# Non-Self-Averaging in Macroeconomic Models: A Criticism of Modern Micro-founded Macroeconomics

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**Abstract**

When the coefficient of variation, namely the ratio of the standard deviation over the mean approaches zero as the number of economic agents becomes large, the system is called self-averaging. Otherwise, it is non-self-averaging. Most economic models take it for granted that economic system is self-averaging. However, they are based on extremely unrealistic assumptions that all the economic agents face the same probability distribution, and that micro shocks are independent. Once these unrealistic assumptions are dropped, non-self-averaging behavior naturally emerges. Using a simple stochastic growth model, this paper demonstrates that the coefficient of variation of aggregate output or GDP does not go to zero even if the number of sectors or economic agents goes to infinity. Non-self-averaging phenomena imply that even if the number of economic agents is large, dispersion could remain significant, and we can not legitimately focus on the means of aggregate variables. It, in turn, means that the standard microeconomic foundations based on representative agents have little value for they are meant to provide

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us with accurate dynamics of the means of aggregate variables. Contrary to the main stream view, micro-founded macroeconomics such as a dynamic general equilibrium model does not provide solid micro foundations.

**Key Words**: Macroeconomics, Microeconomic foudnations, Non-self averaging, Ewens distribution.

# 1 Introduction

Modern *micro-founded macroeconomics* is built on optimization of representative economic agent such as consumer and firm. The contribution of endogenous growth ranging from Romer (1986) to Grossman and Helpman (1991), and Aghion and Howitt (1992), was to *endogenize* the underlying sources of sustained growth in per-capita income. The main analytical exercises in these papers are to consider explicitly the optimization by representative agents in such activities as education, on-the-job training, basic scientific research, and process/product innovations. This approach is not confined to the study of economic growth, of course. It actually originated in the theory of business cycles. Arguably, the rational expectations model by Lucas (1972, 73) opened the door to modern micro-founded macroeconomic theory. In the field of the theory of business cycles, it is now represented by the real business cycle theory (Kydland and Prescott (1982)):

> "Real business cycle models view aggregate economic variables as the outcomes of the decisions made by many individual agents acting to maximize their utility subject to production possibilities and resource constraints. As such, the models have an explicit and firm foundation in microeconomics. (Plosser, 1989, p.53)."

This is the basic tenor which applies not only to the theory of business cycles, but also to the endogenous growth literature, or for that matter to the whole macroeconomic theory. Lucas (1987) made the following declaration against the old macroeconomics.

> "The most interesting recent developments in macroeconomic theory seem to me describable as the reincorporation of aggregative problems such as inflation and the business cycle within the general framework of "microeconomic" theory. If these developments succeed, the term 'macroeconomic' will simply disappear from use and the modifier 'micro' will become superfluous. We will simply speak, as did Smith, Ricardo, Marshall and Walras, of *economic* theory (Lucas, (1987; p.107-108))."

There are, of course, economists who are skeptical of this modern micro-founded macroeconomics. Kirman (1992) is a seminal work in which he advanced a forceful criticism of the representative agent model (See also Hartley (1997)). Turnovsky (1995) in his basically standard textbook makes the following remarks.

> "Any model employed as widely as the representative agent model begins to take on a life of its own and to be accepted almost as an axiom. It is therefore useful to remind ourselves

> periodically of its limitations. ··· It [the representative agent framfork] should be viewed as a step in the continuing development and understanding of macroeconomic theory, ··· Over time, models become superseded, and indeed the extension to heterogeneous agents seems like a promissing avenue for future research. (Turnovsky, (1995; p.275))."

More recently, Solow (2004), in his article entitled "the Tobin approach to monetary economics," states as follows:

> "The other big difference you will notice between Tobin's approach and today's fashion is the absence of a representative agent. That personage is not needed with the common-sense approach to microfoundations. One can take it for granted that agents are heterogeneous, because they are. They differ in their preferences, their expectations, their access to information, their beliefs about the way the economy works, and their notions of "proper" behavior in the economic sphere. ···
>
> It would cut no ice with Tobin or with me to say that the only respectable microfoundations are those deduced from a model with agents who optimize fully, conditional on the usual things. That is just jive talk unless you can make a good case that real agents can carry out a decent approximation to the suggested optimization, and that the agents in the optimizimg model have been endowed with preferences, information, beliefs, and physical constraints that could reasonably be imputed to real consumers, workers, and managers. And on top of it all, some reason would have to be given why a reasonable person should believe that a model with one agent or identical agents could possibly give a decent representation of a world in which agents differ among themselves in all those ways I just talked about, not to mention that some of them are big and others small. (An argument about robustness would be acceptable but I don't recall one ever being made and such an argument would be intrinsically very unlikely to succeed if it were tried.) ···
>
> In short, it is not the general appeal to "microfoundations" that Tobin would have rejected in 1968 or 2002; it is rather the extraordinarily limiting and implausible microfoundations that the literature seems to be willing to accept. One could even question whether a representative-agent model qualifies as microfoundation at all. (Solow, 2004, pp.659-660)."

In this paper, we formalize Solow's criticism with the help of the concept of *non-self-averaging*, and argue that the research program based on representative agents which prevails in modern macroeconomics is misguided. Whether in growth or business cycle models, the fundamental cause for often complex optimization exercises is that they are expected to lead us to our better understanding dynamics of the *means* of *aggregate* stochastic variables. The standard model thus begins with optimization of representative agent, and translates it homothetically into the analysis of the economy as a whole.

Economists doing these exercises are, of course, well aware that economic agents differ, and that they are subject to idiosyncratic (or microeconomic) shocks. As is well known, micro/idiosyncratic shocks are indeed the key factor in Lucas (1972, 73)'s theory of business cycles which originates in the Phelps (1970) "island model". Unlike real business cycle (RBC) model which makes crude representative agent assumption, some of the modern micro-founded macro models apparently presume heterogeneous agents. However, these analyses are based on the extremely unrealistic assumptions that all the economic agents share the same unchanged probability distribution, and that those microeconomic shocks and differences cancel out each other. These assumptions entail that the behaviors of aggregate variables are represented by their means which, in turn, can be well captured by the analysis based on a representative agent. The same assumption is commomly made in the standard labor search models such as Mortensen and Pissarides (1994), Lucas and Prescott (1994).

