamma}(t) = (\gamma^0(t), \gamma^1(t), \gamma^2(t))$.

(4-party Case): In the 4-party case with $\mathcal{C} = \{0, 1, 2, 3\}$, we have

$$h_t'^0 = \mathbf{1}_{\{\tau^3 > t\}} \frac{\partial_0 C(\vec{\gamma}(t))}{C(\vec{\gamma}(t))} + \mathbf{1}_{\{\tau^3 \leq t\}} \frac{\partial_0 \partial_3 C(\vec{\gamma}^{\mathcal{C} \setminus \{3\}}(t), \gamma^3(\tau^3))}{\partial_3 C(\vec{\gamma}^{\mathcal{C} \setminus \{3\}}(t), \gamma^3(\tau^3))} \tag{5.12}$$

$$h_t'^3 = \mathbf{1}_{\{\tau^3 > t\}} \frac{\partial_3 C(\vec{\gamma}(t))}{C(\vec{\gamma}(t))} \tag{5.13}$$

under $(Q', \mathbb{F}')$, and

$$h_t''^0 = \mathbf{1}_{\{\tau^2 > t\}} \frac{\partial_0 C(\vec{\gamma}(t))}{C(\vec{\gamma}(t))} + \mathbf{1}_{\{\tau^2 \leq t\}} \frac{\partial_0 \partial_2 C(\vec{\gamma}^{\mathcal{C} \setminus \{2\}}(t), \gamma^2(\tau^2))}{\partial_2 C(\vec{\gamma}^{\mathcal{C} \setminus \{2\}}(t), \gamma^2(\tau^2))} \tag{5.14}$$

$$h_t''^2 = \mathbf{1}_{\{\tau^2 > t\}} \frac{\partial_2 C(\vec{\gamma}(t))}{C(\vec{\gamma}(t))} \tag{5.15}$$

under $(Q'', \mathbb{F}'')$.

5.2 Numerical Examples

In this paper, we adopt Clayton copula just for its analytical tractability and easy interpretation of its parameter [14]. Clayton copula belongs to Archimedean copula family, whose general form is given by

$$C(\vec{u}) = \phi^{[-1]} \left(\sum_{i=0}^{n} \phi(u^i) \right) \tag{5.16}$$

where the function $\phi(\cdot)$ is called the generator of the copula, and $\phi^{[-1]}(\cdot)$ is its pseudo-inverse function. For Clayton copula, the generator function is given by

$$\phi(u) = (u^{-\alpha} - 1)/\alpha \tag{5.17}$$

for $\alpha > 0$, and hence we have

$$C(\vec{u}) = \left(-n + \sum_{i=0}^{n} (u^i)^{-\alpha} \right)^{-1/\alpha} . \tag{5.18}$$

For this copula, one can easily check that

$$\gamma^i \frac{\partial_i C(\vec{\gamma})}{C(\vec{\gamma})} = \left(\frac{C(\vec{\gamma})}{\gamma^i} \right)^\alpha$$

$$\gamma^i \frac{\partial_i \partial_j C(\vec{\gamma})}{\partial_j C(\vec{\gamma})} = (1 + \alpha) \left(\frac{C(\vec{\gamma})}{\gamma^i} \right)^\alpha \tag{5.19}$$

which means that the hazard rate jumps to $(1+\alpha)$ times of its value just before the default of any other party.
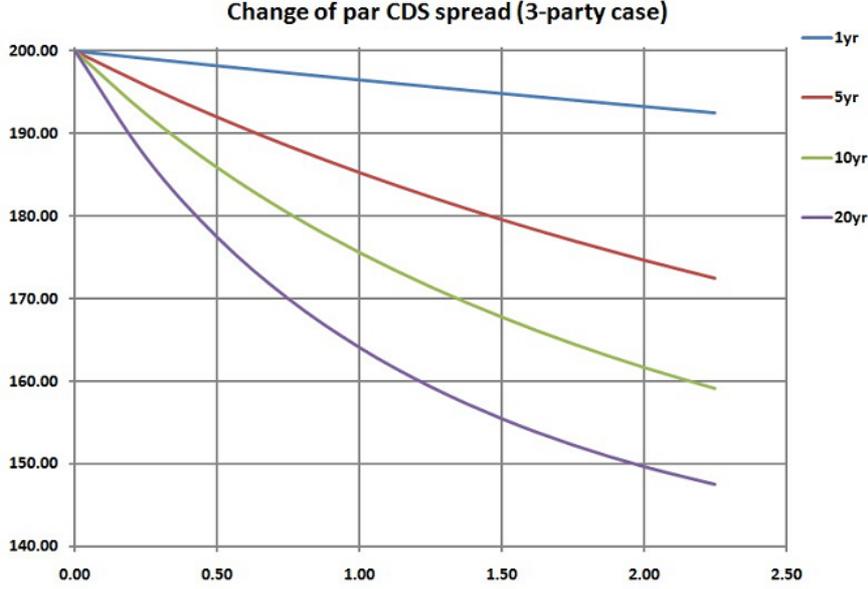
Figure 1: Change of par CDS spread with Clayton copula parameter α.

