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# A New Improvement Scheme for Approximation Methods of Probability Density Functions \*

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## Abstract

This paper develops a new scheme for improving an approximation method of a probability density function, which is inspired by the idea in *best approximation in an inner product space*. Moreover, we apply “Dykstra’s cyclic projections algorithm” for its implementation. Numerical examples for application to an asymptotic expansion method in option pricing demonstrate the effectiveness of our scheme under Black-Scholes and SABR models.

**Keywords:** Density approximation, Probability density function, Asymptotic expansion, Best approximation in inner product spaces, Dykstra’s algorithm, Option pricing, SABR model

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# 1 Introduction

An approximation for a probability density function is a very interesting topic in various research fields. In fact, it seems so useful that a precise analytical approximation for a density would lead to substantial reduction of computational burden so that the subsequent analyses could be very easily implemented. Particularly, in finance the approximations for the densities of the asset prices have drawn much attention for at least more than two decades since fast and precise computation is so important in terms of competition and risk management: for instance, it is crucial in evaluation of derivatives.

An example among a large number of the related researches is an asymptotic expansion approach initiated by [19] Yoshida [63] and [41], which is mathematically justified by Watanabe theory (Watanabe [59]) in Malliavin calculus (e.g. Malliavin [28], Chapter V-8 in Ikeda and Watanabe [17], Nualart [32]). Actually, the asymptotic expansion have been applied to a broad class of problems in finance: See for instance, [42], [43], [20], [21], [22], [30], [35], [36], [37], [38], [39], [44], [45], [53], [54] [56], [57], [52], [46], [47], [48], Li [27], Osajima [33].

For other approximation methods in mathematical finance/financial engineering see for example, Bayer and Laurence [3], Ben Arous and Laurence [4], Gatheral, Hsu, Laurence, Ouyang, and Wang [14], Fouque, Papanicolaou and Sircar [13], Henry-Labordere [25], Kusuoka and Osajima [23], Osajima [34], Siopacha and Teichmann [40], and [10], [11], [12], [18], [55], [60], [61].

Although the asymptotic expansion up to the fifth order is known to be sufficiently accurate for option pricing (e.g. [37], [49], [50], [51]), one of the main criticisms against the method would be that the approximate density function admits negative values typically at its tails that is, some region of the deep Out-of-The-Money (OTM), which could create an arbitrage opportunity in option trading. Also, even if the domain of a true density is restricted to be positive, the domain of its approximation may include negative values unless an appropriate boundary condition is assigned. However, it seems that those and some other problems exist, at least implicitly in other approximation methods.

This paper develops a new scheme for improving density approximation methods, which also contributes to precise approximation of option values efficiently. Specifically, our scheme is inspired by the idea in *a best approximation method in an inner product space*, and so called “Dykstra’s cyclic projections algorithm” is applied for its implementation.

Firstly, we introduce an inner product with some appropriate weight function: the function is in fact a density of a certain probability measure. Then, we represent a closed convex space satisfying the required conditions in terms of the inner product. Thus, we are able to obtain the unique best approximation from a given approximate density to the space in terms of the norm induced by the inner product, which is also shown to be a better approximation for the true density than the original one measured by the norm.

More concretely, we put two basic conditions for a density function such that its total mass is one and its range is nonnegative. It should be noted that if we consider the density of the underlying asset price under a risk-neutral or a forward probability measure, the nonnegative density is necessary for arbitrage-free option prices.

Also, we put *calibration conditions* for the underlying asset prices and option values. That is, the average value under the approximate density of the underlying asset price at maturity should be equal to the current underlying forward price, which is generally known in advance. Moreover, as it is known that the asymptotic expansion method gives precise approximate values of the options close to ATM, a new approximate density may be matched to the density obtained by the asymptotic expansion method for a certain range close to ATM. In addition, option prices around ATM based on the asymptotic expansion method may be used for calibration by a new approximate density.

Further, if we have some information on the tail behavior of the target density such as those in Lee [26] and Benaim et al. [5], we may set *tail conditions* which will be given concretely in the next section.

We reflect those conditions in terms of an appropriate inner product space. Then, by using the corresponding norm we implement *the best approximation* from the density obtained by the asymptotic expansion to the closed convex space created by the required conditions. Consequently, we get a better estimate (in

terms of the norm) for the true density of the underlying asset price than the one based on the original asymptotic expansion. In computation, we utilize so called *Dijkstra's cyclic projections algorithm*. (e.g. see pp.207-214 of Deutsch [8] for the detail.)

Furthermore, numerical experiments for pricing plain-vanilla options under Black-Scholes([6]) and SABR ([15], [16]) models demonstrate the validity of our scheme. In fact, our scheme improves the third order asymptotic expansion preserving the required conditions such as nonnegative densities under an appropriate forward measure.

