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# Asset Bubbles and Bailout\*

Tomohiro Hirano<sup>†</sup> and Noriyuki Yanagawa<sup>‡</sup>

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## Abstract

This paper theoretically investigates the relationship between asset price bubbles and bailout. We show that although bailout may mitigate adverse effects of bubbles' bursting ex-post, it is more likely to cause asset price bubbles by encouraging risk-taking behavior ex-ante. In other words, bubbles are more likely to occur, the more government bailout is anticipated. Moreover, we examine the effects of the anticipated bailout on boom-bust cycles. We find that when productivity is relatively low, the anticipated bailout accelerates output booms and creates large bubbles, thus destabilizing the economy. On the other hand, when productivity is relatively high, the anticipated bailout dampens output booms, thus stabilizing the economy. Finally, we analyze a desirable ex-post bailout policy.

*Key words:* Stochastic Bubbles, Anticipated Bailout, Emergence of Asset Price Bubbles, Bubbly Booms

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# 1 Introduction

Many countries have experienced bubble-like dynamics. Associated with the bursting part of asset price bubbles are significant contractions in real economic activity. Notable examples include the recent U.S. experiences after the financial crisis of 2007/2008 as well as Japan's experiences in the 1990s. To mitigate severe contractions, government tends to take various types of bailouts such as recapitalization through buying equity or through the purchase of troubled assets at inflated prices. Although these policies may mitigate the contractions ex-post, what happens if these policies are anticipated ex-ante? In this paper, we analyze ex-ante effects of bailout. In particular, we ask: how does the anticipated government bailout affect the emergence of asset price bubbles? What are the effects of the anticipated bailout on boom-bust cycles?

To tackle with these questions, we develop a macroeconomic model with stochastic bubbles.<sup>1</sup> Since seminal papers of Farhi and Tirole (2009a) and Ventura (2011), the recent development on rational bubbles have provided a theoretical framework to analyze macroeconomic effects of asset price bubbles (Caballero and Krishnamurthy, 2006; Kocherlakota, 2009; Hirano and Yanagawa, 2010a, 2010b; Martin and Ventura, 2010a, 2010b; Sakuragawa, 2010; Aoki and Nikolov, 2011; Miao and Wang, 2011).<sup>2</sup> The contribution of this paper is that we consider the effects of bailout within a rational bubbles model. The bailout we consider is a transfer policy from workers to entrepreneurs: recapitalization through buying legacy assets at inflated prices as discussed in Farhi and Tirole (2011). When bubbles collapse, the net worth of entrepreneurs decreases significantly, which in turn causes a free-fall in real economic activity. The primary purpose of bailout is to boost the net worth of entrepreneurs and to mitigate the free-fall.

Although the bailout is effective in mitigating the free-fall ex-post, there are side effects. In our framework, since bubble assets are risky in the sense that bubbles may collapse, risk-averse entrepreneurs want to hedge themselves by investing in safe assets. As we show, the entrepreneurs' portfolio decision depends on not only the probability of bursting of bubbles, but also expectations about government bailout. We will show that the anticipated

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<sup>1</sup>Weil (1987) is the first study that analyzes stochastic bubbles in a general equilibrium model.

<sup>2</sup>Kocherlakota (1992), Santos and Woodford (1997), and Hellwig and Lorenzoni (2009) analyze asset price bubbles in an endowment economy with infinitely lived agents.

bailout induces entrepreneurs to take on more risk ex-ante.

What is interesting is that through the change in risk-taking behaviors, the anticipated bailout affects the emergence of asset price bubbles. That is, when the bailout is not expected, bubbles cannot occur in low-productivity economy, because in that economy, the wealth's growth rate of the economy is too low to sustain growing bubbles.<sup>3</sup> However, when the bailout is expected, entrepreneurs are willing to buy more bubble assets and required rate of return of bubbles declines. This decline in turn lowers the growth rate of bubbles. As a result, even low-productivity economy enters bubble regions. As we show, although bailout may mitigate adverse effects of bursting bubbles ex-post, it is more likely to cause asset price bubbles. In other words, bubbles are more likely to occur, the more government bailout is anticipated.

Moreover, the anticipated bailout also greatly influences aggregate economic activity and asset price before the bubble bursts. As we show, when productivity is relatively low, the anticipated bailout induces entrepreneurs to take on more risk ex-ante, which accelerates bubbly booms and creates large bubbles, thus resulting in a large scale government intervention during bubbles' collapsing. In other words, the bailout policy that aims at stabilizing the economy during bubble bursts ends up with destabilizing the economy. On the other hand, when productivity is relatively high, the anticipated bailout dampens bubbly booms, thus stabilizing the economy.

To be sure, our paper is related to theoretical literature that examines government bailouts and risk-taking. For example, Chari and Kehoe (2010), Diamond and Rajan (2011), and Farhi and Tirole (2009b, 2011) stress moral hazard consequences of bailouts and other credit market interventions. Our paper is mainly different from these papers in the point that we analyze the ex-ante effects of bailout within a full blown dynamic macroeconomic model. In this respect, our paper is closely related to Gertler et al. (2011) and Brunnermeier and Sannikov (2011). There are differences in two respects. First, both of these papers analyze the ex-ante effects of bailout in terms of monetary policy, while our paper examines them from the perspective of capital injection policy. Second, Gertler et al. (2011) focus on the effects on the financial system, and Brunnermeier and Sannikov (2011) analyze the effects on risk-taking by the intermediary sector, while we examine the effects

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<sup>3</sup>Tirole (1985) is a seminal paper on rational bubbles. In the paper, Tirole shows that for the emergence of asset price bubbles, the economy's growth rate must be sufficiently high and it is indeed greater than interest rate in the bubbleless economy. This holds true in our model too.

on asset price bubbles.

## 2 The Model

### 2.1 Framework

Consider a discrete-time economy with one homogeneous good and a continuum of entrepreneurs and households (workers). A typical entrepreneur and a representative worker have the following expected discounted utility,

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t \log c_t^i \right], \quad (1)$$

where  $i$  is the index for each entrepreneur, and  $c_t^i$  is the consumption of him/her at date  $t$ .  $\beta \in (0, 1)$  is the subjective discount factor, and  $E_0[a]$  is the expected value of  $a$  conditional on information at date 0.

Let us start with the entrepreneurs, who play a central role in this paper. At each date, each entrepreneur meets high productive investment projects (hereinafter H-projects) with probability  $p$ , and low productive ones (L-projects) with probability  $1 - p$ .<sup>4</sup> The investment projects produce capital. The investment technologies are as follows:

$$k_{t+1}^i = \alpha_t^i z_t^i, \quad (2)$$

where  $z_t^i (\geq 0)$  is the investment level at date  $t$ , and  $k_{t+1}^i$  is the capital at date  $t + 1$ .  $\alpha_t^i$  is the marginal productivity of investment at date  $t$ .  $\alpha_t^i = \alpha^H$  if the entrepreneur has H-projects, and  $\alpha_t^i = \alpha^L$  if he has L-projects. We assume  $\alpha^H > \alpha^L$ . For simplicity, we assume that capital fully depreciates in one period.<sup>5</sup> The probability  $p$  is exogenous, and independent across entrepreneurs and over time. At the beginning of each date  $t$ , the entrepreneur knows his/her own type at date  $t$ , whether he has H-projects or L-projects. Assuming that the initial population measure of each type of the entrepreneur is  $p$  and  $1 - p$  at date 0, the population measure of each type after date 1

<sup>4</sup>A similar setting is used in Woodford (1990), Kiyotaki (1998), Kiyotaki and Moore (2008), Kocherlakota (2009), Nikolov (2010), and Aoki and Nikolov (2011).

<sup>5</sup>As in Kocherlakota (2009), we can consider a situation where some fraction of capital depreciate, and consumption goods can be converted one-for-one into capital at each date, and vice-versa. In this setting, we can also obtain the same results as in the present paper.

is  $p$  and  $1 - p$ , respectively. Throughout this paper, we call the entrepreneurs with H-projects (L-projects) "H-entrepreneurs" ("L-entrepreneurs").

We assume that because of frictions in a financial market, the entrepreneur can pledge at most a fraction  $\theta$  of the future return from his investment to creditors.<sup>6</sup> In such a situation, in order for debt contracts to be credible, debt repayment cannot exceed the pledgeable value. That is, the borrowing constraint becomes:

$$r_t b_t^i \leq \theta q_{t+1} \alpha_t^i z_t^i, \quad (3)$$

where  $q_{t+1}$  is the relative price of capital to consumption goods at date  $t + 1$ .<sup>7</sup>  $r_t$  and  $b_t^i$  are the gross interest rate and the amount of borrowing at date  $t$ , respectively. The parameter  $\theta \in (0, 1]$ , which is assumed to be exogenous, can be naturally taken to be the degree of imperfection of the financial market.

In this economy, there are bubble assets denoted by  $x$ . Total supply is assumed to be constant over time  $X$ . As in Tirole (1985), we define bubble assets as the assets that produce no real return, i.e., the fundamental value of the assets is zero. Here we consider stochastic bubbles. Following Weil (1987), we assume that in each period  $t$ , bubble price becomes zero (bubble bursts) with probability  $1 - \pi$ , given positive bubble price at date  $t - 1$ . Once they burst, they never arise again. This implies that bubbles persist with probability  $\pi (< 1)$  and their prices are positive until they switch to being equal to zero forever. Let  $P_t$  be the per unit price of bubble assets at date  $t$  in terms of consumption goods in the case where bubbles do not collapse at date  $t$ .

