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Pricing Multi-Asset Cross Currency Options

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Abstract

This paper develops a general pricing method for multi-asset cross currency options, whose underlying asset consists of multiple different assets, and the evaluation currency is different from the ones used in the most liquid market of each asset; the examples include cross currency options, cross currency basket options and cross currency average options. We also demonstrate that our scheme is able to evaluate options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment. Moreover, in practice, fast calibration is necessary in the option markets relevant for the underlying assets and the currency, which is also achieved in this paper. Furthermore, we investigate the implied correlations in the cross currency markets on the dates before and after the events, *Lehman Shock* and *Tohoku Earthquake*.

1 Introduction

This paper presents a general framework for pricing multi-asset cross currency options under a broad class of multi-dimensional diffusion models. We notice that the underlying assets of a multi-asset cross currency option are related with multiple underlying asset markets as well as at least one currency market. Moreover, it is necessary for practical use of a pricing model to take the information of each underlying asset's option market into account. Then, calibration to each option market needs a more complex model than Black-Scholes model such as stochastic volatility models to reflect the skew/smile and term structure of implied volatilities observed in the option market. Thus, a multi-dimensional diffusion model should be applied to pricing a multi-asset cross currency option and relevant calibrations, where an analytical valuation method is necessary for fast computation. On the other hand, it is almost impossible to obtain a closed-form option pricing formula under a multi-dimensional diffusion setting. An effective method for overcoming this problem is an asymptotic expansion scheme which is a unified method in order to achieve accurate approximations of option prices and Greeks in

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multi-dimensional models. (For instance, please see [37], [28], [29], [21], [32], [33], [30], [31] for the detail.) We also remark that the Mathematical foundation of this method relies on Watanabe theory in Malliavin calculus. (For instance, please see [34], [36], [22] for the detail.) Applying the scheme, this paper derives an approximation formula for pricing multi-asset cross currency options, and presents several practical examples with numerical analysis which takes actual option markets into consideration.

As the first example, we evaluate cross currency basket options in stochastic volatility models based on the calibration to the relevant option markets of USD quoted currency pairs. Moreover, for examination of the accuracy of our approximation method, we also evaluate basket options whose underlying asset consists of 100 different assets under Black-Scholes [3], Constant elasticity of variance (CEV) [8] and λ-SABR [23] models. Particularly, we demonstrate that our scheme can be applied to pricing options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment. This feature is an advantage of this method comparing to other analytical (approximation) schemes.

There are several existing literatures (e.g. [5], [6], [18], [25] and [38]) that derive approximate formulas for pricing basket options where each underlying asset price follows Black-Scholes model. [14] derives an approximation formula in jump-diffusion model, and [35] provides an approximation formula in a local volatility and jump-diffusion model. Also, [4] derives closed form formulas for the option price and the Greeks of Asian(average) basket options in energy markets under log-normal (Black-Scholes) model of each underlying asset price with moment-matching method. [10] shows the analytic bounds for Asian basket options under Black-Scholes model with comonotonic approach. [28] and [29] proposes a new pricing formula for basket options under general diffusion setting by applying the asymptotic expansion scheme. Recently, [24] develops a new symbolic algorithm for the asymptotic expansion scheme, and applies it to pricing options on VIX under the Gatheral double log-normal stochastic volatility models, where the underlying asset is expressed as square-root of a linear combination of a stochastic variance and its stochastic mean reversion level. [19] approximated basket option under SABR model using Markovian projection. Moreover, [1] derives a very accurate formula for pricing basket options under a general class of local volatility models including CEV and Black-Scholes models, and demonstrates the accuracy using 100 underlying assets. (For instance, see [2] and [15] for the related articles.) Our work is the first one which derives an approximation formula and implements numerical experiments in stochastic volatility environment for pricing basket options with 100 underlying assets, as well as pricing currency basket options based on calibration to the real vanilla option markets.

The second example is cross currency average options. [20] proposes approximations of average option prices under Black-Scholes model. However, it is almost impossible for one parameter set under the Black-Scholes option pricing model to reproduce market prices with various strikes for a given maturity. Then, [37] [28], [29], [27] apply an asymptotic expansion method to pricing average options under the general diffusion process of the underlying asset price; they consider a continuous average of an asset price, which does not represent the underlying price of a contract in the real world precisely.

Hence, the formula they derived can not be used directly for valuation of the options traded in actual markets. Thus, [26] develops an approximation scheme with stochastic volatility models that takes specific features of commodity average price options into consideration. (See the paper for the detail.) [9], [11] and [13] derive approximation formula for cross currency average basket options under Black-Scholes model.

This paper extends [26] to the cross currency correspondents which is very useful for firms outside United States importing energies such as crude oils traded mainly with the U.S. dollar. Especially, we evaluate average options on Japanese-yen based West Texas Intermediate (WTI) futures, where SABR model is applied to WTI futures price processes while an extended λ-SABR model is used for the JPYUSD spot foreign exchange rate [1]process. We show its numerical examples based on the calibration to the WTI futures option market as well as to the USDJPY currency option market. To the best of our knowledge, this is the first work for pricing cross currency average options including numerical analysis which reflects the actual option markets.

Furthermore, we investigate the implied correlations in the various cross currency markets. In particular, we apply SABR model to each USD quoted foreign exchange rate, as opposed to models such as *Double Heston model* (e.g. [7], [16]), in which the two volatility processes for the USD quoted currency pairs relevant with a cross currency pair are perfectly correlated. Clearly, the performance of calibration by our model seems much better, which is confirmed in our numerical analysis. As a related work, [12] analyzes the currency option markets in fairly detail by applying his intrinsic currency framework under stochastic volatility environment. As for the data of the numerical analysis, we use the ones on the dates before and after the events such as *Lehman Shock* and *Tohoku Earthquake*. Moreover, the sensitivities of implied volatilities with respect to the correlation parameters are also examined.

The organization of the paper is as follows: Section 2 discusses a general diffusion model used for pricing multi-asset cross currency options and shows several examples included in this class of options. Section 3 derives an approximation formula for multi-asset cross currency options under a multi-factor extension of the λ-SABR stochastic volatility model. Section 4 presents pricing cross currency basket and cross currency average options with numerical examples. Section 5 investigates the implied correlations in the various cross currency option markets. Appendix shows the derivation of the pricing formula and the calibration results associated with Section 5.

2 Multi-Asset Cross Currency Options

Let 0-th asset $S_0^{k,l}$ be a spot foreign exchange rate that stands for the price of the unit amount of the currency k in terms of currency l. Without loss of generality, we assume that k denotes Japanese yen (JPY) while l denotes US dollar (USD), and write S_0 for $S_0^{k,l}$ that is, 1 Japanese yen = S_0 US dollars. Also, let S_i $(i = 1, \cdots, n)$ denotes the price of the asset i in terms of USD.

[1]price of Japanese yen in terms of U.S. dollars

Next, suppose that the dynamics of S_i $(i = 0, 1, \cdots, n)$ are expressed under the USD risk-neutral measure as follows:

$$
\begin{aligned}
dS_0(t) &= S_0(t)[\alpha_0(t)dt + \hat{\sigma}_0(t, S_0)dZ(t)], &(1)\\
d\sigma_0(t) &= f_0(t, \sigma_0)dt + \bar{\nu}_0(t, \sigma_0)\sigma_0(t)dZ(t), &(2)
\end{aligned}
$$

$$
\begin{aligned}
dS_i(t) &= S_i(t)[\alpha_i(t)dt + \hat{\sigma}_i(t, S_i)]dZ(t), &(3)\\
d\sigma_i(t) &= f_i(t, \sigma_i)dt + \bar{\nu}_i(t, \sigma_i)\sigma_i(t)dZ(t), &(4)
\end{aligned}
$$

where $\alpha_0(t) = r_{USD}(t) - r_{JPY}(t)$, $\alpha_i(t) = r_{USD}(t) - \delta_i(t)(i = 1, \cdots, n)$, and Z denotes the $2(n+1)$-dimensional Brownian motion under the USD risk-neutral measure; r_{USD}, r_{JPY} and δ_i denote the risk-free interest rates of USD, that of JPY and the dividend rate of i-th asset, respectively. Moreover, 2(n+1)-dimensional parameters, $\hat{\sigma}_i(t, S_i)$ and $\bar{\nu}_i(t, \sigma_i)(i = 0, 1, \cdots, n)$ are defined by

$$
\begin{aligned}
\hat{\sigma}_i(t, S_i) &:= \sigma_i(t)g_i(t, S_i)(c_{i,0}(t), c_{i,1}(t), \cdots, c_{i,i}(t), 0, \cdots, 0) &(5)\\
\bar{\nu}_i(t, \sigma_i) &:= (\nu_{i,0}(t, \sigma_i), \cdots, \nu_{i,2n+1}(t, \sigma_i))\\
&:= \nu_i(t, \sigma_i)(c_{n+1+i,0}(t), c_{n+1+i,1}(t), \cdots, c_{n+1+i,n+1+i}(t), 0, \cdots, 0), &(6)
\end{aligned}
$$

where $f_i(t, x)$, $g_i(t, x)$ and $\nu_i(t, x)$, $(i = 0, \cdots, n)$ are some $[0, T] \times \mathbf{R}_+ \mapsto \mathbf{R}$ functions. $c_{i,j}(t)$, $(0 \leq j \leq i \leq 2n+1)$ are $[0, T] \mapsto \mathbf{R}$ functions which are obtained by the Cholesky decompositions of the relevant correlation matrices.

Next, let Y_i, $(i = 1, \cdots, n)$ denote the price of the asset i in terms of JPY:

$$
Y_i(t) = \frac{S_i(t)}{S_0(t)}. \qquad (7)
$$

Then, the dynamics of S_i, σ_i and Y_i under JPY risk-neutral measure are given as follows:

$$
\begin{aligned}
dS_i(t) &= S_i(t)[(\alpha_i(t) + \hat{\sigma}_0(t, S_0)\hat{\sigma}_i(t, S_i))dt + \hat{\sigma}_i(t, S_i)dW(t)] &(8)\\
d\sigma_i(t) &= [f_i(t, \sigma_i) + \hat{\sigma}_0(t, S_0)\sigma_i(t, S_i)\bar{\nu}_i(t, \sigma_i)]dt + \bar{\nu}_i(t, \sigma)dW(t), &(9)\\
dY_i(t) &= Y_i(t)[(\alpha_i(t) - \alpha_0(t))\, dt + (-\hat{\sigma}_0(t, S_0) + \hat{\sigma}_i(t, S_i))\, dW(t)], &(10)
\end{aligned}
$$

where W denotes the $2(n+1)$-dimensional Brownian motion under the JPY risk-neutral measure.

Under this setting, in the following sections we will consider approximations for pricing options whose underlying asset price process X defined by

$$
X(t) = \sum_{i=1}^{n} w_i(t)Y_i(t), \qquad (11)
$$

where $w_i(t)$ is a deterministic function of the time parameter t; for instance, the payoff of a call option with strike price K and maturity T is expressed as $\max\{X(T) - K, 0\}$. Hereafter, we will call this type of options multi-asset cross currency options. We remark that setting $S_0 \equiv 1$, we can treat USD denominated options as a special case. We are also able to consider *quanto*-type products when the underlying asset is given by

$$X(t) = \sum_{i=1}^{n} w_i(t) S_i(t), \tag{12}$$

under the JPY risk-neutral measure. We also assume that the interest rates r_{USD}, r_{JPY} and dividend rates δ_i are non-random just for simplicity, which implies that $\alpha_0(t)$ and $\alpha_i(t)$ in the above equations become deterministic. [2]

Next, let us see several well-known option products whose underlying asset prices are described by (11) with specific $w_i(t)$.

1. Cross Currency Options

   The simplest example may be cross currency options such as EUR/JPY options, where S_1 stands for the exchange rate USD/EUR. Then, we set the weights $w_i(t)$ as $w_1(t) = 1$ and $w_i(t) = 0$ $(i = 2, \cdots, n)$.

2. Spread Options

   The underlying asset prices of spread options are the difference of futures prices/interest rates with different maturities, or the difference of the prices of different assets. In this case, setting the weights as $w_1(t) = 1$, $w_2(t) = -1$ and $w_i(t) = 0$ $(i = 2, \cdots, n)$, we have

$$X(t) = Y_1(t) - Y_2(t). \tag{13}$$

3. Basket Options

   The underlying asset of a basket option is the weighted average of the prices of different assets, where the weights are typically some prespecified (positive) constants, that is $w_i(t) \equiv w_i > 0$ for all i:

$$X(t) = \sum_{i=1}^{n} w_i Y_i(t). \tag{14}$$

4. Average Options

   Average options are one of popular products especially in the commodity markets; the futures contracts with several consecutive maturities may become the underlying assets of an average option as in OTC oil market(e.g. WTI market). (See [26] for the detail of the structure of products.)

   More generally, let us introduce new processes $Z_i(t)$ defined by

$$Z_i(t) = \sum_{j=1}^{m_i} Y_i\left(t_j^{(i)}\right) I\left(\{t_j^{(i)} \le t\}\right),$$

[2] When the underlying asset is a futures contract (denominated by USD) with no dividends, the asset price process is a martingale under the USD risk-neutral measure. Then, $\alpha_i = 0$ in (3).

where $0 \leq t_1^{(i)} < \cdots < t_{m_i}^{(i)} \leq T$, m_i denotes the number of the cross currency price Y_i to which the average option refers, and the dynamics of each Y_i is described by the stochastic differential equation (10) with (9). Then, we can deal with a cross currency(e.g. Japanese yen-based) average option whose underlying asset price $X(t)$ is given by the following:

$$X(t) = \frac{1}{M}\sum_{i=1}^{n} Z_i(t) = \sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{1}{M} I\left(\{t_j^{(i)} \leq t\}\right) Y_i\left(t_j^{(i)}\right), \tag{15}$$

where $M = \sum_{i=1}^{n} m_i$. Note that if each $Y_i\left(t_j^{(i)}\right)$, $i = 1, \cdots, n$, $j = 1, \cdots, m_i$ is regarded as a different asset's price, its weight function in (11) is given by

$$w_i^{(j)}(t) = \frac{1}{M} I\left(\{t_j^{(i)} \leq t\}\right).$$

3 Approximation formula

This section will derive an approximation formula which is useful for analysis of multi-asset cross currency options. In particular, we take an extended λ-SABR model as the underlying asset price dynamics, while the similar formula can be derived for the general model, (1)-(4) in the same procedure.

First, we briefly explain the extended λ-SABR model that we apply to our analysis in the following sections.

We specify the coefficient functions for (1)-(6) in the previous section as follows:

$$\begin{aligned} f(t, \sigma_i) &= \lambda_i(\theta_i(t) - \sigma_i(t)), && (16)\\ g(t, S_i) &= S_i^{\beta_i - 1}, \text{ (a constant } \beta_i \in [0, 1]), && (17)\\ \hat{\sigma}_i(t, S_i) &= \sigma_i(t) g(t, S_i)(c_{i,0}, c_{i,1}, \cdots, c_{i,i}, 0, \cdots, 0) && (18)\\ \bar{\nu}_i(t, \sigma_i) &= \bar{\nu}_i(t) = (\nu_{i,0}(t), \cdots, \nu_{i,2n+1}(t)) && \\ &= \nu_i(c_{(n+1+i),0}(t), c_{(n+1+i),1}(t), \cdots, c_{(n+1+i),(n+1+i)}(t), 0, \cdots, 0), && (19) \end{aligned}$$

where λ_i and ν_i, $i = 0, 1, \cdots, n$ are some positive constants.
Also, $c_{i,j}(t)$, $(0 \leq j \leq i \leq 2n + 1)$ are $[0, T] \mapsto \mathbf{R}$ functions which are obtained by the Cholesky decompositions of the relevant correlation matrices.