We finally remark that our scheme is general and flexible enough to be applied to approximation methods other than the asymptotic expansion approach. For example, it is well known that the density of the approximation formula for SABR model (Hagan et al. [15], [16]) has the negative values for the left tail. Hence, a number of researches have been going on in order to extend SABR model with fixing the problem of the negative densities. (For instance, see [1], [2], [9].) we note that our scheme is a candidate for handling this issue, too.

The organization of the paper is as follows: After the next section describes the setup of the problem, Section 3 provides a concrete formulation of our method as well as the algorithm for the implementation. Section 4 shows numerical examples under Black-Scholes and SABR models. Section 5 concludes.

## 2 Setup

Let  $S_t$  be the spot price of the underlying asset at time  $t \in [0, T]$  and consider a density  $f$  of  $S_T$ , where  $S_T$  takes a value in  $I \subseteq \mathbb{R}$ , such as  $I = \mathbb{R}$ ,  $[0, +\infty)$  or  $(0, +\infty)$ . Clearly, the density function  $f$  of the price  $S_T$  must satisfy the following property. Hereafter,  $\eta$  stands for a density function under a risk-neutral or an appropriate forward probability measure.

**Property 1.** (*Density Condition*) : for a function  $\eta$  on  $I \subseteq \mathbb{R}$ ,

- (1)  $\int_I \eta(x) dx = 1$
- (2)  $\eta \geq 0$ .

Suppose that we have an approximation  $\tilde{f}$  of the density function  $f$  by some method such as asymptotic expansion scheme. Note that the approximation  $\tilde{f}$  does not necessarily have the properties 1. Actually, it has been found through our numerical experience under Black-Scholes model that  $\tilde{f}$  possibly takes a negative value at its tail, namely "deep out of the money."

Also, generally the forward price is given independently of models, and hence the average value of the underlying asset price at  $T$  should be equal to the given forward price with maturity  $T$ . Moreover, it is known that the asymptotic expansion method provides rather precise approximations for the values close to At-The-Money(ATM) options. Thus, it is reasonable that the option prices around ATM under a new approximate density function are calibrated to those computed based on the asymptotic expansion, and that a new density is equal to the one obtained by the asymptotic expansion for a certain range around ATM of the underlying asset price. We call those properties by *Calibration Condition*:

**Property 2.** (*Calibration Condition*)

- (3)  $\int_I x\eta(x) dx = S_0$
- (4)  $\int_I (x - K_n)_+ \eta(x) dx = C_{K_n}$  for some given strikes  $\{K_n\}_{n=1}^N$
- (5)  $\eta = \tilde{f}$  on some subset  $I_0$  of  $I$

Here, the risk-free interest rate as well as the dividend rate of the underlying asset are assumed to be zero for simplicity.  $C_{K_n}$  denotes the option price with strike  $K_n$  and maturity  $T$  computed by the asymptotic expansion method.

In contrast to the accuracy around ATM, the values of the approximated density  $\tilde{f}$  may not be reliable around deep out of the money. However, how fast a density decreases to zero is known under some models or through a moment formula for the implied volatility. Namely, the following quantities are known:

$$\tilde{p} := \sup\{p > 0 : \mathbb{E}S_T^p < +\infty\} \quad (2.1)$$

$$\tilde{q} := \sup\{q > 0 : \mathbb{E}S_T^{-q} < +\infty\} \quad (2.2)$$

under some models or through the moment formula derived by Lee [26]:

$$\tilde{p} = \frac{1}{2\beta_R} + \frac{\beta_R}{8} + \frac{1}{2}, \quad \tilde{q} = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}, \quad (2.3)$$

where

$$\beta_R := \limsup_{x \rightarrow +\infty} \frac{IV^2(x)}{|x|/T}, \quad (2.4)$$

$$\beta_L := \limsup_{x \rightarrow -\infty} \frac{IV^2(x)}{|x|/T}. \quad (2.5)$$

Here,  $IV(x)$  is an implied volatility function in terms of the log-moneyness that is,  $x = \log(S_0/K)$ .

Now, let us assume that  $\tilde{p}$  and  $\tilde{q}$  are known, and suppose  $\chi : I \rightarrow (0, +\infty)$  be a density function which has the same order of the tail condition as  $f$ :

$$\int_{S_0}^{+\infty} x^p \chi(x) dx < +\infty \quad (p < \tilde{p}), \quad \int_0^{S_0} x^{-q} \chi(x) dx < +\infty \quad (q < \tilde{q}). \quad (2.6)$$

Then, it seems natural to impose the following condition:

**Property 3.** (*Tail Condition*)

(6)  $\eta$  has the same tail slopes as  $\chi$ .