The entrepreneur's flow of funds constraint is given by

$$c_t^i + z_t^i + P_t x_t^i = q_t \alpha_{t-1}^i z_{t-1}^i - r_{t-1} b_{t-1}^i + b_t^i + P_t x_{t-1}^i. \quad (4)$$

where  $x_t^i$  be the level of bubble assets purchased by a type  $i$  entrepreneur at date  $t$ . The left hand side of (4) is expenditure on consumption, investment, and the purchase of bubble assets. The right hand side is the available funds at date  $t$ , which is the return from investment in the previous period minus debts repayment, plus borrowing and the sales of bubble assets. We define the net worth of the entrepreneur at date  $t$  as  $e_t^i \equiv q_t \alpha_{t-1}^i z_{t-1}^i - r_{t-1} b_{t-1}^i + P_t x_{t-1}^i$ .

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<sup>6</sup>See Hart and Moore (1994) and Tirole (2006) for the foundations of this setting.

<sup>7</sup>On an equilibrium path we consider,  $q_{t+1}$  is not affected by whether bubbles collapse or not. Hence, there is no uncertainty with regard to  $q_{t+1}$ .

We also impose the short sale constraint on bubble assets:<sup>8</sup>

$$x_t^i \geq 0. \quad (5)$$

Next let us turn to the households. There is a unit measure of households. Each household is endowed with one unit of labor endowment in each period, which is supplied inelastically in labor markets, and earns wage rate,  $w_t$ . The flow of funds constraint, the borrowing constraint, and the short sale constraint for households are given by

$$c_t^u + P_t(x_t^u - x_{t-1}^u) = w_t - r_{t-1}b_{t-1}^u + b_t^u - T_t, \quad (6)$$

$$r_t b_t^u \leq 0, \quad (7)$$

$$x_t^u \geq 0, \quad (8)$$

where  $^u$  represents households. Government levies a lump sum tax  $T_t$  on households.  $T_t > 0$  only when bubbles collapse.  $T_t = 0$  if they survive. The tax revenues are used for bailout of entrepreneurs.<sup>9</sup> Households (tax payers) pay the cost of the bubble bursts. Equation (7) says that households cannot borrow, since they do not have any collateralizable assets such as returns from investment projects.

There are competitive firms which produce final consumption goods using capital and labor.<sup>10</sup> The aggregate production function is

$$Y_t = K_t'^{\sigma} N_t^{1-\sigma}, \quad (9)$$

where  $K_t'$  and  $N_t$  are the aggregate capital stock and labor input at date  $t$ .  $Y_t$  is the aggregate output at date  $t$ . Factors of production are paid their

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<sup>8</sup>Kocherlakota (1992) shows that the short sale constraint plays an important role for the emergence of asset bubbles in an endowment economy with infinitely lived agents.

<sup>9</sup>In the present paper, in order to focus on a transfer policy from workers to entrepreneurs as in Farhi and Tirole (2011), we do not consider the capital income tax. Even if we consider the capital income tax, we can obtain the same results as in the main text, although the borrowing constraint for entrepreneurs and the investment function would be complicated as analyzed in Aoki et al. (2009).

<sup>10</sup>We assume that each firm is operated by the workers. Since the final goods market is competitive, the net profit from operating the firm is zero, so that the flow of funds constraint of the workers is unchanged as equation (6) in equilibrium.

marginal product:

$$q_t = \sigma K_t'^{\sigma-1} \quad \text{and} \quad w_t = (1 - \sigma) K_t'^{\sigma}. \quad (10)$$

## 2.2 Bailout

A bailout we consider is a transfer policy from households to entrepreneurs whose net worth decreases by bubbles collapse. As in Farhi and Tirole (2011), the aim of this policy is to boost the net worth of entrepreneurs.

We assume that each entrepreneur as well as each household anticipates government's bailout with probability  $\lambda \in [0, 1]$ . Since we assume rational expectations, a fraction  $\lambda$  of the entrepreneurs is indeed rescued ex-post. Suppose that bubbles collapse at date  $t$ . The entrepreneurs who hold  $x_{t-1}^i$  amount of bubble assets at the beginning of date  $t$  lose their net worth by  $P_t x_{t-1}^i$ . Here we consider the bailout that fully recovers the net worth of the rescued entrepreneurs, i.e., denote the public funds injected into each entrepreneur by  $m_t^i$ ,  $m_t^i = P_t x_{t-1}^i$ . Thus, total public funds injected into the entrepreneurial sector at date  $t$  is  $M_t = \lambda P_t X$ . In this paper, we want to focus on how a change in  $\lambda$  affects asset price bubbles, ex-ante efficiency, and macro-dynamics.<sup>11</sup> Because of this policy, the net worth of entrepreneurs at date  $t$  is restored by  $M_t$ , which can mitigate adverse effects of bubbles' collapsing.

## 2.3 Equilibrium

Let us denote the aggregate consumption of H-and L-entrepreneurs and the workers at date  $t$  as  $\sum_{i \in H_t} c_t^i \equiv C_t^H$ ,  $\sum_{i \in L_t} c_t^i \equiv C_t^L$ ,  $C_t^u$ , where  $H_t$  and  $L_t$  mean a family of H-and L-entrepreneurs at date  $t$ . Similarly, let  $\sum_{i \in H_t} z_t^i \equiv Z_t^H$ ,  $\sum_{i \in L_t} z_t^i \equiv Z_t^L$ ,  $\sum_{i \in H_t} b_t^i \equiv B_t^H$ ,  $\sum_{i \in L_t} b_t^i \equiv B_t^L$ ,  $B_t^u$ ,  $\sum_{i \in H_t \cup L_t} k_t^i \equiv K_t$ ,  $(\sum_{i \in H_t \cup L_t} x_t^i + X_t^u) \equiv X_t$  be the aggregate investment, the aggregate borrowing, the aggregate capital stock, and the aggregate demand for bubble

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<sup>11</sup>Of course, we can change the amount of public money injected into each entrepreneur. For example, we can consider a situation where a fraction  $\eta \leq 1$  of the lost net worth by bubbles' burst is injected. By so doing, we may be able to analyze how changes in both  $\lambda$  and  $\eta$  affect asset price bubbles and boom-bust cycles. Although this case may be more realistic, it is hard to solve analytically. Hence in the present paper, we assume  $\eta = 1$ , and we want to focus on how a change in  $\lambda$  influences asset price bubbles and boom-bust cycles.



assets. Then the market clearing condition for goods, credit, capital, labor, and bubble assets are

$$C_t^H + C_t^L + C_t^u + Z_t^H + Z_t^L = Y_t, \quad (11)$$

$$B_t^H + B_t^L + B_t^u = 0, \quad (12)$$

$$K_t' = K_t, \quad (13)$$

$$N_t = 1, \quad (14)$$

$$X_t = X. \quad (15)$$

The competitive equilibrium is defined as a set of prices  $\{r_t, w_t, P_t\}_{t=0}^\infty$  and quantities  $\{C_t^H, C_t^L, C_t^u, B_t^H, B_t^L, B_t^u, Z_t^H, Z_t^L, G_t, X_t, K_{t+1}, Y_t\}_{t=0}^\infty$ , such that (i) the market clearing conditions, (11)-(15), are satisfied in each period, and (ii) each entrepreneur chooses consumption, borrowing, investment, capital stock, and the amount of bubble assets,  $\{c_t^i, b_t^i, z_t^i, k_{t+1}^i, x_t^i\}_{t=0}^\infty$ , to maximize his expected discounted utility (1) under the constraints (2)-(5). (iii) each household chooses consumption, borrowing, and the amount of bubble assets,  $\{c_t^u, b_t^u, x_t^u\}_{t=0}^\infty$ , to maximize his/her expected discounted utility (1) under the constraints (6)-(8).

## 2.4 Entrepreneur's Behavior

We now characterize the equilibrium behavior of entrepreneurs and households. We consider the case

$$q_{t+1}\alpha^L \leq r_t < q_{t+1}\alpha^H.$$

In equilibrium, interest rate must be at least as high as  $q_{t+1}\alpha^L$ , since nobody lends to the projects if  $r_t < q_{t+1}\alpha^L$ .

Households consume all the wage income in each period. That is,  $c_t^w = w_t$  and  $b_t^w = 0$ . Hence,  $C_t^w = w_t$  and  $B_t^w = 0$  hold. We later verify this in Appendix. In this analysis, workers play little role, except to soak up the returns to labor.

For the entrepreneurs, both the borrowing constraint and the short sale constraint simultaneously become binding for H-entrepreneurs, but not binding for L-entrepreneurs. Since the utility function is log-linear, each entrepreneur consumes a fraction  $1 - \beta$  of the net worth in each period, that is,

$c_t^i = (1 - \beta)e_t^i$ .<sup>12</sup> Then, by using (3), (4), and (5), the investment function of H-entrepreneurs at date  $t$  can be written as

$$z_t^i = \frac{\beta e_t^i}{1 - \frac{\theta q_{t+1} \alpha^H}{r_t}}. \quad (16)$$

This is a popular investment function under financial constraint problems, except that the presence of bubble assets affects the net worth.<sup>13</sup> We see that the investment equals the leverage,  $1/[1 - (\theta q_{t+1} \alpha^H / r_t)]$ , times a fraction  $\beta$  of the net worth. From this investment function, we also understand that for the entrepreneurs who purchased bubble assets in the previous period, they are able to sell those assets at the time they encounter H-projects. In our analysis, the entrepreneurs buy bubble assets when they have L-projects, and sell those bubbles when they have opportunities to invest in H-projects. As a result, their net worth increases (compared to the bubbleless case), which relaxes the borrowing constraint and boosts their investments. That is, bubbles generate "balance sheet effects". Moreover, the expansion level of the investment is more than the direct increase of the net worth because of the leverage effect.