Then, in sum we obtain the dynamics of S_i and σ_i for $i = 0, 1, \cdots, n$ as follows:

$$\begin{aligned} dS_i(t) &= \alpha_i S_i(t)dt + \hat{\sigma}_i(t) S_i(t) dZ(t);\ S_i(0) \text{ given, } \alpha_i \text{ is a constant.} && (20)\\ d\sigma_i(t) &= \lambda_i(\theta_i(t) - \sigma_i(t))dt + \bar{\nu}_i(t)\sigma_i(t)dZ(t);\ \sigma_i(0) \text{ given,} && (21) \end{aligned}$$

where $\hat{\sigma}_i(t)$ and $\bar{\nu}_i(t)$ are defined by (18) and (19), respectively; λ_i is a positive constant, and $\theta(t)$ is a deterministic function of the time parameter t; Z denotes $2(n + 1)$-dimensional Brownian motion under the USD risk-neutral measure. Then, the dynamics of S_i, σ_i and Y_i under the JPY risk-neutral measure are given by (8)-(10).

Next, we will present the pricing formula under the λ-SABR model. For the SABR model [17], it is enough to set $\lambda = 0$ in the λ-SABR model. In particular, the following theorem shows a formula for a European-type call option whose payoff is given by

$$C(T) = \max\{X(T) - K, 0\}, \tag{22}$$

where

$$X(t) = \sum_{i=1}^{n} w_i(t) Y_i(t).$$

Although the detail of the derivation of the formula is quite lengthy and hence is omitted, let us briefly explain the idea. In the first place, in order to get perturbative random processes of (8)-(10), we replace $\hat{\sigma}_0$, $\hat{\sigma}_i$ and $\bar{\nu}_i$ in (8)-(10) by $\epsilon\hat{\sigma}_0$, $\epsilon\hat{\sigma}_i$ and $\epsilon\bar{\nu}_i$, respectively to obtain $S_i^{(\epsilon)}$, $\sigma_i^{(\epsilon)}$ and $Y_i^{(\epsilon)}$. As a result, we obtain $X^{(\epsilon)}(T)$, a perturbative random variable of $X(T)$, and then expand $X^{(\epsilon)}(T)$ around $X^{(0)}(T)$ which is $X^{(\epsilon)}(T)$ evaluated at $\epsilon = 0$. (Due to the zero volatilities of $S_i^{(0)}$, $\sigma_i^{(0)}$ and $Y_i^{(0)}$, the process $\{X^{(0)}(t) : 0 \le t \le T\}$ has no volatility term, and hence $X^{(0)}(T)$ corresponds to the initial price of the T-maturity forward contract for the underlying asset.)

Next, we implement an asymptotic expansion of the density function for a normalized random variable, $\hat{X}^{(\epsilon)}(T) := \frac{X^{(\epsilon)}(T) - X^{(0)}(T)}{\epsilon}$ of which limiting density is Gaussian. For example, the asymptotic expansion up to the ϵ^2-order for the density function of $\hat{X}^{(\epsilon)}(T)$ denoted by $f_{\hat{X}^{(\epsilon)}}(x)$ is obtained as follows:

$$\begin{aligned} f_{\hat{X}^{(\epsilon)}}(x) &= n[x;0,\Sigma]\Bigg\{1 + \epsilon C_1 \frac{H_3(x;\Sigma)}{\Sigma^3} \\ &\quad + \epsilon^2 \left(C_2 \frac{H_6(x;\Sigma)}{\Sigma^6} + C_3 \frac{H_4(x;\Sigma)}{\Sigma^4} + C_4 \frac{H_2(x;\Sigma)}{\Sigma^2} \right) \Bigg\} + o(\epsilon^2), \end{aligned} \tag{23}$$

where $n[x;0,\Sigma]$ is the normal density function with mean 0 and variance Σ, that is $n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(\frac{-x^2}{2\Sigma}\right)$. The coefficients $C_1, \cdots, C_4$ are some constants. Moreover, $H_k(x;\Sigma)$ denotes the k-th order Hermite polynomial, $H_k(x;\Sigma) = (-\Sigma)^k e^{x^2/2\Sigma} \frac{d^k}{dx^k} e^{-x^2/2\Sigma}$. Also, notice that the call payoff on $X^{(\epsilon)}$ with maturity T and strike K can be expressed by

$$\max\left\{X^{(\epsilon)}(T) - K, 0\right\} = \epsilon \max\left\{\left(\hat{X}^{(\epsilon)}(T) + y\right), 0\right\}, \tag{24}$$

where $y = \frac{X^{(0)}(T) - K}{\epsilon}$. Then, by using the expansion of the density and the call payoff's expression, we obtain an approximation of the call option price through an expansion of the right hand side of the equation below:

$$e^{-rT}\mathbf{E}\left[\max\left\{X^{(\epsilon)}(T) - K, 0\right\}\right] = \epsilon e^{-rT}\mathbf{E}\left[\max\left\{\left(\hat{X}^{(\epsilon)}(T) + y\right), 0\right\}\right]. \tag{25}$$

Particularly, setting $\epsilon = 1$ in this approximation, we are able to get an approximation for the call option price with payoff (22), $C(T) = \max\{X(T) - K, 0\}$. Consequently, we obtain the following theorem.

Theorem 3.1. *Let $C(0)$ be the time-0 price of the European call option with payoff (22). Then, an approximation formula of $C(0)$ obtained by an asymptotic expansion up to the ϵ^3-order is given by setting $\epsilon = 1$ in the following expression:*

$$e^{-rT}\left\{\epsilon\left(yN\left(\frac{y}{\sqrt{\Sigma}}\right) + \Sigma n[y;0,\Sigma]\right) - \epsilon^2 C_1 \frac{H_1(y;\Sigma)}{\Sigma} n[y;0,\Sigma]\right.$$
$$\left. +\epsilon^3\left(C_2\frac{H_4(y;\Sigma)}{\Sigma^4} + C_3\frac{H_2(y;\Sigma)}{\Sigma^2} + C_4\right)n[y;0,\Sigma]\right\}, \tag{26}$$

where r is a constant risk-free rate, $y = \frac{X^{(0)}(T)-K}{\epsilon}$, $X^{(0)}(T)$ stands for the initial price of the T-maturity forward contract for the underlying asset, $n[x;0,\Sigma]$ is the normal density function with mean 0 and variance Σ, and $N(x)$ denotes the standard normal distribution function. The coefficients $C_1, \cdots, C_4$ are some constants. Moreover, $H_k(x;\Sigma)$ denotes the k-th order Hermite polynomial, and particularly, $H_1(x;\Sigma) = x$, $H_2(x;\Sigma) = x^2 - \Sigma$ and $H_4(x;\Sigma) = x^4 - 6\Sigma x^2 + 3\Sigma^2$.

The detail of the derivation as well as the coefficients $C_1, \cdots, C_4$ are in Appendix A.

4 Numerical Examples

After showing an approximation result in a simple Black-Scholes model, this section will describe numerical examples of basket options and cross currency average options. We remark that in applying the approximation formula (26) in Theorem 3.1 to all the numerical examples, the perturbation parameter ϵ in (26) is set to be 1.

Generally speaking, our expansion formula is able to approximate option prices very well for all the examples. In particular, Subsection 4.2 will show that high dimensionality is not a big issue in a sense that the same formula (26) can be applied to pricing options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment. This feature is an advantage of this method comparing to other analytical (approximation) schemes.

On the other hand, because the expansion is made around $\epsilon = 0$, that is zero volatility, the approximation is expected to become less accurate as the underlying asset volatility and the volatility on volatility are larger. Moreover, as we remark in the end of the previous section, since the limiting density of the normalized random variable $\hat{X}^{(\epsilon)}(T)$ is Gaussian, as the distribution of the underlying asset price is closer to a normal distribution, the approximation is expected to be better and vice versa.

Furthermore, in option pricing, the difference of the underlying asset price distribution has the larger effect for pricing more OTM options because the shape of the tail in the distribution becomes more important. Hence, an approximation for OTM option prices is expected to become more difficult when the underlying distribution is less close to normal.

These points will be shown or/and discussed in details with concrete numerical experiments in the following subsections, which will indicate a robustness of our method in terms of accuracy even under difficult situations for this approximation scheme.

4.1 Plain-Vanilla Option under Black-Scholes Model

The first example is on the Black-Scholes model with no drift:

$$dS(t) = \sigma S(t)dW(t);\ S(0) = x > 0. \tag{27}$$

In this simplest case, $\Sigma = \sigma^2 x^2 T$, and C_i, $i = 1, \cdots, 4$ in (26) are easily obtained as follows:

$$C_1 = \frac{\sigma^4 x^3 T^2}{2},\ C_2 = \frac{\sigma^8 x^6 T^4}{8},\ C_3 = \frac{2\sigma^6 x^4 T^3}{3},\ C_4 = \frac{\sigma^4 x^2 T^2}{4}. \tag{28}$$

The next table compares the approximation result for call options with the exact result by Black-Scholes formula (BS in the table), where the parameters are set as $r = 0$, $x = 10,000$, $\sigma = 0.15$ and $T = 1$. We observe that the third order approximation (denoted by AE 3rd in the table) is rather well.

Table 1: European Call Option (Black-Scholes model)

Strike	8000	9000	10000	11000	12000
AE 3rd	2,040.1	1,202.1	597.9	250.1	89.3
BS	2,040.4	1,202.2	597.9	250.0	89.1
Difference	-0.2	-0.0	-0.0	0.0	0.2
Relative Difference (%)	0.0%	0.0%	0.0%	0.0%	0.2%

4.2 Basket Options with 100 Underlying Assets

First, we examine the accuracy of our method by pricing basket call options with 100 underlying assets. Due to its high dimensionality (e.g. 200 dimensions under the stochastic volatility model used for Table 4), this product is hard to be evaluated by other existing analytical methods especially under stochastic volatility models. Even numerical methods such as Monte Carlo simulations are very time-consuming to obtain accurate prices. We also note that our method is fast enough for practical usage. [3] For instance, by using one core of Xeon X5675 3.07GHz, it takes 3.8 seconds by our approximation formula (26) to obtain the result reported in Table 4 below while it does 7,300 seconds by Monte Carlo simulation. That is, the computational speed of our method is around 1,900 times faster than that of Monte Carlo simulation. Moreover, we remark that when the dimension of the underlying state variables is lower, our method has more advantage than Monte Carlo simulation in terms of the computational speed. (For pricing a plain vanilla call option under λ-SABR model, it takes only 4.2×10^{-5} seconds by the method while 18.4 seconds by Monte Carlo simulation, that is $438,000$ times faster.)

[3] Please see for instance, [1] for other fast accurate analytical method for pricing basket options.

Let us consider the payoff: for $n = 100$,

$$\max\left\{\left(\sum_{i=1}^{n} S_i(T)\right) - K, 0\right\},$$

where we do not consider cross currency options that is, $S_0 \equiv 1$. As for the underlying asset prices S_i ($i = 1, \cdots, n$), we take λ-SABR, CEV and Black-Scholes models. Specifically, λ-SABR model is expressed in the following: for $i = 1, 2, \cdots, n$,

$$\begin{aligned} dS_i(t) &= \alpha_i S_i(t)dt + \sigma_i(t) S_i(t)^\beta \bar{c}_i dZ(t);\ S_i(0) \text{ given}, \quad \alpha_i \text{ is a constant.} && (29) \\ d\sigma_i(t) &= \lambda_i(\theta_i - \sigma_i(t))dt + \bar{\nu}_i \sigma_i(t) dZ(t);\ \sigma_i(0) \text{ given}, && (30) \end{aligned}$$

where λ_i and θ_i are positive constants, and Z is a $2n$-dimensional Brownian motion; c_i and $\bar{\nu}_i$ are defined by

$$\begin{aligned} \bar{c}_i &= (c_{i,1}, \cdots, c_{i,i}, 0, \cdots, 0) && (31) \\ \bar{\nu}_i &= \nu_i(c_{(n+i),1}, \cdots, c_{(n+i),(n+i)}, 0, \cdots, 0), && (32) \end{aligned}$$

where ν_i is a positive constant, and $c_{i,j}$, $(1 \leq j \leq i \leq 2n)$ are obtained by the Cholesky decomposition of the relevant correlation matrix. In λ-SABR model, the details for parameters' specification in the equations above are as follows.

1. The sum of the underlying asset's initial price is 10000; Each initial asset price is generated by the following procedure:

   Firstly, we generate 99 random variables, $a_i^S (i = 1, \cdots, 99)$ from a uniform distribution in $[0, 1]$, that is $U[0, 1]$. Then, we arrange them in ascending order and relabel those by b_i^S so that $b_1^S < \cdots < b_{99}^S$. Next, we take the difference of the 100 ordered couples to define $c_i^S = b_i^S - b_{i-1}^S$ with setting $b_0^S = 0$ and $b_{100}^S = 1$. Finally, we define $S_i(0) = 5000 c_i^S + 50$ to get $S_i(0)$ $(i = 1, \cdots, 100)$.

2. We set $\alpha_i \equiv 0$ without loss of generality, The average of initial volatility $\sigma_i(0)$ and that of the constant mean-reversion level $\theta_i(t) \equiv \theta_i$ are 15%, and the average of volatility on volatility ν_i is 30%. We define each initial volatility and volatility on volatility by $\sigma_i(0) = 50 c_i^\sigma + 1$ and $\nu_i = 5 c_i^\nu + 0.25$," where c_i^σ and c_i^ν are obtained in the similar way as c_i^S above.

3. The parameters λ_i and β_i are set as $\lambda_i = 1$ and $\beta_i = 0.5$, respectively.

4. The correlation between two different asset prices is 0.8; the correlation between an asset price and a volatility is -0.4; the correlation between two different volatilities is 0.3.

CEV and Black-Scholes models have no stochastic volatility processes (21), of course. Moreover, in Black-Scholes and CEV models the parameters are specified as those corresponding ones in λ-SABR model, except that in Black-Scholes model, we set $\beta_i = 1$, and determine each asset price volatility as $\sigma_i^{BS} = \frac{\sigma_i(0)}{100}$ by using $\sigma_i(0)$ in λ-SABR model.

By applying the 3rd order expansion formula in Theorem 3.1, we evaluate the basket call options with 1 year maturity and strike prices 8000, 9000, 10000, 11000 and 12000. Then, we compare those with the prices calculated by Monte Carlo, where the random number generator is Mersenne Twister, the number of trials is 3 million with the antithetic variable method and time steps are 64/year. We remark that in Monte Carlo simulations the convergence becomes very slow for this large number of the underlying assets, and hence, we provide the standard errors in the tables. The results of Black-Scholes, CEV and λ-SABR models are given in Table 2 - Table 4, respectively, where "AE3" stands for the approximate price by the third order asymptotic expansion based on the formula (26) in Theorem 3.1.

