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A Characterization of the Plurality Rule

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Abstract

We consider an axiomatic characterization of the plurality rule, which selects the alternative(s) most preferred by the largest number of individuals. We strengthen the characterization result of Yeh (Economic Theory 34: 575–583, 2008) by replacing *efficiency* axiom by the weaker axiom called *faithfulness*. Formally, we show that the plurality rule is the only rule satisfying *anonymity, neutrality, reinforcement, tops-only*, and *faithfulness*.

Keywords: Plurality rule; Faithfulness; Reinforcement; Tops-only.

JEL Classification Numbers: D70; D71; D72.

1 Introduction

We consider the problem of choosing alternatives from a fixed set of finitely many alternatives. A *social choice function* assigns chosen alternative(s) to each profile of preferences of individuals in a society. We consider the case where the number of individuals may vary and each individual's preference is a linear order (no indifference between any two alternatives).

In this setting Yeh (2008) characterized the *plurality rule*, which selects the alternative(s) most preferred by the largest number of individuals: the plurality rule is the only rule satisfying *anonymity, neutrality, reinforcement, tops-only*, and *efficiency*.[1] *Anonymity* and *neutrality* are standard symmetric

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[1] Other characterizations of the plurality rule are found in Richelson (1978) and Ching (1996).

axioms: the former requires that the names of individuals should not matter, and the latter requires that the names of alternatives should not matter. *Reinforcement* is an invariance axiom which requires that if two disjoint groups of individuals choose the same alternative, then their union should also choose this alternative.[2] *Tops-only* requires that the choice should depend only on the information about top-ranked alternatives of individuals.[3] *Efficiency* requires that inefficient alternatives should not be chosen.

We strengthen Yeh (2008)'s characterization result by replacing *efficiency* by the weaker axiom called *faithfulness*: when there is only one individual, her most preferred alternative should be uniquely chosen (Young, 1974). Namely, we show that the plurality rule is the only rule satisfying *anonymity, neutrality, reinforcement, tops-only*, and *faithfulness* (Theorem 1).

Our characterization result is related to Young (1974)'s characterization of Borda's rule (Borda, 1781).[4] Young (1974) considered *cancellation* axiom: if for all x and y in X, the number of individuals preferring x to y equals the number of individuals preferring y to x, then all alternatives are chosen. And Young (1974) showed that Borda's rule is the unique rule satisfying *cancellation* and our axioms excepting *tops-only*. That is,

$$\begin{array}{l} anonymity,\ neutrality,\ reinforcement,\ faithfulness \\ \qquad\qquad + \left\{ \begin{array}{llll} \textit{tops-only} & \Longleftrightarrow & \text{plurality rule} & \text{(Theorem 1)} \\ \textit{cancellation} & \Longleftrightarrow & \text{Borda's rule} & \text{(Young 1974)} \end{array} \right. \end{array}$$

2 Definitions

There is a countable set $\mathbb{N}$ of "potential" individuals. Let $\mathcal{N}$ be the family of all nonempty and finite subsets of $\mathbb{N}$. Let X be the finite set of alternatives. We denote $\mathcal{P}$ the set of all linear orders (transitive, antisymmetric, and complete binary relations) on X. Given $N \in \mathcal{N}$ and $i \in N$, we denote individual i's preference by $P_i \in \mathcal{P}$, a preference profile by $P_N = (P_i)_{i \in N}$, and the set of all preference profiles by $\mathcal{P}^N$. A **social choice function** is a mapping $f : \bigcup_{N \in \mathcal{N}} \mathcal{P}^N \to 2^X \setminus \{\emptyset\}$.

Let $T(P_i)$ be i's top-ranked alternative, and $T(P_N) = (T(P_i))_{i \in N}$ the profile of top-ranked alternatives. Let $T(x, P_N)$ be the number of individuals whose top-ranked alternative is x, that is, $T(x, P_N) = |\{i \in N | T(P_i) = x\}|$.

[2] This axiom was proposed independently by Smith (1973), Fine and Fine (1974a, b) and Young (1974, 1975).

[3] This axiom has been studied in various fields. See, for example, Moulin (1980), Barbera et al. (1991), Koray (2000), Mihara (2000), and Masso and Neme (2004).

[4] See also Hansson and Sahlquist (1976).

The **plurality rule** f_P is defined by

$$f_P(P_N) = \arg\max_{x \in X} T(x, P_N).$$

Yeh (2008) showed that the plurality rule is characterized by the following five axioms.

Given $N \in \mathcal{N}$, let Π_N be set of all permutations on N. Given $\pi \in \Pi_N$, let $P_{\pi(N)} = (P_{\pi(i)})_{i \in N}$.

Anonymity: $\forall N \in \mathcal{N}$, $\forall P_N \in \mathcal{P}^N$, $\forall \pi \in \Pi_N$, $f(P_{\pi(N)}) = f(P)$.

Let Σ be the set of all permutations on X. Given $P \in \mathcal{P}$ and $\sigma \in \Sigma$, let $\sigma(P)$ be a linear order defined by $\sigma(x)\sigma(P)\sigma(y)$ iff xPy. Given $N \in \mathcal{N}$, let $\sigma(P_N) = (\sigma(P_i))_{i \in N}$.

Neutrality: $\forall N \in \mathcal{N}$, $\forall P_N \in \mathcal{P}^N$, $\forall \sigma \in \Sigma$, $f(\sigma(P_N)) = \sigma(f(P_N))$.