For L-entrepreneurs at date  $t$ , their portfolio problem is more complicated because they take the government rescues into account. Since bubble assets deliver no return with probability  $1 - \pi$ , risk-averse L-entrepreneurs may want to hedge themselves by investing in L-projects as well as lending to other entrepreneurs. Since  $c_t^i = (1 - \beta)e_t^i$ , the budget constraint (4) becomes

$$z_t^i + P_t x_t^i - b_t^i = \beta e_t^i. \quad (17)$$

Each L-entrepreneur allocates his/her savings,  $\beta e_t^i$ , into  $z_t^i$ ,  $P_t x_t^i$ , and  $b_t^i$ . Since investing in L-projects and lending are perfect substitutes,  $z_t^i \geq 0$  if  $r_t = q_{t+1} \alpha^L$ , and  $z_t^i = 0$  if  $r_t > q_{t+1} \alpha^L$ . That is, the following conditions must be satisfied:

$$(r_t - q_{t+1} \alpha^L) z_t^i = 0, \quad z_t^i \geq 0, \quad \text{and} \quad r_t - q_{t+1} \alpha^L \geq 0.$$

Each L-entrepreneur chooses optimal amounts of  $b_t^i$ ,  $x_t^i$ , and  $z_t^i$ , so that the expected marginal utility from investing in three assets respectively is

<sup>12</sup>See, for example, chapter 1.7 of Sargent (1988).

<sup>13</sup>See, for example, Bernanke and Gertler (1989), Bernanke et al. (1999), Holmstrom and Tirole (1998), Kiyotaki and Moore (1997), and Matsuyama (2007, 2008).

equalized. The first order conditions with respect to  $b_t^i$  and  $x_t^i$  are

$$(b_t^i) : \frac{1}{c_t^i} = \pi\beta\frac{r_t}{c_{t+1}^{i,\pi}} + (1-\pi)\lambda\beta\frac{r_t}{c_{t+1}^{i,(1-\pi)\lambda}} + (1-\pi)(1-\lambda)\beta\frac{r_t}{c_{t+1}^{i,(1-\pi)(1-\lambda)}}, \quad (18)$$

$$(x_t^i) : \frac{1}{c_t^i} = \pi\beta\frac{1}{c_{t+1}^{i,\pi}}\frac{P_{t+1}}{P_t} + (1-\pi)\lambda\beta\frac{1}{c_{t+1}^{i,(1-\pi)\lambda}}\frac{P_{t+1}}{P_t}, \quad (19)$$

where  $c_{t+1}^{i,\pi} = (1-\beta)(q_{t+1}\alpha^L z_t^i - r_t b_t^i + P_{t+1} x_t^i)$  is the date  $t+1$  consumption of him/her when bubbles persist. The first term of the right hand side in equation (18) and (19) represents the expected marginal utility from lending a unit and from buying a unit of bubble assets in this case.  $c_{t+1}^{i,(1-\pi)\lambda} = (1-\beta)(q_{t+1}\alpha^L z_t^i - r_t b_t^i + m_{t+1}^i)$  is the date  $t+1$  consumption of him/her when bubbles collapse and the government rescues him/her. The second term of equation (18) and (19) represents the expected marginal utility from lending a unit and from buying a unit of bubble assets in this case, respectively.  $c_{t+1}^{i,(1-\pi)(1-\lambda)} = (1-\beta)(q_{t+1}\alpha^L z_t^i - r_t b_t^i)$  is the date  $t+1$  consumption when bubbles collapse and the government does not rescue him/her. The third term of equation (18) represents the expected marginal utility from lending a unit in this case.<sup>14</sup>  $P_{t+1}/P_t$  is the rate of return of bubbles when bubbles persist.

From (17), (18), and (19), we can derive the demand function for bubble assets of a type  $i$  agent:

$$P_t x_t^i = \frac{\delta \frac{P_{t+1}}{P_t} - r_t}{\frac{P_{t+1}}{P_t} - r_t} \beta e_t^i, \quad (20)$$

where  $\delta = \pi + (1-\pi)\lambda$ .

The remaining fraction of savings is split across  $z_t^i$  and  $b_t^i$ :

$$z_t^i - b_t^i = \frac{(1-\delta)\frac{P_{t+1}}{P_t}}{\frac{P_{t+1}}{P_t} - r_t} \beta e_t^i.$$

From (20), we see that an entrepreneurs's portfolio decision depends on

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<sup>14</sup>Since the entrepreneur consumes a fraction  $1-\beta$  of the current net worth in each period, the optimal consumption level at date  $t+1$  is independent of the entrepreneur's type at date  $t+1$ . It only depends on whether bubbles collapse or whether government rescues the entrepreneur.

its perceptions of risk, which in turn depends on both the probability of bursting of bubbles ( $\pi$ ) and expectations about the government bailout ( $\lambda$ ). Here we obtain the following Proposition.

**Proposition 1** *Ceteris paribus,  $x_t^i$  is an increasing function of  $\lambda$ .*

When  $\lambda$  rises, the type  $i$  agent is willing to buy more bubble assets. This means that the anticipated bailout induces the entrepreneurs to take on more risk, even though the potential probability of the bubble bursts remain unchanged ( $\pi$  is unchanged).

## 2.5 Aggregation

We are now in a position to consider the aggregate economy. The great merit of the expressions for each entrepreneur's investment and the demand for bubble assets,  $z_t^i$  and  $x_t^i$ , is that they are linear in period- $t$  net worth,  $e_t^i$ . Hence aggregation is easy: we do not need to keep track of the distributions. From (16), we can derive the aggregate H-investments:

$$Z_t^H = \frac{\beta p A_t}{1 - \frac{\theta q_{t+1} \alpha^H}{r_t}}, \quad (21)$$

where  $A_t \equiv \sigma K_t^\sigma + P_t X$  is the aggregate wealth of entrepreneurs at date  $t$ , and  $\sum_{i \in H_t} e_t^i = p A_t$  is the aggregate wealth of H-entrepreneurs at date  $t$ . Recall that probability of meeting H-projects is independently distributed. We learn that the aggregate investments of H-entrepreneurs depend on asset price,  $P_t$ , as well as cash flows from the investment projects in the previous period,  $\sigma K_t^\sigma$ . In this respect, this investment function is similar to the one in Kiyotaki and Moore (1997). There is a significant difference. In the Kiyotaki-Moore model, the investment function depends on land prices which reflect fundamentals (cash flows from the present to the future), while in our model, it depends on bubble price.

When  $r_t = q_{t+1} \alpha^L$ , L-entrepreneurs may invest. Their aggregate investments are determined by the goods market clearing condition (11). Since H- and L-entrepreneurs consume a fraction  $1 - \beta$  of their net worth, (11) can be written as

$$Z_t^H + Z_t^L + P_t X = \beta A_t \quad (22)$$

Thus,

$$Z_t^L = \beta A_t - \frac{\beta p A_t}{1 - \frac{\theta q_{t+1} \alpha^H}{r_t}} - P_t X. \quad (23)$$

The aggregate L-investments are equal to the aggregate savings of entrepreneurs minus the aggregate H-investments and the purchase of bubble assets.

The aggregate counterpart to (20) is

$$P_t X_t = \frac{\delta \frac{P_{t+1}}{P_t} - r_t}{\frac{P_{t+1}}{P_t} - r_t} \beta (1-p) A_t, \quad (24)$$

where  $\sum_{i \in L_t} e_t^i = (1-p)A_t$  is the aggregate net worth of L-entrepreneurs at date  $t$ . (24) is the aggregate demand function for bubble assets at date  $t$ .

## 2.6 Dynamics

Using (21) and (23), we can derive evolution of capital:

$$K_{t+1} = \begin{cases} \alpha^H \frac{\beta p A_t}{1 - \frac{\theta \alpha^H}{\alpha^L}} + \alpha^L \left( \beta A_t - \frac{\beta p A_t}{1 - \frac{\theta \alpha^H}{\alpha^L}} - P_t X \right) & \text{if } r_t = q_{t+1} \alpha^L, \\ \alpha^H \frac{\beta p A_t}{1 - \frac{\theta q_{t+1} \alpha^H}{r_t}} = \alpha^H [\beta A_t - P_t X] & \text{if } r_t > q_{t+1} \alpha^L. \end{cases} \quad (25)$$

When  $r_t = q_{t+1} \alpha^L$ , L-entrepreneurs may invest in equilibrium, i.e.,  $Z_t^L \geq 0$ . The first term and the second term in the first line represent the capital at date  $t + 1$  produced by H- and L-entrepreneurs. When  $r_t > q_{t+1} \alpha^L$ , L-entrepreneurs never invest, i.e.,  $Z_t^L = 0$ , and only H-entrepreneurs invest. From (22), we know  $Z_t^H = \beta A_t - P_t X$ . ( $-P_t X$ ) in the first and the second lines captures a traditional crowd-out effect of bubbles (Tirole, 1985), i.e., some of entrepreneurs' savings flow to bubble assets (since L-entrepreneurs buy bubbles), which crowds investments out.

The aggregate wealth of entrepreneurs evolves over time as

$$A_{t+1} = \begin{cases} q_{t+1} \left[ \alpha^H \frac{\beta p A_t}{1 - \theta \alpha^H} + \alpha^L \left( \beta A_t - \frac{\beta p A_t}{1 - \theta \alpha^H} - P_t X \right) \right] + P_{t+1} X & \text{if } r_t = q_{t+1} \alpha^L, \\ q_{t+1} \alpha^H [\beta A_t - P_t X] + P_{t+1} X & \text{if } r_t > q_{t+1} \alpha^L. \end{cases} \quad (26)$$

The first term in the first line represent the returns from H-and L-investments undertaken at date  $t$ . The second term is the value of bubbles at date  $t + 1$ .

Defining  $\phi_t \equiv P_t X / \beta A_t$  as the size of bubbles (the share of the value of bubbles),  $\phi_t$  evolves over time as

$$\phi_{t+1} = \frac{\frac{P_{t+1}}{A_{t+1}}}{\frac{P_t}{A_t}} \phi_t. \quad (27)$$

The evolution of the size of bubbles depends on the relation between wealth's growth rate (denominator) and bubbles' growth rate (numerator).