Though apparently heterogeneous agents, firms, or sectors are introduced in these models, they are in essence *homogenous* in the sense that they face the *same unchanged* probability that an "event" occurs to them. Microsoft and small grocery store on the street face "idyosyncratic" or micro shocks which come from the *same* probability distribution! We maintain that this standard assumption does not correctly describe true micro or idyosyncratic shocks in the real economy.

In the next section, we first explain the concept of *non-self-averaging*, the crucial concept for our purpose in the present paper. After discussing the fundamental problem of the standard micro-founded models in Section 3, Section 4 demonstrates that non-self-averaging emerges very naturally in a simple model of economic growth.

In non-self-averaging models, even if the number of economic agents is large, the behavior of the macroeconomy can not be generally well approximated by the means. The implication is that analyses based on representative agent which generate the means of stochastic time paths of aggregate variables, have little value. Put it another way, the standard mico-foundations are not true micro-foundations. The final section offers concluding remarks.

# 2 Non-self-averaging

In this section, we explain the concept of "non-self-averaging", the crucial concept for our present purpose. It is not known in economics. However, the term "non-self-averaging" is extensively used in the physics literature (see

Sornette (2000, p.369)). Kadanoff (2000), for example, states as follows:

> "In statistical physics we distinguish between two kinds of statistical behavior produced by events with many steps and parts. The simpler kind of behavior is described by the phrase *self-averaging.* A self-averaging behavior is one in which the effects of the individual events add up to produce an almost deterministic outcome. An example of this is the pressure on a macroscopic surface. This pressure is produced by huge numbers of individual collisions, which produce a momentum transfer per unit time which seems essentially without fluctuations. The larger the number of collisions, the less is the uncertainty in the pressure.
>
> In contrast multiplicative random processes all have a second behavior called *non-self-averaging.* When different markets or different securities are described by any kind of multiplicative random process, as we have described in this section, then the different securities can have huge and (mostly) unpredictable price swings. The larger the number of steps in the process, the more the uncertainty in price. There are other examples of measurements which do not self-average. For example the electrical resistance of a disordered quantum system at low temperature is determined in awful detail by the position of each atom. Change one atomic position and you might change the resistance by a factor of two. Here too the large number of individual units does not guarantee a certainly defined output(Kadanoff (2000, p.85-86)."

We consider the coefficient of variation *(C.V.)* of a size-dependent (*i.e.* "extensive" in physics) random variable, $X_n$ defined by

$$C.V.(X_n) = \frac{\sqrt{variance(X_n)}}{mean(X_n)} = \frac{\sqrt{E(X_n - E(X_n))^2}}{E(X_n)} \tag{1}$$

where $n$ refers to the size of system such as the number of economic agents. If $E(X_n)$ does not vanish, and $C.V.(X_n)$ converges to zero as model size $n$ goes to infinity, we say that the system is *self-averaging.* Otherwise, it is *non-self-averaging.*

Because we have

$$C.V.(X_n) = \sqrt{\frac{E(X_n^2)}{(E(X_n))^2} - 1}, \tag{2}$$

the coefficient of variation goes to zero as $n \to \infty$ if and only if

$$\frac{E(X_n^2)}{(E(X_n))^2} \to 1, \qquad n \to \infty. \tag{3}$$

This is equivalent to

$$E[(\frac{X_n}{(E(X_n)} - 1)^2] \to 0, \qquad n \to \infty. \tag{4}$$

Hence, $X_n/E(X_n)$ converges to 1 in probability:

$$P_r((1-\epsilon)E(X_n) < X_n < (1+\epsilon)E(X_n)) \to 1 \quad for\ any\ \epsilon > 0, \qquad n \to \infty. \tag{5}$$

Thus, as Kadanoff says, if the system is self-averaging, "the effects of the individual events add up to produce an almost deterministic outcome." In other words, in the case of self-averaging, we can replace stochastic time evolution by deterministic time evolution (Darling and Norris (2008)), and legitimately focus on the means to understand behavior of macro system.

First of all, we have the following theorem on self-averaging. It is trivial, but nontheless is very important.

**Theorem**: If a sequence of identically distributed random variables $x_1$, $x_2$, $\cdots$, $x_i$, $\cdots$ has finite second moment, and is uncorrelated for any pair, *i.e.*

$$E(x_i, x_j) = E(x_i)E(x_j) \qquad for\ any\ i \neq j,$$

then, their sum $X_n = x_1 + x_2 + \cdots + x_n$ is self-averaging.

In this case, $C.V.(X_n)$ converges to zero as $n \to \infty$ in the order of $1/\sqrt{n}$. As a corollary, if a sequence of random variables $x_1$, $x_2$, $\cdots$ with finite second moment, is identically and *independently* distributed, then the sum is self-averaging.

Virtually all the micro-founded macro models rest on the above theorem. The point is best illustrated by the Poisson model which is so widely used in economics ranging from labor search theory to endogenous growth models (*e.g.* Aghion and Howitt (1992)). Suppose that the Poisson parameter is $\lambda$ which designates the instantaneous probability that an "event" such as technical progress and job arrival occurs. This probability which pertains to one economic agent is assumed to *commonly apply to all the agents and also exogenously given* —— the crucial assumption! Then, given the same Poisson process with parameter $\lambda$ for each individual agent, we obtain the Poisson process with the parameter $\lambda n$ for the economy as a whole where there are $n$ economic agents. The mean and the standard deviation of the number of "events" in the macroeconomy are $\lambda n$ and $\sqrt{\lambda n}$, respectively. The coefficient of variation defined as the standard deviation divided by the mean, is, therefore, $\sqrt{\lambda n}/\lambda n = 1/\sqrt{\lambda n}$. Thus, in the Poisson model, when the number of economic agents $n$ becomes large ($n \to \infty$), the coefficient of variation approaches zero in the order of $1/\sqrt{n}$. This property known as self-averaging provides us with justification for our concentrating on the means of variables in macro models. Now, because the mean depends basically on $\lambda$, it is natural to explore how $\lambda$ is determined in models. Indeed, in standard models, $\lambda$ is endogenously determined as an outcome of economic agents' optimization and market equilibrium. Exactly the same story holds for the case of the normal distribution.