Reinforcement: $\forall N, N' \in \mathcal{N}$ with $N \cap N' = \emptyset$, $\forall P_N \in \mathcal{P}^N$, $\forall P_{N'} \in \mathcal{P}^{N'}$, $f(P_N) \cap f(P_{N'}) \neq \emptyset \Rightarrow f(P_N, P_{N'}) = f(P_N) \cap f(P_{N'})$.

Tops-only: $\forall N \in \mathcal{N}$, $\forall P_N, P'_N \in \mathcal{P}^N$, $T(P_N) = T(P'_N) \Rightarrow f(P_N) = f(P'_N)$.

Efficiency: $\forall N \in \mathcal{N}$, $\forall P_N \in \mathcal{P}^N$, $\exists x, y \in X$, $\forall i \in N$, $yP_i x \Rightarrow x \notin f(P_N)$.

3 Result

We strengthen Yeh (2008)'s characterization result by replacing *efficiency* by the weaker axiom called *faithfulness*.

Faithfulness: $\forall N \in \mathcal{N}$ with $N = \{i\}$, $\forall P_i \in \mathcal{P}$, $f(P_i) = T(P_i)$.

Faithfulness requires that when there is only one individual, her top-ranked alternative should be uniquely chosen, or equivalently, inefficient alternatives should not be chosen. Thus, *faithfulness* is weaker than *efficiency*.

Theorem 1 A social choice function f is the plurality rule f_P if and only if it satisfies *anonymity, neutrality, reinforcement, tops-only*, and *faithfulness*.

To prove our theorem, we begin with the following simple observation.

Lemma 1 Suppose that f is a social choice function satisfying *anonymity, neutrality*, and *tops-only*. For each $N \in \mathcal{N}$ and each $P_N \in \mathcal{P}^N$, if $|\bigcup_{i \in N} T(P_i)| = |N|$, then

$$f(P_N) = \begin{cases} X & \text{or} \\ \bigcup_{i \in N} T(P_i) & \text{or} \\ X \setminus \bigcup_{i \in N} T(P_i). & \end{cases}$$

Proof Suppose that f is a social choice function satisfying the three axioms. Let $N \in \mathcal{N}$ and $P_N \in \mathcal{P}^N$. Since f satisfies *tops-only*, $f(P_N)$ depends only on $T(P_N)$. Furthermore, *anonymity* and *neutrality* implies that for all x and y in X such that $T(x, P_N) = T(y, P_N)$, $x \in f(P_N)$ if and only if $y \in f(P_N)$.

Suppose that $|\bigcup_{i\in N} T(P_i)| = |N|$. Then, we have

$$T(x, P_N) = 1 \quad \Longleftrightarrow \quad x \in \bigcup_{i\in N} T(P_i) \quad \text{and}$$
$$T(x, P_N) = 0 \quad \Longleftrightarrow \quad x \notin \bigcup_{i\in N} T(P_i),$$

which completes the proof. ■

The following lemma is a variant of Lemma 1 in Yeh (2008). It states that if f satisfies our five axioms and the most preferred alternatives of all individuals differ, then the chosen alternatives are the most preferred ones.

Lemma 2 Suppose that f is a social choice function satisfying *anonymity, neutrality, reinforcement, tops-only*, and *faithfulness*. For each $N \in \mathcal{N}$ and each $P_N \in \mathcal{P}^N$, if $|\bigcup_{i\in N} T(P_i)| = |N|$, then $f(P_N) = \bigcup_{i\in N} T(P_i)$.

Proof Suppose that f is a social choice function satisfying the five axioms. Let $N \in \mathcal{N}$ and $P_N \in \mathcal{P}^N$. Suppose, to the contrary, that $|\bigcup_{i\in N} T(P_i)| = |N|$ and there exists $x \notin \bigcup_{i\in N} T(P_i)$ such that $x \in f(P_N)$. Take any $P \in \mathcal{P}$ such that $T(P) = x$. Then, by *faithfulness*, $f(P) = \{x\}$. However, *reinforcement* implies that $f(P_N, P) = f(P_N) \cap f(P) = \{x\}$, which contradicts Lemma 1. Thus, if $x \notin \bigcup_{i\in N} T(P_i)$, then $x \notin f(P_N)$. By Lemma 1, we have $f(P_N) = \bigcup_{i\in N} T(P_i)$. ■

Proof of Theorem 1 Obviously, the plurality rule f_P satisfies the five axioms. Conversely, suppose that f is a social choice function satisfying the five axioms. Due to Lemma 2, it suffices to consider only the case where $|\bigcup_{i\in N} T(P_i)| < |N|$.[5] Suppose that $m = \max_{x\in X} T(x, P_N)$. Consider a partition $\{N_1, \cdots, N_m\}$ of N such that for each component N_k if $i, j \in P_{N_k}$, then $T(P_i) \neq T(P_j)$. Then, for each N_k, $|\bigcup_{i\in N_k} T(P_i)| = |N_k|$, and hence Lemma 2 implies that $f(P_{N_k}) = \bigcup_{i\in N_k} T(P_i)$. Thus, if $T(x, P_N) = m$, then $\forall N_k$, $x \in f(P_{N_k})$; if $T(x, P_N) < m$, then $\exists N_k$, $x \notin f(P_{N_k})$. Hence, by *reinforcement*

$$f(P_N) = \bigcap_{k\in\{1,\cdots,m\}} f(P_{N_k}) = \arg\max_{x\in X} T(x, P_N) = f_P(P_N),$$

which completes the proof. ■

[5]The remainder of the proof is the same as the proof in Yeh (2008), since neither *efficiency* nor *faithfulness* is not used in the remainder. The same proof is also found in Ching (1996).

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