(24) can be solved for  $P_{t+1}/P_t$  using  $\phi_t$ ,

$$\frac{P_{t+1}}{P_t} = \frac{r_t(1 - p - \phi_t)}{\delta(1 - p) - \phi_t}. \quad (28)$$

It follows that  $P_{t+1}/P_t$  is a decreasing function of  $\lambda$ . When  $\lambda$  rises, ceteris paribus, the entrepreneur's required rate of return on bubble assets becomes lower, because bubble assets become safer.

When  $r_t > q_{t+1} \alpha^L$ , from (22), interest rate is determined by

$$r_t = \frac{q_{t+1} \theta \alpha^H (1 - \phi_t)}{1 - p - \phi_t}.$$

Thus equilibrium interest rate is

$$r_t = q_{t+1} \text{Max} \left[ \alpha^L, \frac{\theta \alpha^H (1 - \phi_t)}{1 - p - \phi_t} \right]. \quad (29)$$

Note that when  $r_t > q_{t+1} \alpha^L$ ,  $r_t$  is an increasing function of  $\phi_t$ , reflecting the tightness of the credit markets.

Moreover, in order that bubbles do not explode, the following condition

must be satisfied:

$$\frac{A_{t+1}}{A_t} \geq \frac{P_{t+1}}{P_t}. \quad (30)$$

This condition means that wealth's growth rate must be greater than the growth rate of bubbles. Otherwise, the size of bubbles explodes and the economy eventually cannot sustain bubbles.

The dynamical system of this economy can be characterized by (25)-(30). We can obtain the following Proposition.

**Proposition 2** *There is a saddle point path on which the economy converges to an equilibrium where all variables  $(K_t, A_t, q_t, r_t, w_t, P_t, \phi_t)$  become constant over time as long as bubbles persist.*

**Proof.** See Appendix. ■

When the economy gets on the saddle path,  $\phi_t$  becomes constant over time, i.e., aggregate wealth of entrepreneurs and asset price grow at the same rate.<sup>15</sup>

Thus, rearranging (25), we can derive simple difference equations concerning the capital stock:

$$K_{t+1} = \begin{cases} \frac{\alpha^L p \beta}{1 - \delta \beta (1 - p)} \frac{\alpha^H (1 - \theta)}{\alpha^L - \theta \alpha^H} \sigma K_t^\sigma & \text{if } r_t = q_{t+1} \alpha^L \\ \alpha^H \frac{\beta [1 - \delta (1 - p)] + (1 - \beta) \theta}{1 - \delta \beta (1 - p)} \sigma K_t^\sigma & \text{if } r_t > q_{t+1} \alpha^L. \end{cases} \quad (31)$$

And equilibrium asset price,  $P_t$ , follows

$$P_t = \frac{\beta \phi}{X (1 - \beta \phi)} \sigma K_t^\sigma. \quad (32)$$

The asset price rises together with capital stock.

As long as bubbles persist, the economy runs according to the above equations, and converges to the equilibrium where all variables  $(K_t, A_t, q_t, r_t, w_t, P_t)$  become constant over time, which Farhi and Tirole (2009a) call a conditional bubbly steady-state.

A feature of bubbly dynamics is that there is a two-way interaction between asset price and aggregate quantity. An increase in cash flows in period

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<sup>15</sup>On the saddle path, aggregate wealth, aggregate output, aggregate consumption, and asset price all grow at the same rate.

$t$  raises period  $t$  bubble price, which in turn increases cash flows and bubble price in period  $t + 1$  even further. These knock-on effects will continue not only in period  $t + 1$ , but also in period  $t + 2, t + 3, \dots$ . Moreover, this anticipated increase in the bubble price is reflected in period  $t$  bubble price, since the bubble price is a forward looking variable. In equilibrium, all these mechanisms occur simultaneously, and capital stock and asset price run according to (31) and (32).

## 2.7 Anticipated Bailout and Asset Bubbles

In order that stochastic bubbles can exist, the following condition must be satisfied:<sup>16</sup>

$$\phi > 0.$$

We can obtain the following Proposition.

**Proposition 3** *The existence condition of stochastic bubbles in the case where government bailout is anticipated with probability  $\lambda$  is*

$$\theta < \delta\beta(1 - p).$$

and

$$\alpha^H > \alpha^L \frac{1 - \delta(1 - p)\beta}{(1 - \delta\beta)\theta + p\beta\delta} \equiv \alpha_1^H,$$

**Proof.** See Appendix. ■

It follows that  $\alpha_1^H$  is a decreasing function of  $\lambda$ . The more  $\lambda$  rises, the wider bubble regions becomes. In other words, bubbles are more likely to occur, the more government bailout is anticipated.

In order to understand the mechanism, let us first explain the case of  $\lambda = 0$ , i.e., no government bailout is anticipated. In this case, for stochastic bubbles to exist,  $\theta$  must be low enough and  $\alpha^H$  must be sufficiently high. Intuitively, in high  $\theta$  regions, since interest rate is so great, the bubbles' growth rate is so high that the economy cannot sustain growing bubbles. Thus, in high  $\theta$  regions, bubbles cannot occur. On the other hand, in low  $\theta$

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<sup>16</sup>If  $\phi \leq 0$ , even the other equilibrium path with bubbles except for the saddle one cannot exist (see footnote 13). Thus, no equilibrium path with bubbles can exist if  $\phi \leq 0$ .



regions, interest rate is lower and so is the bubbles' growth rate. As long as  $\alpha^H$  is sufficiently high, wealth's growth rate becomes sufficiently high that the economy can sustain growing bubbles.<sup>17</sup> From these observations, we learn that bubbles cannot occur in low-productivity economy or in economy with more efficient financial market.<sup>18</sup>

When bailout is expected, required rate of return of bubbles becomes lower and so does their growth rate. Thus, even economy with lower productivity or with more efficient financial market enters bubble regions. Proposition 3 says that although bailout mitigates adverse effects of bubbles' collapsing on the economy ex-post, it is more likely to cause asset price bubbles.<sup>19</sup> Figure 1 illustrates the relationship between bubble regions and  $\lambda$ .

Moreover, we learn the relationship between  $\lambda$  and  $\phi$ .

**Proposition 4**  *$\phi$  is an increasing function of  $\lambda$ .*

**Proof.** See Appendix. ■

Proposition 4 says that the anticipated bailout leads to large size bubbles. Together with (32), we learn that when bailout is expected with higher probability at date  $t$ , the asset price jumps up at date  $t$  instantaneously.

Before going further, let us mention how interest rate is determined in equilibrium. The following Proposition summarizes this.

**Proposition 5** *Equilibrium interest rate in the bubble economy depends on the level of productivity,  $\alpha^H$ .*

$$r_t = \begin{cases} q_{t+1}\alpha^L & \text{if } \alpha^H \in (\alpha_1^H, \alpha_2^H], \\ \theta q_{t+1}\alpha^H\gamma & \text{if } \alpha^H > \alpha_2^H, \end{cases}$$

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<sup>17</sup>We can use the structure of the bubbleless economy to characterize the existence condition. The existence condition also says that the interest rate is sufficiently lower than the economy's growth rate in the bubbleless economy. This condition is similar to the existence condition in Tirole(1985). In our model, since we consider stochastic bubbles, interest rate must be sufficiently low in the bubbleless economy. Otherwise, stochastic bubbles cannot arise.

<sup>18</sup>Here we should note that if

$$\alpha^H < \frac{\alpha^L}{\delta\beta},$$

then, stochastic bubbles cannot exist. In other words, if productivity is too low, bubbles never arise for any  $\theta$ .

<sup>19</sup>We can consider the same thought experiment concerning the relationship between  $\lambda$  and  $p$ .

where  $\gamma \equiv \frac{\beta[1-\delta(1-p)]+(1-\beta)\theta}{\beta(1-p)(1-\delta)+(1-\beta+p\beta)\theta}$  and  $\alpha_2^H = \alpha^L/\theta\gamma$ .  $\alpha_2^H$  is a decreasing function of  $\lambda$ .

**Proof.** See Appendix. ■

Intuitively, when productivity is low, L-entrepreneurs cannot lend enough to H-entrepreneurs because collateral value is low. L-entrepreneurs hold idle savings, but they cannot invest all of those savings in bubble assets, because bubble assets are risky. They end up with investing in their own L-projects for risk-hedge. On the other hand, when productivity is high, since they can lend enough to H-entrepreneurs, they do not need to invest in L-projects for risk-hedge.

### 3 Macroeconomic Effects of Anticipated Bailout

In this section, we investigate how the anticipated bailout affects macroeconomic performance before the bubble bursts.

#### 3.1 Case with $\lambda = 0$

Let us first analyze the case with  $\lambda = 0$ , i.e., no government bailout is anticipated. We compare the bubble economy and the bubbleless economy. The following Proposition summarizes macroeconomic impacts of stochastic bubbles.

**Proposition 6** *Suppose that the economy is in the steady-state of the bubbleless economy until date  $t = t' - 1$ . Then, at date  $t = t'$ , bubbles occur unexpectedly. Macroeconomic impacts of stochastic bubbles are different depending on the value of  $\alpha^H$ . There are two regions.*

- In region 1 where  $\alpha^H \in (\alpha_1^H, \alpha_3^H]$ ,  $Y_t$  increase after date  $t = t' + 1$  onwards until bubbles collapse. In the conditional steady-state, the aggregate output under the stochastic bubbly economy is higher than that under the bubbleless economy.  $\alpha_3^H$  is the greater value of the following quadratic equation:  $\alpha^H \frac{\beta[1-\pi(1-p)]+(1-\beta)\theta}{1-\pi\beta(1-p)} = \alpha^H \frac{\alpha^L\beta p}{\alpha^L-\theta\alpha^H} + \alpha^L(\beta - \frac{\alpha^L\beta p}{\alpha^L-\theta\alpha^H})$ .
- In region 2 when  $\alpha^H > \alpha_3^H$ ,  $Y_t$  decrease after date  $t = t' + 1$  onwards until bubbles collapse. In the conditional steady-state, the aggregate

*output under the stochastic bubbly economy is lower than that under the bubbleless economy.*

**Proof.** See Appendix. ■

In region 1 where productivity is relatively low, the crowd-in effect dominates the crowd-out effect, i.e., bubbles are expansionary until they burst. Intuitively, when  $\alpha^H$  is relatively low and without bubbles, enough savings cannot be transferred to H-projects, because the value of collateral is low. As a result, L-entrepreneurs end up with investing their idle savings in their L-projects with low returns. Bubbles provide a high return vehicle for them, thus improving efficiency in production by reducing L-projects.