However, for ease of computation, the condition may be replaced with the following:

**Property 4.** (*Weak Tail Condition*)

(6-1)  $\eta \leq \chi$  on  $(0, K_L]$  for some positive number  $K_L$

(6-2)  $\eta \leq \chi$  on  $[K_R, +\infty)$  for some positive number  $K_R$

Thus, we state our problem formally as follows:

**Definition 1** (Problem). *Find a new approximate density function  $f^*$  for the target density  $f$  such that it satisfies the properties 1, 2 and 4, and*

$$\|f - f^*\| \leq \|f - \tilde{f}\|, \quad (2.7)$$

where the norm  $\|\cdot\|$  will be defined in the next section.

### 3 Formulation and Algorithm

This section concretely formulates the previous discussion and provides an algorithm for the implementation.

### 3.1 Formulation

Firstly, suppose a probability space  $(\mathbb{R}, \mathcal{M}, \mu)$ , where the measure  $\mu$  is assumed to have a density which is equal to  $\chi$  given in the previous section on  $I$  and to 0 on  $I^c$ . Next, we define the set of square integrable functions on  $(\mathbb{R}, \mathcal{M}, \mu)$  denoted by  $\mathcal{H} := L^2(\mathbb{R}, \mathcal{M}, \mu)$ , and introduce an inner product for  $f, g \in \mathcal{H}$  by

$$\langle f, g \rangle = \int_I f(x)g(x)\chi(x)dx.$$

With this preparation, the properties (1) to (5), (6-1) and (6-2) in Section 2 are restated as follows:

- (1)  $\langle \eta/\chi, 1_I \rangle = 1$
- (2)  $\langle \eta/\chi, \delta_x \rangle \geq 0$  for each  $x \in I$
- (3)  $\langle \eta/\chi, id_I \rangle = S_0$
- (4)  $\langle \eta/\chi, g_{K_n} \rangle = C_{K_n}$
- (5)  $\langle \eta/\chi, \delta_x \rangle = \tilde{f}(x)$  for each  $x \in I_0$
- (6-1)  $\langle \eta/\chi, \delta_x \rangle \leq \chi(x)$  for each  $x \in (0, K_L]$
- (6-2)  $\langle \eta/\chi, \delta_x \rangle \leq \chi(x)$  for each  $x \in [K_R, +\infty)$ ,

where  $1_I$ ,  $id_I$  and  $g_{K_n}$  are elements of  $\mathcal{H}$  such that  $1_I(x) = 1$ ,  $id_I(x) = x$  and  $g_{K_n}(x) = (x - K_n)_+$ .

Next, let us define some subsets of  $\mathcal{H}$  as  $\mathcal{K}_D$ ,  $\mathcal{K}_C$  and  $\mathcal{K}_T$  which stand for the properties (1)-(2), (3)-(5) and (6-1)-(6-2) above, respectively:

$$\mathcal{K}_D := \{\varphi \in \mathcal{H} \mid \langle \varphi, 1_I \rangle = 1\} \cap \bigcap_{x \in I} \{\varphi \in \mathcal{H} \mid \langle \varphi, \delta_x \rangle \geq 0\}, \quad (3.1)$$

$$\mathcal{K}_C := \{\varphi \in \mathcal{H} \mid \langle \varphi, id_I \rangle = S_0\} \cap \bigcap_{n \leq N} \{\varphi \in \mathcal{H} \mid \langle \varphi, g_{K_n} \rangle = C_{K_n}\} \cap \bigcap_{x \in I_0} \{\varphi \in \mathcal{H} \mid \langle \varphi, \delta_x \rangle = \tilde{f}(x)\} \quad (3.2)$$

and

$$\mathcal{K}_T := \bigcap_{x \in (0, K_L]} \{\varphi \in \mathcal{H} \mid \langle \eta/\chi, \delta_x \rangle \leq \chi(x)\} \cap \bigcap_{x \in [K_R, +\infty)} \{\varphi \in \mathcal{H} \mid \langle \eta/\chi, \delta_x \rangle \leq \chi(x)\}. \quad (3.3)$$

Moreover, define  $\mathcal{K}$  as the intersection of  $\mathcal{K}_D$ ,  $\mathcal{K}_C$  and  $\mathcal{K}_T$ :

$$\mathcal{K} := \mathcal{K}_D \cap \mathcal{K}_C \cap \mathcal{K}_T, \quad (3.4)$$

which is assumed to be nonempty.