Table 2: Basket Option (Black-Scholes model)

Strike	8000	9000	10000	11000	12000
AE 3rd	2,006.4	1,089.1	433.2	121.2	24.4
Monte Carlo	2,006.6	1,089.3	433.1	120.8	24.2
Difference	-0.2	-0.2	0.2	0.4	0.3
Relative Difference (%)	0.0%	0.0%	0.0%	0.3%	1.1%
MC Std Error	0.8	0.7	0.5	0.3	0.2

Table 3: Basket Option (CEV model)

Strike	8000	9000	10000	11000	12000
AE 3rd	2,007.0	1,084.5	413.2	99.5	14.5
Monte Carlo	2,007.1	1,084.6	413.0	99.3	14.4
Difference	-0.1	-0.0	0.2	0.2	0.1
Relative Difference (%)	0.0%	0.0%	0.0%	0.2%	0.5%
MC Std Error	0.7	0.6	0.5	0.3	0.1

Table 4: Basket Option (λ-SABR model)

Strike	8000	9000	10000	11000	12000
AE 3rd	2,038.0	1,169.6	520.5	162.6	32.2
Monte Carlo	2,037.7	1,169.2	520.4	162.7	32.4
Difference	0.4	0.3	0.1	-0.1	-0.2
Relative Difference (%)	0.0%	0.0%	0.0%	-0.1%	-0.7%
MC Std Error	0.7	0.6	0.4	0.2	0.1

From these results, we observe that the 3rd order expansion can approximate basket option prices rather well.

4.3 Currency Basket Options

This subsection will examine currency basket options using actual data. In particular, we evaluate basket options with three month maturity, whose underlying assets consist of five currency pairs; JPYUSD, EURUSD, AUDUSD, GBPUSD and CADUSD, where the quote currencies are USD. Moreover, the model parameters are obtained by the calibration to plain-vanilla option prices with three month maturity as of the first business day of Nov. 2011; particularly, we remark that the correlation parameters among currency pairs should be estimated through the relevant 10 cross currency option markets whose procedure will be described below.

Also, for pricing vanilla and basket options we apply our formula in Theorem 3.1 to (the extended) SABR model with $\beta = 1$: for $i = 0, 1, 2, 3, 4$,

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i(t)S_i(t)\bar{c}_i dZ(t);\ S_i(0) \text{ given}, \quad \alpha_i \text{ is a constant.} \tag{33}$$

$$d\sigma_i(t) = \bar{\nu}_i \sigma_i(t)dZ(t);\ \sigma_i(0) \text{ given}, \tag{34}$$

where c_i and $\bar{\nu}_i$ are defined by

$$\bar{c}_i = (c_{i,0}, c_{i,1}, \cdots, c_{i,i}, 0, \cdots, 0) \tag{35}$$

$$\bar{\nu}_i = \nu_i(c_{(n+1+i),0}, c_{(n+1+i),1}, \cdots, c_{(n+1+i),(n+1+i)}, 0, \cdots, 0), \tag{36}$$

where ν_i is a positive constant, and $c_{i,j}$, $(0 \leq j \leq i \leq 2n+1)$ are obtained by the Cholesky decomposition of the relevant correlation matrix.

In the following, let us briefly describe the calibration procedure:

(**Calibration Procedure**)

- (Step 1) Suppose that each USD quoted exchange rate follows (33)-(34) under the USD risk-neutral measure. Then, given α_i and $S_i(0)$ by observation of the market, through the calibration to each European plain-vanilla currency option market as of the first business day of Nov. 2011, we estimate $\sigma_i(0)$, ν_i and the correlation between the exchange rate and its volatility, that is $(\bar{c}_i \cdot \bar{\nu}_i)$.

- (Step 2) Each exchange rate of 10 cross currency pairs[4] follows the stochastic differential equation for $S_i/S_j (i \neq j)$ which is obtained through (10) with (9), where we replace S_j with S_0 for the definition of Y_i in (7) and specify the parameters by using (33)-(34) instead of (1)-(4). By calibration to each European plain-vanilla cross currency option market, we estimate the remaining 4 correlations that is, $(\bar{c}_i \cdot \bar{c}_j)$ $(\bar{c}_i \cdot \bar{\nu}_j)$, $(\bar{c}_j \cdot \bar{\nu}_i)$, $(\bar{\nu}_i \cdot \bar{\nu}_j)$, while $(\bar{c}_i \cdot \bar{\nu}_i)$ and $(\bar{c}_j \cdot \bar{\nu}_j)$ are already obtained in (Step 1) above; note that the estimation is subject to satisfying the positive definite of the relevant 4×4 correlation matrix.

The results are listed in Table 5 and 6. However, we find that the 10×10 correlation matrix in Table 6 does not satisfy the positive definite condition. In order to fix it,

[4] EURJPY, GBPJPY, AUDJPY, CADJPY, EURCAD, GBPCAD, AUDCAD, EURAUD, GBPAUD, EURGBP.

through simultaneous calibration to five cross currency option markets, we re-estimate the correlation parameters subject to satisfying the positive definiteness of the 10×10 correlation matrix. The result is given in Table 7. We note that the fitting to the market prices based on the correlation matrix in Table 7 becomes worse than before.

Table 5: Volatility Parameters

	$\sigma_i(0)$	ν_i
JPYUSD	0.098	1.703
EURUSD	0.155	1.232
GBPUSD	0.102	1.333
AUDUSD	0.174	1.586
CADUSD	0.124	1.286

Table 6: Correlation Matrix I.

	JPY	EUR	GBP	AUD	CAD	JPY Vol	EUR Vol	GBP Vol	AUD Vol	CAD Vol
JPY	1	0.198	0.060	0.280	0.175	0.256	0.479	0.379	0.238	0.014
EUR	0.198	1	0.796	0.734	0.746	-0.007	-0.570	-0.277	-0.393	-0.294
GBP	0.060	0.796	1	0.683	0.571	-0.269	-0.617	-0.451	-0.547	-0.733
AUD	0.280	0.734	0.683	1	0.801	0.176	-0.870	-0.803	-0.568	-0.597
CAD	0.175	0.746	0.571	0.801	1	-0.394	-0.481	-0.552	-0.531	-0.499
JPY Vol	0.256	-0.007	-0.269	0.176	-0.394	1	0.701	0.966	0.083	0.738
EUR Vol	0.479	-0.570	-0.617	-0.870	-0.481	0.701	1	0.664	0.895	0.489
GBP Vol	0.379	-0.277	-0.451	-0.803	-0.552	0.966	0.664	1	0.889	0.734
AUD Vol	0.238	-0.393	-0.547	-0.568	-0.531	0.083	0.895	0.889	1	0.995
CAD Vol	0.014	-0.294	-0.733	-0.597	-0.499	0.738	0.489	0.734	0.995	1

Table 7: Correlation Matrix II.

	JPY	EUR	GBP	AUD	CAD	JPY Vol	EUR Vol	GBP Vol	AUD Vol	CAD Vol
JPY	1	0.206	0.053	0.249	0.190	0.256	0.419	0.357	0.044	0.124
EUR	0.206	1	0.827	0.767	0.736	-0.083	-0.570	-0.272	-0.731	-0.381
GBP	0.053	0.827	1	0.750	0.628	-0.318	-0.604	-0.451	-0.720	-0.570
AUD	0.249	0.767	0.750	1	0.848	-0.354	-0.309	-0.487	-0.568	-0.495
CAD	0.190	0.736	0.628	0.848	1	-0.320	-0.480	-0.290	-0.544	-0.499
JPY Vol	0.256	-0.083	-0.318	-0.354	-0.320	1	0.562	0.823	0.371	0.436
EUR Vol	0.419	-0.570	-0.604	-0.309	-0.480	0.562	1	0.462	0.692	0.556
GBP Vol	0.357	-0.272	-0.451	-0.487	-0.290	0.823	0.462	1	0.473	0.285
AUD Vol	0.044	-0.731	-0.720	-0.568	-0.544	0.371	0.692	0.473	1	0.749
CAD Vol	0.124	-0.381	-0.570	-0.495	-0.499	0.436	0.556	0.285	0.749	1

Next, we evaluate the 5 currency basket option with three month maturity, where the weights of the exchange rates other than JPYUSD are equal to 1, while the weight of JPYUSD is 100 so as to be of the same order. The weighted average of the initial forward prices of five currencies as of the first business day of Nov. 2011 is 6.2541. We compute the prices of put options with strikes 6.00 and 6.15, as well as of call options with strikes 6.25, 6.35 and 6.50. We use the parameters given in Table 5 and Table 7. For comparison, the benchmark prices are calculated by Monte Carlo simulations, where the random number generator is Mersenne Twister, the number of trials is 5 millions with the antithetic variable method and time steps are 250 per year.

The result is given in Table 8, where "AE 3rd" stands for the approximate price by the third order asymptotic expansion based on the formula (26) in Theorem 3.1: we observe that the approximations are less accurate than those for λ-SABR model in Table 4, which is partly because the calibrated volatility parameters on the volatilities in Table 5 are much higher than 30 % for Table 4. Also, we set $\beta = 1$ in (33), while $\beta = 0.5$ in (29), from which we recall the following observation: because the asymptotic expansion method is based on an expansion around a normal distribution and the distribution of the underlying asset price is closer to a normal when β is closer to zero, the smaller is β, the approximation becomes more accurate in general.

Table 8: Currency Basket Option

Strike	6	6.15	6.25	6.35	6.5
Monte Carlo	0.0522	0.0909	0.1335	0.0847	0.0367
AE 3rd	0.0534	0.0913	0.1334	0.0842	0.0354
Difference	0.0012	0.0004	-0.0001	-0.0006	-0.0014
Relative Difference (%)	2.3%	0.4%	-0.1%	-0.7%	-3.7%

4.4 Cross Currency Average Options

This subsection describes the evaluation of cross currency average options. In particular, we evaluate average options on Japanese yen(JPY)-based WTI future prices, which refers to a JPY-based WTI futures price every business day.

Let us briefly describe the specific features of the WTI average price option: the underlying price of an average option is the average of the settlement prices of the first nearby WTI futures contract during the last one month prior to the maturity of the average option. Note also that the expiration of an average option is the last business day of a calendar month, while trading of a WTI futures contract usually ceases on the third business day prior to the twenty-fifth calendar day. Hence, as the expiration of trading a futures contract is about a week before the end of a calendar month, the futures contracts with two consecutive maturities become the underlying assets of an average option.

More precisely, let the reference dates $t_1^{(1)}, \cdots, t_{m_1}^{(1)}$ for the first contract and $t_1^{(2)}, \cdots, t_{m_2}^{(2)}$ for the subsequent contract ($t_1^{(1)} < \cdots < t_{m_1}^{(1)} < t_1^{(2)} < \cdots < t_{m_2}^{(2)} = T$).

Also, denote the relevant two underlying futures prices as $S_i (i = 1, 2)$, and the spot (USD quoted) JPYUSD exchange rate as S_0. Then, the JPY-based average price at option's maturity T is expressed as follows:

$$X(T) = \frac{1}{M} \left(\sum_{j=1}^{m_1} Y_1 \left(t_j^{(1)} \right) + \sum_{j=1}^{m_2} Y_2 \left(t_j^{(2)} \right) \right), \tag{37}$$

where $M = m_1 + m_2$ and $Y_i = S_i / S_0 (i = 1, 2)$.

Thus, the payoff functions of average call and put options with strike price $K$and maturity T are given by $\max\{X(T) - K, 0\}$ and $\max\{K - X(T), 0\}$, respectively.

We take the following specifications for our numerical examples with the data on July 1, 2009.

- The maturities of average options: two month and four month.
- The relevant WTI futures traded on the NYMEX division of the CME Group: SEP09(until August 18) and OCT09(from August 19) for the two month maturity, and NOV09(until October 19) and DEC09(from October 20) for the four month maturity.
- The maturities of the relevant currency options: one month, two month, three month and six month.

As for the JPYUSD exchange rate process S_0, we apply the λ-SABR model with $\beta = 1$ because we implement simultaneous calibration to currency options with 1,2,3,6 month maturities, in which we need to take the term structure of the JPYUSD exchange rate volatility process into account.

For the WTI futures price processes $S_i (i = 1, 2)$, We take SABR model with $\beta = 0.5, 1$, since our previous analysis in [26] has found that SABR model can achieve good calibration to the WTI futures option market. (See [26] for the detail.)

In sum, the relevant stochastic processes are described by the solutions to the following stochastic differential equations:

$$
\begin{aligned}
dS_0(t) &= \alpha_0 S_0(t)dt + \sigma_0(t)S_0(t)\bar{c}_0 dZ(t);\ S_0(0) \text{ given}, \quad \alpha_0 \text{ is a constant.} & (38)\\
d\sigma_0(t) &= \lambda_0(\theta_0 - \sigma_0(t))dt + \bar{\nu}_0\sigma_0(t)dZ(t);\ \sigma_0(0) \text{ given}, & (39)\\
&\quad \lambda_0 \text{ and } \theta_0 \text{ are positive constants.}
\end{aligned}
$$

For $i = 1, 2$,

$$
\begin{aligned}
dS_i(t) &= \sigma_i(t)S_i(t)^{\beta}\bar{c}_i dZ(t);\ S_i(0) \text{ given}, & (40)\\
d\sigma_i(t) &= \bar{\nu}_i\sigma_i(t)dZ(t);\ \sigma_i(0) \text{ given.} & (41)
\end{aligned}
$$

Here, $c_i (i = 0, 1, 2)$ and $\bar{\nu}_i (i = 0, 1, 2)$ are defined by

$$
\begin{aligned}
\bar{c}_i &= (c_{i,0}, c_{i,1}, \cdots, c_{i,i}, 0, \cdots, 0) & (42)\\
\bar{\nu}_i &= \nu_i(c_{(n+1+i),0}, c_{(n+1+i),1}, \cdots, c_{(n+1+i),(n+1+i)}, 0, \cdots, 0), & (43)
\end{aligned}
$$

where ν_i is a positive constant, and $c_{i,j}$, $(0 \leq j \leq i \leq 2n + 1)$ are obtained by the Cholesky decomposition of the relevant correlation matrix. Note also that the futures price processes have no drifts in (40).

Each JPY-based WTI price $Y_i = S_i/S_0 (i = 1 \text{ or } 2)$ follows the stochastic differential equation for Y_i which is obtained through (10) with (9), where we specify the parameters by using (38)-(41) instead of (1)-(4).

The correlations between the JPYUSD rate and WTI futures prices, as well as those between different futures contracts' prices are estimated from the Historical estimation data of the previous two month(four month) for the two-month(four-month) maturity options, which are reported in Table 9 and Table 10 below.

Table 9: Correlation I (2M)

	JPYUSD	Sep09	Oct09
JPYUSD	1	-0.234	-0.243
Sep09	-0.234	1	0.999
Oct09	-0.243	0.999	1

Table 10: Correlation I (4M)

	JPYUSD	Nov09	Dec09
JPYUSD	1	-0.227	-0.224
Nov09	-0.227	1	0.999
Dec09	-0.224	0.999	1

The other correlations necessary for the evaluation of cross currency average options are given in Table 11 - Table 14, which are determined in the following way:

- The correlation between the NOV09 price and the DEC09 volatility is set to be the same as the one between the DEC09 price and DEC09 volatility which is obtained by the calibration and is reported in Table 15 and Table 16 below.

