CIRJE-F-840

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February 2012; Revised in September 2012

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First version: February 3, 2012
This version: June 25, 2012

*Forthcoming in *Quarterly Journal of Finance*. This research is supported by CARF (Center for Advanced Research in Finance) and the global COE program "The research and training center for new development in mathematics." All the contents expressed in this research are solely those of the authors and do not represent any views or opinions of any institutions. The authors are not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research.

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**Abstract**

In this work, we apply our newly proposed perturbative expansion technique to a quadratic growth FBSDE appearing in an incomplete market with stochastic volatility that is not perfectly hedgeable. By combining standard asymptotic expansion technique for the underlying volatility process, we derive explicit expression for the solution of the FBSDE up to the third order of volatility-of-volatility for its level, and the fourth order for its diffusion part that can be directly translated into the optimal investment strategy. We compare our approximation with the exact solution, which is known to be derived by the Cole-Hopf transformation in this popular setup. The result is very encouraging and shows good accuracy of the approximation up to quite long maturities. Since our new methodology can be extended straightforwardly to multi-dimensional setups, we expect it will open real possibilities to obtain explicit optimal portfolios or hedging strategies under realistic assumptions.

**Keywords :** FBSDE, optimal portfolio, incomplete markets, quadratic growth, perturbative expansion, asymptotic expansion

# 1 Introduction

In the last couple of decades, forward-backward stochastic differential equations (FBSDE) have attracted significant academic interests. They were first introduced by Bismut (1973) [1], and then later extended by Pardoux and Peng (1990) [14] for general non-linear cases. They were found particularly relevant for optimal portfolio and indifference pricing issues in incomplete and/or constrained markets. Their financial applications are discussed in details in, for example, El Karoui, Peng and Quenez (1997) [5], Ma and Yong (2000) [13] and a recent book edited by Carmona (2009) [2] . Various topics regarding recursive utilities are thoroughly reviewed in the article written by Skiadas (2008) [15] and references therein.

FBSDEs have become also relevant in practical problems, too. Intensive research on counterparty credit risk, collateral cost, funding rate asymmetry has made clear that one has to handle complicated FBSDEs for these problems (See, for example, [4, 6, 3].). Furthermore, forthcoming regulations on the balance sheets of financial firms and increasing demand of cash collateral both for centrally-cleared and OTC trades are expected to constrain trader's position severely, and may even turn a part of financial products effectively nontradable. These new developments in the financial market will make deeper understanding of FBSDEs a more pressing issue in the coming years.

In the previous work [7], we have presented a simple analytical approximation scheme for generic non-linear FBSDEs. By treating the interested system as the linear decoupled FBSDE perturbed by a non-linear driver and feedback terms, the problem of each order of approximation turns out to be equivalent to those for pricing of standard European contingent claims. In this work, we consider its application to a particular type of FBSDEs with a quadratic growth driver. This type of system is receiving strong attention because it appears in the optimal portfolio problems for very popular utilities of exponential and power forms. In particular, we study the optimal portfolio problem in an incomplete market with one risky asset whose stochastic volatility is not perfectly hedgeable. We derive the explicit solution of the corresponding FBSDE up to the third order of volatility of volatility (vol-of-vol) for the first "level" component, and the fourth order of vol-of-vol for the second "diffusion" component. It allows us to have the explicit expression of the optimal strategy, which is of great importance for practical applications.

In the particular setup we use in this paper, a special transformation of variable known as *the Cole-Hopf transformation* gives the closed form expression [20][1], which allows us to test accuracy of the perturbative expansion for both of the backward components. We shall see that the comparisons to the solution are quite encouraging. Since our approximation scheme is easily extended to multi-dimensional setups, we expect it will open real possibilities to obtain explicit optimal portfolios or hedging strategies in more realistic situations, which is so far limited to very simplistic models.

[1] It still requires numerical simulation to evaluate the expectation.

## 2 Setup

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the augmented filtration generated by two dimensional Brownian motion $(B_1, B_2)$. The market consists of one risk-free money market account with zero interest rate, and one risky asset with stochastic volatility. The SDEs of the risky asset $S$ and its volatility $X$ are assumed to follow

$$\begin{aligned} dS_t/S_t &= \mu dt + \sqrt{X_t}\Big(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}\Big) &(2.1)\\ dX_t &= k\big(m - X_t)dt + c\sqrt{X_t}dB_{1t} &(2.2)\end{aligned}$$

where $\rho \in (-1, 1)$ is a constant correlation parameter and $\mu, k, m$ and $c$ are all positive constants. Let us denote $\pi_t$ is the invested amount to the risky asset. Then, the investor's wealth dynamics follows

$$dW_t^\pi = \mu\pi_t dt + \pi_t\sqrt{X_t}\Big(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}\Big) \tag{2.3}$$

with the initial endowment $w_0$. We assume that the utility of an agent is given by the exponential form with risk aversion parameter $\gamma > 0$ and only dependent on the terminal wealth at time $T$. Let us denote a function $U$ as

$$U(x) = -\exp\Big(-\gamma x\Big)\ , \tag{2.4}$$

and then the agent's problem is given by

$$J(w_0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[U(W_T^\pi)\Big] \tag{2.5}$$

where $\mathcal{A}$ is the set of all the admissible strategies.

It is well known that the above problem can be represented by a quadratic growth FBSDE. Particularly simple and clear derivation of the relevant FBSDE are given in Hu, Imkeller and Müller (2005) [9] for exponential and power utilities, and in Horst et al. (2011) [10] for generic form of utilities. It can be shown that the optimal strategy $\pi^*$ is specified by

$$\pi_t^* = \frac{1}{\gamma X_t}\Big(\mu - \gamma\rho\sqrt{X_t}Z_t\Big) \tag{2.6}$$

where $Z$ is a solution of the following FBSDE:

$$\begin{aligned} dV_t &= -f(Z_t, X_t)dt + Z_t dB_{1t}\\ V_T &= 0 &(2.7)\end{aligned}$$

with a quadratic growth driver:

$$f(Z_t, X_t) = -\frac{\gamma}{2}(1-\rho^2)Z_t^2 - \frac{\mu}{\sqrt{X_t}}\rho Z_t + \frac{1}{2\gamma}\frac{\mu^2}{X_t}\ . \tag{2.8}$$

One can concentrate on the FBSDE system composed by $X$ and $V$ since the dynamics of $S$ itself drops off from the system. In the following, we denote $B_t$ instead of $B_{1t}$ for simplicity.