Then, let  $\tilde{\varphi} := \tilde{f}/\chi \notin \mathcal{K}$ , and the best approximation set from  $\tilde{\varphi}$  to  $\mathcal{K}$  is defined as

$$P_{\mathcal{K}}(\tilde{\varphi}) := \{\varphi^* \in \mathcal{K} \mid \|\tilde{\varphi} - \varphi^*\| = \inf_{\eta \in \mathcal{K}} \|\tilde{\varphi} - \eta\|\}. \quad (3.5)$$

Note that the set  $P_{\mathcal{K}}(\tilde{\varphi})$  has the only one element since  $\mathcal{K}$  is a closed convex set in a Hilbert space. Hereafter, we may use the notation  $P_{\mathcal{K}}(\tilde{\varphi})$  for the unique element of  $P_{\mathcal{K}}(\tilde{\varphi})$  without any confusion.

Thus, it is easily shown that  $P_{\mathcal{K}}(\tilde{\varphi})$  is a better approximation for  $f/\chi \in \mathcal{K}$  than  $\tilde{\varphi} := \tilde{f}/\chi \notin \mathcal{K}$ , because we have

$$\begin{aligned} \|f/\chi - P_{\mathcal{K}}(\tilde{\varphi})\|^2 &= \|f/\chi - \varphi^\perp\|^2 + \|\varphi^\perp - P_{\mathcal{K}}(\tilde{\varphi})\|^2 \\ &\leq \|f/\chi - \varphi^\perp\|^2 + \|\varphi^\perp - \tilde{\varphi}\|^2 \\ &= \|f/\chi - \tilde{\varphi}\|^2, \end{aligned} \quad (3.6)$$

where  $\varphi^\perp$  is the foot of a perpendicular line through  $\tilde{\varphi}$  and  $P_{\mathcal{K}}(\tilde{\varphi})$  from  $f/\chi$ .

Finally, we are able to obtain a better approximated density function as

$$f^* := P_{\mathcal{K}}(\tilde{\varphi})\chi. \quad (3.7)$$

This is actually better than the original one  $\tilde{f}$  in the following sense:

$$\int_I |f(x) - f^*(x)|^2 \frac{1}{\chi(x)} dx \leq \int_I |f(x) - \tilde{f}(x)|^2 \frac{1}{\chi(x)} dx. \quad (3.8)$$

## 3.2 Algorithm

In order to compute (3.5), we apply an iterative algorithm called *Dykstra's algorithm*. (See pp. 207-214 of Deutsch [8] for the detail of the algorithm and its convergence discussion.)

In particular, let  $\tilde{\mathcal{K}}$  be a convex set obtained by discretization of  $\mathcal{K}$ , which is an intersection of finite many closed convex sets  $\mathcal{K}_i$  ( $i = 1, 2, \dots, r$ ) in the Hilbert space  $\mathcal{H}$ :

$$\tilde{\mathcal{K}} = \bigcap_{i=1}^r \mathcal{K}_i. \quad (3.9)$$

Here, we assume  $\tilde{\mathcal{K}}$  to be nonempty.

First, for each  $n \in \mathbb{N}$ , let  $[n]$  denote  $n \bmod r$ ; that is,

$$[n] := \{1, 2, \dots, r\} \cap \{n - kr : k = 0, 1, 2, \dots\}. \quad (3.10)$$

For instance,  $[1] = 1, [2] = 2, \dots, [r] = r, [r+1] = 1, \dots, [2r] = r, \dots$

Next, for  $\tilde{\varphi} \in \mathcal{H}$ , set

$$\begin{aligned} \varphi_0 &:= \tilde{\varphi}, \quad e_{-(r-1)} = \dots = e_{-1} = e_0 = 0, \\ \varphi_n &:= P_{\mathcal{K}_{[n]}}(\varphi_{n-1} + e_{n-r}), \\ e_n &:= \varphi_{n-1} + e_{n-r} - \varphi_n \\ &= \varphi_{n-1} + e_{n-r} - P_{\mathcal{K}_{[n]}}(\varphi_{n-1} + e_{n-r}). \end{aligned} \quad (3.11)$$

Then, we have

$$\lim_{n \rightarrow +\infty} \|\varphi_n - P_{\mathcal{K}}(\tilde{\varphi})\| = 0. \quad (3.12)$$

See Theorem 9.24 (Boyle-Dykstra Theorem) in p. 213 of Deutsch [8] for the proof.

## 4 Numerical Example

This section examines the validity of our method through numerical experiments under Black-Scholes and SABR models.