  The same rule is applied to the correlation between the DEC09 price and the NOV09 volatility, the correlation between the SEP09 price and the OCT09 volatility, and the correlation between the OCT09 price and the SEP09 volatility.

- The following correlations are set as 0:

  the JPYUSD rate's volatility and the futures prices' volatilities;

  the JPYUSD rate's volatility and the futures prices;

  the JPYUSD rate's volatility and the futures prices' volatilities.

- All the correlations between the futures prices' volatilities with different contracts are set as 0.999.

Table 11: Correlation II ($\beta = 1$, 2M)

	JPYUSD	Sep09	Oct09	vol of JPYUSD	vol of Sep09	vol of Oct09
JPYUSD	1	-0.234	-0.243	0.525	0	0
Sep09	-0.234	1	0.999	0	-0.369	-0.369
Oct09	-0.243	0.999	1	0	-0.363	-0.363
vol of JPYUSD	0.525	0	0	1	0	0
vol of Sep09	0	-0.369	-0.363	0	1	0.9999
vol of Oct09	0	-0.369	-0.363	0	0.9999	1

Table 12: Correlation II ($\beta = 0.5$, 2M)

	JPYUSD	Sep09	Oct09	vol of JPYUSD	vol of' Sep09	vol of' Oct09
JPYUSD	1	-0.234	-0.243	0.525	0	0
Sep09	-0.234	1	0.999	0	-0.225	-0.225
Oct09	-0.243	0.999	1	0	-0.188	-0.188
vol of JPYUSD	0.525	0	0	1	0	0
vol of Sep09	0	-0.225	-0.188	0	1	0.9999
vol of Oct09	0	-0.225	-0.188	0	0.9999	1

Table 13: Correlation II ($\beta = 1$, 4M)

	JPYUSD	Nov09	Dec09	vol of JPYUSD	vol of Nov09	vol of Dec09
JPYUSD	1	-0.227	-0.224	0.525	0	0
Nov09	-0.227	1	0.999	0	-0.383	-0.383
Dec09	-0.224	0.999	1	0	-0.365	-0.365
vol of JPYUSD	0.525	0	0	1	0	0
vol of Nov09	0	-0.383	-0.365	0	1	0.9999
vol of Dec09	0	-0.383	-0.365	0	0.9999	1

Table 14: Correlation II ($\beta = 0.5$, 4M)

	JPYUSD	Nov09	Dec09	vol of JPYUSD	vol of Nov09	vol of Dec09
JPYUSD	1	-0.227	-0.224	0.525	0	0
Nov09	-0.227	1	0.999	0	-0.190	-0.190
Dec09	-0.224	0.999	1	0	-0.172	-0.172
vol of JPYUSD	0.525	0	0	1	0	0
vol of Nov09	0	-0.190	-0.172	0	1	0.9999
vol of Dec09	0	-0.190	-0.172	0	0.9999	1

The results of calibration are reported in Table 15 and Table 16.

Table 15: Calibrated Parameters ($\beta = 0.5$)

	$S(0)$	α_0	$\sigma(0)$	λ	θ	ν	ρ
JPYUSD	0.010319	0.004	0.132	1.170	0.121	1.295	0.525
SEP 09	70.27	-	3.607	-	-	1.427	-0.225
OCT 09	71.08	-	3.623	-	-	1.140	-0.188
NOV 09	71.78	-	3.606	-	-	1.000	-0.190
DEC 09	72.36	-	3.519	-	-	0.993	-0.172

Table 16: Calibrated Parameters ($\beta = 1.0$)

	$S(0)$	α_0	$\sigma(0)$	λ	θ	ν	ρ
JPYUSD	0.010319	0.004	0.132	1.170	0.121	1.295	0.525
SEP 09	70.27	-	0.433	-	-	1.519	-0.369
OCT 09	71.08	-	0.433	-	-	1.252	-0.363
NOV 09	71.78	-	0.430	-	-	1.109	-0.383
DEC 09	72.36	-	0.419	-	-	1.098	-0.365

Given the parameters above, we finally evaluate cross currency average options that is, put options with strikes 6000 and 6500, and call options with strikes 7000, 7500 and 8000; also, the ATM prices for the two-month maturity and for the four-month maturity are given by 6832.4 and 6969.0, respectively.

In addition, in order for investigation of the accuracy of our approximations, benchmark prices are computed by Monte Carlo simulations, where the random number generator is Mersenne Twister, and the number of trials is 2.5 million with the antithetic variable method and with 256 time-steps for calculation of each price.

The results of the third order approximate prices are reported in Table 17 - Table 20, where "AE 3rd" stands for the approximate price by the third order asymptotic expansion based on the formula (26) in Theorem 3.1. We observe that our third-order formula (26) provides very accurate approximations. Also, we remark that the approximations with $\beta = 0.5$ are more accurate than those with $\beta = 1$ especially for OTM options, since as noted at the end of the previous subsection, our asymptotic expansion is made around the normal distribution, and the underlying price's distribution under $\beta = 0.5$ is closer to a normal distribution than the one under $\beta = 1$.

Moreover, we would like to stress that the computational speed in calibration and pricing is very fast [5], which implies our formula is so useful in practice especially for high dimensional pricing models.

Table 17: Third-order Approximate Prices ($\beta = 0.5, 2M$)

Strike	6000	6500	7000	7500	8000
Monte Carlo	148.1	297.6	366.8	193.8	94.1
AE 3rd	149.2	298.2	367.3	194.0	94.0
Difference	1.1	0.6	0.5	0.2	-0.1
Relative Difference (%)	0.7%	0.2%	0.1%	0.1%	-0.1%

[5]The advantage of our method in computational speed is demonstrated for pricing *basket options with 100 underlying assets.* in Subsection 4.2

Table 18: Third-order Approximate Prices ($\beta = 1.0, 2M$)

Strike	6000	6500	7000	7500	8000
Monte Carlo	148.8	299.0	368.5	195.4	95.5
AE 3rd	150.5	299.8	368.8	195.1	94.7
Difference	1.8	0.8	0.3	-0.2	-0.8
Relative Difference (%)	1.2%	0.3%	0.1%	-0.1%	-0.8%

Table 19: Third-order Approximate Prices ($\beta = 0.5, 4M$)

Strike	6000	6500	7000	7500	8000
Monte Carlo	315.3	487.9	686.8	477.4	322.4
AE 3rd	316.9	488.6	687.6	477.8	322.5
Difference	1.5	0.7	0.8	0.4	0.1
Relative Difference (%)	0.5%	0.2%	0.1%	0.1%	0.0%

Table 20: Third-order Approximate Prices ($\beta = 1.0, 4M$)

Strike	6000	6500	7000	7500	8000
Monte Carlo	314.4	488.0	688.0	479.1	324.5
AE 3rd	318.4	490.2	689.3	479.3	323.8
Difference	4.0	2.2	1.3	0.2	-0.7
Relative Difference (%)	1.3%	0.5%	0.2%	0.0%	-0.2%

5 Correlations in the Currency Option Market

This section investigates the correlations implied in the currency option market. A correlation between different currency pairs is widely traded as a correlation swap in the current Over-the-counter(OTC) market. On the other hand, the correlations such as between currency pairs' volatilities are not explicitly observable in the market. In this section, through calibration to the currency option market, we extract the implied correlations including the directly unobservable ones in the market.

In calibration, we use SABR model with $\beta = 1$ for the dynamics of each USD quoted currency pair and apply the same method as in Section 4.3 to obtain that of a cross currency pair. For empirical investigation, we use the currency pairs, EURUSD, USDJPY and EURJPY for highly liquid ones and USDKRW, JPYKRW, USDSGD, SGDJPY for relatively illiquid ones. Moreover, we take the dates before and after the events such as *Lehman Shock* and *Tohoku Earthquake* in Japan. In particular, as for Lehman Shock, we use the data on September 8th and 22nd 2008 while we use the data on March 3rd and 17th 2011 for Tohoku Earthquake; For JPYKRW, as the smile data of its option market in 2008 is not available, we implement the analysis only for the data in 2011. On the other hand, as the smile data for the SGDJPY option market is not available for the period around the March 11th, 2011 was not available, we take the data of March 1st and 29th, 2011, instead. In addition, we use the data for currency options with maturities 1M, 2M, 3M and 6M for our analysis.

Furthermore, although we mainly apply SABR model with its own volatility process for each USD quoted pair as in (33)-(34), we also test the model, in which the volatility processes for the USD quoted currency pairs relevant with a cross currency pair are perfectly correlated as in *Double Heston model*(e.g. Gauthier-Possamaire (2010)). In the following tables, the former model is denoted by (4f) while the latter one is denoted by (3f).

For (4f), the way of calibration is exactly the same as (**Calibration Procedure**) described in Section 4.3. As for (3f), while (Step 1) in the calibration procedure is the same, the only remaining correlation used for the calibration to a cross currency option market is the one between two relevant USD quoted foreign exchange rates.

Table 21 - Table 23 show the calibration results for cross currency option markets, from which we observe that the calibration by (4f) is much more accurate than that by (3f) as expected. We also report the results for the calibration to the 3 month option markets of the USD quoted currency pairs in Table 31 - Table 33 of Appendix, which shows fairly good performance in general; the results for the calibration to the other maturities' options are quite similar, and thus they are omitted.

Table 21: Volatility Smile (%)

		2008/9/8					2008/9/22				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
EURJPY	Calibrated Vol (4f)	17.59	14.94	12.81	11.40	10.87	19.86	16.83	14.50	13.00	12.47
	Calibrated Vol (3f)	17.76	14.96	12.88	11.43	10.64	19.46	16.50	14.37	13.20	12.97
	Market Vol	17.59	14.92	12.84	11.38	10.88	19.86	16.81	14.53	12.97	12.48
	Difference (4f)	0.00	0.02	-0.03	0.03	-0.01	0.00	0.02	-0.03	0.03	-0.01
	Difference (3f)	0.17	0.04	0.04	0.06	-0.24	-0.40	-0.30	-0.16	0.23	0.49
SGDJPY	Calibration Vol (4f)		12.90	10.85	9.15			15.86	13.44	11.76	
	Calibrated Vol (3f)		13.20	10.78	9.00			16.43	13.49	11.31	
	Market Vol		12.90	10.85	9.15			15.82	13.50	11.73	
	Difference (4f)		0.00	0.00	0.00			0.04	-0.06	0.03	
	Difference (3f)		0.31	-0.07	-0.16			0.61	-0.01	-0.41	

Table 22: Volatility Smile (%)

		2011/3/3					2011/3/17				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
EURJPY	Calibration Vol (4f)	14.71	13.05	11.78	11.14	11.16	22.60	19.57	17.28	15.90	15.65
	Calibrated Vol (3f)	15.29	13.38	11.91	10.92	10.53	23.52	20.04	17.41	15.58	14.84
	Market Vol	14.71	13.06	11.76	11.15	11.15	22.60	19.58	17.26	15.91	15.65
	Difference (4f)	0.00	-0.01	0.02	-0.02	0.01	0.00	-0.01	0.02	-0.01	0.00
	Difference (3f)	0.58	0.32	0.15	-0.24	-0.62	0.92	0.46	0.15	-0.33	-0.81
JPYKRW	Calibration Vol (4f)	11.92	12.14	13.41	16.37	19.36	15.86	16.20	17.76	21.45	25.15
	Calibrated Vol (3f)	11.05	11.88	13.54	16.81	20.39	14.40	15.69	17.92	22.22	26.91
	Market Vol	11.94	12.09	13.46	16.34	19.37	15.90	16.09	17.86	21.39	25.17
	Difference (4f)	-0.02	0.05	-0.05	0.03	-0.01	-0.04	0.10	-0.11	0.06	-0.02
	Difference (3f)	-0.89	-0.21	0.08	0.47	1.01	-1.50	-0.40	0.05	0.84	1.75

Table 23: Volatility Smile (%)

		2011/3/1					2011/3/29				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
SGDJPY	Calibrated Vol (4f)		10.60	9.51	9.05			11.71	10.55	10.06	
	Calibrated Vol (3f)		11.04	9.60	8.62			12.32	10.67	9.49	
	Market Vol		10.61	9.50	9.06			11.72	10.55	10.07	
	Difference (4f)		-0.01	0.01	-0.01			0.00	0.00	0.00	
	Difference (3f)		0.44	0.10	-0.43			0.60	0.12	-0.58	

Table 24 and Table 25 show the correlation sensitivities associated with EURJPY and JPYKRW options as examples, where each correlation sensitivity is defined by the change in the implied volatility(%) caused by 0.01 change of a correlation inside our model(4f).

Firstly, we first observe that the correlation between two foreign exchange rates has the largest impact as expected, and the direction of the impact is the same for each moneyness. On the other hand, the correlations between an exchange rate and the volatility of another exchange rate have impacts mainly for the skewness of the implied volatility curve. Although the correlation between volatilities has the smallest impact, it has a reasonable effect on the smile shape of the implied volatility curve since its (absolute) sensitivity at ATM is the largest among the different moneynesses. Note also that those observations are unchanged between the dates before and after the events.