The above theorem implies that for a sequence of identically distributed random variables, non-self-averaging may arise for at least two reasons. First is that the second moment of the distribution is not finite. This is the case, for example, for power distribution $P(x)$ whose power is less than two:

$$P(x) = \frac{C}{x^{\alpha}} \qquad (C > 0, \quad 0 < \alpha \leq 3). \tag{6}$$

Because *multiplicative processes* often generate power distribution, as Kadanoff (2000) states, multiplicative process can lead us to non-self-averaging.

Secondly, non-self-averaging may arise if random variables $x_1$, $x_2$, $\cdots$ are correlated. In this case, the variance of $X_n = x_1 + x_2 + \cdots + x_n$, $V(X_n)$ is

$$V(X_n) = nV(x) + \sum_{i,j} Cov(x_i, x_j). \tag{7}$$

Therefore, if there exists a positive number $r$ such that all the $Cov(x_i, x_j)$ is greater than or equal to $r$, then the following inequality holds.

$$V(X_n) \geq nV(x) + n(n-1)r. \tag{8}$$

Because $E(X_n)$ is of the order of $n$, non-self-averaging obtains in this case. Thus, the presence of correlations among random variables is plainly a source of non-self-averaging.

Given these propositions, we can understand that non-self-averaging is not a pathological phenomenon. It is instructive to see how non-self-averaging emerges in natural phenomena with the help of a simple example (Sornette(2000, p.370)).

Consider breaking a stick of size 1 into an infinite number of smaller pieces of size $W_n$ ($n$=1, 2, ...). We first break the stick into two pieces of sizes 1 - $p$ and $p$, respectively ($0 < p < 1$). Then, we do the same for the piece of size $p$ obtaining two pieces of sizes $(1-p)p$ and $p^2$. We repeat this procedure for the pieces of sizes $p^2$, $p^3$, $\cdots$. In this way, we obtain a set of pieces of sizes

$$W_1 = 1 - p$$

$$W_2 = (1-p)p$$

$$\vdots$$

$$W_n = (1-p)p^{n-1} \tag{9}$$

where $W_n$ is the size of the piece kept at the $n$-th step.

Now, suppose that $p$ is a random variable whose probability density function is $f(p)$. The random variable $p$ is drawn and fixed for each fragmentation history. We will consider a variable $Y$ defined by

$$Y = \sum_n W_n^2 = \sum_n (1-p)^2 p^{2(n-1)} = \frac{1-p}{1+p} \tag{10}$$

Note that unlike $\sum_n W_n^2$, $\sum_n W_n$ is always equal to 1. $Y$ is a measure of the degree of fragmentation ($0 < Y < 1$). $Y$ close to one indicates the presence of a large piece whereas $Y$ close to zero means that all the pieces are minute. $Y$ is a random variable because $p$ is random. It can be easily shown that $Y$, and consequently, the size of piece $W_n$ is non-self-averaging. We note that the process of fragmentation is path-dependent. As Kadanoff (2000) states, there are many similar examples in natural sciences.

More generally, Garibaldi and Scalas (2010, p.101-102) explicitly show how non-self-averaging emerges in *the Pólya process.* The Pólya process $X_1$, $X_2$, $\cdots$ is characterized by the following conditional probability for $X_{n+1}$ given realizations of $X_1$, $X_2$, $\cdots$, $X_n$:

$$P(X_{n+1} = j | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \frac{\alpha_j + n_j}{\alpha + n}. \tag{11}$$

Here, realizations of $X_i$'s take one of 1, 2, $\ldots$, $g$ ; That is, $X_i$ belongs to one of $g$ different "types". $n_j$ is the number of $X_i$'s whose realizations are of type $j$. $n_j$'s sum up to $n$, namely $\sum_{j=1}^g n_j = n$. $\alpha$ and $\alpha_j$ are the positive parameters which satisfy $\sum_{j=1}^g \alpha_j = \alpha$.

For finite $\alpha (0 < \alpha < \infty)$, Garibaldi and Scalas show that the coefficient of variation of $S_n^j = n_j/n$ is non-self-averaging [1]. Specifically, we obtain

$$\lim_{n\to\infty} C.V.(S_n^j) = \sqrt{\frac{\alpha - \alpha_j}{\alpha_j} \cdot \frac{1}{\alpha + 1}}. \tag{12}$$

We note that the Pólya process is path-dependent, and, therefore, that random variables are correlated. It is a generic model which accommodates wide applications.

Any non-self-averaging phenomenon has very important implications because some degree of imprecision or dispersion remains about the time trajectories in non-self-averaging models even when the number of economic agents goes to infinity. This means that focus on the mean path behavior of macroeconomic variables is not justified. It, in turn, means that sophisticated optimization exercises which are meant to provide us with information on the means have little value.

In Section 4, we will observe that non-self-averaging behavior very naturally arises in macroeconomics as well. Before doing so, we will discuss the fundamental problem of the standard micro-founded macroeconomics.

[1] If $\alpha$ is infinite, the coefficient of variation approaches zero when $n$ goes to infinity. Namely, the sequence is self-averaging. See Garibaldi and Scalas (2010, p.102).

# 3 Standard Micro-founded Macroeconomics

Almost all the economic opportunities such as job offer, discoveries of new technology, market, and resources are stochastic. The standard micro-founded macroeconomics — Lucas' rational expectations model, real business cycle theory, labor search theory, and endogenous growth theory — rightly takes into account stochastic events. However, in these micro-founded models, it is taken for granted that as the number of agents goes to infinity, any micro or "idyosyncratic" fluctuations vanish, and that well defined deterministic macroeconomic relations prevail. That is, self-averaging is tacitly presumed.