### 4.1 Black-Scholes Model

Firstly, let us consider Black-Scholes model, where analytical formulas for the density of the underlying asset price and option values are known. Under Black-Scholes model, the dynamics of the underlying price process under an appropriate forward measure is given as follows:

$$dS_t = \sigma S_t dW_t, \quad (4.1)$$

where  $\sigma$  is a constant. Clearly, the density function of  $S_T$ ,  $f(x)$  is given by a log-normal:

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x\sigma\sqrt{T}} e^{-\frac{(\log x - m)^2}{2\sigma^2 T}}, \quad (4.2)$$

where  $m := \log S_0 - \frac{1}{2}\sigma^2 T$ . For this case, we take  $\chi(x)$  for  $I := (0, +\infty)$  as follows:

$$\chi(x) = \begin{cases} A_L \frac{1}{\sqrt{2\pi}} \frac{1}{x\sigma\sqrt{T}} e^{-\frac{(\log x - m)^2}{2\sigma^2 T}} & x \in (0, K_L) \\ f_3(x) & x \in [K_L, K_R) \\ A_R \frac{1}{\sqrt{2\pi}} \frac{1}{x\sigma\sqrt{T}} e^{-\frac{(\log x - m)^2}{2\sigma^2 T}} & x \in [K_R, +\infty), \end{cases} \quad (4.3)$$

and  $K_L$ ,  $K_R$ ,  $A_L$  and  $A_R$  positive constants and  $f_3$  stands for the density function obtained by the third order asymptotic expansion.

With this setup, we calculate option values in the three different ways below:

- (a) exact Black-Scholes formula (Benchmark)
- (b) asymptotic expansion up to the third order
- (c) our algorithm with  $K_i = 80, 100, 120$  ( $N = 3$  in  $\mathcal{K}_C$  of (3.2)) and  $f^* = \tilde{f}$  on  $[70, 150)$ ,

where we set  $\tilde{f} = f_3$  and specify Black-Scholes model parameters as  $S_0 = 100$ ,  $\sigma = 0.15$ ,  $T = 2$ ,  $K_L = 70$  and  $K_R = 150$ ;  $A_L$  and  $A_R$  are determined so that  $\chi$  is continuous at  $K_L$  and  $K_R$ , respectively.

The results for the option prices and the density functions obtained by three different methods above are shown in Table 2 and Figure 1-3, respectively.

In the table, the comparison between the third order expansion (b) and our method (c) shows that our method improves accuracies for the deep OTM prices, where the strike prices are in the range of 10-50 and 160-200. On the other hand, for the strikes in the range of 60-150, both methods provide the same order of good accuracies: it is plausible because in our method (c) the option values around ATM and the density between 70 and 150 of the underlying asset price are calibrated to those based on the third order expansion.

Although Figure 1 shows that both the third order expansion and our method produce good approximations in total, the close look at the left tail in Figure 2 reveals that the third order expansion suffers the negative densities in the range of 31-48 of the underlying asset price. It also puts very small but nonzero densities on the negative underlying asset values, which is not the case for the log-normal model with a positive initial value.

On the other hand, our method keeps the density nonnegative due to the property (2) in Section 3, reflected in  $\mathcal{K}_D$  of (3.1). Moreover, by construction it does not put any densities on the negative underlying asset prices.

Furthermore, Figure 3 for the right tail of the density clarifies that our method improves the approximation, especially around 160 of the underlying price.

Consequently, we can conclude that our scheme works very well for this case.

## 4.2 SABR Model

The second example takes SABR model as a stochastic volatility model, where the underlying price process under a forward measure follows:

$$dS_t = \sigma_t S_t^c dW_t^1 \quad (4.4)$$

$$d\sigma_t = \varepsilon \sigma_t dW_t^2, \quad (4.5)$$

where  $\varepsilon > 0$ ,  $c \in (0, 1]$  and  $W^1$  and  $W^2$  are Brownian motions with a constant correlation  $\rho$ . Firstly, let us consider the case for  $c = 1$  and  $\rho < 0$ . According to the result by Benaim et al. [5], it holds that in

$$\tilde{p} = \frac{1}{1 - \rho^2}, \quad \tilde{q} = 0. \quad (4.6)$$



In this case, the left tail is so fat that any moment is infinite, that is  $\mathbb{E}(S_T^{-q}) = +\infty$  for  $q > 0$ . Hence, taking the observation into consideration, we may set  $\chi$  for  $I := (0, +\infty)$  as follows:

$$\chi(x) = \begin{cases} \gamma x + A_L & x \in (0, K_L) \\ f_3(x) & x \in [K_L, K_R) \\ \beta(x - \alpha)^{-\tilde{p}} & x \in [K_R, +\infty), \end{cases} \quad (4.7)$$

where  $K_L, K_R, A_L, \alpha, \beta$  and  $\gamma$  are positive constants and  $f_3$  stands for the density function obtained by the third order asymptotic expansion.