## 3 Perturbative Expansion

We now introduce a perturbative expansion parameter $\epsilon$ to render the original system linear decoupled FBSDE in each order of $\epsilon$. We write

$$dV_t^{(\epsilon)} = -\frac{\mu^2}{2\gamma X_t}dt - \epsilon g(Z_t^{(\epsilon)}, X_t)dt + Z_t^{(\epsilon)}dB_t \tag{3.1}$$

$$V_T^{(\epsilon)} = 0 \tag{3.2}$$

where

$$g(z,x) = -\frac{\gamma}{2}(1-\rho^2)z^2 - \frac{\mu\rho}{\sqrt{x}}z \ . \tag{3.3}$$

We suppose that the solution is given by a perturbative expansion in terms of $\epsilon$ as

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{3.4}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots \ . \tag{3.5}$$

Although it is possible to eliminate the linear term of $z$ from the driver function $g(z,x)$ by using the change of probability measure, we treat it directly here since it is not always a practical method in the presence of complicated state dependencies in its coefficient in more realistic situations.

Once we obtain the solution up to the certain order of $\epsilon$, then putting $\epsilon = 1$ will provide a reasonable approximation as long as the contribution from $g(z,x)$ is small enough. In economic terms, the above approximation corresponds to an expansion of the optimal strategy around the myopic mean-variance portfolio. It is expected to be naturally fit to our perturbative assumption as long as the hedging contribution is only sub-dominant. In the reminder of this work, we consider the expansion up to the third order of $\epsilon$.

**Proposition 1** *$(V^{(i)}, Z^{(i)})$ with $i = \{0,1,2,3\}$ follow the linear FBSDEs given below:*

$$dV_t^{(0)} = -\frac{\mu^2}{2\gamma}\frac{1}{X_t}dt + Z_t^{(0)}dB_t \tag{3.6}$$

$$dV_t^{(1)} = -g(Z_t^{(0)}, X_t)dt + Z_t^{(1)}dB_t \tag{3.7}$$

$$dV_t^{(2)} = -\partial_z g(Z_t^{(0)}, X_t)Z_t^{(1)}dt + Z_t^{(2)}dB_t \tag{3.8}$$

$$dV_t^{(3)} = -\left\{\partial_z g(Z_t^{(0)}, X_t)Z_t^{(2)} + \frac{1}{2}\partial_z^2 g(Z_t^{(0)}, X_t)(Z_t^{(1)})^2\right\}dt + Z_t^{(3)}dB_t, \tag{3.9}$$

*where the terminal values are all zero, $V_T^{(i)} = 0$ with $i \in \{0,1,2,3\}$, and $\partial_z$ denotes partial derivative with respect to the first argument of function $g(z,x)$.*

*Proof*: It follows from a straightforward application of the method given in [7]. ■

From Proposition 1, one can see that each pair of $(V^{(i)}, Z^{(i)})$ is a solution of a linear decoupled FBSDE and thus easy to integrate. One obtains
zeroth order:

$$V_t^{(0)} = \frac{\mu^2}{2\gamma}\int_t^T \mathbb{E}\left[\frac{1}{X_u}\bigg|\mathcal{F}_t\right]du \tag{3.10}$$

$$Z_t^{(0)} = \frac{\mu^2}{2\gamma}\int_t^T \mathbb{E}\left[\mathcal{D}_t\left(\frac{1}{X_u}\right)\bigg|\mathcal{F}_t\right]du \tag{3.11}$$

first order:

$$V_t^{(1)} = \int_t^T \mathbb{E}\Big[g(Z_u^{(0)}, X_u)\Big|\mathcal{F}_t\Big]du \tag{3.12}$$

$$Z_t^{(1)} = \int_t^T \mathbb{E}\Big[\mathcal{D}_t g(Z_u^{(0)}, X_u)\Big|\mathcal{F}_t\Big]du \tag{3.13}$$

second order:

$$V_t^{(2)} = \int_t^T \mathbb{E}\Big[\partial_z g(Z_u^{(0)}, X_u)Z_u^{(1)}\Big|\mathcal{F}_t\Big]du \tag{3.14}$$

$$Z_t^{(2)} = \int_t^T \mathbb{E}\Big[\mathcal{D}_t\Big(\partial_z g(Z_u^{(0)}, X_u)Z_u^{(1)}\Big)\Big|\mathcal{F}_t\Big]du \tag{3.15}$$

third order:

$$V_t^{(3)} = \int_t^T \mathbb{E}\left[\partial_z g(Z_u^{(0)}, X_u)Z_u^{(2)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0)}, X_u)(Z_u^{(1)})^2\bigg|\mathcal{F}_t\right]du \tag{3.16}$$

$$Z_t^{(3)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t\Big(\partial_z g(Z_u^{(0)}, X_u)Z_u^{(2)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0)}, X_u)(Z_u^{(1)})^2\Big)\bigg|\mathcal{F}_t\right]du \tag{3.17}$$

respectively, where $\mathcal{D}_t$ is a Malliavin derivative with respect to $B$.

# 4 Asymptotic Expansion

Although, in the previous section, we have formally expanded the original non-linear FBSDE in terms of a series of linear decoupled FBSDEs, we need to explicitly evaluate the involved expectations to obtain a quantitative result. As explained in [7], this can be done by making use of standard asymptotic expansion technique, which is now widely used for pricing of various European contingent claims and also for computation of the optimal portfolio in complete markets (See, for examples [12, 16, 17, 18, 19] and references therein for concrete examples.).

We introduce a different parameter $\delta$ to expand the forward component $X$ in terms of the vol-of-vol, ie, $c$:

$$dX_u^{(\delta)} = k(m - X_u^{(\delta)})du + \delta c\sqrt{X_u^{(\delta)}}dB_u \ . \tag{4.1}$$

We expand $X$ up to the third order of $\delta$ as

$$X_u^{(\delta)} = X_u^{(0)} + \delta D_{tu} + \frac{\delta^2}{2} E_{tu} + \frac{\delta^3}{3!} F_{tu} + o(\delta^3) \tag{4.2}$$

$$X_t^{(\delta)} = x_t \tag{4.3}$$

where each term is defined by

$$D_{tu} = \left.\frac{\partial X_u^{(\delta)}}{\partial \delta}\right|_{\delta=0}, \qquad E_{tu} = \left.\frac{\partial^2 X_u^{(\delta)}}{\partial \delta^2}\right|_{\delta=0}, \qquad F_{tu} = \left.\frac{\partial^3 X_u^{(\delta)}}{\partial \delta^3}\right|_{\delta=0}. \tag{4.4}$$

The relevant formulas regarding the above expansions are summarized in Appendix B.