In region 2 where productivity is relatively high, the crowd-out effect dominates the crowd-in effect. Hence, bubbles are contractionary until they burst. Intuition is that when  $\alpha^H$  is relatively high, the value of collateral is high enough and enough savings can be transferred to H-projects without bubbles. In such a situation, when bubbles occur, they crowd savings away from H-projects, thereby reducing H-investments.

Farhi and Tirole (2009a) also analyze whether bubbles are expansionary or contractionary. They find that it depends on what they call outside liquidity, while, in our model, it depends on productivity.<sup>20</sup> Figure 1 illustrates the effects of stochastic bubbles on capital stock in the conditionally steady-state.

### 3.2 Case with $\lambda > 0$

We now compare the bubble economy with  $\lambda = 0$  and with  $\lambda > 0$ . From (31) and Proposition 5,

$$K_{t+1} = \begin{cases} \frac{\alpha^L p \beta}{1 - \delta \beta (1 - p)} \frac{\alpha^H (1 - \theta)}{\alpha^L - \theta \alpha^H} \sigma K_t^\sigma & \text{if } \alpha^H \in (\alpha_1^H, \alpha_2^H], \\ \alpha^H \frac{\beta [1 - \delta (1 - p)] + (1 - \beta) \theta}{1 - \delta \beta (1 - p)} \sigma K_t^\sigma & \text{if } \alpha^H > \alpha_2^H, \end{cases} \quad (33)$$

where  $K_{t+1}$  is an increasing function of  $\lambda$  in  $\alpha^H \in (\alpha_1^H, \alpha_2^H]$ , while it is a decreasing function of  $\lambda$  in  $\alpha^H > \alpha_2^H$ .

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<sup>20</sup>We can also characterize the relationship between macroeconomic effects of stochastic bubbles and  $\theta$ , given  $\alpha^H$ . Hirano and Yanagawa (2010a) characterize the relationship between the effects of bubbles on long-run economic growth rate and  $\theta$  within an endogenous growth framework.

From (33), we can understand how a change in  $\lambda$  affects output dynamics. The following Proposition summarizes this.

**Proposition 7** *Suppose that until date  $t = s - 1$ ,  $\lambda = 0$ . Then, at date  $t = s$ ,  $\lambda$  changes to any  $0 < \lambda \leq 1$  unexpectedly, and  $0 < \lambda \leq 1$  for all  $t \geq s$ . The impact of the change in  $\lambda$  on aggregate output after date  $t = s$  onwards depends on the level of productivity,  $\alpha^H$ . There are three regions.*

- *In region 3 where  $\alpha^H \in (\alpha_1^H, \alpha_1^H]$ , the economy enters bubble region after date  $t = s$ .*
- *In region 4 where  $\alpha^H \in (\alpha_1^H, \alpha_4^H]$ , suppose that until date  $t = s - 1$ , the economy is in the conditional steady-state of the bubble economy. Then,  $Y_t$  increase after date  $t = s + 1$  onwards until bubbles collapse.  $\alpha_4^H = \frac{\alpha^L}{\theta} \left[ 1 - p\beta(1 - \theta) / \frac{\beta[1 - \delta(1 - p)] + (1 - \beta)\theta}{1 - \delta\beta(1 - p)} \right]$ .*
- *In region 5 where  $\alpha^H > \text{Max} [\alpha_1^H, \alpha_4^H]$ , suppose that until date  $t = s - 1$ , the economy is in the conditional steady-state of the bubble economy. Then,  $Y_t$  decrease after date  $t = s + 1$  onwards.*

**Proof.** See Appendix. ■

In region 3 where productivity is low enough, the economy is in the bubbleless region before the government bailout is anticipated. However, once the bailout is expected at date  $t = s$ , the economy enters bubble region. When bubbles occur, they are expansionary (see Proposition 6).<sup>21</sup>

In region 4 where productivity level is in middle range, the anticipated bailout accelerates output booms. When bailout is expected at date  $t = s$ , L-entrepreneurs take on risk by buying more bubble assets instead of investing in L-projects. The asset price,  $P_t$ , jumps up at date  $t = s$ , although aggregate output does not respond immediately. Together with the increase in the asset price, the net worth of H-entrepreneurs increases and their investments also jump up at date  $t = s$ , while the share of L-projects is reduced. In other words, efficiency in investment improves at date  $t = s$ . Thus, aggregate

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<sup>21</sup>When  $\lambda = 1$ , bubbles can occur even in the economy with  $\alpha^H = \alpha_{1,bail}^H$ . Entrepreneurs behaves as if bubbles never collapsed. Equilibrium in this case is equivalent to the one in deterministic bubbles. In this case, even if bubbles occur, all macroeconomic variables in the bubble economy is the same as the ones in the bubbleless economy, because the rate of return of bubbles equals the rate of return of L-projects.

capital stock, output, and wage rate all jump up at date  $t = s + 1$ , and they all continue to increase over time until bubbles collapse. Moreover, because of the acceleration of aggregate output, the asset price continues to rise even further and large bubbles are created, which in turn requires a large scale government intervention when bubbles collapse.<sup>22</sup> Figure 2 captures how the change in  $\lambda$  affects aggregate output's dynamics in region 4, and Figure 3 summarizes the impulse response of macroeconomic variables when bailout is expected in region 4.<sup>23</sup>

In region 5 where productivity is high enough, more savings are allocated to H-projects even before government intervention is anticipated, because the value of collateral is high. When bailout is expected at date  $t = s$ , interest rate rises substantially, which crowds savings away from H-projects, thus dampening output booms.<sup>24</sup> Figure 4 illustrates dampening effects on output booms.<sup>25</sup>

Figure 5 summarizes the effect of the change in  $\lambda$  on the level of aggregate capital stock in the conditional steady-state.<sup>26</sup>

## 4 Ex-Post Bailout that Maximizes Output

In this final section, we examine ex-post bailout that maximizes output. We know that output gets higher, the more capital is installed. We also know that if capital stock before the bubble bursts is high, the capital stock after the bubble bursts is also high, and vice versa. Thus, the ex-post bailout policy that maximizes output is to maximize the level of capital stock before the bubble bursts. Indeed, depending on the level of productivity, there is an

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<sup>22</sup>Large bubbles are created, Large amount of public money,  $\lambda P_t X$ , needs to be injected into the entrepreneurial sector to rescue the economy. Otherwise, a large free-fall occurs. This suggests that the bailout policy that aims at stabilizing the economy during bubble bursts ends up with increasing boom-bust cycles.

<sup>23</sup>Figure 5, we consider a situation where until date  $s$ , the economy runs on the bubbly dynamics toward the conditional steady-state.

<sup>24</sup>In region 3, although the anticipated bailout dampens aggregate capital stock, aggregate output, and wage rate after date  $t = s' + 1$ , the impacts on asset price and entrepreneurs' consumption are ambiguous and depend on parameter values.

<sup>25</sup>In region 5, the anticipated bailout decreases boom-bust cycles, thus stabilizing the economy.

<sup>26</sup>We can also characterize the effects of the change in  $\lambda$  on aggregate economic activity and asset price with  $\theta$ , given  $\alpha^H$ .

optimal value of  $\lambda \equiv \lambda^*$  that maximizes output. The following Proposition summarizes this.

**Proposition 8** *Ex-post bailout that maximizes output depends on the level of productivity,  $\alpha^H$ . In  $\alpha^H \geq \alpha_3^H$ ,  $\lambda^* = 0$ . In  $\alpha^H \in (\alpha_1^H(\lambda = 1), \alpha_3^H)$ ,  $\lambda^* = \hat{\lambda}$ , where  $\hat{\lambda}$  is the value of  $\lambda$  at which  $\alpha^H = \alpha_2^H$ .*

This proposition suggests that in high productivity regions, the anticipated bailout ends up with decreasing output. Thus, no bailout policy is desirable in terms of output, i.e.,  $\lambda^* = 0$ . On the other hand, in low productivity regions, a rise in  $\lambda$  initially increases output by crowding L-projects out and then decreases it by crowding H-projects out. Thus, desirable policy is to set  $\lambda^* = \hat{\lambda}$  at which L-projects are just eliminated.

Before concluding, we should emphasize limitations of our analysis. Here we only focus on the ex-post bailout that maximizes output. The policy maximizes ex-ante welfare for workers, but in order to derive an optimal ex-post bailout policy, we need to conduct welfare analysis for entrepreneurs. These are left for future research.

## 5 Conclusion

This paper theoretically investigates the relationship between asset price bubbles and bailout. We show that although bailout may mitigate adverse effects of bubbles' bursting ex-post, it is more likely to cause asset price bubbles by encouraging risk-taking behavior ex-ante. In other words, bubbles are more likely to occur, the more government bailout is anticipated. Moreover, we examine the effects of the anticipated bailout on boom-bust cycles. We find that when productivity is relatively low, the anticipated bailout accelerates output booms and creates large bubbles, thus destabilizing the economy. On the other hand, when productivity is relatively high, the anticipated bailout dampens output booms, thus stabilizing the economy. Finally, we analyze an ex-post bailout policy that maximizes output.