Next, let us consider the case for  $c < 1$ . For this case, it is known that  $S_T$  can reach 0 with positive probability. According to the result by Benaim et al. [5], it holds that

$$\tilde{p} = +\infty, \quad \tilde{q} = 0. \quad (4.8)$$

Then, while the density decreases so fast to zero on the right tail, the left tail is so fat that any moment is infinite:  $\mathbb{E}(S_T^p) < +\infty$  for  $p > 0$  and  $\mathbb{E}(S_T^{-q}) = +\infty$  for  $q > 0$ . Thus, taking this observation into account, we may specify  $\chi$  for  $I := [0, +\infty)$  in the following:

$$\chi(x) = \begin{cases} A_L & x = 0, \\ f_3(x) & x \in (0, K_R) \\ A_R \frac{1}{\sqrt{2\pi}} \frac{1}{x\tilde{\sigma}\sqrt{T}} e^{-\frac{(\log x - m)^2}{2\tilde{\sigma}^2 T}} & x \in [K_R, +\infty), \end{cases} \quad (4.9)$$

where  $A_L$  is a positive constant and  $m = \log S_0 - \frac{1}{2}\tilde{\sigma}^2 T$ . Note that we do not impose the weak tail condition (6-1) for this case.

With this setup, we calculate OTM option prices by the following methods:

- (a) Monte Carlo simulation (Benchmark)
- (b) asymptotic expansion up to the fifth order
- (c) asymptotic expansion up to the third order
- (d) our algorithm with  $K_i = 80, 100, 120$  ( $N = 3$  in  $\mathcal{K}_C$  of (3.2)), and  $f^* = \tilde{f}$  on  $[K_L, K_R)$  for  $c = 1$  and on  $(0, K_R)$  for  $c = \frac{1}{2}$ , where  $\tilde{f}$  is equivalent to (c) the third order expansion.

For both cases of  $c = 1$  and  $c = \frac{1}{2}$ , we set  $\tilde{f} = f_3$ ,  $K_R = 150$  and  $A_L = \tilde{f}(0)$ . Moreover, for the case of  $c = 1$ , we set  $K_L = 50$ ;  $\gamma$  is determined so that  $\chi$  is continuous;  $\alpha$  and  $\beta$  are determined by  $\chi(x) = \tilde{f}(x)$  for some two points around  $x = K_R$  so that the function  $\chi$  decreases so fast to zero as the function  $\tilde{f}$  does around  $x = K_R$ . For the case of  $c = \frac{1}{2}$ , we set  $\tilde{\sigma} = 0.45$ , which is the same level as the implied volatility around  $K_R$ ;  $A_R$  is determined so that  $\chi$  is continuous. We set the other parameters as in Table 1 below.

The results for the option values by four methods (a)-(d) above are given in Table 3 for  $c = 1$  and Table 4 for  $c = \frac{1}{2}$ . Also, the density functions obtained by four methods (a)-(d) are shown in Figure 4 - 6 for  $c = 1$  and Figure 7 - 9 for  $c = \frac{1}{2}$ .

Table 3 for  $c = 1$  shows that comparing to the third order expansion (c), our method (d) provides the better approximations for 12 out of 20 cases: it substantially improves the accuracies for the deep OTM prices for the strikes in the range of 10-50 and 160-200, while the accuracies of (c) and (d) are comparable in the range of 60-150, which is expected because in our method (d) the option values around ATM and the density between 50 and 150 of the underlying asset price are calibrated to those based on the third order expansion.

On the other hand, the fifth order expansion (b) still produces the best approximations, which is reasonable since our new method is designed based on the third order expansion. In terms of the density approximations, although Figure 5 shows that our method and the fifth order expansion look better than the third order expansion in total, the examination of the left tail in Figure 5 reveals that the fifth order expansion suffers the negative densities in some range below 5 of the underlying asset price, while our method

does not suffer those for all the range by construction. Also, as in Black-Scholes model, the third and fifth order expansions put some nonzero (negative) densities on the negative values of the underlying asset price, which is not the case for the benchmark Monte Carlo and our method.

On the other hand, Figure 6 shows that the fifth order expansion has no problem for the right tail, and our method improves the third order expansion around between 160 and 220 of the underlying asset price.

Moreover, we expect that applying our method to the fifth order expansion will improve the approximation accuracies of option prices with the conditions such as the nonnegative densities and no densities on the negative underlying asset values.

In the approximation of the option values for  $c = \frac{1}{2}$ , Table 4 confirms that the similar observation holds as for  $c = 1$ , except that our method improves the third order expansion more substantially than for  $c = 1$ , and produces sufficient accuracies comparable to the fifth order expansion.

Further, thanks to the flexibility of our choice of  $\chi(x)$  in (4.9), our density approximation is able to take the absorption barrier at  $x = 0$  into consideration, at least to a certain extent, which cannot be achieved by the original asymptotic expansion method only: the asymptotic expansion itself puts nonzero densities on some negative values of the underlying asset prices as in the cases of Black-Scholes and SABR with  $c = 1$ . We observe these points in Figure 7 - 9. Note finally that the approximation is expected to become more precise if we can add other accurate information such as for the probability of  $S_T = 0$  (e.g. Doust [9]) to  $\mathcal{K}$  in (3.4) through its inner product expression.