Now, in each order of $\epsilon$, we try to expand the backward components in terms of $\delta$. More concretely, we are going to approximate each pair of $(V^{(i)}, Z^{(i)})$ with $i \in \{0, 1, 2, 3\}$ as

$$V_t^{(i,\delta)} = V_t^{(i,0)} + \delta V_t^{(i,1)} + \frac{\delta^2}{2} V_t^{(i,2)} + \frac{\delta^3}{3!} V_t^{(i,3)} + o(\delta^3) \tag{4.5}$$

$$Z_t^{(i,\delta)} = Z_t^{(i,0)} + \delta Z_t^{(i,1)} + \frac{\delta^2}{2} Z_t^{(i,2)} + \frac{\delta^3}{3!} Z_t^{(i,3)} + \frac{\delta^4}{4!} Z^{(i,4)} + o(\delta^4) \tag{4.6}$$

As we shall see, the required calculation to obtain $V^{(i,j)}$ is to take expectation value of a polynomial function of $X^{(k)}$ with $k \in \{1, 2, 3\}$. Since each $X^{(k)}$ is given by a multiple Wiener integral, the evaluation of the expectation for $V^{(i,j)}$ can be easily calculated. Once $V^{(i,j)}$ is obtained explicitly in terms of $x_t$, simple application of Itô's formula gives us the expression of $Z^{(i,j+1)}$ by

$$Z_t^{(i,j+1)} = (j+1) c \sqrt{x_t} \frac{\partial}{\partial x_t} V_t^{(i,j)}(x_t) . \tag{4.7}$$

It is easy to see that $Z^{(i,0)}$ is zero. The reason why we expand $Z$ by one higher order is to study the convergence of $Z$ itself. As long as the vol-of-vol (or $c$) is small relative to the other parameters, putting $\delta = 1$ is expected to give a reasonable approximation to the original model.

### 4.1 Asymptotic Expansion of $V^{(0,\delta)}$

In the zero-th order of $\epsilon$, we want to expand

$$V_t^{(0,\delta)}(x_t) = \frac{\mu^2}{2\gamma} \int_t^T \mathbb{E}\Big[ v_u^{(\delta)} \Big| \mathcal{F}_t \Big] du \tag{4.8}$$

in terms of $\delta$, where

$$v_u^{(\delta)} = \frac{1}{X_u^{(\delta)}} . \tag{4.9}$$

One can show that

$$v_u^{(\delta)} = v_u^{(0)} + \delta v_u^{(1)} + \frac{\delta^2}{2} v_u^{(2)} + \frac{\delta^3}{3!} v_u^{(3)} + o(\delta^3) \tag{4.10}$$

where each term is given by

$$
\begin{aligned}
v_u^{(0)} &= (X_u^{(0)})^{-1} && (4.11)\\
v_u^{(1)} &= -(X_u^{(0)})^{-2} D_{tu} && (4.12)\\
v_u^{(2)} &= 2(X_u^{(0)})^{-3} D_{tu}^2 - (X_u^{(0)})^{-2} E_{tu} && (4.13)\\
v_u^{(3)} &= -6(X_u^{(0)})^{-4} D_{tu}^3 + 6(X_u^{(0)})^{-3} D_{tu} E_{tu} - (X_u^{(0)})^{-2} F_{tu} \ . && (4.14)
\end{aligned}
$$

Let us define

$$
v_u^{(i)}(x_t) := \mathbb{E}\left[ v_u^{(i)} \middle| \mathcal{F}_t \right] \quad (4.15)
$$

then, from the results of Appendix, one can check that

$$
v_u^{(1)}(x_t) = v_u^{(3)}(x_t) = 0 \quad (4.16)
$$

and also

$$
\begin{aligned}
v_u^{(0)}(x_t) &= (X_u^{(0)}(x_t))^{-1} && (4.17)\\
v_u^{(2)}(x_t) &= 2(X_u^{(0)}(x_t))^{-3} D_{tu}^2(x_t) \ . && (4.18)
\end{aligned}
$$

Integration in (4.8) can be performed explicitly as

$$
V_t^{(0,\delta)}(x_t) = V_t^{(0,0)}(x_t) + \frac{\delta^2}{2} V_t^{(0,2)}(x_t) + o(\delta^3) \quad (4.19)
$$

where

$$
\begin{aligned}
V_t^{(0,0)}(x_t) &= -\frac{\mu^2}{2\gamma}\frac{1}{km} \ln\left( \frac{Y_{tT} x_t}{X_T^{(0)}(x_t)} \right) && (4.20)\\
V_t^{(0,2)}(x_t) &= -\frac{\mu^2}{2\gamma}\frac{c^2}{k^2} \left\{ \frac{(1-Y_{tT})\big[m(1-Y_{tT}) + 2Y_{tT} x_t\big]}{2m(X_T^{(0)}(x_t))^2} + \frac{1}{m^2} \ln\left( \frac{Y_{tT} x_t}{X_T^{(0)}(x_t)} \right) \right\} \ . && (4.21)
\end{aligned}
$$

The relevant definitions of variables are given in Appendix.

## 4.2 Asymptotic Expansion of $Z^{(0,\delta)}$

Although we have considered the dynamics of Malliavin derivative $\mathcal{D}_t X_u^{(\delta)}$ directly in [7], it is easier to simply apply Itô's formula to the result of $V^{(0,\delta)}$, since we already have its explicit expression in terms of $x_t$. One can easily confirm that

$$
Z_t^{(0,\delta)}(x_t) = \delta Z_t^{(0,1)}(x_t) + \frac{\delta^3}{3!} Z_t^{(0,3)}(x_t) + o(\delta^4) \quad (4.22)
$$

where

$$
\begin{aligned}
Z_t^{(0,1)}(x_t) &= -\frac{\mu^2 c}{2\gamma k} \frac{1 - Y_{tT}}{\sqrt{x_t}(X_T^{(0)}(x_t))} && (4.23)\\
Z_t^{(0,3)}(x_t) &= -\frac{3\mu^2 c^3}{2\gamma k^2} \frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \Big[ m(1-Y_{tT}) + 2Y_{tT}\ x_t \Big] \ . && (4.24)
\end{aligned}
$$

### 4.3 Asymptotic Expansion of $V^{(1,\delta)}$

In the first order of $\epsilon$, we need to expand

$$V_t^{(1,\delta)}(x_t) = \int_t^T \mathbb{E}\Big[g(Z_u^{(0,\delta)}, X_u^{(\delta)})\Big|\mathcal{F}_t\Big]du \tag{4.25}$$

$$= -\frac{\gamma}{2}(1-\rho^2)\int_t^T \mathbb{E}\Big[(Z_u^{(0,\delta)})^2\Big|\mathcal{F}_t\Big]du - \mu\rho\int_t^T \mathbb{E}\Big[(X_u^{(\delta)})^{-\frac{1}{2}}Z_u^{(0,\delta)}\Big|\mathcal{F}_t\Big]du\ . \tag{4.26}$$