In the following two figures, we have shown the numerical examples of the par premium of the perfectly collateralized CDS under Clayton copula. For simplicity, we have assumed continuous payment of the premium, and also assumed that the recovery rate $R^i = 40\%$, the collateral rate $c = 0.02$, and the marginal intensity λ^i of each name is constant.

In Fig. 1, we have shown the results of 3-party case for the set of maturities; 1yr, 5yr, 10yr and 20yr. Here, the effective marginal intensity of each party $\overline{\lambda}^i = (1 - R^i)\lambda^i$ is given as follows:

$$\left(\overline{\lambda}^0, \overline{\lambda}^1, \overline{\lambda}^2\right) = \left(200\text{bp}, \ 100\text{bp}, \ 120\text{bp}\right) .$$

The horizontal axis denotes the value of the copula parameter α. Considering the situation where major financial institutions are involved, and recalling the events just after the Lehman collapse, the significant jump of hazard rates seems possible. One can see that there is meaningful deviation of the par CDS premium from the marginal intensity $\overline{\lambda}^0$ even within the reasonable range of the jump size, or α.

If Fig. 2, we have shown the corresponding results for the 4-party case. Here, we have set the effective marginal intensities as

$$\left(\overline{\lambda}^0, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\lambda}^3\right) = \left(200\text{bp}, \ 30\text{bp}, \ 150\text{bp}, \ 75\text{bp}\right) .$$

In this case, we have modeled the situation where the investor has very high credit quality, which enters back-to-back trades with the two firms that have quite different credit worthiness. In the figure, we have used the solid lines for the trade with party-2, and the dashed lines for the offsetting trade with party-3. The result tells us that the back-to-back trades have non-zero mark-to-market value if the investor applies the same premium. This fact is very important for a CCP. It tells us that, even under the very stringent collateral management, the CCP has to recognize that it is not free from the "risk". It is, in fact, free from the credit risk of the counter party, but still suffers from the contagious effects from the defaults of its counter parties.
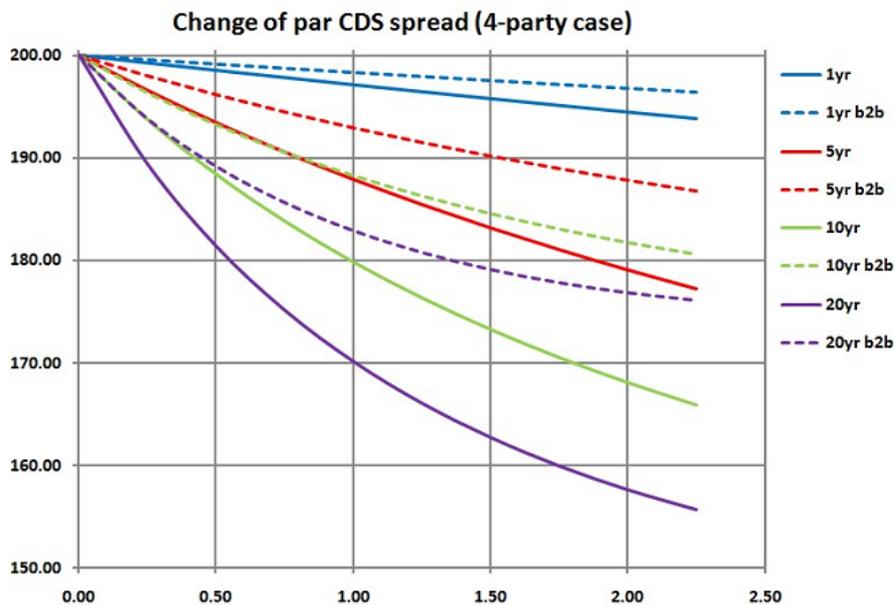
13

Figure 2: Change of par CDS spread with Clayton copula parameter α.

6 Conclusions

In this paper, we have studied the pricing of CDS under continuous collateralization. We have made use of the "survival measure" to avoid the "no-jump" assumption required in the previous work [9]. It allows us straightforward derivation of pricing formula of CDS.

In the main body of the paper, we have focused on the situation where the CDS is perfectly collateralized. We have shown that there exists irremovable trace of the two participating firms in the CDS price through their default dependence with the reference name. For numerical examples, we have adopted Clayton copula to show the change of the par CDS premium according to the dependence parameter. The results have shown that there exists significant deviation of the par premium from the marginal intensity of the reference entity when the default dependence is high.

We should emphasize that these numerical calculations are solely for demonstrative purpose. Because of the simplicity of the chosen copula, there is no way to assign different dependence among the relevant names. The detailed analysis with dynamic underlying processes, and more realistic dependence structure is left for our future research. However, we still think that our results are enough to demonstrate the importance of default dependence or contagious effects among the relevant names even under the very stringent collateral management. This fact seems particularly important for the CCPs, where the members are usually major broker-dealers that are expected to have very significant impacts on all the other names just we have experienced in this crisis.

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