Table 24: Correlation Sensitivities (EURJPY)

		2011/3/3				2011/3/17			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	-10D	-0.0747%	-0.0176%	0.0093%	-0.0019%	-0.0935%	-0.0251%	0.0087%	-0.0003%
	-25D	-0.0830%	-0.0035%	0.0089%	-0.0062%	-0.0974%	-0.0128%	0.0050%	-0.0058%
	ATM	-0.0919%	0.0076%	0.0030%	-0.0078%	-0.1085%	0.0030%	-0.0003%	-0.0077%
	25D	-0.0876%	0.0130%	-0.0086%	-0.0066%	-0.1160%	0.0231%	-0.0074%	-0.0063%
	10D	-0.0706%	0.0113%	-0.0205%	-0.0037%	-0.1141%	0.0407%	-0.0137%	-0.0033%
2M	-10D	-0.0767%	-0.0187%	0.0108%	-0.0017%	-0.0618%	-0.0273%	0.0043%	0.0007%
	-25D	-0.0823%	-0.0041%	0.0096%	-0.0073%	-0.0782%	-0.0105%	0.0086%	-0.0063%
	ATM	-0.0911%	0.0078%	0.0028%	-0.0092%	-0.1026%	0.0068%	0.0060%	-0.0090%
	25D	-0.0889%	0.0145%	-0.0102%	-0.0077%	-0.1170%	0.0262%	-0.0065%	-0.0071%
	10D	-0.0744%	0.0139%	-0.0230%	-0.0040%	-0.1100%	0.0397%	-0.0220%	-0.0027%
3M	-10D	-0.0757%	-0.0206%	0.0110%	-0.0017%	-0.0662%	-0.0285%	0.0059%	0.0014%
	-25D	-0.0831%	-0.0038%	0.0110%	-0.0083%	-0.0795%	-0.0102%	0.0096%	-0.0071%
	ATM	-0.0945%	0.0093%	0.0040%	-0.0106%	-0.1021%	0.0080%	0.0060%	-0.0103%
	25D	-0.0912%	0.0162%	-0.0114%	-0.0088%	-0.1154%	0.0283%	-0.0084%	-0.0079%
	10D	-0.0718%	0.0141%	-0.0267%	-0.0044%	-0.1071%	0.0415%	-0.0254%	-0.0024%
6M	-10D	-0.0719%	-0.0239%	0.0105%	-0.0010%	-0.0445%	-0.0344%	0.0004%	0.0006%
	-25D	-0.0830%	-0.0040%	0.0126%	-0.0096%	-0.0702%	-0.0084%	0.0126%	-0.0104%
	ATM	-0.0992%	0.0111%	0.0060%	-0.0126%	-0.1076%	0.0139%	0.0125%	-0.0145%
	25D	-0.0976%	0.0204%	-0.0127%	-0.0102%	-0.1193%	0.0346%	-0.0078%	-0.0112%
	10D	-0.0743%	0.0189%	-0.0321%	-0.0041%	-0.0881%	0.0395%	-0.0333%	-0.0031%

Table 25: Correlation Sensitivities (JPYKRW)

		2011/3/3				2011/3/17			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	-10D	-0.0688%	-0.0383%	0.0257%	-0.0056%	-0.1035%	-0.0259%	0.0493%	-0.0196%
	-25D	-0.1084%	-0.0368%	0.0051%	-0.0093%	-0.1164%	-0.0192%	0.0237%	-0.0309%
	ATM	-0.1042%	-0.0248%	-0.0079%	-0.0105%	-0.1123%	-0.0096%	0.0030%	-0.0354%
	25D	-0.0625%	0.0002%	-0.0069%	-0.0065%	-0.1005%	0.0060%	-0.0122%	-0.0215%
	10D	-0.0479%	0.0345%	-0.0032%	0.0015%	-0.1119%	0.0235%	-0.0299%	0.0062%
2M	-10D	-0.0975%	-0.0500%	0.0263%	-0.0072%	-0.0788%	-0.0359%	0.0520%	-0.0264%
	-25D	-0.1162%	-0.0408%	0.0078%	-0.0113%	-0.1117%	-0.0321%	0.0232%	-0.0418%
	ATM	-0.1062%	-0.0248%	-0.0048%	-0.0127%	-0.1131%	-0.0202%	0.0005%	-0.0477%
	25D	-0.0742%	0.0018%	-0.0065%	-0.0075%	-0.0917%	0.0068%	-0.0113%	-0.0262%
	10D	-0.0697%	0.0375%	-0.0073%	0.0022%	-0.1111%	0.0400%	-0.0250%	0.0110%
3M	-10D	-0.1083%	-0.0560%	0.0270%	-0.0074%	-0.1287%	-0.0463%	0.0519%	-0.0244%
	-25D	-0.1176%	-0.0413%	0.0110%	-0.0116%	-0.1305%	-0.0277%	0.0266%	-0.0442%
	ATM	-0.1069%	-0.0232%	-0.0009%	-0.0131%	-0.1181%	-0.0093%	0.0057%	-0.0512%
	25D	-0.0815%	0.0038%	-0.0060%	-0.0073%	-0.1017%	0.0114%	-0.0109%	-0.0259%
	10D	-0.0843%	0.0393%	-0.0120%	0.0029%	-0.1224%	0.0352%	-0.0302%	0.0158%
6M	-10D	-0.1093%	-0.0469%	0.0316%	-0.0072%	-0.1272%	-0.0541%	0.0573%	-0.0098%
	-25D	-0.1210%	-0.0174%	0.0169%	-0.0126%	-0.1360%	-0.0329%	0.0295%	-0.0180%
	ATM	-0.1128%	0.0067%	0.0044%	-0.0144%	-0.1238%	-0.0117%	0.0072%	-0.0206%
	25D	-0.0897%	0.0209%	-0.0063%	-0.0071%	-0.1049%	0.0136%	-0.0110%	-0.0097%
	10D	-0.1024%	0.0342%	-0.0190%	0.0041%	-0.1412%	0.0419%	-0.0330%	0.0058%

Finally, the calibrated correlation parameters for cross currency option data are given in the rows, "Calibration(4f)" and "Calibration(3f)" of Table 26 - Table 30. Also, the calibrated parameters associated with option markets of the USD quoted currency pairs are reported in Table 34 - Table 36 of Appendix. In addition, historically estimated correlations are shown in Table 26 - Table 30 for comparative purpose. In order to estimate historical correlations, we use the spot prices and ATM options' volatilities, where the reference period is the same as the corresponding option's maturity.

Generally speaking, it is observed that a correlation between two foreign exchange rates(S_1-S_2) moves similarly for the implied and historically estimated ones, while a correlation between a exchange rate and another exchange rate's volatility(S_1-V_2 or V_1-S_2) does not.

For EURJPY, the implied correlations between the foreign exchange rates(S_1-S_2) after *Lehman Shock* went up; it was consistent with the historically estimated ones, which was caused by the rise of both EUR and JPY against USD after the shock. After *Tohoku Earthquake*, the implied and historically estimated correlations between two foreign exchange rates(S_1-S_2) fell down, which is consistent to the rise of both EUR and USD against JPY after the earthquake.

On the other hand, the implied correlations between two volatilities(V_1-V_2) for the one month(1M) rose up to 90%, while the corresponding historically estimated correlations fell down: this difference may be caused by the serious accidents of the Fukushima nuclear plant right after the earthquake, which leads to the substantial rise in the short-term implied volatility much quicker than in the historically estimated one. For the correlation between a exchange rate and another exchange rate's volatility(S_1-V_2 or V_1-S_2), there seems little relation in size and behavior between the implied and historically estimated ones.

As for SGDJPY, with no data of 10 delta, we have to calibrate four parameters based on three volatility points(for ATM and +25 delta/-25delta), whose results are not stable except the correlations between two exchange rates(S_1-S_2). The correlations between two foreign exchange rates(S_1-S_2) do not move similarly for the implied and historically estimated ones; also, the levels of the correlations are rather different among the implied and historical ones. Note also that we need to care about the low liquidity of the currency pair SGDJPY, especially after big events, which makes the option prices move quickly with few trades.

JPYKRW volatility's data is not available for the period around the Lehman Shock, so that we examine the data in March 2011. The implied correlations between two exchange rates rose after the earthquake, that is JPY rose up and KRW fell down against USD, which was consistent with the behavior of the historically estimated ones. However, the levels were different among the implied and historical correlations.

Finally, we remark that for the case of the model (3f), the correlations between two exchange rates(S_1-S_2) are lower than those for the case of the model (4f).

Table 26: $S_1 = EUR, S_2 = JPY$

		2008/9/8				2008/9/22			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	Calibration(4f)	0.339	-0.553	0.017	-0.062	0.419	-0.093	0.516	0.743
	Calibration(3f)	0.297				0.416			
	Historical estimation	-0.086	-0.295	-0.154	0.234	0.254	0.161	0.304	0.666
2M	Calibration(4f)	0.334	-0.525	0.016	-0.116	0.441	-0.058	0.581	0.763
	Calibration(3f)	0.281				0.435			
	Historical estimation	0.122	0.007	0.227	0.441	0.274	0.179	0.366	0.631
3M	Calibration(4f)	0.409	-0.459	0.187	0.080	0.442	-0.063	0.585	0.772
	Calibration(3f)	0.345				0.434			
	Historical estimation	0.244	0.112	0.156	0.428	0.314	0.227	0.318	0.571
6M	Calibration(4f)	0.437	-0.292	0.396	0.466	0.479	-0.001	0.724	0.893
	Calibration(3f)	0.379				0.478			
	Historical estimation	0.422	0.227	0.335	0.249	0.391	0.296	0.321	0.521

Table 27: $S_1 = EUR, S_2 = JPY$

		2011/3/3				2011/3/17			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	Calibration(4f)	0.466	-0.634	-0.264	0.239	0.304	-0.157	0.202	0.900
	Calibration(3f)	0.422				0.278			
	Historical estimation	0.539	0.159	0.227	0.650	0.289	-0.029	0.288	0.440
2M	Calibration(4f)	0.435	-0.531	-0.212	0.438	0.280	-0.546	-0.534	0.154
	Calibration(3f)	0.391				0.227			
	Historical estimation	0.268	0.045	0.066	0.474	0.286	-0.001	0.205	0.407
3M	Calibration(4f)	0.436	-0.593	-0.254	0.315	0.263	-0.521	-0.430	0.272
	Calibration(3f)	0.376				0.207			
	Historical estimation	0.501	-0.016	-0.033	0.542	0.279	-0.014	0.211	0.356
6M	Calibration(4f)	0.392	-0.668	-0.285	0.317	0.330	-0.633	-0.623	-0.215
	Calibration(3f)	0.331				0.231			
	Historical estimation	0.419	-0.086	-0.031	0.379	0.482	-0.013	0.001	0.301

Table 28: $S_1 = SGD, S_2 = JPY$

		2008/9/8				2008/9/22			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	Calibration(4f)	0.463	-0.168	0.417	0.311	0.389	-0.043	0.379	0.022
	Calibration(3f)	0.388				0.298			
	Historical estimation	0.367	0.312	-0.180	0.055	0.039	0.122	0.391	0.529
2M	Calibration(4f)	0.486	-0.155	0.340	0.119	0.443	0.034	0.364	-0.326
	Calibration(3f)	0.371				0.276			
	Historical estimation	0.347	0.293	0.208	0.299	0.214	0.232	0.201	0.275
3M	Calibration(4f)	0.507	-0.132	0.253	-0.104	0.516	0.110	0.351	-0.312
	Calibration(3f)	0.337				0.287			
	Historical estimation	0.334	0.246	0.087	0.176	0.220	0.250	0.279	0.314
6M	Calibration(4f)	0.578	0.001	0.369	0.009	0.618	0.177	0.387	-0.123
	Calibration(3f)	0.356				0.618			
	Historical estimation	0.323	0.408	0.198	0.151	0.290	0.368	0.171	0.166

Table 29: $S_1 = SGD, S_2 = JPY$

		2011/3/1				2011/3/29			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	Calibration(4f)	0.580	-0.050	-0.017	-0.307	0.629	0.167	0.105	0.086
	Calibration(3f)	0.422				0.516			
	Historical estimation	0.325	-0.365	0.168	0.444	0.196	-0.355	0.586	0.766
2M	Calibration(4f)	0.505	-0.045	0.175	0.669	0.645	0.001	0.007	0.120
	Calibration(3f)	0.450				0.514			
	Historical estimation	0.197	-0.149	-0.135	0.189	0.150	-0.321	0.392	0.521
3M	Calibration(4f)	0.414	0.070	0.369	0.854	0.492	-0.049	-0.084	0.901
	Calibration(3f)	0.378				0.453			
	Historical estimation	0.329	-0.003	-0.249	0.299	0.182	-0.310	0.442	0.491
6M	Calibration(4f)	0.470	0.036	0.147	0.552	0.563	0.041	0.039	0.308
	Calibration(3f)	0.376				0.428			
	Historical estimation	0.298	-0.029	-0.122	0.320	0.265	-0.177	0.071	0.343

Table 30: $S_1 = JPY, S_2 = KRW$

		2011/3/3				2011/3/17			
		S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2	S_1-S_2	S_1-V_2	V_1-S_2	V_1-V_2
1M	Calibration(4f)	0.353	0.579	0.381	-0.001	0.292	0.426	0.055	0.413
	Calibration(3f)	0.276				0.119			
	Historical estimation	0.134	0.076	-0.540	0.610	-0.288	0.483	-0.492	0.752
2M	Calibration(4f)	0.327	0.339	0.103	0.638	0.284	0.656	-0.006	0.294
	Calibration(3f)	0.271				0.204			
	Historical estimation	0.041	0.052	-0.326	0.492	0.001	0.245	-0.305	0.616
3M	Calibration(4f)	0.293	0.212	-0.181	0.853	0.235	0.170	-0.297	0.946
	Calibration(3f)	0.251				0.189			
	Historical estimation	0.353	-0.102	-0.171	0.393	0.006	0.281	-0.267	0.531
6M	Calibration(4f)	0.242	0.013	-0.492	0.950	0.234	0.211	-0.163	0.945
	Calibration(3f)	0.199				0.160			
	Historical estimation	0.262	-0.025	-0.122	0.280	0.215	0.070	-0.169	0.329

6 Conclusion

This paper has developed a general pricing method for multi-asset cross currency options under high-dimensional diffusion models, and presented a series of practical examples with numerical analysis; pricing cross currency options, cross currency basket options and cross currency average options under multiple (λ-)SABR models. We have also demonstrated that our scheme is capable of evaluating options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment.

Moreover, we have examined the correlations implied in the cross currency option markets by using data on the dates before and after Lehman Shock and Tohoku Earthquake in Japan. In order to find a model which is able to explain the currency option markets historically well, we need to implement more thorough empirical analysis, where we will make use of our general analytical method for option pricing. This is one of the next topics for our research.

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A Asymptotic Expansion Approach

For given $\epsilon \in (0,1]$, the extended λ-SABR model in an asymptotic expansion approach is described as follows:

$$
\begin{aligned}
S_i^{(\epsilon)}(T) &= S_i(0) + \int_0^T (\alpha_i(t) + \epsilon^2 \hat{\sigma}_0^{(\epsilon)}(t)\hat{\sigma}_i^{(\epsilon)}(t))S_i^{(\epsilon)}(t)dt + \epsilon \int_0^T \hat{\sigma}_i^{(\epsilon)}(t)S_i^{(\epsilon)}(t)dW \\
&= S_i(0) + \int_0^T \left(\alpha_i(t) + \epsilon^2 c_{0,i}\sigma_0^{(\epsilon)}\sigma_i^{(\epsilon)} S_0^{(\epsilon)\beta-1} S_i^{(\epsilon)\beta-1}\right) S_i^{(\epsilon)} dt \\
&\quad + \epsilon \int_0^T \sigma_i^{(\epsilon)} S_i^{(\epsilon)\beta} \sum_{j=0}^{i} c_{i,j} d\mathbf{W}_j
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
\sigma_i^{(\epsilon)}(T) &= \int_0^T \lambda_i(t)(\theta_i(t) - \sigma_i^{(\epsilon)}(t))dt + \epsilon^2 \int_0^T \hat{\sigma}_0^{(\epsilon)}(t)\sigma_i^{(\epsilon)}(t)\bar{\nu}_i(t)dt + \epsilon \int_0^T \bar{\nu}_i(t)\sigma_i^{(\epsilon)}(t)dW, \\
&= \sigma_i(0) + \int_0^T \lambda_i \left(\theta_i - \sigma_i^{(\epsilon)}\right) dt + \epsilon^2 \int_0^T \left(\nu_{i,0}\sigma_0^{(\epsilon)}\sigma_i^{(\epsilon)} S_0^{(\epsilon)\beta-1}\right) dt \\
&\quad + \epsilon \sum_{j=0}^{n+i+1} \int_0^T \nu_{i,j}\sigma_i^{(\epsilon)} dW_j
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
Y_i^{(\epsilon)}(t) &= \int_0^T (\alpha_i(t) - \alpha_0(t))\, Y_i^{(\epsilon)}(t)dt + \epsilon \int_0^T \left(-\hat{\sigma}_0^{(\epsilon)}(t) + \hat{\sigma}_i^{(\epsilon)}(t)\right) Y_i^{(\epsilon)}(t)dW \\
&= \int_0^T (\alpha_i(t) - \alpha_0(t))\, Y_i^{(\epsilon)}dt + \epsilon \int_0^T Y_i^{(\epsilon)} \left(-\sigma_0^{(\epsilon)} S_0^{(\epsilon)\beta-1} dW_0 + \sum_{j=0}^{i} c_{i,j}\sigma_i^{(\epsilon)} S_i^{(\epsilon)\beta-1} dW_j\right).
\end{aligned}
\tag{46}
$$