From the previous results, we have

$$Z_u^{(0,\delta)} = \delta Z_u^{(0,1)}(X_u^{(\delta)}) + \frac{\delta^3}{3!}Z_u^{(0,3)}(X_u^{(\delta)}) + o(\delta^4) \tag{4.27}$$

and hence both of the integrands in (4.26) can be explicitly written as a function of $X_u^{(\delta)}$. Therefore, we can follow the same procedures in Section 4.1: Firstly apply $\partial_\delta$, ie, partial derivative with respect to $\delta$, and then express the integrand as a function of $X_u^{(0)}$, $D_{tu}$ etc.. The evaluation of its expectation is now easily performed using the results given in Appendix. After straightforward but lengthy calculation, we obtain

$$V_t^{(1,\delta)}(x_t) = \delta V_t^{(1,1)}(x_t) + \frac{\delta^2}{2}V_t^{(1,2)}(x_t) + \frac{\delta^3}{3!}V_t^{(1,3)}(x_t) + o(\delta^3) \tag{4.28}$$

where

$$V_t^{(1,1)}(x_t) = -\frac{\rho\mu^3 c}{2\gamma k^2}\left\{\frac{(1-Y_{tT})}{mX_T^{(0)}(x_t)} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.29}$$

$$V_t^{(1,2)}(x_t) = (1-\rho^2)\frac{\mu^4c^2}{4\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}\ x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.30}$$

$$\begin{aligned}V_t^{(1,3)}(x_t) = \frac{3\rho\mu^3c^3}{2\gamma k^3}\Bigg\{&\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^2}\big[m(1-Y_{tT}) - 2Y_{tT}\ x_t\big] - \frac{2}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\\ &-\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^3}\Big[5m^2(1-Y_{tT})^2 + 9m(1-Y_{tT})(Y_{tT}x_t) + 2(Y_{tT}x_t)^2\Big]\Bigg\}\ .\end{aligned} \tag{4.31}$$

### 4.4 Asymptotic Expansion of $Z^{(1,\delta)}$

By applying Itô's formula to the expanded $V^{(1,\delta)}$, one obtains the volatility component easily as before:

$$Z_t^{(1,\delta)}(x_t) = \frac{\delta^2}{2}Z_t^{(1,2)}(x_t) + \frac{\delta^3}{3!}Z_t^{(1,3)}(x_t) + \frac{\delta^4}{4!}Z_t^{(1,4)}(x_t) + o(\delta^4) \tag{4.32}$$

where

$$Z_t^{(1,2)}(x_t) \;=\; -\frac{\rho\mu^3c^2}{\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^2} \tag{4.33}$$

$$Z_t^{(1,3)}(x_t) \;=\; (1-\rho^2)\frac{3\mu^4c^3}{4\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \tag{4.34}$$

$$Z_t^{(1,4)}(x_t) \;=\; -\frac{6\rho\mu^3c^4}{\gamma k^3}\frac{(1-Y_{tT})^3[2m(1-Y_{tT})+5Y_{tT}\;x_t]}{\sqrt{x_t}(X_T^{(0)}(x_t))^4}\;. \tag{4.35}$$

## 4.5 Asymptotic Expansion of $V^{(2,\delta)}$

In the second order of $\epsilon$, we have to evaluate

$$V_t^{(2,\delta)}(x_t) = \int_t^T \mathbb{E}\Big[\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)})Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big]du \tag{4.36}$$

$$= -\gamma(1-\rho^2)\int_t^T \mathbb{E}\Big[Z_u^{(0,\delta)}Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big]du - \mu\rho\int_t^T \mathbb{E}\Big[(X_u^{(\delta)})^{-\frac{1}{2}}Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big]du\;. \tag{4.37}$$

Following the same arguments in Section 4.3, we can express the above expectation explicitly. After tedious calculation, one obtains

$$V_t^{(2,\delta)}(x_t) = \frac{\delta^2}{2}V_t^{(2,2)}(x_t) + \frac{\delta^3}{3!}V_t^{(2,3)}(x_t) + o(\delta^3) \tag{4.38}$$

where

$$V_t^{(2,2)}(x_t) \;=\; -\frac{\rho^2\mu^4c^2}{\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\;x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.39}$$

$$V_t^{(2,3)}(x_t) \;=\; \rho(1-\rho^2)\frac{9\mu^5c^3}{4\gamma k^4}\left\{\frac{(1-Y_{tT})\big[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\big]}{6m^3(X_T^{(0)}(x_t))^3} + \frac{1}{m^4}\ln\left(\frac{Y_{tT}\;x_t}{X_T^{(0)}(x_t)}\right)\right\}\;. \tag{4.40}$$

## 4.6 Asymptotic Expansion of $Z^{(2,\delta)}$

As before, simple application of Itô's formula yields

$$Z_t^{(2,\delta)}(x_t) = \frac{\delta^3}{3!}Z_t^{(2,3)}(x_t) + \frac{\delta^4}{4!}Z_t^{(2,4)} + o(\delta^4) \tag{4.41}$$

where

$$
\begin{aligned}
Z_t^{(2,3)}(x_t) &= -\frac{3\rho^2\mu^4c^3}{\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} && (4.42)\\
Z_t^{(2,4)}(x_t) &= \rho(1-\rho^2)\frac{9\mu^5c^4}{\gamma k^4}\frac{(1-Y_{tT})^4}{\sqrt{x_t}(X_T^{(0)}(x_t))^4}\ . && (4.43)
\end{aligned}
$$

## 4.7 Asymptotic Expansion of $(V^{(3,\delta)}, Z^{(3,\delta)})$

We have

$$
V_t^{(3,\delta)}(x_t) = \int_t^T \mathbb{E}\left[\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)})Z_u^{(2,\delta)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0,\delta)}, X_u^{(\delta)})\big(Z_u^{(1,\delta)}\big)^2\bigg|\mathcal{F}_t\right]du \quad (4.44)
$$

and we can easily confirm that the contribution of $O(\delta^3)$ comes only from the first term. The result is

$$
V_t^{(3,\delta)}(x_t) = \frac{\delta^3}{3!}V_t^{(3,3)}(x_t) + o(\delta^3) \quad (4.45)
$$

where

$$
\begin{aligned}
V_t^{(3,3)}(x_t) &= -\frac{3\rho^3\mu^5c^3}{\gamma k^4}\Bigg\{\frac{(1-Y_{tT})\Big[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\Big]}{6m^3(X_T^{(0)}(x_t))^3}\\
&\qquad + \frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\Bigg\}\ . && (4.46)
\end{aligned}
$$

It is clear to see

$$
Z_t^{(3,\delta)}(x_t) = \frac{\delta^4}{4!}Z_t^{(3,4)} + o(\delta^4) \quad (4.47)
$$

where

$$
Z_t^{(3,4)} = -\frac{12\rho^3\mu^5c^4}{\gamma k^4}\frac{(1-Y_{tT})^4}{\sqrt{x_t}(X_T^{(0)}(x_t))^4}\ . \quad (4.48)
$$

## 4.8 Asymptotic Expansion of $(V^{(i,\delta)}, Z^{(i,\delta)})$ with $(i \geq 4)$

Let us consider what happens when we proceed further to a higher order of $\epsilon$. In the fourth order, we see that $V^{(4,\delta)}$ has contributions from

$$
\begin{aligned}
&\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)})Z_u^{(3,\delta)} && (4.49)\\
&\partial_z^2 g(Z_u^{(0,\delta)}, X_u^{(\delta)})Z_u^{(1,\delta)}Z_u^{(2,\delta)} && (4.50)\\
&\partial_z^3 g(Z_u^{(0,\delta)}, X_u^{(\delta)})\big(Z_u^{(1,\delta)}\big)^3 && (4.51)
\end{aligned}
$$

where the last term vanishes and all the others have $o(\delta^3)$. Therefore we have $V^{(4,\delta)} = o(\delta^3)$ and hence obviously, $Z^{(4,\delta)} = o(\delta^4)$. By repeating the same arguments, we can conclude

$$V_t^{(i,\delta)} = o(\delta^3) \tag{4.52}$$
$$Z_t^{(i,\delta)} = o(\delta^4) \tag{4.53}$$

for all $i \geq 4$.