Self-averaging emerges when *all the agents are assumed to face the same unchanged "well-behaved" probability distribution* such as the normal and Poisson distributions, and their respective stochastic events are assumed to be independent. Lucas (1972, 73)'s famous model of business cycles is a primary example. It is instructive to trace his model in detail. Lucas begins to model a supplier's behavior in each individual market as follows;

> "Quantity supplied in each market will be viewed as the product of a normal (or secular) component common to all markets and a cyclical component which varies from market to market. Letting $z$ index markets, and using $y_{nt}$ and $y_{ct}$ to denote the *logs* of these components, supply in market $z$ is:
>
> (1) $$y_t(z) = y_{nt} + y_{ct}(z)$$
>
> ...... The cyclical component varies with perceived, *relative* prices and with its own lagged value:
>
> (3) $$y_{ct}(z) = \gamma\,[P_t(z) - E(P_t \mid I_t(z))] + \lambda y_{c,t-1}(z)$$
>
> Where $P_t(z)$ is the actual price in $z$ at $t$ and $E(P_t \mid I_t(z))$ is the mean current, general price level, conditioned on information available in $z$ at $t$, $I_t(z)$."

Given this framework, he goes on to the information structure of the economy.

> "The information available to suppliers in $z$ at $t$ comes from two sources. First, traders enter period $t$ with knowledge of the past course of demand shifts, of normal supply $y_{nt}$, and of past deviations $y_{c,t-1}, y_{c,t-2}, \cdots$. While this information does not permit exact inference of the *log* of the current general price level, $P_t$, it does determine a "prior" distribution on $P_t$, common to traders in all markets. We assume that this distribution is knwon to be normal, with mean $\bar{P}_t$ (depending in a known way on the above history) and a constant variance $\sigma^2$. Second, we suppose that the actual price deviates form the (geometric) economy-wide

average by an amount which is distributed independently of $P_t$. Specifically, let the percentage deviation of the price in $z$ from the average $P_t$ be denoted by $z$ (so that markets are indexed by their price deviations from average) where $z$ is normally distributed, independent of $P_t$, with mean zero and variance $\tau^2$. Then the observed price in $z, P_t(z)$ (in *logs*) is the sum of independent, normal variates

$$(4) \qquad P_t(z) = P_t + z$$

The information $I_t(z)$ relevant for estimation of the unobserved (by suppliers in $z$ at $t$), $P_t$ consists then of the observed price $P_t(z)$ and the history summarized in $\bar{P}_t$ (Lucas(1973; p.328))".

It is extremely important to note that though $z$ varies from market to market, the distribution of $z$ is uniquely given and shared by all the markets. We also note that in this model, markets are indexed not by intrinsic identity such as steel industry but by their price deviations from average. Each market finds a realization of $z$ each period given the same distribution of $z$. The assumption of rational expectations then permits suppliers in individual markets to make efficient inferences on the relative prices. This leads to micro supply functions. Given micro supply functions, the aggregate supply function, is trivially derived.

"To utilize this information, suppliers use (4) to calculate the distribution of $P_t$, conditional on $P_t(z)$ and $\bar{P}_t$. This distribution is (by straightforward calculation) normal with mean:

$$(5) \qquad E(P_t \mid I_t(z)) = E(P_t \mid P_t(z), \bar{P}_t) = (1-\theta)P_t(z) + \theta \bar{P}_t$$

where $\theta = \gamma^2/(\sigma^2 + \gamma^2)$, and variance $\theta\sigma^2$. Combining (1), (3), and (5) yields the supply function for market $z$:

$$(6) \qquad y_t(z) = y_{nt} + \theta\gamma[P_t(z) - \bar{P}_t] + \lambda y_{c,t-1}(z)$$

Averaging over markets (integrating with respect to the distribution of $z$) gives the aggregate supply function:

$$(7) \qquad y_t = y_{nt} + \theta\gamma(P_t - \bar{P}_t) + \lambda[y_{t-1} - y_{n,t-1}]$$

(Lucas(1973; p.328))"

The aggregate supply function is the core of Lucas' rational expectations model of business cycles. In this model, the crucial assumption is his equation (4) above. More specifically, the random variable $z$ is assumed to be normally distribution with mean zero and variance $\tau^2$. As pointed out earlier, each supplier faces the same probability distribution of micro shock although a realization of such a shock, of course, differs across suppliers.

The same assumption is routinely made, and taken by most economists as innocuous. It is as if there were no other way of defining *the micro or idyosyncratic shocks*. However, the standard assumption means that Microsoft and small grocery store on the street face micro shocks drawn from the *same unchanged* probability distribution! It presumes *homogeneity with respect to the probability distribution of micro shocks.* Though it is routinely made in standard micro-founded models, this assumption is extremely unrealistic, and as Solow (2004) notes, almost absurd.

Lucas' model emphasizes the role of micro shocks which by definition differ across sectors or agents. In this sense, it apparently rejects representative agent. However, like other micro-founded macro models, it is built on the crucial premise that every agent faces the same unchanged probability distribution of micro shocks. This assumption entails self-averaging. Specifically, in his model, one can easily obtain the aggregate supply function by "averaging over markets (integrating with respect to the distribution of $z$.)" Note that his aggregate output $y_t$ (his equation (7)) is nothing but the mean of stochastic aggregate output.

The fundamental problem is not confined to Lucas (1972,73) rational expectations model but also applies to search theory. Lucas and Prescott (1974)'s model of equilibrium search and unemployment is a primary example. The model, in the authors' own description, is as follows:

> "We think of an economy in which production and sale of goods occur in a large number of spatially distinct markets. Product demand in each market shifts stochastically, driven by shocks which are independent over markets (so that aggregate demand is constant) but autocorrelated within a single market. Output to satisfy current period demand is produced in the current period, with labor as the only input. Each product market is competitive.
>
> There is a constant workforce which at the beginning of a period is distributed in some way over markets. In each market, labor is allocated over firms competitively with actual money wages being market clearing. Each worker may either work at this wage rate, in which case he will remain in this market into the next period, or leave. If he leaves, he earns nothing this period but enters a "pool" of unemployed workers which are distributed in some way over markets for the next period. In this way, a new workforce distribution is determined, new demands are "drawn", and the process continues.
>
> In this process, all agents are assumed to behave optimally in light of their objectives and the information available to them. For firms, this means simply that labor is employed to the point at which its marginal value product equals the wage rate. For workers, the decision to work or to search is taken so as to maximize the expected, discounted present value of the earnings stream. In

carrying out this calculation, workers are assumed to be aware of the values of the variables affecting the market where they currently are (i.e., demand and workforce) and of the true probability distributions governing the future state of this market and the present and future states of all others. That is, expectations are taken to be *rational*. (Lucas and Prescott, 1974; p.190) "

Markets are all competitive, so that the marginal value product of labor equals the wage in every market. However, the state of demand represented by a realization of a stochastic variable $s$ differs across markets while at the same time, mobility of labor is not instantaneous. As a consequence, the marginal value products of labor and wages differ across markets. In contrast to the standard general equilibrium model, productivity dispersion exists in equilibrium as actually observed in the economy. Once again, the problem is the nature of stochastic equilibrium in their model.