$S_i^{(\epsilon)}(T)$ is expanded around 0 as

$$
S_i^{(\epsilon)} = S_i^{(0)} + \epsilon S_i^{(1)} + \epsilon^2 \frac{S_i^{(2)}}{2} + \cdots . \tag{47}
$$

The coefficients $S_i^{(m)} = \frac{\partial^m S_i^{(\epsilon)}}{\partial \epsilon^m}|_{\epsilon=0}(m = 0, 1, 2, 3, ...)$ are derived by substitution (44) for (45), and m-times differentiation by ϵ around 0. The results of calculations are follows:

$$
Y_i^{(0)}(T) = Y_i(0)e^{\int_0^T (\alpha_i(t)-\alpha_0(t))dt} = \frac{S_i(0)}{S_0(0)} e^{\int_0^T (\alpha_i(t)-\alpha_0(t))dt}. \tag{48}
$$

$$
S_i^{(0)}(T) = S_i(0)e^{\int_0^T \alpha_i(s)dt} \tag{49}
$$

$$
\sigma_i^{(0)}(T) = e^{\int_0^T \lambda_i(t)dt} \left(\sigma_i(0) + \int_0^T e^{\int_0^t \lambda_i(s)ds} \lambda_i(t)\theta_i(t)dt\right). \tag{50}
$$

Hereafter, we omit the time parameter t for simplicity:

$$
\begin{aligned}
Y_i^{(1)}(T) &= -\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)}\sigma_0^{(0)} S_0^{(0)\beta_0-1} dW_0 \\
&\quad + \sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)} c_{i,j}\sigma_i^{(0)} S_i^{(0)\beta_i-1} dW_j \qquad (51)\\
S_i^{(1)}(T) &= \sum_{j=0}^{i} c_{i,j}\int_0^T e^{\int_t^T \alpha_i(s)ds} S_i^{(0)\beta_i}\sigma_i^{(0)} dW_j \qquad (52)\\
\sigma_i^{(1)}(T) &= \sum_{j=0}^{n+1+i}\int_0^T \nu_{i,j} e^{-\int_t^T \lambda_i(s)ds}\sigma_i^{(0)} dW_j \qquad (53)\\
Y_i^{(2)}(T) &= 2\Bigg\{-\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(1)}\sigma_0^{(0)} S_0^{(0)\beta_0-1} dW_0 \\
&\quad + \sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(1)} c_{i,j}\sigma_i^{(0)} S_i^{(0)\beta_i-1} dW_j \\
&\quad - \int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)}\left(\sigma_0^{(1)} S_0^{(0)\beta_0-1} + (\beta_0-1)\sigma_0^{(0)} S_0^{(1)} S_0^{(0)\beta_0-2}\right) dW_0 \\
&\quad + \sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)}\left(c_{i,j}\sigma_i^{(1)} S_i^{(0)\beta_i-1} + c_{i,j}(\beta_i-1)\sigma_i^{(0)} S_i^{(1)} S_i^{(0)\beta_i-2}\right) dW_j\Bigg\}, \qquad (54)\\
S_i^{(2)}(T) &= 2\Bigg\{c_{i,0}\int_0^T e^{\int_t^T \alpha_i(s)ds}\sigma_0^{(0)}\sigma_i^{(0)} S_0^{(0)\beta_0-1} S_i^{(0)\beta_0} dt \\
&\quad + 2\sum_{j=0}^{i} c_{i,j}\int_0^T e^{\int_t^T \alpha_i(s)ds}\left(\sigma_i^{(1)} S_i^{(0)\beta_i} + \sigma_i^{(0)}\beta_i S_i^{(0)\beta_i-1} S_i^{(1)}\right) dW_j\Bigg\}, \qquad (55)\\
\sigma_i^{(2)}(T) &= 2\Bigg\{c_{i,0}\int_0^T e^{-\int_t^T \lambda_i(s)ds}\sigma_0^{(0)}\nu_i\sigma_i^{(0)} S_0^{(0)\beta_0-1} dt \\
&\quad + \sum_{j=0}^{n+1+i}\int_0^T \nu_{i,j} e^{-\int_t^T \lambda_i(s)ds}\sigma_i^{(1)} dW_j\Bigg\}, \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
Y_i^{(3)}(T) &= 3\Big\{-\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(2)}\sigma_0^{(0)}S_0^{(0)\beta_0-1}dW_0 \\
&+\sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(2)}c_{i,j}\sigma_i^{(0)}S_i^{(0)\beta_i-1}dW_j \\
&-2\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(1)}\left(\sigma_0^{(1)}S_0^{(0)\beta_0-1}+\sigma_0^{(0)}(\beta_0-1)S_0^{(0)\beta_0-2}S_0^{(1)}\right)dW_0 \\
&+2\sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(1)}c_{i,j}\left(\sigma_i^{(1)}S_i^{(0)\beta_i-1}+\sigma_i^{(0)}(\beta_i-1)S_i^{(0)\beta_i-2}S_i^{(1)}\right)dW_j \\
&-\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(0)}\Big(\sigma_0^{(2)}S_0^{(0)\beta_0-1}+\sigma_0^{(1)}2(\beta_0-1)S_0^{(0)\beta_0-2}S_0^{(1)} \\
&\qquad+\sigma_0^{(0)}(\beta_0-1)(\beta_0-2)S_0^{(0)\beta_0-3}S_0^{(1)2}+\sigma_0^{(0)}(\beta_0-1)S_0^{(0)\beta_0-2}S_0^{(2)}\Big)dW_0 \\
&+\sum_{j=0}^{i}\int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds}Y_i^{(0)}c_{i,j}\Big(\sigma_i^{(2)}S_i^{(0)\beta_i-1}+\sigma_i^{(1)}2(\beta_i-1)S_i^{(0)\beta_i-2}S_i^{(1)} \\
&+\sigma_i^{(0)}(\beta_i-1)(\beta_i-2)S_i^{(0)\beta_i-3}S_i^{(1)2}+\sigma_i^{(0)}(\beta_i-1)S_i^{(0)\beta_i-2}S_i^{(2)}\Big)dW_i\Big\}. \qquad (57)
\end{aligned}
$$

Let X represent the underlying asset price of a multi-asset cross currency option as

$$
X = \sum_{i=1}^{n} w_i Y_i. \qquad (58)
$$

Then, the expansion of X for the option pricing is given as follows:

Proposition A.1. *Let $W=(W_1,\cdots,W_n)'$. Then, $X^{(m)}(T)$, $m=0,1,2,3$ which stand for the m-th order asymptotic expansion of X are derived as follows:*

$$
\begin{aligned}
X^{(0)} &= \sum_{i=1}^{n} w_i Y_i(0), \qquad (59)\\
X^{(1)} &= \sum_{i=1}^{n}\int_0^T f_{1,i}(s)'dW(s), \qquad (60)\\
X^{(2)} &= 2\sum_{i=1}^{5n}\int_0^T\int_0^s f_{2,i}(u)'dW(u)g_{2,i}(s)'dW(s), \qquad (61)\\
X^{(3)} &= 6\sum_{i=1}^{11n}\int_0^T\int_0^s\int_0^u f_{3,i}(v)'dW(v)g_{3,i}(u)'dW(u)h_{3,i}(s)'dW(s) \\
&\quad+6\sum_{i=1}^{8n}\int_0^T\left(\int_0^s g_{4,i}(u)'dW(u)\right)\left(\int_0^s f_{4,i}(u)'dW(u)\right)h_{4,i}(s)'dW(s) \\
&\quad+6\sum_{i=1}^{4n}\int_0^T\int_0^s f_{5,i}(u)'du g_{5,i}(s)'dW(s). \qquad (62)
\end{aligned}
$$

$f_{1,1}(t)$, $f_{2,i}(t)$, $g_{2,i}(t)$ $(i=1,\cdots,5n)$, $f_{3,i}(t)$, $g_{3,i}(t)$, $h_{3,i}(t)$ $(i=1,\cdots,11n)$, $f_{4,i}$, $g_{4,i}$, $h_{4,i}$ $(i=1,\cdots,8n)$, $f_{5,i}$, $g_{5,i}$, $(i=1,\cdots,4n)$, are expressed as follows:

$$f_{1,i}(t)=w_ie^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}\begin{pmatrix} e^{-\int_0^t(\alpha_i(t)-\alpha_0(t))ds}Y_i^{(0)}\left(-\sigma_0^{(0)}S_0^{(0)\beta_0-1}+c_{i,0}\sigma_i^{(0)}S_i^{(0)\beta_i-1}\right)\\ \vdots \\ e^{-\int_0^t(\alpha_i(t)-\alpha_0(t))ds}Y_i^{(0)}c_{i,i}\sigma_i^{(0)}S_i^{(0)\beta_i-1}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$g_{2,i}=w_ie^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}\frac{e^{\int_0^t(\alpha_i(t)-\alpha_0(t))ds}}{Y_i^{(0)}}f_{1,i}(t),$$

$$g_{2,n+i}(t)=w_ie^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}\begin{pmatrix} -e^{-\int_0^t(-\alpha_0(s)+\alpha_i(s)+\lambda_i(s))ds}c_{0,0}Y_i^{(0)}S_0^{(0)\beta_0}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$g_{2,2n+i}(t)=w_ie^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}\begin{pmatrix} e^{-\int_0^t(-\alpha_0(s)+\alpha_i(s)+\lambda_i(s))ds}c_{i,0}Y_i^{(0)}S_i^{(0)\beta_i}\\ \vdots\\ e^{-\int_0^t(-\alpha_0(s)+\alpha_i(s)+\lambda_i(s))ds}c_{i,i}Y_i^{(0)}S_i^{(0)\beta_i}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$f_{2,n+i}(t)=\begin{pmatrix} \nu_{0,0}e^{\int_0^t\lambda_0(s)ds}\sigma_0^{(0)}\\ \vdots\\ \nu_{i,n+1}e^{\int_0^t\lambda_0(s)ds}\sigma_0^{(0)}\\ 0\\ \vdots\\ 0\end{pmatrix},\ f_{2,2n+i}(t)=\begin{pmatrix} \nu_{i,0}e^{\int_0^t\lambda_i(s)ds}\sigma_i^{(0)}\\ \vdots\\ \nu_{i,n+1+i}e^{\int_0^t\lambda_i(s)ds}\sigma_i^{(0)}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$g_{2,3n+i}(t)=w_ie^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}\begin{pmatrix} -e^{\int_0^t\alpha_0(s)ds}c_{0,0}Y_i^{(0)}(\beta_0-1)\sigma_0^{(0)}S_0^{(0)\beta_0-2}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$g_{2,4n+i}(t) = w_i e^{\int_0^T (\alpha_i(t)-\alpha_0(t))dt} \begin{pmatrix} e^{-\int_0^t \alpha_i(s)ds} c_{i,0} Y_i^{(0)} (\beta_i - 1) \sigma_i^{(0)} S_i^{(0)\beta_i - 2} \\ \vdots \\ e^{-\int_0^t \alpha_i(s)ds} c_{i,i} Y_i^{(0)} (\beta_i - 1) \sigma_i^{(0)} S_i^{(0)\beta_i - 2} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$f_{2,3n+i}(t) = \begin{pmatrix} c_{0,0} e^{\int_0^t \alpha_0(s)ds} S_0^{(0)\beta_0} \sigma_0^{(0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, f_{2,4n+i}(t) = \begin{pmatrix} c_{i,0} e^{-\int_0^t \alpha_i(s)ds} S_i^{(0)\beta_i} \sigma_i^{(0)} \\ \vdots \\ c_{i,i} e^{-\int_0^t \alpha_i(s)ds} S_i^{(0)\beta_i} \sigma_i^{(0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$g_{3,7n+i}(t) = \begin{pmatrix} c_{0,0} e^{-\int_0^t (-\alpha_0(s)+\lambda_0(s))ds} S_0^{(0)\beta} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, g_{3,8n+i}(t) = \begin{pmatrix} c_{i,0} e^{-\int_0^t (\alpha_i(s)+\lambda_i(s))ds} S_i^{(0)\beta} \\ \vdots \\ c_{i,i} e^{-\int_0^t (\alpha_i(s)+\lambda_i(s))ds} S_i^{(0)\beta} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$g_{3,9n+i}(t) = \begin{pmatrix} c_{0,0} \beta S_0^{(0)\beta_0 - 1} \sigma_0^{(0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, g_{3,10n+i}(t) = \begin{pmatrix} c_{i,0} \beta S_i^{(0)\beta_i - 1} \sigma_i^{(0)} \\ \vdots \\ c_{i,i} \beta S_i^{(0)\beta_i - 1} \sigma_i^{(0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$h_{4,6n+i}(t) = w_i e^{\int_0^T (\alpha_i(t)-\alpha_0(t))dt} \begin{pmatrix} -e^{\int_0^t (\alpha_0(s)-\alpha_i(s))ds} c_{0,0} Y_i^{(0)} (\beta_0 - 1)(\beta_0 - 2) \sigma_0^{(0)} S_0^{\beta_0 - 3} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$h_{4,7n+i}(t) = w_i e^{\int_0^T (\alpha_i(t)-\alpha_0(t))dt} \begin{pmatrix} e^{-\int_0^t (-\alpha_0(s)+\alpha_i(s))ds} c_{i,0} Y_i^{(0)} (\beta - 1)(\beta - 2) \sigma_i^{(0)} S_i^{\beta_i - 3} \\ \vdots \\ e^{-\int_0^t (-\alpha_0(s)+\alpha_i(s))ds} c_{i,i} Y_i^{(0)} (\beta - 1)(\beta - 2) \sigma_i^{(0)} S_i^{\beta_i - 3} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$f_{5,i}(t) = e^{\int_0^t \lambda_0(s)ds}\nu_{n+1,0}\sigma_0^{(0)2}, f_{5,n+i}(t) = e^{\int_0^t \lambda_i(s)ds}\nu_{n+1+i,0}\sigma_0^{(0)}\sigma_i^{(0)},$$
$$f_{5,2n+i}(t) = \sigma_0^{(0)2}S_0^{(0)2\beta_i-1},$$
$$f_{5,3n+i}(t) = c_{i,0}\sigma_0^{(0)}\sigma_i^{(0)}S_0^{(0)\beta_0-1}S_i^{(0)\beta_i},$$
$$f_{3,i} = f_{4,i} = f_{4,n+i} = f_{4,2n+i} = f_{4,3n+i} = f_{1,i},$$
$$-g_{3,i} = e^{-\int_0^T(\alpha_i(t)-\alpha_0(t))dt}g_{2,i},$$
$$h_{3,i} = h_{3,n+i} = h_{3,2n+i} = h_{3,3n+i} = h_{3,4n+i} = g_{2,i},$$
$$f_{3,n+i} = f_{3,5n+i} = f_{3,7n+i} = g_{4,i} = f_{4,4n+i} = f_{2,n+i},$$
$$f_{3,2n+i} = f_{3,6n+i} = f_{3,8n+i} = g_{4,2n+i} = f_{4,5n+i} = f_{2,2n+i},$$
$$f_{3,3n+i} = f_{3,9n+i} = g_{4,2n+i} = g_{4,4n+i} = f_{4,6n+i} = g_{4,6n+i} = f_{2,3n+i},$$
$$f_{3,4n+i} = f_{3,10n+i} = g_{4,3n+i} = g_{4,5n+i} = f_{4,7n+i} = g_{4,7n+i} = f_{2,4n+i},$$
$$-g_{3,n+i} = g_{5,i} = e^{-\int_0^T(\alpha_i(t)-\alpha_0(t))dt}g_{2,n+i},$$
$$h_{3,5n+i} = g_{2,n+i},$$
$$-g_{3,2n+i} = g_{5,n+i} = e^{-\int_0^T(\alpha_i(t)-\alpha_0(t))dt}g_{2,2n+i},$$
$$h_{3,6n+i} = g_{2,2n+i},$$
$$-h_{4,i} = w_i e^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}2f_{2,n+i}e^{\int_0^t(-\alpha_0+\alpha_i)ds}/Y_i^{(0)},$$
$$-h_{4,n+i} = w_i e^{\int_0^T(\alpha_i(t)-\alpha_0(t))dt}2f_{2,2n+i}e^{\int_0^t(-\alpha_0+\alpha_i)ds}/Y_i^{(0)},$$
$$-g_{3,3n+i} = g_{5,2n+i} = e^{-\int_0^T(\alpha_i(t)-\alpha_0(t))dt}g_{2,3n+i},$$
$$h_{3,7n+i} = h_{3,9n+i} = g_{2,3n+i},$$
$$-g_{3,4n+i} = g_{5,3n+i} = e^{-\int_0^T(\alpha_i(t)-\alpha_0(t))dt}g_{2,4n+i},$$
$$h_{3,8n+i} = h_{3,10n+i} = g_{2,4n+i},$$
$$h_{4,2n+i} = 2g_{2,3n+i}e^{\int_0^t(-\alpha_0+\alpha_i)ds}/Y_i^{(0)}, h_{4,3n+i} = 2g_{2,4n+i}e^{\int_0^t(-\alpha_0+\alpha_i)ds}/Y_i^{(0)},$$
$$h_{4,4n+i} = 2g_{2,3n+i}e^{-\int_0^t \lambda_0 ds}/\sigma_0^{(0)}, h_{4,5n+i} = 2g_{2,4n+i}e^{-\int_0^t \lambda_i ds}/\sigma_i^{(0)},$$
$$g_{3,5n+i} = f_{2,n+i}e^{-\int_0^t \lambda_0 ds}/\sigma_0^{(0)},$$
$$g_{3,6n+i} = f_{2,2n+i}e^{-\int_0^t \lambda_i ds}/\sigma_i^{(0)}.$$