### 4.9 Summary of Expansion and its Interpretation

Let us suppose, as we have hypothesized at the beginning, that the perturbative expansions

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \epsilon^3 V_t^{(3)} + \cdots \tag{4.54}$$
$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \epsilon^3 Z_t^{(3)} + \cdots \tag{4.55}$$

really converges to the true solution. From the previous observation, it is easy to see that there is no contribution to the solution of FBSDE from the fourth or higher order terms of $\epsilon$ as long as we work in $O(\delta^3)$ for $V$ and $O(\delta^4)$ for $Z$ components, respectively. Therefore, the results we have obtained can be interpreted as the asymptotic expansion of the true solution of the FBSDE in $O(\delta^3)$ for the level component $V$ and in $O(\delta^4)$ for the diffusion component $Z$.

As a summary, whole of the discussion in Section 4 leads to the next proposition:

**Proposition 2** *The solution $(V, Z)$ of the following FBSDE:*

$$dV_t = -\left\{-\frac{\gamma}{2}(1-\rho^2)Z_t^2 - \frac{\mu}{\sqrt{X_t}}\rho Z_t + \frac{1}{2\gamma}\frac{\mu^2}{X_t}\right\}dt + Z_t dB_t;\ V_T = 0, \tag{4.56}$$
$$dX_t = k(m - X_t)dt + c\sqrt{X_t}dB_t;\ X_0 = x \tag{4.57}$$

*can be asymptotically expanded in terms of vol-of-vol that is c, as:*

$$\begin{aligned} V_t(x_t) &= V_t^{(0,0)}(x_t) + \frac{1}{2}V_t^{(0,2)}(x_t) + V_t^{(1,1)}(x_t) + \frac{1}{2}V_t^{(1,2)}(x_t) + \frac{1}{3!}V_t^{(1,3)}(x_t) \\ &\quad + \frac{1}{2}V_t^{(2,2)}(x_t) + \frac{1}{3!}V_t^{(2,3)}(x_t) + \frac{1}{3!}V_t^{(3,3)}(x_t) + o(c^3) \end{aligned} \tag{4.58}$$

$$\begin{aligned} Z_t(x_t) &= Z_t^{(0,1)}(x_t) + \frac{1}{3!}Z_t^{(0,3)}(x_t) + \frac{1}{2}Z_t^{(1,2)}(x_t) + \frac{1}{3!}Z_t^{(1,3)}(x_t) + \frac{1}{4!}Z_t^{(1,4)} \\ &\quad + \frac{1}{3!}Z_t^{(2,3)}(x_t) + \frac{1}{4!}Z_t^{(2,4)} + \frac{1}{4!}Z_t^{(3,4)} + o(c^4)\ , \end{aligned} \tag{4.59}$$

*where each term is given by*

$$V_t^{(0,0)}(x_t) = -\frac{\mu^2}{2\gamma}\frac{1}{km}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right)$$

$$V_t^{(0,2)}(x_t) = -\frac{\mu^2}{2\gamma}\frac{c^2}{k^2}\left\{\frac{(1-Y_{tT})\left[m(1-Y_{tT}) + 2Y_{tT}x_t\right]}{2m(X_T^{(0)}(x_t))^2} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(1,1)}(x_t) = -\frac{\rho\mu^3 c}{2\gamma k^2}\left\{\frac{(1-Y_{tT})}{mX_T^{(0)}(x_t)} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(1,2)}(x_t) = (1-\rho^2)\frac{\mu^4 c^2}{4\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}\ x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(1,3)}(x_t) = \frac{3\rho\mu^3 c^3}{2\gamma k^3}\left\{\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^2}\left[m(1-Y_{tT})-2Y_{tT}\ x_t\right] - \frac{2}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right.$$
$$\left. -\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^3}\left[5m^2(1-Y_{tT})^2+9m(1-Y_{tT})(Y_{tT}x_t)+2(Y_{tT}x_t)^2\right]\right\}$$

$$V_t^{(2,2)}(x_t) = -\frac{\rho^2\mu^4 c^2}{\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(2,3)}(x_t) = \rho(1-\rho^2)\frac{9\mu^5 c^3}{4\gamma k^4}\left\{\frac{(1-Y_{tT})\left[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\right]}{6m^3(X_T^{(0)}(x_t))^3}\right.$$
$$\left. +\frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(3,3)}(x_t) = -\frac{3\rho^3\mu^5 c^3}{\gamma k^4}\left\{\frac{(1-Y_{tT})\left[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\right]}{6m^3(X_T^{(0)}(x_t))^3}\right.$$
$$\left. +\frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

*and*

$$Z_t^{(0,1)}(x_t) = -\frac{\mu^2 c}{2\gamma k}\frac{1-Y_{tT}}{\sqrt{x_t}(X_T^{(0)}(x_t))}$$

$$Z_t^{(0,3)}(x_t) = -\frac{3\mu^2 c^3}{2\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^3}\left[m(1-Y_{tT})+2Y_{tT}\ x_t\right]$$

$$Z_t^{(1,2)}(x_t) = -\frac{\rho\mu^3 c^2}{\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^2}$$

$$Z_t^{(1,3)}(x_t) = (1-\rho^2)\frac{3\mu^4 c^3}{4\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3}$$

$$Z_t^{(1,4)}(x_t) = -\frac{6\rho\mu^3 c^4}{\gamma k^3}\frac{(1-Y_{tT})^3[2m(1-Y_{tT})+5Y_{tT}\ x_t]}{\sqrt{x_t}(X_T^{(0)}(x_t))^4}$$

$$
\begin{aligned}
Z_t^{(2,3)}(x_t) &= -\frac{3\rho^2\mu^4c^3}{\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \\
Z_t^{(2,4)}(x_t) &= \rho(1-\rho^2)\frac{9\mu^5c^4}{\gamma k^4}\frac{(1-Y_{tT})^4}{\sqrt{x_t}(X_T^{(0)}(x_t))^4} \\
Z_t^{(3,4)} &= -\frac{12\rho^3\mu^5c^4}{\gamma k^4}\frac{(1-Y_{tT})^4}{\sqrt{x_t}(X_T^{(0)}(x_t))^4} \ . \qquad (4.60)
\end{aligned}
$$

*It then specifies the optimal strategy $\pi_t^*$ in (2.6) up to the fourth order of vol-of-vol.*