The stochastic disturbances in Lucas and Prescott (1974) are the demand shifts $s$. They are assumed to be independent across markets and the number of markets is large.

"By large, we mean either a continuum of markets or a countable infinity. Economically, then, the assumption of independent demand shifts means that *aggregate* demand is taken to be constant through time. (Lucas and Prescott, 1974; Footnote 8 on p.192) "

The micro disturbances are assumed to cancel each other. Self-averaging ensues. In this way, Lucas and Prescott can describe the determination of the stationary distribution of employment, workforce, and wages or marginal value products in a *representative market*. On their own assumption, they state as follows:

"The distribution of the workforce over locations (indexed by (s, y)) would in this case be the same as the stationary distribution of (s, y) in any one market. (This follows from our assumptions that the number of markets is large and that demand shifts are independent across markets.) (Lucas and Prescott, 1974; p.202)"

The same assumption allows Lucas and Prescott to focus on the means characteristics of which are described in a representative market. Specifically, worker's search depends on the expected present value of search, $\lambda$. The maximization exercises (Section 3 of their paper) are done on the assumption that $\lambda$ is common to all the markets, and that "the search process eliminate rents on average."

Modern micro-founded macroeconomics emphasizing the role of micro / idiosyncratic shocks is built on Phelps (1970) island-paradigm. We can usefully interpret Phelps island model in terms of *ultrametric trees* (see chapter 5 of Aoki and Yoshikawa (2007)). Ultrametric trees measure the "distance" between markets or agents. What the "distance" measures depends on model, of course. For example, the distance between two states a and b may indicate transition rate from a to b in Markov model. Now, in general, the distance between two markets/agents differs depending on a pair. In Figure 1 (a), an example of three-level tree is shown. In this example, the distance between markets/agents 1 and 2 is much closer than that between 1 and 3; Grocery (agent1) and laundry on the same street (agent 2) are "closer" to each other than to Microsoft (agent 3). Or GM(agent 1) and Ford (agent 2) are closer to each other than to grocery (agent 3).

In contrast, in the one-level tree shown in Figure 1 (b), all the markets/agents are symmetric. One may think of $N$ markets in the standard micro-founded models discussed in the present section as leaves of a one-level tree with $N$ branches from the root. This organization is a very special case of multi-level trees. In the one-level tree arrangement of $N$ markets, each branch is the same as any other branch because markets are identical by assumption. Then, every one of the $N$ markets can serve as a representative market. Mixing these markets randomly by introducing a probability distribution, as Lucas and Prescott do in their paper (to be specific, their probability distribution $\Phi$ on p.198) does nothing to the model. The mixture is identical to any one of the branches; that is, the mixture is again a representative market. This is the essence of Phelps (1970) island-paradigm on which modern micro-founded macroeconomics is built.

To sum up, standard micro-founded models other than real business cycle models all introduce apparent heterogeneity of agents and markets, and emphasize the role of micro or idiosyncratic shocks. However, contrary to the impression they give, the tacit assumption of self-averaging effectively makes such frameworks models of a representative agent or market.

# 4 Non-self-averaging in a Growth Model

In this section, we present a simple innovation driven growth model in which aggregate output or GDP is non-self-averaging. The main purpose of this exercise is not to advance a realistic model of economic growth, but rather to demonstrate how naturally non-self-averaging emerges in economic model.

## The Model

Following the literature on endogenous growth, we assume that the economy grows by innovations. Innovations are shochastic events. There are two kinds of innovations in our model. Namely, an innovation, when it occurs, either raises productivity of one of the existing sectors, or creates a new sector. Thus, the number of sectors is not given, but increases over time.

By the time $n$th innovation occurs, the total of $K_n$ sectors are formed in the economy wherein the $i$-th sector has experienced $n_i$ innovations ($i = 1, 2, \ldots, K_n$). By definition, the following equality holds:

$$n_1 + n_2 + \cdots + n_k = n \tag{13}$$

when $K_n = k$. If $n$-th innovation creates a new sector (secotor $k$), then $n_k = 1$.

The aggregate output or GDP when $n$ innovations have occured is denoted by $Y_n$. $Y_n$ is simply the sum of outputs in all the sectors, $y_i$.

$$Y_n = \sum_{i}^{K_n} y_i. \tag{14}$$

Output in secotr $i$ grows thanks to innovations which stochastically occur in that sector. Specifically, we assume

$$y_i = \eta\gamma^{n_i}. \qquad (\eta > 0, \gamma > 1) \tag{15}$$

For our purpose, it is convenient to rewrite equation (13) as follows.

$$n = \sum_{j}^{n} j a_j(n) \tag{16}$$

In equation (16), $a_j(n)$ is the number of sectors where $j$ innovations have occurred. The vector $a(n)$ consisting of $a_j(n)$, is called *partition vector*[2]. With this partition vector, $a(n)$, $K_n$ can be expressed as

$$K_n = \sum_{j}^{n} a_j(n). \tag{17}$$

Using the following approximation

$$\gamma^{n_i} = exp(n_i \ln\gamma) \approx 1 + \ln(\gamma) n_i,$$

we can rewrite equation (15) as

$$y_i = \eta + \eta\ln(\gamma) n_i. \tag{18}$$

Thus, from equations (13), (14), (16), (17) and (18), we obtain

$$Y_n \approx K_n + \beta \sum_{j}^{n} j a_j(n). \tag{19}$$

where $\beta = \ln(\gamma) > 0$. Here, without loss of generality, we assume that $\eta$ is one. Obviously, the behavior of the aggregate output, $Y_n$ depends on how innovations occur.

[2]See chapter 2 of Aoki and Yoshikawa (2007) for partition vector.