Proof. First, Note that

$$X^{(0)} = \sum_{i=1}^n w_i Y_i^{(0)}(t),$$

$$
\begin{aligned}
Y_i^{(0)}(t) &= \int_0^t (-\alpha_0(s)+\alpha_i(s))(Y_0^{(0)}(s)+\epsilon Y_0^{(1)}(s)+\cdots)ds \\
&+\epsilon\Bigg(-\int_0^t (Y_i^{(0)}(s)+\epsilon Y_i^{(1)}(s)+\cdots)(\sigma_0^{(0)}(s)+\epsilon\sigma_0^{(1)}(s)+\cdots)(S_0^{(0)}(s)+\epsilon S_0^{(1)}(s)+\cdots)^{\beta_0-1}dW_0(s) \\
&+\int_0^t \sum_{j=0}^{i} c_{i,j}(Y_i^{(0)}(s)+\epsilon Y_i^{(1)}(s)+\cdots)(\sigma_i^{(0)}(s)+\epsilon\sigma_i^{(1)}(s)+\cdots) \\
&(S_i^{(0)}(s)+\epsilon S_i^{(1)}(s)+\cdots)^{\beta_i-1}dW_j(s)+O(\epsilon)\Bigg)\Bigg|_{\epsilon=0} \\
&= \int_0^t (-\alpha_0(s)+\alpha_i(s))Y_i^{(0)}(s)ds.
\end{aligned}
$$

Here, $Y_i^{(0)}(t)$ is easily solved as $Y_i^{(0)}(t)=e^{\int_0^t -\alpha_0(s)+\alpha_i(s)ds}Y_i(0)=e^{\int_0^t -\alpha_0(s)+\alpha_i(s)ds}S_0(0)S_i(0)$. Then substitute $Y_i^{(0)}(t)$ for $X^{(0)}$.

Next, $X^{(1)}$ is calculated:

$$
X^{(1)}=\frac{\partial X^{(\epsilon)}}{\partial\epsilon}\Big|_{\epsilon=0}=\sum_{i=1}^{n}\frac{\partial Y_i^{(\epsilon)}(t)}{\partial\epsilon}\Big|_{\epsilon=0}=\sum_{i=1}^{n}Y_i^{(1)}(t),
$$

$$
Y_i^{(1)}(t)=\frac{\partial Y_i^{(\epsilon)}(t)}{\partial\epsilon}\Big|_{\epsilon=0}
$$

$$
\begin{aligned}
&= \int_0^t (-\alpha_0(s) + \alpha_i(s))(Y_i^{(1)}(s) + \epsilon Y_i^{(2)}(s) + \cdots)ds \\
&- \int_0^t (Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_0^{(0)}(s) + \epsilon \sigma_0^{(1)}(s) + \cdots) \\
&(S_0^{(0)}(s) + \epsilon S_0^{(1)}(s) + \cdots)^{\beta_0 - 1} dW_0(s) \\
&+ \int_0^t \sum_{j=0}^{i} c_{i,j}(Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_i^{(0)}(s) + \epsilon \sigma_i^{(1)}(s) + \cdots) \\
&(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)^{\beta_i - 1} dW_j(s) \\
&+\epsilon \Bigg(- \int_0^t (Y_i^{(1)}(s) + \epsilon Y_i^{(2)}(s) + \cdots)(\sigma_0^{(0)}(s) + \epsilon \sigma_0^{(1)}(s) + \cdots) \\
&(S_0^{(0)}(s) + \epsilon S_0^{(1)}(s) + \cdots)^{\beta_0} dW_0(s) \\
&+ \int_0^t \sum_{j=0}^{i} c_{i,j}(Y_i^{(1)}(s) + \epsilon Y_i^{(2)}(s) + \cdots)(\sigma_i^{(0)}(s) + \epsilon \sigma_i^{(1)}(s) + \cdots) \\
&(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)^{\beta_i - 1} dW_j(s) \\
&- \int_0^t (Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_0^{(1)}(s) + \epsilon \sigma_0^{(2)}(s) + \cdots) \\
&(S_0^{(0)}(s) + \epsilon S_0^{(1)}(s) + \cdots)^{\beta_0 - 1} dW_0(s) \\
&+ \int_0^t \sum_{j=0}^{i} c_{i,j}(Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_i^{(1)}(s) + \epsilon \sigma_i^{(2)}(s) + \cdots) \\
&(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)^{\beta_i - 1} dW_j(s) \\
&- \int_0^t (Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_0^{(0)}(s) + \epsilon \sigma_0^{(1)}(s) + \cdots)(\beta_0 - 1) \\
&(S_0^{(0)}(s) + \epsilon S_0^{(1)}(s) + \cdots)^{\beta_0 - 2}(S_0^{(2)}(s) + \epsilon S_0^{(2)}(s) + \cdots) dW_0(s) \\
&+ \int_0^t \sum_{j=0}^{i} c_{i,j}(Y_i^{(0)}(s) + \epsilon Y_i^{(1)}(s) + \cdots)(\sigma_i^{(0)}(s) + \epsilon \sigma_i^{(1)}(s) + \cdots) \\
&(\beta_i - 1)(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)^{\beta_i - 2}(S_0^{(1)}(s) + \epsilon S_0^{(2)}(s) + \cdots) dW_j(s) \Big) \Bigg) \Big|_{\epsilon = 0} \\
&= \int_0^t (-\alpha_0(s) + \alpha_i(s)) Y_i^{(1)}(s) ds - \int_0^t Y_i^{(0)}(s) \sigma_0^{(0)}(s) S_0^{(0)}(s)^{\beta_0 - 1} dW_0(s) \\
&+ \int_0^t \sum_{j=0}^{i} c_{i,j} Y_i^{(0)}(s) \sigma_i^{(0)}(s) S_i^{(0)}(s)^{\beta_i - 1} dW_j(s).
\end{aligned}
$$

$$S_i^{(0)}(t) = \Bigg(\int_0^t \alpha_i(s)(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)ds$$

$$+\epsilon \int_0^t \sum_{j=0}^{i} c_{i,j}(\sigma_i^{(0)}(s) + \epsilon\sigma_i^{(1)}(s) + \cdots)(S_i^{(0)}(s) + \epsilon S_i^{(1)}(s) + \cdots)^{\beta_i} dW_j(s) + O(\epsilon^2) \Bigg)\Bigg|_{\epsilon=0}$$

$$= \int_0^t \alpha_i(s) S_i^{(0)}(s)ds,$$

$$\sigma_i^{(0)}(t) = \Bigg(\sigma_i(0) + \int_0^t \lambda_i(\theta_i - (\sigma_i^{(0)}(s) + \epsilon\sigma_i^{(1)}(s) + \cdots))ds$$

$$+\epsilon \sum_{j=0}^{n+1+i} \int_0^t \nu_{i,j}(\sigma_i^{(0)}(s) + \epsilon\sigma_i^{(1)}(s) + \cdots)dW_j(t) + O(\epsilon^2) \Bigg)\Bigg|_{\epsilon=0}$$

$$= \sigma_i(0) + \int_0^t \lambda_i(\theta_i - \sigma_i^{(0)}(s))ds.$$

Here, in order to calculate $Y_i^{(m)}(t)$, $S_i^{(m)}(t)$ and $\sigma_i^{(m)}(t)$ $(m = 0, 1, \cdots)$, it is sufficient to solve the equation:

$$dA(t) = (a(t)A(t) + b(t))dt + c(t)dW(t). \tag{63}$$

This stochastic differential equation can be easily solved. Apply the Ito's lemma to $f(A(t), t) = A(t)e^{-\int_0^t a(s)ds}$,

$$\begin{aligned} df(A(t), t) &= -a(t)A(t)e^{-\int_0^t a(s)ds}dt + e^{-\int_0^t a(s)ds}dA(t) \\ &= b(t)e^{-\int_0^t a(s)ds}dt + c(t)e^{-\int_0^t a(s)ds}dW(t). \end{aligned}$$

Integrate it from 0 to t, we obtain

$$A(t)e^{-\int_0^t a(s)ds} = A(0) + \int_0^t b(s)e^{-\int_0^s a(u)du}ds + \int_0^t c(s)e^{-\int_0^s a(u)du}dW(s).$$

Multiplying $e^{\int_0^t \alpha(s)ds}$ to the both sides, it is shown that the stochastic differential equation (63) has the solution as follows:

$$A(t) = A(0)e^{-\int_0^t a(s)ds} + \int_0^t b(s)e^{\int_s^t a(u)du}ds + \int_0^t c(s)e^{\int_s^t a(u)du}dW(s).$$

The dynamics of $Y_i^{(1)}(t)$, $S_i^{(0)}(t)$ and $\sigma_i^{(0)}(t)$ have the same form as (63). Thus, those equations can be easily solved by the method of variation of constants as:

$$\sigma_i^{(0)}(t) = e^{\int_0^t \lambda(s)ds} \left(\sigma_i(0) + \int_0^t e^{\int_0^s \lambda(u)du} \lambda(s)\theta(s)ds \right).$$

$$
\begin{aligned}
Y_i^{(1)}(T) &= \int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)} \sigma_0^{(0)} S_0^{(0)\beta-1} dW_0 \\
&\quad + \sum_{j=0}^{i} \int_0^T e^{\int_t^T(\alpha_i(s)-\alpha_0(s))ds} Y_i^{(0)} c_{i,j} \sigma_i^{(0)} S_i^{(0)\beta-1} dW_j,
\end{aligned}
$$

gives the expressions of $f_{1,i}$.

The remaining coefficients can be easily derived in the same way, and hence the details of the derivation are omitted. □

Thus, the price of multi-asset cross currency call option is expressed as follows:

$$
\begin{aligned}
C(0) &= e^{-rT}\Bigg(\epsilon\left(y\int_{-y}^{\infty} n[x;0,\Sigma]dx + \int_{-y}^{\infty} x\ n[x;0,\Sigma]dx\right) \\
&\quad + \epsilon^2 \frac{1}{2}\int_{-y}^{\infty} \mathbf{E}\left[X^{(2)}(T)\middle|X^{(1)}(T)=x\right] n[x;0,\Sigma]dx \\
&\quad + \epsilon^3\Bigg(\frac{1}{6}\int_{-y}^{\infty} \mathbf{E}\left[X^{(3)}(T)\middle|X^{(1)}(T)=x\right] n[x;0,\Sigma]dx \\
&\qquad + \frac{1}{8}\mathbf{E}\left[\left(X^{(2)}(T)\right)^2 \middle| X^{(1)}(T)=y\right] n[y;0,\Sigma]\Bigg)\Bigg) + o(\epsilon^3).
\end{aligned}
$$

Where, $y = \frac{X^{(0)}-K}{\epsilon}$, $\Sigma = \int_0^T F_{11}(s)' F_{11}(s)ds$, $F_{11} = \sum_{i=1}^n f_{1,i}$, $n[x;0,\Sigma] = \frac{1}{\sqrt{2\pi\Sigma}}\exp\left(\frac{-x^2}{2\Sigma}\right)$. A more concrete approximation of the price is obtained by the application of conditional expectation formulas shown in Appendix B.