Obviously, our analysis can be extended in several directions. Let us discuss some of them here. First, in the present paper, we have only considered the bailout of buying bubble assets. We can also think of different types of bailouts such as low interest rate policy. As a number of authors such as Rajan (2010) suggested, one of the causes of the recent financial crisis

in the U.S. lies in what is called “Greenspan Put”, which says that the ex-post low interest rate policy by central bank after bubbles’ collapse induced banks or investment banks to take on more risk ex-ante, which ended up with causing bubbles in financial markets. Our framework would be extended to examine the ex-ante effects of an ex-post interest rate policy in a full blown macroeconomic framework.

The second direction would be related to the first direction. We can examine the relationship between monetary policy and asset price bubbles. Our model suggests that bubbles cannot arise in low productivity economy ( $\alpha^H$  is too low), because in those regions, wealth’s growth rate is too low compared to interest rate. Our model’s prediction is that if monetary policy can lower interest rate, even low productivity economy will enter bubble regions. This suggests that expansionary monetary policy can be a cause for the emergence of asset price bubbles. A recent paper by Gali (2011) analyzes the impact of monetary policy rule on asset price bubbles by using an OLG model with nominal rigidities. These are left for future research.

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## 6 Appendix

### 6.1 Proof of Proposition 1

Rearranging (??) by using (??), (??), and (29), we can obtain the dynamic equation concerning  $\phi_t$  :

$$\phi_{t+1} = \begin{cases} \frac{\frac{(1-p-\phi_t)}{\pi(1-p)-\phi_t}}{(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p)\beta+\frac{(1-\pi)(1-p)}{\pi(1-p)-\phi_t}\beta\phi_t} \phi_t & \text{if } r_t = q_{t+1}\alpha^L, \\ \frac{\theta}{\beta} \frac{1}{\pi(1-p)-(1-\theta)\beta\phi_t} \phi_t & \text{if } r_t > q_{t+1}\alpha^L. \end{cases} \quad (34)$$

In addition, (??) can be rearranged as

$$K_{t+1} = \begin{cases} \frac{[(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p)\beta\alpha^L-\alpha^L\beta\phi_t]}{1-\beta\phi_t} \sigma K_t^\sigma & \text{if } r_t = q_{t+1}\alpha^L, \\ \frac{\alpha^H\beta[1-\phi_t]}{1-\beta\phi_t} \sigma K_t^\sigma & \text{if } r_t > q_{t+1}\alpha^L. \end{cases} \quad (35)$$

The dynamical system of this economy can be characterized by (34) and (35).

Given an initial state variable,  $K_0$ , there is a unique initial bubble price,  $P_0$  which satisfies

$$\phi_0 \equiv \frac{P_0 X}{\beta(\sigma K_0^\sigma + P_0 X)} = \begin{cases} \frac{\pi-[1-\pi\beta(1-p)]/v}{1-[1-\pi\beta(1-p)]/v} (1-p) & \text{if } r_t = q_{t+1}\alpha^L, \\ \frac{\pi\beta(1-p)-\theta}{\beta(1-\theta)} & \text{if } r_t > q_{t+1}\alpha^L, \end{cases} \quad (36)$$

Once  $P_0$  is determined,  $\phi_t$  becomes constant over time after date 1 onwards until bubbles collapse.

Then, the dynamics of the capital stock is governed by

$$K_{t+1} = \begin{cases} \frac{[(1+\frac{\alpha^H-\alpha^L}{\alpha^L-\theta\alpha^H}p)\beta\alpha^L-\alpha^L\beta\phi]}{1-\beta\phi} \sigma K_t^\sigma & \text{if } r_t = q_{t+1}\alpha^L, \\ \frac{\alpha^H\beta[1-\phi]}{1-\beta\phi} \sigma K_t^\sigma & \text{if } r_t > q_{t+1}\alpha^L, \end{cases}$$

which is equivalent to (31).

## 6.2 Proof of Proposition 2

From (36), in order that  $\phi > 0$ , two conditions must be satisfied:

$$\frac{\pi - [1 - \pi\beta(1 - p)]/v}{1 - [1 - \pi\beta(1 - p)]/v}(1 - p) > 0,$$

and

$$\frac{\pi\beta(1 - p) - \theta}{\beta(1 - \theta)} > 0.$$

When we solve for  $\alpha^H$  and  $\theta$ , we can obtain the existence conditions in Proposition 2.

## 6.3 Proof of Proposition 3

The evolution of the capital stock in the bubbleless economy,

$$K_{t+1}^* = \begin{cases} (1 + \frac{\alpha^H - \alpha^L}{\alpha^L - \theta\alpha^H} p)\beta\alpha^L\sigma K_t^{*\sigma} & \text{if } \alpha^H \in (\alpha_1^H, \alpha^L(1 - p)/\theta], \\ \alpha^H\beta\sigma K_t^{*\sigma} & \text{if } \alpha^H > \alpha^L(1 - p)/\theta. \end{cases} \quad (37)$$

where  $K^*$  denotes the capital stock in the bubbleless economy.

We know  $\alpha_2^H < \alpha^L(1 - p)/\theta$  if  $\theta < \pi\beta(1 - p)$ . Then, given the same initial capital stock at date  $t$ ,  $K_t = K_t^*$ , by comparing (31) to (37), we see  $K_{t+1} > K_{t+1}^*$  for any  $\alpha^H \in (\alpha_1^H, \alpha_2^H]$ . And when  $\alpha^H = \alpha^L(1 - p)/\theta$ ,  $K_{t+1} < K_{t+1}^*$  if  $\theta < \pi\beta(1 - p)$ . Hence, by continuity, there is a threshold value of  $\alpha^H \equiv \alpha_3^H$ . For any  $\alpha^H \in (\alpha_2^H, \alpha_3^H)$ ,  $K_{t+1} > K_{t+1}^*$ . For  $\alpha^H = \alpha_3^H$ ,  $K_{t+1} = K_{t+1}^*$ . For any  $\alpha^H > \alpha_3^H$ ,  $K_{t+1} < K_{t+1}^*$ .

## 6.4 Proof of Proposition 4

By substituting the steady-state value of  $\phi = [\pi\beta(1 - p) - \theta]/\beta(1 - \theta)$  into  $\theta q_{t+1}\alpha^H(1 - \phi_t)/(1 - p - \phi_t)$ , (29) can be rewritten as

$$r_t = q_{t+1} \text{Max} [\alpha^L, \theta\alpha^H\xi],$$

where Since  $\theta\alpha^H\xi$  is a linear increasing function of  $\alpha^H$ , there is a threshold value of  $\alpha^H \equiv \alpha_2^H = \alpha^L/\theta\xi$  below which  $r_t = q_{t+1}\alpha^L$  and above which  $r_t = q_{t+1}\theta\alpha^H\xi$ .

## 6.5 Proof of Proposition 5

On the saddle point path,  $\phi_t$  becomes constant over time and satisfies

$$\phi = \begin{cases} \frac{[\pi+(1-\pi)\lambda]-\{1-[\pi+(1-\pi)\lambda]\beta(1-p)\}/v}{1-\{1-[\pi+(1-\pi)\lambda]\beta(1-p)\}/v}(1-p) & \text{if } r_t = q_{t+1}\alpha^L, \\ \frac{[\pi+(1-\pi)\lambda]\beta(1-p)-\theta}{\beta(1-\theta)} & \text{if } r_t > q_{t+1}\alpha^L. \end{cases} \quad (38)$$

From (38), when we solve for  $\alpha^H$  and  $\theta$  as in the proof of Proposition 2, we can obtain the existence conditions in Proposition 5.

## 6.6 Proof of Proposition 6

From (38), it is obvious that  $\phi$  is an increasing function of  $\lambda$ .

## 6.7 Proof of Proposition 7

We know that there is a value of  $\alpha^H = \alpha_4^H$ , where  $\alpha_4^H$  is the greater value of  $\alpha^H$  which satisfies  $\alpha^H F(\lambda) = \alpha^L v / [1 - \pi\beta(1-p)]$ . We know that  $K_{t+1}$  is an increasing function of  $\lambda$  when  $r_t = q_{t+1}\alpha^L$ , and  $K_{t+1}$  is a decreasing function of  $\lambda$  when  $r_t > q_{t+1}\alpha^L$ , and  $\alpha_{2,bail}^H$  is a decreasing function of  $\lambda$ . Hence, given the same initial capital stock at date  $t$ ,  $K_{t+1}^{\lambda>0} \geq K_{t+1}$  for any  $\alpha^H \in (\alpha_1^H, \alpha_4^H]$ , where  $K_{t+1}^{\lambda>0}$  denotes the capital stock at date  $t+1$  when bailout is expected with probability  $\lambda$ . For any  $\alpha^H > \text{Max}[\alpha_1^H, \alpha_4^H]$ ,  $K_{t+1}^{\lambda>0} < K_{t+1}$ .