## The Ewens Distribution of Innovations

We now describe how innovations stochastically occur in the model. An innovation follows *Ewens (E) distribution* or the Chinese restaurant process due to Hansen and Pitman (2000)[3]; See also Capter 9 of Garibaldi and Scalas (2010).

Given the two-parameter $E(\alpha, \theta)$ distribution, when there are $k$ clusters of sizes $n_i$, $(i = 1, 2, \dots, k)$, and $n = n_1 + n_2 + \cdots + n_k$, an innovation occurs in one of the existing sectors of "size" $n_i$ with probability rate $p_i$:

$$p_i = \frac{n_i - \alpha}{n + \theta}. \tag{20}$$

The "size" of sector $i$, $n_i$ is equal to the number of innovations that have already occurred in sector $i$. The two parameters $\alpha$ and $\theta$ satisfy the following conditions:

$$\theta + \alpha > 0, \quad \text{and} \quad 0 < \alpha < 1.$$

With $\alpha = 0$ there is a single parameter $\theta$, and the distribution boils down to the one-parameter $E$ distribution, $E(\theta)$.

$p_i$ is the probability that an innovation occurs in one of the existing sectors. Now, a new sector emerges with probability rate[4] $p$:

$$p = 1 - \sum_{1}^{k} \frac{n_i - \alpha}{n + \theta} = \frac{\theta + k\alpha}{n + \theta}. \tag{21}$$

In the two-parameter $E(\alpha, \theta)$ distribution, the probabilitiy that the number of sectors increases by one from $n$ to $n+1$ conditional on $K_n = k$, is given by[5]

[3]Kingman invented the one-parameter Poisson-Dirichlet distribution to describe random partitions of populations of heterogeneous agents into distinct clusters. The one-parameter Poisson-Dirichlet model is also known as Ewens model, (Ewens (1972)); See Aoki (2000a, 2000b) for further explanation. The one-parameter model was then extended to the two-parameter Poisson-Dirichlet distributions by Pitman; See Kingman (1978, 1993), Carlton (1999), Feng and Hoppe (1998), Pitman (2006), Feng (2010), among others. Aoki (2006) has shown that the two-parameter Poisson-Dirichlet models are qualitatively different from the one-parameter version because the former is not self-averaging while the latter is. These models are therefore not exponential growth models familiar to economists but they belong to a broader class of models without steady state constant exponential growth rate. None of the previous works, however, have comparatively examined the asymptotic behavior of the coefficient of variation of these two classes of models.

[4]Probabilities of new types entering Ewens model, are discussed in Aoki (2002, Sec.10.8, App. A.5).

[5]Because the following inequality holds:

$$\frac{\theta + k\alpha}{n + \theta} > \frac{\theta}{n + \theta},$$

we observe that the probability that a new sector emerges is higher in the two-parameter *PD* model than in the one-parameter *PD* model.

$$\Pr(K_{n+1} = k + 1 | K_1, \ldots, K_n = k) = p = \frac{\theta + k\alpha}{n + \theta}. \tag{22}$$

On the other hand, the correspondinting probability that the number of sectors remains unchanged is

$$\Pr(K_{n+1} = k | K_1, \ldots, K_n = k) = \sum_i p_i = \frac{n - k\alpha}{n + \theta}. \tag{23}$$

It is important to note that in this model, sectors are *not homogeneous with respect to the probability that an innovation occurs.* The larger sector $i$ is, the greater the probability that an innovation occurs in sector $i$ becomes. Moreover, these probabilities *change endogenously* as $n_i$ changes over time. *In this respect, the Ewens distribution is in marked contrast to the common assumption on micro shocks routinely made in modern micro-founded models.*

Given the model, we are interested in the behavior of GDP, namely $Y_n$. We obtain the following proposition.

**Proposition**
In the two-parameter Ewens model, the aggregate output $Y_n$ is non-self averaging.

The proof is given in the Appendix.

The two-parameter Ewens model is non-self averaging. Interestingly, the one parameter Ewens model ($\alpha = 0$) is self-averaging. It is, therefore, worth inquiring why the two-parameter $E$ model is non-self averaging. The answer lies in (22) and (23). In this model, innovations occur in one of the two different types of sectors, one, the new type and the other, known or pre-existing types. The probability that an innovation generates a new sector is $(\theta + K_n\alpha)/(n + \theta)$, and the probability that an innovation occurs in one of the existing sectors is $(n - K_n\alpha)/(n + \theta)$. Here, $K_n$ is the number of types of sectors in the model by the time $n$ innovations occurred. These probabilities and their ratio vary endogenously, depending on the histories of how innovations occured. In other words, the mix of old and new sectors evolve endogenously, and is path-dependent. This is the reason why non-self averaging emerges in the two parameter $E$ model. In one parameter $E$ model in which $\alpha = 0$, two probabilities (22) and (23) become independent of $K_n$, and that the model becomes self-averaging. The two-parameter Ewens model is fundamentally different from standard micro-founded models we discussed in the previous section.

# 5 Conclusion

Macroeconomics is meant to analyze the real economy. Keynesian macroeconomics was indeed born amid the Great Depression in the serious endeavor to meet the grave challenges facing the world economy. After the financial crisis and global recession during 2008-10, modern micro-founded macroeconomics has been justifyably subject to serious criticisms. For example, Paul Krugman in his Lionel Robbins Lecture held at London School of Economics in June, 2009 said that "most macroeconomics of the past 30 years was spectacularly useless at best, and positively harmful at worst."[6] Stiglitz (2010) also criticizes micro-founded macroeconomics as follows:

> "The blame game continues over who is responsible for the worst recession since the Great Depression — the financiers who did such a bad job of managing risk or the regulators who failed to stop them. But the economics profession bears more than a little culpability. ⋯
>
> It is hard for non-economists to understand how peculiar the predominant macroeconomic models were. Many assumed demand had to equal supply — and that meant there could be no unemployment. (Right now a lot of people are just enjoying an extra dose of leisure; why they are unhappy is a matter for psychiatry, not economics.) Many used "representative agent models" — all individuals were assumed to be identical, and this meant there could be no meaningful financial markets (who would be lending money to whom?). ⋯
>
> Changing paradigms is not easy. Too many have invested too much in the wrong models. Like the Ptolemaic attempts to preserve earth-centric views of the universe, there will be heroic efforts to add complexities and refinements to standard paradigm. The resulting models will be an improvement and policies based on them may do better, but they too are likely to fail. Nothing less than a paradigm shift will do. (Stiglitz, 2010)."