# 5 Numerical Comparison to the Exact Solution

In [20], it is shown that the Cole-Hopf transformation allows the closed form solution for our problem. We define $K_t = e^{\eta V_t}$ with some constant $\eta \in \mathbb{R}$. Then, the dynamics of $K$ is given by

$$
\begin{aligned}
dK_t/K_t &= \left(\frac{\gamma\eta}{2}(1-\rho^2)+\frac{\eta^2}{2}\right)Z_t^2dt \\
&\quad + \left\{\frac{\mu\eta}{\sqrt{X_t}}\rho Z_t - \frac{\mu^2\eta}{2\gamma}\frac{1}{X_t}\right\}dt + \eta Z_t dB_t \ . \qquad (5.1)
\end{aligned}
$$

Thus, by choosing $\eta^* = -\gamma(1-\rho^2)$ one can eliminate the quadratic term. By defining $Q_t = \eta^* K_t Z_t$, the above equation becomes

$$
dK_t = \left(\frac{\mu\rho}{\sqrt{X_t}}Q_t - \frac{\mu^2\eta^*}{2\gamma}\frac{K_t}{X_t}\right)dt + Q_t dB_t \ , \qquad (5.2)
$$

which is a linear FBSDE with terminal value $K_T = 1$.

Now, let us introduce a new measure $\mathbb{P}^*$ for which Brownian motion is related to that in the original measure $\mathbb{P}$ by

$$
dB_t^* = dB_t + \frac{\mu\rho}{\sqrt{X_t}}dt \ . \qquad (5.3)
$$

Then, we have

$$
dK_t = \frac{\mu^2(1-\rho^2)}{2X_t}K_t dt + Q_t dB_t^* \ , \qquad (5.4)
$$

which can be integrated easily. Thus, the solution of the original FBSDE is given by

$$
V_t = -\frac{1}{\gamma(1-\rho^2)}\ln\left\{\mathbb{E}^{\mathbb{P}^*}\left[\exp\left(-\frac{\mu^2}{2}(1-\rho^2)\int_t^T\frac{ds}{X_s}\right)\middle|\mathcal{F}_t\right]\right\} \qquad (5.5)
$$

where $X$ follows

$$
dX_t = k(n-X_t)dt + c\sqrt{X_t}dB_t^* \qquad (5.6)
$$

under the new measure, where the adjusted mean $n$ denotes $n = m - \rho\mu c/k$.

The diffusion part $Z$ is given by

$$Z_t = c\sqrt{X_t}\left(\frac{\partial V_t}{\partial x_t}\right) \tag{5.7}$$

where the partial derivative by the initial value can be easily estimated by taking the delta of $V$ relative to the shift of $x_t$. Although $Z$ can also be written with a Malliavin derivative of $X$, the higher order terms $\propto 1/X_s^2$ $(s > t)$ and the dynamics of stochastic flow makes it difficult to achieve stable results of Monte Carlo simulation when it is directly applied to its expression.

*Remark: Note that the Cole-Hope transformation cannot always be used to derive exact solutions in more generic situations, such as cases including multi-dimensional risk factors, time or state dependent correlation parameters, e.t.c.. Our scheme can be extended easily, at least in principle, for these cases, too.*

## 5.1 Numerical Comparison

We now numerically estimate the the solution in Eq.(5.5) by Monte Carlo (MC) simulation. In order to guarantee the positivity of $X$, we use the implicit Milstein scheme [11]:

$$X(t_n) = \frac{X(t_{n-1}) + kn\Delta t + c\sqrt{X(t_{n-1})}\xi_n\sqrt{\Delta t} + \frac{1}{4}c^2\Delta t\big(\xi_n^2 - 1\big)}{1 + k\Delta t} \tag{5.8}$$

where $(t_n)_{n\geq 1}$ is equally spaced time grids and $\Delta t = t_n - t_{n-1}$. $(\xi_n)_{n\geq 1}$ is a sequence of independent random variable with standard normal distribution $\mathbf{N}(0,1)$. We have run 1-million plus 1-million antipathetic scenarios with step size $\Delta t = 0.005$ to obtain the numerical estimate of $V_0$ in Eq. (5.5). We have compared it to the results of our asymptotic expansion up to the third order of vol-of-vol. Furthermore, for the diffusion part, we have run another $(1+1)$-million scenarios to obtain $V_0$ with the initial value of $X$ shifted by a small amount $\Delta x_0 = 5 \times 10^{-4}$ to estimate $(\partial V_0/\partial x_0)$. We have then multiplied it by $c\sqrt{x_0}$ to obtain the numerical estimate of $Z_0$. We have compared it with the analytical approximation up to the fourth order of vol-of-vol.

Table 3 in Appendix A gives the comparison of $V_0$ with $m = 6.25\%$ and $c = 5\%$, which corresponds to roughly $\sqrt{m} = 25\%$ implied volatility of the risky asset with $c/\sqrt{m} = 20\%$ vol-of-vol in log-normal terms. The each column represents the maturity $T$, the result of MC simulation, its standard deviation, $\epsilon$-0th, $\epsilon$-1st, $\epsilon$-2nd and $\epsilon$-3rd order approximation, respectively. All the parameters used are provided in the caption. One can see that the approximation is quite accurate even for 10-year maturity. Table 1 gives the comparison of $Z_0$ with the same parameters in Table 3. The column with the label "err" gives the expected error of $Z_0$ implied from the standard deviation in the estimation of $V_0$. Consistently with the convergence of $V_0$, one can see that the diffusion part converges nicely to the estimated true value of $Z_0$.

Since the analytical approximation is given by the power series of vol-of-vol "$c$", one can expect that its performance deteriorates when the larger $c$ is used. One can see this in Table 4 given in Appendix A where we have used $m = 6.25\%$ and $c = 12\%$, which corresponds to $\sqrt{m} = 25\%$ and $c/\sqrt{m} = 48\%$. Especially for longer maturities, one can observe that the zero-th and first order expansions significantly over/under estimate $V_0$. Although $\epsilon$-2nd and 3rd order approximations still provide reasonable estimation of the true value in this example, one needs higher order expansions or some new devise to improve the approximations for larger values of $c$, in general. For example, it would be better to introduce the expansion parameter $\delta$ also in the drift term of $X$ to avoid the appearance of small parameters in denominators of the resultant formulas. These possibilities may be pursed in a separate paper [2]. Table 2 gives the corresponding comparison for $Z_0$ with the same parameters used in Table 4.