## 6.8 Proof of Proposition 8

In  $\alpha^H \in [\alpha_2^H, \alpha_3^H)$ , the optimal value of  $\lambda$  is to set  $\lambda^* = 0$ . In this region, compare two types of economies. One is the economy with  $\lambda = 0$ , and the other is the economy with  $\lambda > 0$ . If the economy starts from the same level of capital stock initially, we know that capital stock in the economy with  $\lambda > 0$  is lower, because from (33),  $K_{t+1}$  is a decreasing function of  $\lambda$ . Thus,  $\lambda^* = 0$ . In  $\alpha^H \in (\alpha_1^H(\lambda = 1), \alpha_2^H)$ , there is a critical value of  $\lambda \equiv \hat{\lambda}$  at which  $\alpha^H = \alpha_2^H$ , where  $\alpha^H < \alpha_2^H$  if  $\lambda \in [0, \hat{\lambda})$  and  $\alpha^H > \alpha_2^H$  if  $\lambda \in (\hat{\lambda}, 1]$ . Together with (33), we learn that  $K_{t+1}$  is an increasing function of  $\lambda$  in  $\lambda \in [0, \hat{\lambda})$ , and is a decreasing function of  $\lambda$  in  $\lambda \in (\hat{\lambda}, 1]$ . So, an increase in  $\lambda$  has

non-monotonic effect on output and wage rate. Thus, the optimal value of  $\lambda$  in  $\alpha^H \in (\alpha_1^H(\lambda = 1), \alpha_2^H)$  is to set  $\lambda^* = \hat{\lambda}$ .<sup>27</sup>

## 6.9 Proof : Behavior of H-entrepreneurs

We verify that H-entrepreneurs never buy bubbles in equilibrium. In order that the short sale constraint binds, we need to show

$$\frac{1}{c_t^i} > \beta E_t \left[ \frac{1}{c_{t+1}^i} \frac{P_{t+1}}{P_t} \right]. \quad (39)$$

We know that when the borrowing constraint is binding,

$$\frac{1}{c_t^i} = \beta E_t \left[ \frac{r_t}{c_{t+1}^i} \frac{q_{t+1} \alpha^H (1 - \theta)}{r_t - \theta q_{t+1} \alpha^H} \right]. \quad (40)$$

We also know that  $c_{t+1}^i = (1 - \beta) \left[ \frac{r_t \alpha^H (1 - \theta)}{r_t - \theta q_{t+1} \alpha^H} \right]$  if (39) is true. Considering this, by inserting (40) into (39), (39) can be rearranged as

$$\beta \frac{1}{c_{t+1}^i} \frac{r_t (q_{t+1} \alpha^H - \pi \frac{P_{t+1}}{P_t}) + \theta q_{t+1} \alpha^H (\pi \frac{P_{t+1}}{P_t} - r_t)}{r_t - \theta q_{t+1} \alpha^H} > 0. \quad (41)$$

We see that the second term in the numerator is positive as long as  $\phi > 0$ . Next we show that the first term is also positive, that is, we show

$$q_{t+1} \alpha^H > \pi \frac{P_{t+1}}{P_t}.$$

By using (10) and (32), the above can be rewritten as

$$\sigma \alpha^H K_t^\sigma > \pi K_{t+1}. \quad (42)$$

First we show that (42) holds true when  $r_t = q_{t+1} \alpha^L$ . By using (31), (42) is

$$\alpha^H > \pi \left[ \left( 1 + \frac{\alpha^H - \alpha^L}{\alpha^L - \theta \alpha^H} p \right) \beta \alpha^L - \beta \alpha^L (1 - p) \right] / [1 - \pi \beta (1 - p)]. \quad (43)$$

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<sup>27</sup>At  $\alpha^H = \alpha_{1,bail}^H(\lambda = 1)$ , wage rate in the bubble economy and that in the bubbleless economy is the same.

When  $\alpha^H = \alpha_1^H$ , (43) is

$$\alpha_1^H > \alpha^L.$$

which is true.

And when  $\alpha^H = \alpha_2^H$ , we know  $\phi = 1 - \alpha^L p(\alpha^L - \theta \alpha_2^H)$ . By using the relation, (42) can be rearranged as

$$\alpha_2^H(1 - \beta) + \frac{\alpha^L p \beta \alpha_2^H}{\alpha^L - \theta \alpha_2^H}(1 - \pi) > 0,$$

which is true. Since the right hand side of (43) is a convex function of  $\alpha^H$ , (42) holds true in  $\alpha^H \in (\alpha_1^H, \alpha_2^H]$ .

Next, we show that (42) holds true when  $r_t > q_{t+1} \alpha^L$ . From (31), by using  $\phi$ , (42) is

$$1 - \beta \phi > \pi \beta (1 - \phi),$$

which is true.

## 6.10 Proof : Behavior of Workers

We verify that both the borrowing constraint and the short sale constraint bind for workers in equilibrium. In order that both constraints bind, the following conditions must be satisfied:

$$\frac{1}{c_t^w} > \beta E_t \left[ \frac{r_t}{c_{t+1}^w} \right]. \quad (44)$$

$$\frac{1}{c_t^i} > \beta E_t \left[ \frac{1}{c_{t+1}^i} \frac{P_{t+1}}{P_t} \right]. \quad (45)$$

We know that  $c_t^w = w_t$  if (44) and (45) are true. (44) can be rewritten as

$$K_{t+1}^\sigma > \beta K_t^\sigma r_t. \quad (46)$$

First we show that (46) holds true when  $r_t = q_{t+1} \alpha^L$ . By using (10) and (31), (46) is

$$p \frac{\alpha^H(1 - \theta)}{\alpha^L - \theta \alpha^H} > 1 - \pi \beta (1 - p). \quad (47)$$

When  $\alpha^H = \alpha_1^H$ , (47) is

$$1 > \pi \beta,$$

which is true. Since the left hand side of (47) is an increasing function of  $\alpha^H$ , (46) holds true in  $\alpha^H \in (\alpha_1^H, \alpha_2^H]$ .

Next, we show that (46) holds true when  $r_t > q_{t+1}\alpha^L$ . By using (31), (46) is rearranged as

$$\beta(1-p)(1-\pi) > \theta \{ \beta [1 - \pi\beta(1-p)] - (1 - \beta + p\beta) \}$$

This inequality condition holds true for any  $\theta$  in  $\theta < \pi\beta(1-p)$ .

Finally, we show that (45) holds true. By using (10) and (32), (45) is

$$1 > \pi\beta,$$

which is true.