The result of pricing formula is shown in Theorem 3.1 when the coefficients $C_1, \cdots, C_4$ are defined as follows:

$$
\begin{aligned}
C_1 &= \sum_{i=1}^{5n} \int_0^T F_{11}(s)' g_{2,i}(s) \int_0^s F_{11}(u)' f_{2,i}(u) du ds, \\
C_2 &= \frac{1}{2}\sum_{i=1}^{5n}\sum_{j=1}^{5n} \left(\int_0^T F_{11}(s)' g_{2,i}(s)\int_0^s F_{11}(s)' f_{2,i}(u) du ds\right) \\
&\quad \times \left(\int_0^T F_{11}(s)' g_{2,j}(s)\int_0^s F_{11}(s)' f_{2,j}(u) du ds\right),
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \sum_{i=1}^{11n}\int_0^T F_{11}(s)'h_{3,i}(s)\int_0^s F_{11}(u)'g_{3,i}(u)\int_0^u F_{11}(v)'f_{3,i}(v)dvduds \\
&+\sum_{i=1}^{8n}\int_0^T F_{11}(s)'h_{4,i}(s)\int_0^s F_{11}(u)'g_{4,i}(u)du\int_0^s F_{11}(u)'f_{4,i}(u)duds, \\
&+\frac{1}{2}\sum_{i=1}^{5n}\sum_{j=1}^{5n}\Bigg(\int_0^T F_{11}(s)'g_{2,j}(s)\int_0^s F_{11}(u)'g_{2,i}(u)\int_0^u f_{2,i}(v)'f_{2,j}(v)dvduds \\
&+\int_0^T F_{11}(s)'g_{2,i}(s)\int_0^s F_{11}(u)'g_{2,j}(u)\int_0^u f_{2,i}(v)'f_{2,j}(v)dvduds \\
&+\int_0^T F_{11}(s)'g_{2,i}(s)\int_0^s f_{2,i}(u)'g_{2,j}(u)\int_0^u F_{11}(v)'f_{2,j}(v)dvduds \\
&+\int_0^T g_{2,i}(s)'g_{2,j}(s)\int_0^s F_{11}(u)'f_{2,j}(u)du\int_0^s F_{11}(u)'f_{2,i}(u)duds \\
&+\int_0^T F_{11}(s)'g_{2,j}(s)\int_0^s g_{2,i}(u)'f_{2,j}(u)\int_0^u F_{11}(v)'f_{2,i}(v)dvduds\Bigg), \\
C_4 &= \sum_{i=1}^{8n}\int_0^T F_{11}(s)'h_{4,i}(s)\int_0^s g_{4,i}(u)'f_{4,i}(u)duds, \\
&+\sum_{i=1}^{4n}\int_0^T g_{5,i}(s)\int_0^s f_{5,i}(u)duds, \\
&+\frac{1}{2}\sum_{i=1}^{5n}\sum_{j=1}^{5n}\int_0^T g_{2,i}(s)'g_{2,j}(s)\int_0^s f_{2,i}(u)'f_{2,j}(u)duds.
\end{aligned}
$$

Remark . *The integrals on the right hand side of (26) are evaluated by the following relation:*

$$
\int_{-y}^{\infty}\frac{1}{\Sigma^k}H_k(x;\Sigma)n[x;0,\Sigma]dx \;=\; \frac{1}{\Sigma^{k-1}}H_{k-1}(-y;\Sigma)n[y;0,\Sigma]\ (k\geq 1).
$$

B Formulas for the Conditional Expectations of the Wiener-Itô Integrals

This appendix summarizes conditional expectation formulas for explicit computation of the asymptotic expansions up to the third order.

In the following, W is a d-dimensional Brownian motion and $q_i = (\hat{q}_{i1},\cdots,\hat{q}_{id})^{'}$ where $\hat{q}_i \in L^2[0,T]$, $i = 1,2,...,5$ and $x^{'}$ denotes the transpose of x. $H_n(x;\Sigma)$ denotes the Hermite polynomial of degree n and $\Sigma = \int_0^T |q_{1t}|^2 dt$. For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda [30].

1.

$$\mathbf{E}\left[\int_0^T q_{2t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] = \left(\int_0^T q_{2t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

2.

$$\mathbf{E}\left[\int_0^T \int_0^t q_{2u}^{'} dW_u q_{3t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T \int_0^t q_{2u}^{'} q_{1u} du q_{3t}^{'} q_{1t} dt\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$

3.

$$\mathbf{E}\left[\left(\int_0^T q_{2u}^{'} dW_u\right)\left(\int_0^T q_{3s}^{'} dW_s\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{2u}^{'} q_{1u} du\right)\left(\int_0^T q_{3s}^{'} q_{1s} ds\right) \frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+ \int_0^T q_{2t}^{'} q_{3t} dt$$

4.

$$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2u}^{'} dW_u q_{3s}^{'} dW_s q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{4t}^{'} q_{1t} \int_0^t q_{3s}^{'} q_{1s} \int_0^s q_{2u}^{'} q_{1u} du ds dt\right) \frac{H_3(x;\Sigma)}{\Sigma^3}$$

5.

$$\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2u}^{'} dW_u\right)\left(\int_0^t q_{3s}^{'} dW_s\right) q_{4t}^{'} dW_t | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left\{\int_0^T \left(\int_0^t q_{2u}^{'} q_{1u} du\right)\left(\int_0^t q_{3s}^{'} q_{1s} ds\right) q_{4t}^{'} q_{1t} dt\right\} \frac{H_3(x;\Sigma)}{\Sigma^3}$$
$$+ \left(\int_0^T \int_0^t q_{2u}^{'} q_{3u} du q_{4t}^{'} q_{1t} dt\right) \frac{H_1(x;\Sigma)}{\Sigma}$$

6.

$$\mathbf{E}\left[\left(\int_0^T \int_0^t q_{2s}^{'} dW_s q_{3t}^{'} dW_t\right)\left(\int_0^T \int_0^r q_{4u}^{'} dW_u q_{5r}^{'} dW_r\right) | \int_0^T q_{1v}^{'} dW_v = x\right] =$$
$$\left(\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2s}^{'} q_{1s} ds dt\right)\left(\int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{4u}^{'} q_{1u} du dr\right) \frac{H_4(x;\Sigma)}{\Sigma^4}$$
$$+ \left\{\int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{5r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt + \int_0^T q_{5t}^{'} q_{1t} \int_0^t q_{3r}^{'} q_{1r} \int_0^r q_{2u}^{'} q_{4u} du dr dt\right.$$
$$+ \int_0^T q_{3t}^{'} q_{1t} \int_0^t q_{2r}^{'} q_{5r} \int_0^r q_{4u}^{'} q_{1u} du dr dt + \int_0^T q_{3t}^{'} q_{5t} \left(\int_0^t q_{2s}^{'} q_{1s} ds\right)\left(\int_0^t q_{4u}^{'} q_{1u} du\right) dt$$
$$\left. + \int_0^T q_{5r}^{'} q_{1r} \int_0^r q_{3u}^{'} q_{4u} \int_0^u q_{2s}^{'} q_{1s} ds du dr\right\} \frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+ \int_0^T \int_0^t q_{2u}^{'} q_{4u} du q_{3t}^{'} q_{5t} dt$$

C Calibration Results for USD quoted Option Markets

Table 31: Volatility Smile I

		2008/9/8					2008/9/22				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
EURUSD	Calibrated Vol	13.40	12.34	11.70	11.55	11.85	13.87	13.16	13.06	13.60	14.52
	Market Vol	13.39	12.38	11.69	11.53	11.87	13.86	13.18	13.05	13.59	14.53
	Difference	0.02	-0.04	0.01	0.02	-0.01	0.01	-0.02	0.01	0.01	-0.01
SGDUSD	Calibrated Vol	7.69	7.40	7.58	8.33	9.52	7.76	7.43	7.62	8.43	9.66
	Market Vol	7.70	7.37	7.58	8.35	9.51	7.77	7.40	7.63	8.45	9.65
	Difference	-0.02	0.03	0.00	-0.03	0.02	-0.02	0.04	-0.01	-0.03	0.02
JPYUSD	Calibrated Vol	15.48	12.95	10.85	9.58	9.06	19.56	16.41	13.73	12.12	11.59
	Market Vol	15.52	12.87	10.91	9.55	9.08	19.64	16.26	13.81	12.09	11.61
	Difference	-0.04	0.08	-0.05	0.03	-0.02	-0.07	0.15	-0.07	0.04	-0.03
KRWUSD	Calibrated Vol	14.54	14.91	16.39	19.21	22.88	14.97	16.33	18.68	22.60	27.56
	Market Vol	14.56	14.88	16.39	19.27	22.85	15.08	16.16	18.67	22.78	27.48
	Difference	-0.02	0.03	0.01	-0.05	0.03	-0.10	0.17	0.01	-0.18	0.08

Table 32: Volatility Smile II

		2011/3/3					2011/3/17				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
EURUSD	Calibrated Vol	13.48	12.15	11.25	10.89	10.99	14.45	12.83	11.68	11.11	11.07
	Market Vol	13.45	12.21	11.22	10.88	11.01	14.41	12.87	11.67	11.11	11.07
	Difference	0.03	-0.06	0.03	0.01	-0.01	0.04	-0.04	0.01	-0.01	0.00
JPYUSD	Calibrated Vol	12.33	10.99	10.06	9.74	10.10	19.82	17.50	15.64	14.86	15.46
	Market Vol	12.36	10.96	10.04	9.78	10.09	19.94	17.33	15.61	14.96	15.42
	Difference	-0.02	0.03	0.02	-0.04	0.02	-0.11	0.17	0.03	-0.10	0.04
KRWUSD	Calibrated Vol	11.44	11.54	12.48	15.58	18.66	12.10	12.37	13.19	16.42	20.62
	Market Vol	11.58	11.39	12.51	15.19	18.97	12.19	12.11	13.41	16.38	20.60
	Difference	-0.14	0.15	-0.03	0.39	-0.31	-0.09	0.27	-0.22	0.04	0.02

Table 33: Volatility Smile III

		2011/3/1					2011/3/29				
		-10D	-25D	ATM	25D	10D	-10D	-25D	ATM	25D	10D
JPYUSD	Calibrated Vol	12.16	10.85	9.86	9.50	9.85	14.66	13.14	12.00	11.65	12.18
	Market Vol	12.21	10.80	9.85	9.55	9.83	14.71	13.07	11.98	11.72	12.15
	Difference	-0.04	0.05	0.01	-0.04	0.02	-0.05	0.07	0.02	-0.07	0.03
SGDUSD	Calibrated Vol	6.73	6.64	6.96	7.81	8.95	6.08	6.20	6.65	7.57	8.73
	Market Vol	6.73	6.64	6.94	7.86	8.93	6.09	6.18	6.64	7.63	8.70
	Difference	-0.01	0.00	0.03	-0.05	0.03	-0.01	0.02	0.02	-0.06	0.03

Table 34: Parameters before/after Lehman Shock

		2008/9/8					2008/9/22				
		$S(0)$	α	$\sigma(0)$	ν	ρ	$S(0)$	α	$\sigma(0)$	ν	ρ
1M	EURUSD	1.4222	-0.0204	0.127	1.41	-0.205	1.4575	-0.0035	0.145	1.53	0.094
	SGDUSD	0.6979	0.0144	0.085	2.05	-0.302	0.7101	0.0177	0.083	2.14	-0.258
	JPYUSD	0.009188	0.0210	0.119	2.04	0.559	0.009406	0.0429	0.162	2.01	0.531
	KRWUSD	0.0009249	-0.0034	0.216	1.50	-0.593	0.0008762	0.0749	0.241	1.54	-0.704
2M	EURUSD	1.4222	-0.0194	0.114	1.07	-0.197	1.4575	-0.0063	0.135	1.19	0.083
	SGDUSD	0.6979	0.0144	0.078	1.50	-0.288	0.7101	0.0167	0.077	1.56	-0.272
	JPYUSD	0.009188	0.0198	0.110	1.62	0.577	0.009406	0.0371	0.144	1.65	0.549
	KRWUSD	0.0009249	-0.0037	0.181	1.37	-0.531	0.0008762	0.0372	0.213	1.40	-0.638
3M	EURUSD	1.4222	-0.0199	0.115	1.00	-0.179	1.4575	-0.0087	0.127	1.11	0.076
	SGDUSD	0.6979	0.0152	0.074	1.34	-0.274	0.7101	0.0168	0.074	1.39	-0.257
	JPYUSD	0.009188	0.0205	0.105	1.50	0.588	0.009406	0.0329	0.132	1.55	0.571
	KRWUSD	0.0009249	-0.0041	0.165	1.14	-0.687	0.0008762	0.0243	0.192	1.11	-0.867
4M	EURUSD	1.4222	-0.0188	0.109	0.87	-0.139	1.4575	-0.0128	0.119	0.91	0.070
	SGDUSD	0.6979	0.0147	0.071	0.96	-0.269	0.7101	0.0172	0.071	0.99	-0.252
	JPYUSD	0.009188	0.0214	0.098	1.27	0.613	0.009406	0.0297	0.114	1.34	0.604
	KRWUSD	0.0009249	-0.0076	0.130	1.15	-0.592	0.0008762	0.0141	0.157	1.07	-0.763

Table 35: Parameters before/after Tohoku Earthquake I

		2011/3/3					2011/3/17				
		$S(0)$	α	$\sigma(0)$	ν	ρ	$S(0)$	α	$\sigma(0)$	ν	ρ
1M	EURUSD	1.3864	-0.0044	0.104	1.53	-0.243	1.4025	-0.0050	0.110	1.58	-0.371
	JPYUSD	0.012223	0.0021	0.091	1.93	0.175	0.012724	0.0018	0.164	2.10	0.297
	KRWUSD	0.0008929	-0.0224	0.108	2.55	-0.540	0.0	-0.0211	0.124	2.17	-0.726
2M	EURUSD	1.3864	-0.0046	0.106	1.24	-0.277	1.4025	-0.0057	0.112	1.27	-0.376
	JPYUSD	0.012223	0.0026	0.093	1.55	0.237	0.012724	0.0025	0.154	1.60	0.327
	KRWUSD	0.0008929	-0.0232	0.114	2.08	-0.497	0.0	-0.0221	0.124	2.04	-0.535
3M	EURUSD	1.3864	-0.0048	0.111	1.10	-0.296	1.4025	-0.0058	0.115	1.12	-0.378
	JPYUSD	0.012223	0.0027	0.097	1.27	0.275	0.012724	0.0027	0.150	1.41	0.339
	KRWUSD	0.0008929	-0.0230	0.119	1.77	-0.478	0.0	-0.0218	0.128	1.78	-0.495
4M	EURUSD	1.3864	-0.0062	0.116	0.87	-0.310	1.4025	-0.0067	0.123	0.86	-0.355
	JPYUSD	0.012223	0.0032	0.107	0.94	0.298	0.012724	0.0036	0.151	1.05	0.356
	KRWUSD	0.0008929	-0.0211	0.128	1.31	-0.481	0.0	-0.0198	0.135	1.36	-0.464

Table 36: Parameters before/after Tohoku Earthquake II

		2011/3/1					2011/3/29				
		$S(0)$	α	$\sigma(0)$	ν	ρ	$S(0)$	α	$\sigma(0)$	ν	ρ
1M	JPYUSD	0.01218	0.0023	0.090	1.95	0.210	0.01220	0.0018	0.105	2.13	0.151
	SGDUSD	0.7852	0.0022	0.057	2.24	-0.312	0.7930	0.0087	0.0623	2.0133	-0.3551
2M	JPYUSD	0.01218	0.0026	0.092	1.55	0.259	0.01220	0.0023	0.1119	1.5379	0.2281
	SGDUSD	0.7852	0.0023	0.067	1.66	-0.293	0.7930	0.0004	0.0623	1.5998	-0.3964
3M	JPYUSD	0.01218	0.0027	0.095	1.30	0.294	0.01220	0.0025	0.1154	1.3192	0.2565
	SGDUSD	0.7852	0.0023	0.068	1.30	-0.349	0.7930	0.0007	0.0654	1.2393	-0.4572
4M	JPYUSD	0.01218	0.0034	0.106	0.96	0.306	0.01220	0.0033	0.1229	0.9926	0.2809
	SGDUSD	0.7852	0.0024	0.072	1.10	-0.352	0.7930	0.0010	0.0697	1.0287	-0.4119