Table 1: Parameter

$S_0$	$c$	$\sigma_0$	$\varepsilon$	$\rho$	$T$
100	0.5 or 1.0	0.5	0.3	-0.5	1

## 5 Conclusion

We have proposed a new improvement scheme of approximation methods for probability density functions, of which applications contain option pricing and computation of *Greeks* in finance.

Particularly, we have made use of an idea of "best approximation in an inner product space from a convex set," and applied an iterative algorithm called *Dijkstra's algorithm* for its implementation. Consequently, we have obtained a better approximate density that satisfies a set of appropriate conditions including the nonnegative density under a risk-neutral or a forward measure necessary for no arbitrage option prices.

Moreover, an application to an asymptotic method has confirmed the validity of our scheme through numerical experiments under Black-Scholes and SABR models.

We finally remark that our scheme is general and flexible enough to be applied to other density approximation methods such as Hagan et al. [16], as well as to include some other accurate information for the true density such as for the probability of the absorption at origin. Further applications and extensions along the line are among our next research topics.

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Table 2: Option prices under Black-Scholes model

strike	(a)	(b)	(c)	(b)-(a)	(c)-(a)
10	-0.00	0.00	0.00	0.001	0.000
20	0.00	0.01	0.00	0.005	-0.000
30	0.00	0.01	0.00	0.013	0.000
40	0.00	0.02	0.00	0.019	-0.000
50	0.00	0.01	0.00	0.011	-0.002
60	0.04	0.04	0.04	-0.006	-0.006
70	0.34	0.33	0.33	-0.010	-0.007
80	1.43	1.42	1.43	-0.004	-0.000
90	3.99	3.99	3.99	-0.001	0.002
100	8.45	8.45	8.45	-0.000	-0.000
110	4.75	4.75	4.74	0.001	-0.002
120	2.50	2.51	2.50	0.003	0.000
130	1.25	1.26	1.26	0.009	0.006
140	0.60	0.61	0.61	0.009	0.010
150	0.28	0.28	0.28	-0.001	0.007
160	0.12	0.11	0.13	-0.013	0.004
170	0.05	0.04	0.06	-0.017	0.002
180	0.02	0.01	0.02	-0.013	0.001
190	0.01	0.00	0.01	-0.008	0.000
200	0.00	0.00	0.00	-0.004	0.000

Figure 1: Densities under Black-Scholes model (total)

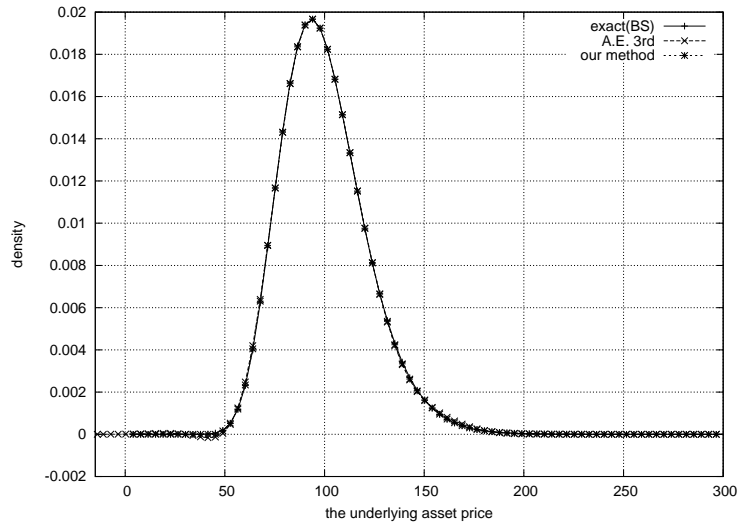


Figure 2: Densities under Black Scholes model (left tail)

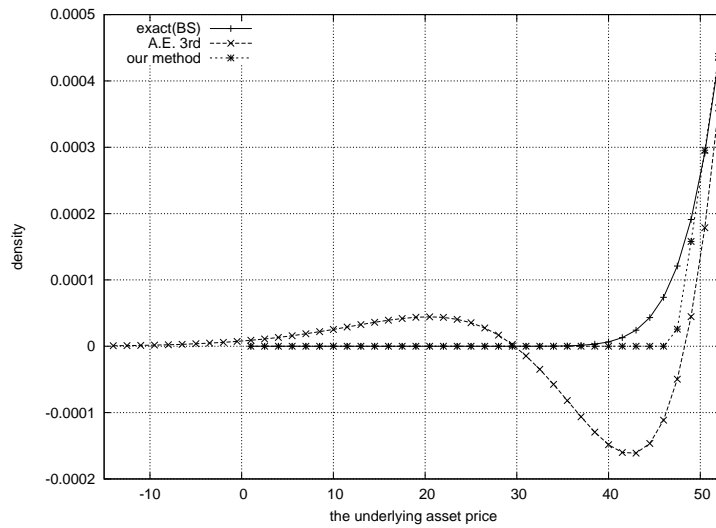




Figure 3: Densities under Black-Scholes model (right tail)