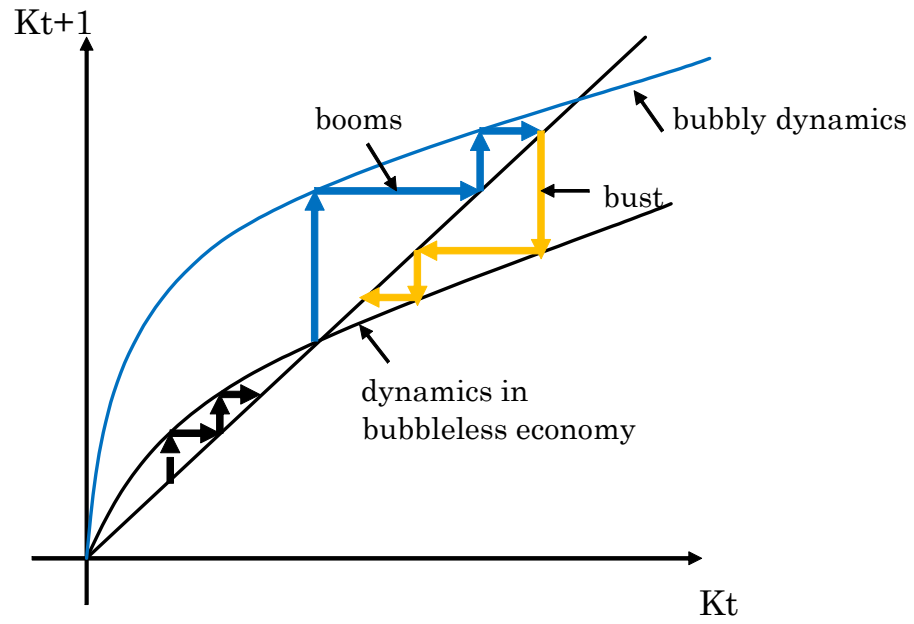


Figure 1. Dynamic Effect of Stochastic Bubbles in Region 1

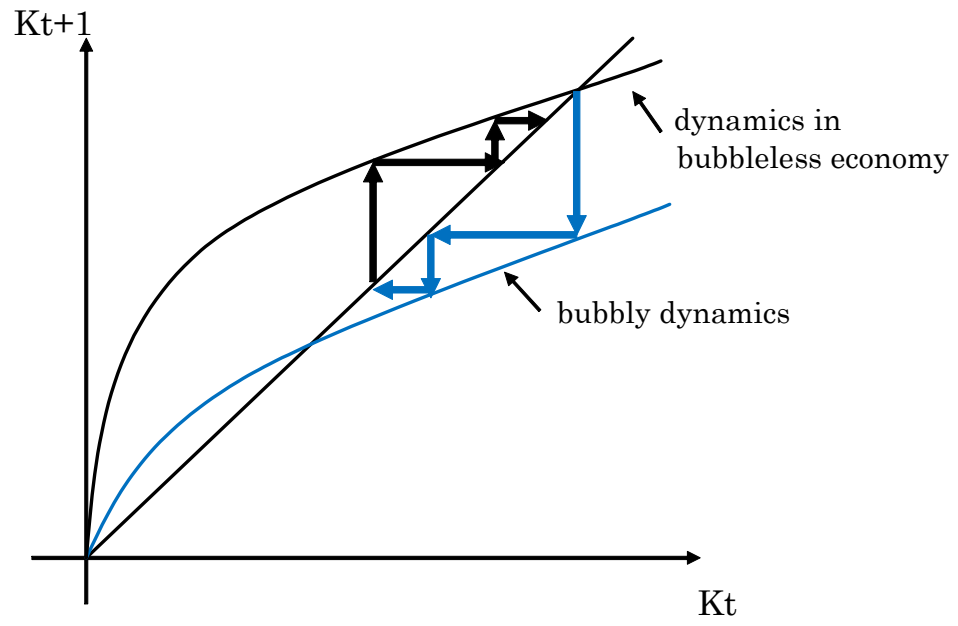


Figure 2. Dynamic Effect of Stochastic Bubbles in Region 2

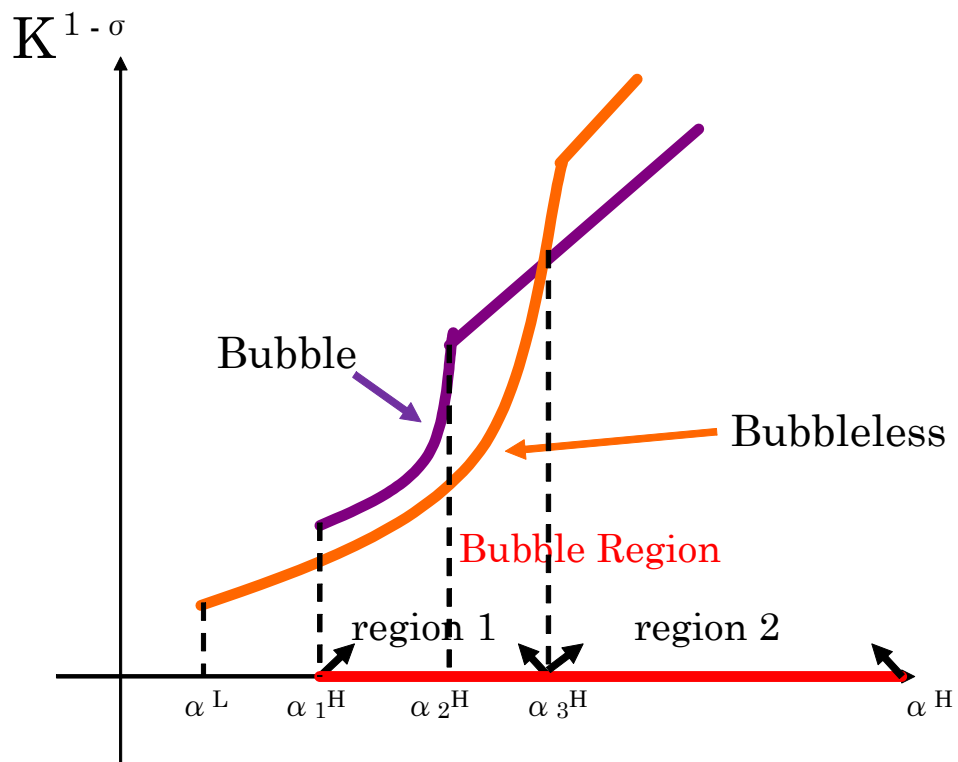


Figure 3. Effect of Stochastic Bubbles on Capital Stock

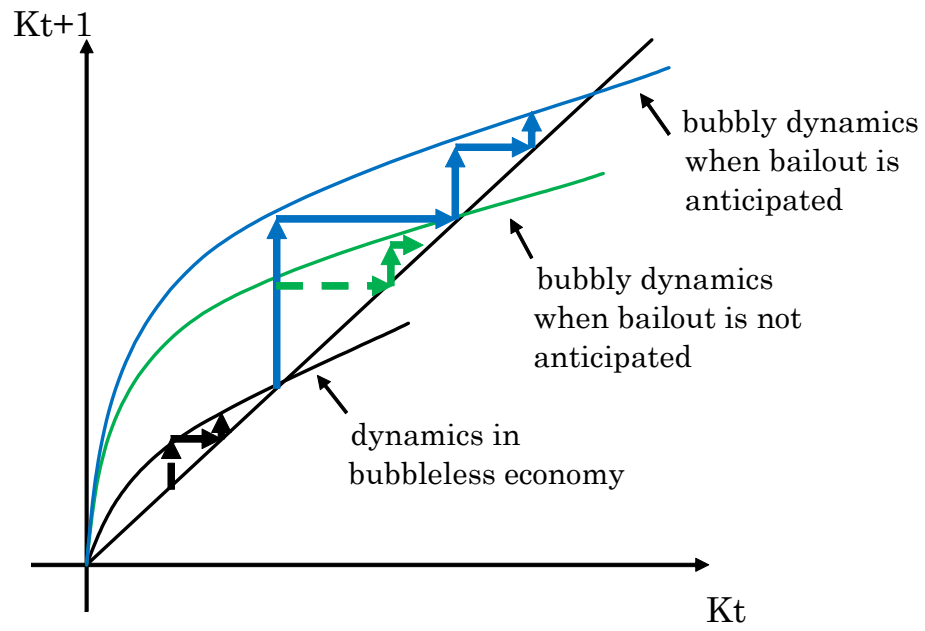


Figure 4. Bubbly Dynamics in Relatively Low  $\alpha^{\mathbb{H}}$

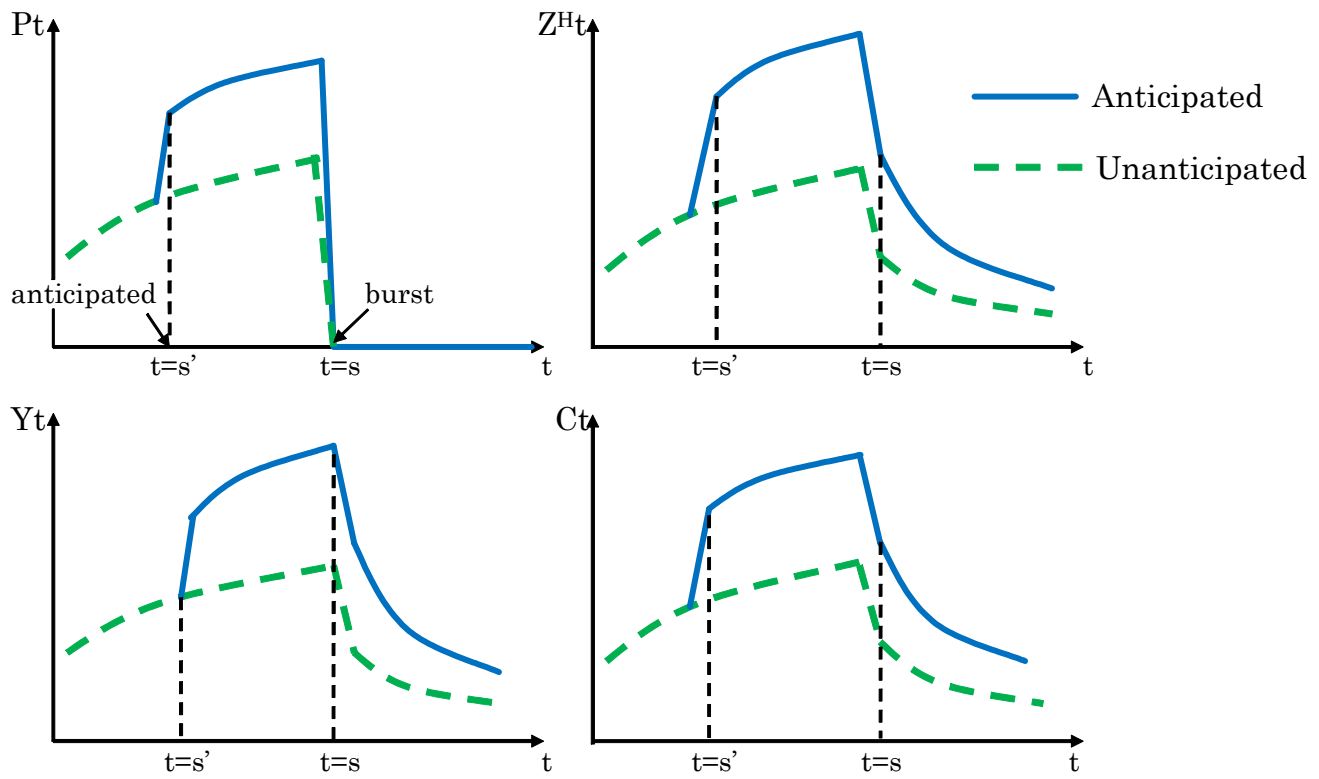


Figure 5. Bubbly Booms : Anticipated vs. Unanticipated

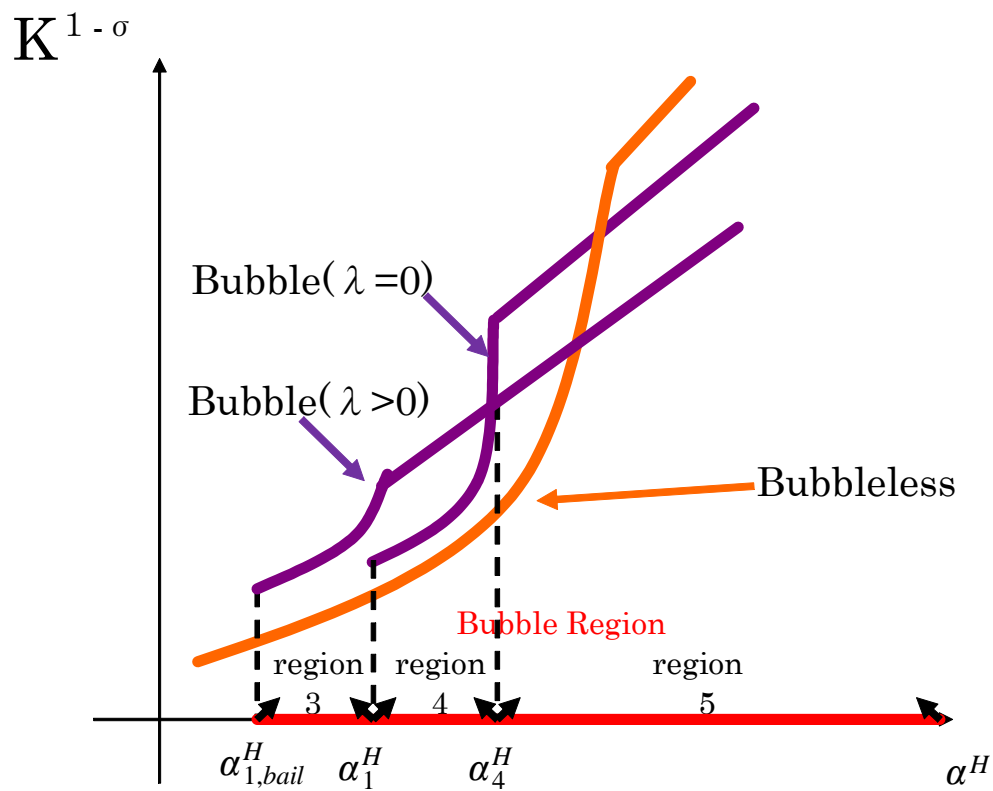


Figure 6. Effect of Anticipated Bailout on Capital Stock