Certainly, changing paradigms is necessary. To achieve the goal, it is imperative to understand precisely what is the fundamental problem of modern micro-founded macroeconomics. As Kirman (1992), Solow (2004) and others all rightly point out, the assumption of representative agents is seriously flawed: See Aoki and Yoshikawa (2007) for our own criticism.

The assumption of representative agents is crystal clear in such models as real business cycle theory where literally the representative consumer and firm are introduced. However, Lucas (1972, 73) rational expectations model, labor search theory such as Lucas/Prescott (1974) and Mortensen/Pissarides (1994), and some of the endogenous growth models such as Aghion and Howitt (1992) all assume apparently heterogeneous economic agents and emphasize the role of micro/idyosyncratic shocks. In this paper, we showed that despite *apparent* heterogeneity introduced, these models are based in

[6] See also "Modern Economic Theory — where it went wrong, and how the crisis is changing it" *The Economist*, July 18-24, 2009.

essence on a representative agent. Put it another way, the so-called micro/idyosynctratic shocks in the standard micro-founded models are *not* true micro shocks.

The crucial concept for our thesis is *non-self-averaging*. Self-averaging is tacitly assumed not only in crude representative agent models such as real business cycle theory but also in Lucas rational expectations models and labor search theory. It justifies us to focus on means. The standard optimization exercises based on representative agents are meant to analyze the behavior of means.

Self-averaging is, however, obtained only on the extremely unrealistic assumptions that all the economic agents share the same probability distribution, and that micro shocks are independent. As we discussed it in deatil, in Lucas rational expectations models and labor search theory, a unique *representative* probability distribution of micro/idyosyncratic shocks shared by all the economic agents is assumed. In Section 4, we demonstrated that once this unrealistic assumption is dropped out, non-self-averaging quite naturally emerges in economic model. A model presented in Section 4 is meant to be nothing but an example. Plainly, we can easily apply the same framework to other economic models.

Non-self-averaging emerges in a wide class of stochastic models. Garibaldi and Scalas (2010, p.101-102) demonstrate that non-self-averaging obtains in the Pólya process. Based on their analyses, they make the following remarks:

> "When considering large macroeconomic aggregates or a long time evolution, fluctuations may become irrelevant and only the deterministic dynamics of expected values is important. In other words, stochastic processes may be replaced by difference equations or even by differential equations for empirical averages.$\cdots$ $\cdots$ (However), we have directed the attention of the reader to the phenomenon of lack of self-averaging, which is often there in the presence of correlations. In other words, when correlations are there, it is not always possible to neglect fluctuations, and a description of economic systems in terms of random variables and stochastic processes becomes necessary. (Garibaldi and Scalas (2010, p.225))."

The presence of correlations is an important reason why we obtain non-self-averaging. We have, in fact, all the reasons why micro shocks are correlated. *Power distribution* is another factor generating non-self-averaging. We might note that many economic variables such as income distribution and firm sizes indeed follow power-laws.

Non-self-averaging deprives us of justification for our focusing on means. It, in turn, means that such sophisticated microeconomic analyses as infinite horizon stochastic dynamic programming so popular in macroeconomic models have little value. Those analyses provide us with no foundations for macroeconomic analyses because time paths of macro variables are sample dependent in any way.

Summing up, macroeconomics must seek different microeconomic foundations from the standard optimization of representative agent. We believe that the right track is the methods of statistical physics and combinational stochastic processes (See Aoki and Yoshikawa (2007)). In any case, as Solow (2004) argues, macroeconomics is better freed from too much of optimization exercises. This is the fundamental implication of non-self-averaging for macroeconomics.

# Appendix

In this appendix, we provide the proof of the proposition presented in Section 4. We show that $Y_n$ is non-self-averaging.

Toward this goal, we first normalize $Y_n$ by $n^\alpha$. Then, from equation (19) in the main text, we obtain

$$\frac{Y_n}{n^\alpha} = \frac{K_n}{n^\alpha} + \beta\Sigma_j^n \frac{a_j(n)}{n^\alpha}. \tag{24}$$

In what follows, we show that $Y_n$ is non-self-averaging. Toward this goal, we first define partial sums of $K_n$ and $Y_n$ up to $l(< n)$, $K_n(1,l)$ and $Y_n(1,l)$, as follows:

$$K_n(1,l) = \Sigma_{j=1}^l a_j(n) \tag{25}$$

and

$$Y_n(1,l) = K_n(1,l) + \beta\hat{Y}_n(1,l) \tag{26}$$

where

$$\hat{Y}_n(1,l) = \Sigma_{j=1}^l j a_j(n). \tag{27}$$

Yamato and Sibuya (2000; p.7 their propositions 4.1 and 4.2) showed that given $l$, $K_n(1,l)/n^\alpha$ and $\hat{Y}_n(1,l)/n^\alpha$ converge in distribution ($\longrightarrow d$) as $n$ approaches infinite as follows:

$$\frac{K_n(1,l)}{n^\alpha} \longrightarrow^d C_1(l)L \tag{28}$$

and

$$\frac{\hat{Y}_n(1,l)}{n^\alpha} \longrightarrow^d C_2(l)L \tag{29}$$

where

$$C_1(l) = 1 - \frac{(1-\alpha)^{[l]}}{l!}$$

$$C_2(l) = \frac{(2-\alpha)^{[l-1]}}{(l-1)!}.$$

Here, $[j]$ in $C_1(l)$ and $C_2(l)$ denotes an ascending factorial:

$$x^{[j]} = x(x+1)\dots(x+j-1).$$

The random variable $L$ in (28) and (29) has the probability density function $g_{\alpha,\theta}(x)$:

$$g_{\alpha\theta}(x) = \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{\frac{\theta}{\alpha}} g_\alpha(x) \tag{30}$$

where $g_\alpha$ is the density of the Mittag-Leffler distribution[7] with parameter $\alpha$. Pitman (2006) also showd the a.s.convergence. See Yamato and Sibuya (2000, p.8).