| maturity (yr) | $Z$-MC (%) | err (%) | $\epsilon$-0th (%) | $\epsilon$-1st (%) | $\epsilon$-2nd (%) | $\epsilon$-3rd (%) |
|---|---|---|---|---|---|---|
| 1 | -4.293 | 0.015 | -4.442 | -4.250 | -4.258 | -4.258 |
| 2 | -7.860 | 0.042 | -8.470 | -7.725 | -7.785 | -7.783 |
| 3 | -10.760 | 0.065 | -12.055 | -10.471 | -10.661 | -10.650 |
| 4 | -13.094 | 0.082 | -15.205 | -12.594 | -12.998 | -12.972 |
| 5 | -14.974 | 0.090 | -17.950 | -14.208 | -14.907 | -14.859 |
| 6 | -16.498 | 0.092 | -20.329 | -15.420 | -16.480 | -16.403 |
| 7 | -17.740 | 0.096 | -22.384 | -16.320 | -17.787 | -17.676 |
| 8 | -18.752 | 0.118 | -24.154 | -16.984 | -18.884 | -18.736 |
| 9 | -19.588 | 0.158 | -25.675 | -17.469 | -19.812 | -19.625 |
| 10 | -20.267 | 0.211 | -26.982 | -17.820 | -20.603 | -20.377 |

Table 1: A comparison to the MC simulation and asymptotic expansion of $Z$ with parameters: $m = 6.25\%, k = 15\%,\ c = 5\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

| maturity (yr) | $Z$-MC (%) | std err (%) | $\epsilon$-0th (%) | $\epsilon$-1st (%) | $\epsilon$-2nd (%) | $\epsilon$-3rd (%) |
|---|---|---|---|---|---|---|
| 1 | -11.197 | 0.241 | -11.968 | -10.460 | -10.593 | -10.586 |
| 2 | -19.537 | 0.571 | -24.096 | -17.689 | -18.749 | -18.672 |
| 3 | -25.954 | 0.730 | -35.113 | -21.253 | -24.521 | -24.252 |
| 4 | -29.546 | 0.786 | -44.602 | -22.002 | -28.764 | -28.165 |
| 5 | -32.782 | 0.797 | -52.544 | -20.930 | -32.174 | -31.134 |
| 6 | -34.411 | 0.870 | -59.084 | -18.839 | -35.154 | -33.602 |
| 7 | -36.211 | 1.116 | -64.420 | -16.287 | -37.886 | -35.788 |
| 8 | -37.186 | 1.507 | -68.754 | -13.629 | -40.427 | -37.785 |
| 9 | -37.565 | 1.931 | -72.265 | -11.071 | -42.778 | -39.617 |
| 10 | -38.079 | 2.382 | -75.108 | -8.723 | -44.925 | -41.285 |

Table 2: A comparison to the MC simulation and asymptotic expansion of $Z$ with parameters: $m = 6.25\%, k = 20\%,\ c = 12\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

Lastly, in Figure 1, we give a sample path each for the mean-variance and the approximated $\epsilon$-3rd order optimal portfolio weight $\pi^*$ with parameters $m = 6.25\%$, $k = 15\%$, $c = 5\%$, $x_0 = m$, $\mu = 17\%$, $\rho = -35\%$ and $\gamma = 1$ for a 10-year investment. One can see

[2] After submitting this work, we have developed the new Monte Carlo scheme inspired by the branching diffusion method to bypass the needs of the asymptotic expansion by allowing simulation of underlying state processes directly [8].

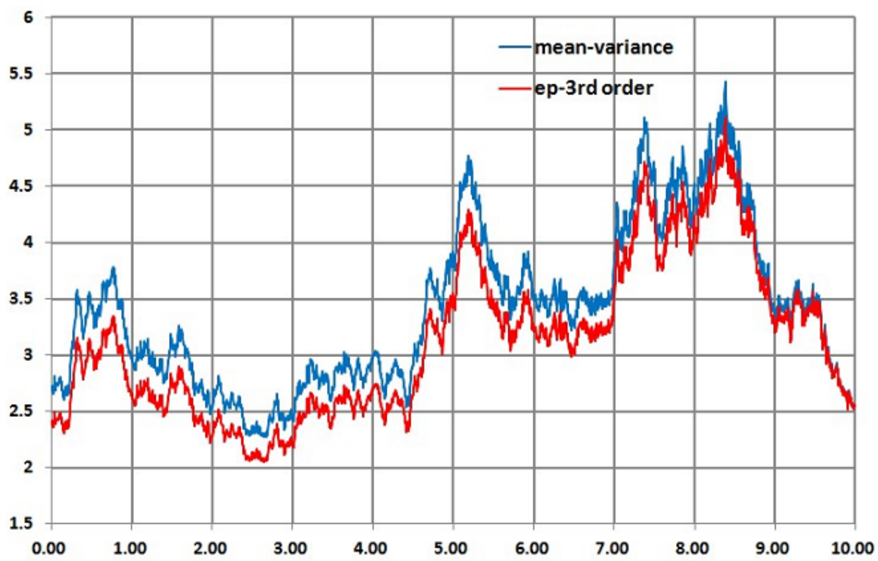

Figure 1: A sample path each for the mean-variance portfolio and approximated ($\epsilon$-2nd order) optimal portfolio weight. The used parameters are $m = 6.25\%$, $k = 15\%$, $c = 5\%$, $x_0 = m$, $\mu = 17\%$, $\rho = -40\%$ and $\gamma = 1$.

that the optimal amount of investment is smaller than that of the mean-variance strategy due to the hedging demand. This relationship flips the sign when the positive correlation $\rho$ is used. The difference between the mean-variance and optimal strategies becomes gradually smaller as the time comes closer to the maturity as expected.

## 6 Conclusion

In this work, we have studied the optimal portfolio problem in an incomplete market with stochastic volatility that is not perfectly hedgeable. We have applied the newly developed perturbative methodology combined with standard asymptotic expansion technique and derived the explicit solution of the corresponding quadratic growth FBSDE up to the third order of vol-of-vol for its level and to the fourth order for its diffusion component. The comparison to the exact solution shows quite encouraging results about its accuracy even for quite long maturities, such as 10 years. As long as we know, the existing numerical techniques, such as regression based Monte Carlo simulations, seem mostly limited to short maturities, say, several months to one year. Furthermore, the great advantage of our method is its ability to provide explicit expressions of the optimal portfolios or hedging strategies, which obviously have great importance for the practical use.

In contrast to the Cole-Hopf transformation, our method can be applied to much more generic setups with multi-dimensional risk factors, which is expected to open real possibilities to obtain explicit expressions of optimal portfolios and hedging strategies in incomplete and/or constrained markets with realistic assumptions. This will be addressed in separate works in the future.