It is shown by Yamato and Sibuya and by Pitman that

$$\frac{K_n}{n^\alpha} \longrightarrow^d L, \tag{31}$$

$$\frac{K_n}{n^\alpha} \longrightarrow L \qquad\qquad a.s. \tag{32}$$

and

$$C.V.\left(\frac{K_n}{n^\alpha}\right) \longrightarrow C.V.(L) > 0. \tag{33}$$

Because $C_1(l)$ and $C_2(l)$ are constant in (28) and (29), for each fixed $l$, and $\alpha > 0$, we obtain

$$C.V.\left(\frac{K_n(1,l)}{n_\alpha}\right) \longrightarrow C.V.(L) > 0, \tag{34}$$

and

$$C.V.\left(\frac{(\hat{Y}_n(1,l)}{n^\alpha}\right) \longrightarrow C.V.(L) > 0. \tag{35}$$

Therefore, given (26), we obtain

$$C.V.\left(\frac{(Y_n(1,l)}{n^\alpha}\right) \longrightarrow C.V.(L) > 0 \tag{36}$$

Thus, for sufficiently large $l$,

$$C.V.\left(\frac{Y_n}{n^\alpha}\right) \longrightarrow C.V.(L). \tag{37}$$

*Mittag-Leffler function* $g_\alpha(x)$ has the property that its $p$ th moment is given by

$$\int_0^\infty x^p g_\alpha(x)dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)} \qquad\qquad (p > -1). \tag{38}$$

[7]See Blumenfeld and Mandelbrot (1997), Erdely (1955), and Pitman (2006) on Mittag-Leffler function.

Thus, using (30) and (38), we can obtain the first and second moments of $L$, $E_{\alpha,\theta}(L)$ and $E_{\alpha,\theta}(L^2)$ as follows:

$$E_{\alpha,\theta}(L) = \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}, \tag{39}$$

and

$$E_{\alpha,\theta}(L^2) = \frac{(\theta+\alpha)\Gamma(\theta+1)}{\alpha^2\Gamma(\theta+2\alpha)}. \tag{40}$$

Moments of $L$ in the large $n$ limit can be also obtained by a different method; See Pitman(2006).

The variance of $L, var(L)$ is, then

$$var(L) = E_{\alpha,\theta}(L^2) - [E_{\alpha,\theta}(L)]^2 = \gamma_{\alpha,\theta}\frac{\Gamma(\theta+1)}{\alpha^2}, \tag{41}$$

where

$$\gamma_{\alpha,\theta} := \frac{\theta+\alpha}{\Gamma(\theta+2\alpha)} - \frac{\Gamma(\theta+1)}{[\Gamma(\theta+\alpha)]^2}. \tag{42}$$

The coefficient of variation of $L$ is given by

$$C.V.(L) = \frac{\sqrt{var(L)}}{E_{\alpha,\theta}(L)} = \sqrt{\frac{\gamma_{\alpha,\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha). \tag{43}$$

Note that $\gamma_{\alpha,\theta}$ defined by (42) is zero when $\alpha = 0$, but that it is positive when $\alpha > 0$. Therefore, $C.V.(L)$ is zero in the one-parameter Ewens model ($\alpha = 0$), but is positive in the two-parameter Ewens model ($\alpha > 0$).

Now, we have shown above that $C.V.(Y_n/n^\alpha)$ converges to $C.V.(L)$. Thus, thanks to (37) and (43), we finally obtain

$$C.V.\left(\frac{Y_n}{n^\alpha}\right) \to \sqrt{\frac{\gamma_{\alpha,\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha). \tag{44}$$

The right-hand side of (44) does not approach zero even if $n$ goes to infinity in the two-parameter Ewens model ($\alpha > 0$). Thus, we have established the proposition.

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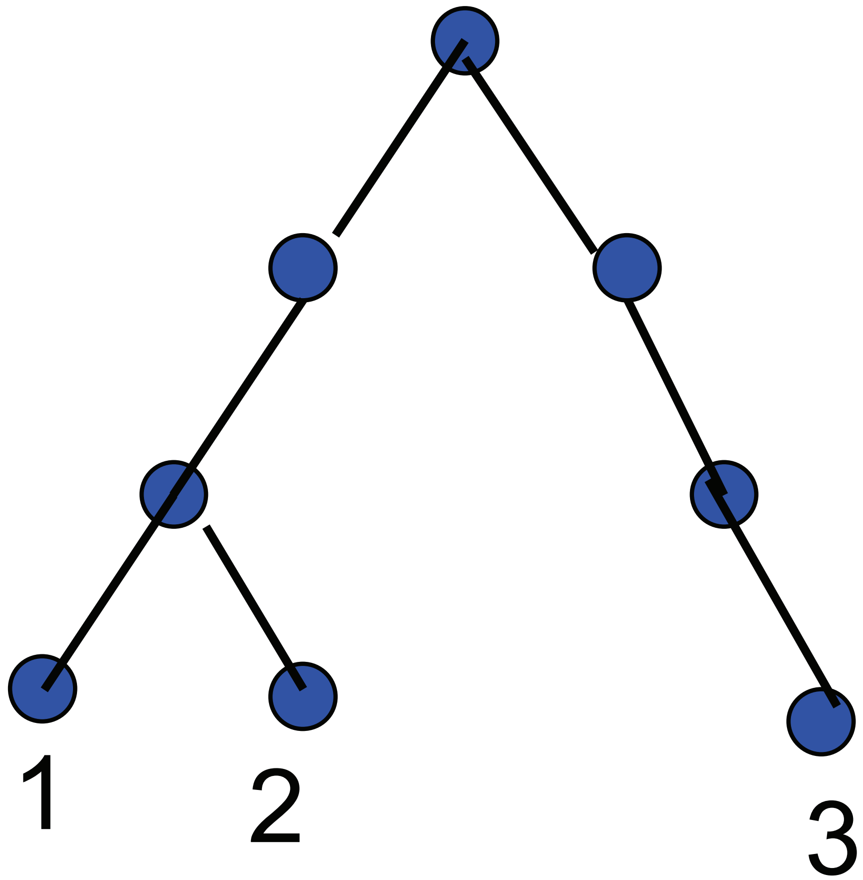
(a) Three-level Tree




(b) One-level Tree