# A Numerical results for the "level" component $V$

| maturity (yr) | $V$-MC (%) | std err (%) | $\epsilon$-0th (%) | $\epsilon$-1st (%) | $\epsilon$-2nd (%) | $\epsilon$-3rd (%) |
|---|---|---|---|---|---|---|
| 1 | 23.061 | 0.0003 | 23.539 | 23.035 | 23.049 | 23.049 |
| 2 | 45.844 | 0.0008 | 47.769 | 45.671 | 45.787 | 45.783 |
| 3 | 68.197 | 0.0013 | 72.510 | 67.691 | 68.086 | 68.067 |
| 4 | 90.067 | 0.0016 | 97.630 | 88.997 | 89.919 | 89.868 |
| 5 | 111.455 | 0.0018 | 123.031 | 109.560 | 111.313 | 111.207 |
| 6 | 132.397 | 0.0018 | 148.639 | 129.398 | 132.317 | 132.128 |
| 7 | 152.938 | 0.0019 | 174.401 | 148.552 | 152.987 | 152.685 |
| 8 | 173.128 | 0.0023 | 200.278 | 167.076 | 173.377 | 172.932 |
| 9 | 193.011 | 0.0031 | 226.239 | 185.028 | 193.537 | 192.918 |
| 10 | 212.630 | 0.0041 | 252.263 | 202.468 | 213.508 | 212.686 |

Table 3: A comparison to the MC simulation and asymptotic expansion of $V$ with parameters: $m = 6.25\%, k = 15\%,\ c = 5\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

| maturity (yr) | $V$-MC (%) | std err (%) | $\epsilon$-0th (%) | $\epsilon$-1st (%) | $\epsilon$-2nd (%) | $\epsilon$-3rd (%) |
|---|---|---|---|---|---|---|
| 1 | 24.340 | 0.0020 | 25.461 | 23.896 | 23.992 | 23.988 |
| 2 | 49.550 | 0.0048 | 54.541 | 47.232 | 48.090 | 48.035 |
| 3 | 73.840 | 0.0061 | 86.046 | 68.269 | 71.261 | 71.038 |
| 4 | 96.840 | 0.0066 | 119.177 | 86.407 | 93.441 | 92.870 |
| 5 | 118.640 | 0.0066 | 153.398 | 101.609 | 114.899 | 113.761 |
| 6 | 139.490 | 0.0072 | 188.350 | 114.097 | 135.960 | 134.016 |
| 7 | 159.560 | 0.0093 | 223.792 | 124.189 | 156.901 | 153.913 |
| 8 | 179.030 | 0.0125 | 259.561 | 132.223 | 177.919 | 173.660 |
| 9 | 198.030 | 0.0161 | 295.551 | 138.520 | 199.144 | 193.404 |
| 10 | 216.650 | 0.0200 | 331.688 | 143.364 | 220.643 | 213.235 |

Table 4: A comparison to the MC simulation and asymptotic expansion of $V$ with parameters: $m = 6.25\%, k = 20\%,\ c = 12\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

# B Formulas for $X$'s Asymptotic Expansion

We assume $(u > t)$ throughout this section. The value $x_t$ is defined as the initial condition at time $t$ by

$$x_t = X_t^{(\delta)}\ . \tag{B.1}$$

## B.1 $\delta$ 0th order

The relevant equation becomes deterministic in this case:

$$dX_u^{(0)} = k(m - X_u^{(0)})du \tag{B.2}$$

and thus

$$X_u^{(0)} = Y_{tu} x_t + m(1 - Y_{tu}) \tag{B.3}$$

where we have defined

$$Y_{tu} = \exp\Big(-k(u-t)\Big) \ . \tag{B.4}$$

## B.2 $\delta$ 1st order

Since we have

$$d(\partial_\delta X_u^{(\delta)}) = -k(\partial_\delta X_u^{(\delta)})du + \left(c\sqrt{X_u^{(\delta)}} + \frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta X_u^{(\delta)})\right) dB_u \tag{B.5}$$

which yields

$$dD_{tu} = -kD_{tu}du + c\sqrt{X_u^{(0)}}dB_u \tag{B.6}$$

and hence

$$D_{tu} = c\int_t^u Y_{us}\sqrt{X_s^{(0)}}dB_s \ . \tag{B.7}$$

## B.3 $\delta$ 2nd order

Since we have

$$\begin{aligned} d(\partial_\delta^2 X_u^{(\delta)}) &= -k(\partial_\delta^2 X_u^{(\delta)})du \\ &+ \left\{c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta X_u^{(\delta)}) - \frac{1}{4}\delta c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})^2 + \frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^2 X_u^{(\delta)})\right\} dB_u \end{aligned}$$

which yields

$$dE_{tu} = -kE_{tu}du + c(X_u^{(0)})^{-\frac{1}{2}}D_{tu}dB_u \tag{B.8}$$

and hence

$$E_{tu} = c\int_t^u Y_{us}(X_s^{(0)})^{-\frac{1}{2}}D_{ts}dB_s \ . \tag{B.9}$$

## B.4 $\delta$ 3rd order

We have

$$\begin{aligned} d(\partial_\delta^3 X_u^{(\delta)}) &= -k(\partial_\delta^3 X_u^{(\delta)})du \\ &+ \Big\{-\frac{3}{4}c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})^2 + \frac{3}{2}c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^2 X_u^{(\delta)}) \\ &+ \frac{3}{8}\delta c(X_u^{(\delta)})^{-\frac{5}{2}}(\partial_\delta X_u^{(\delta)})^3 - \frac{3}{4}\delta c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})(\partial_\delta^2 X_u^{(\delta)}) \\ &+ \frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^3 X_u^{(\delta)})\Big\}dB_u \end{aligned} \tag{B.10}$$

thus,

$$dF_{tu} \;=\; -kF_{tu}du + \frac{3}{2}c\left\{(X_u^{(0)})^{-\frac{1}{2}}E_{tu} - \frac{1}{2}(X_u^{(0)})^{-\frac{3}{2}}D_{tu}^2\right\}dB_u \tag{B.11}$$

and then

$$F_{tu} = \frac{3}{2}c\int_t^u Y_{us}\Big\{(X_s^{(0)})^{-\frac{1}{2}}E_{ts} - \frac{1}{2}(X_s^{(0)})^{-\frac{3}{2}}D_{ts}^2\Big\}dB_s \ . \tag{B.12}$$

### B.5 Relevant expectation values

It is easy to check that

$$\mathbb{E}\Big[D_{tu}\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[E_{tu}\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[F_{tu}\Big|\mathcal{F}_t\Big] = 0 \ . \tag{B.13}$$

On the other hand, we have

$$\begin{aligned} d(D_{tu})^2 &= 2D_{tu}dD_{tu} + d\langle D_{t\cdot}\rangle_u \\ &= -2kD_{tu}^2du + c^2X_u^{(0)}du + 2cD_{tu}\sqrt{X_u^{(0)}}dB_u \end{aligned} \tag{B.14}$$

and hence

$$\begin{aligned} D_{tu}^2(x_t) &:= \mathbb{E}\Big[D_{tu}^2\Big|\mathcal{F}_t\Big] = c^2\int_t^u e^{-2k(u-s)}X_s^{(0)}(x_t)ds \\ &= \frac{c^2}{2k}(1-Y_{tu})\Big[(1-Y_{tu})m + 2Y_{tu}x_t\Big] \ . \end{aligned} \tag{B.15}$$

By following the similar procedures, it is easy to confirm that

$$\mathbb{E}\Big[D_{tu}^3\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[D_{tu}E_{tu}\Big|\mathcal{F}_t\Big] = 0 \ . \tag{B.16}$$

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