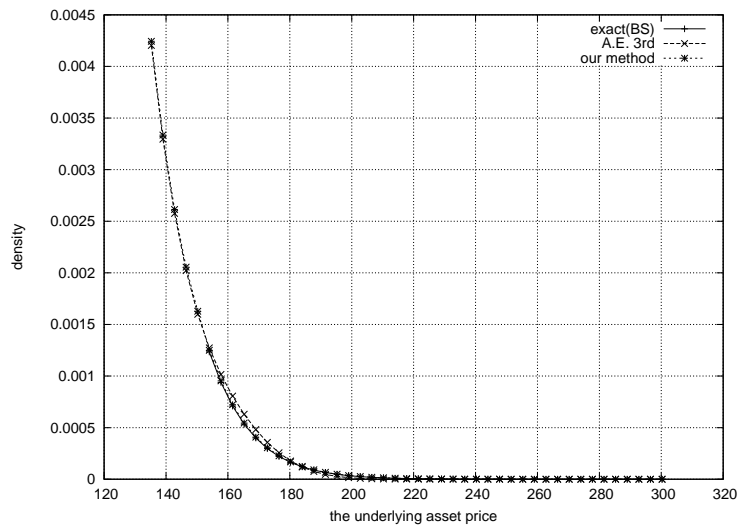


Table 3: Option prices under SABR model with  $c = 1$

strike	(a)	(b)	(c)	(d)	(b)-(a)	(c)-(a)	(d)-(a)
10	0.00	0.00	-0.14	0.01	0.00	0.14	-0.00
20	0.05	0.03	-0.13	0.06	0.02	0.18	-0.02
30	0.25	0.23	0.09	0.28	0.02	0.16	-0.03
40	0.79	0.78	0.68	0.83	0.02	0.12	-0.04
50	1.86	1.86	1.80	1.93	0.00	0.06	-0.06
60	3.64	3.64	3.61	3.71	-0.00	0.03	-0.07
70	6.25	6.25	6.24	6.31	-0.01	0.01	-0.07
80	9.75	9.76	9.76	9.80	-0.01	-0.00	-0.05
90	14.17	14.17	14.17	14.20	-0.01	-0.01	-0.03
100	19.44	19.45	19.46	19.46	-0.01	-0.01	-0.01
110	15.54	15.53	15.54	15.55	0.01	0.00	-0.01
120	12.33	12.32	12.33	12.33	0.01	-0.01	-0.01
130	9.72	9.72	9.74	9.73	0.01	-0.02	-0.01
140	7.64	7.63	7.67	7.65	0.01	-0.03	-0.01
150	5.98	5.98	6.03	6.00	0.00	-0.05	-0.02
160	4.68	4.68	4.76	4.70	-0.00	-0.08	-0.03
170	3.65	3.65	3.76	3.68	-0.00	-0.11	-0.03
180	2.85	2.85	2.97	2.87	-0.00	-0.12	-0.02
190	2.23	2.22	2.35	2.24	0.00	-0.12	-0.01
200	1.74	1.73	1.84	1.72	0.01	-0.10	0.02

Figure 4: Densities under SABR model with  $c = 1$ (total)

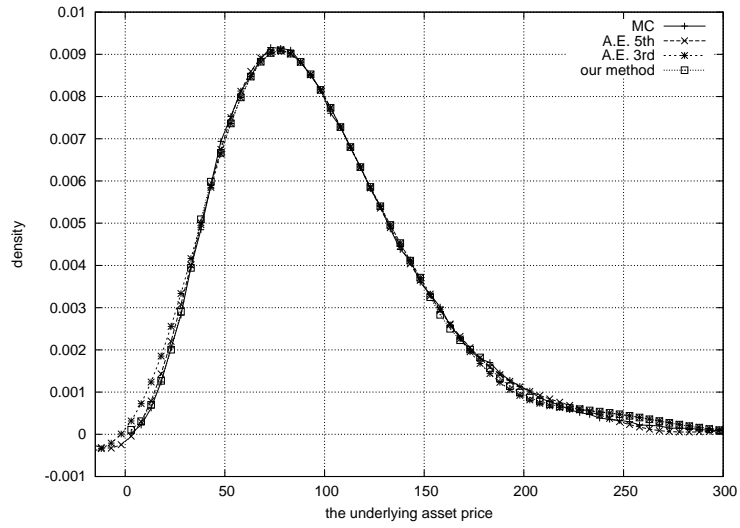


Figure 5: Densities under SABR model with  $c = 1$ (left tail)

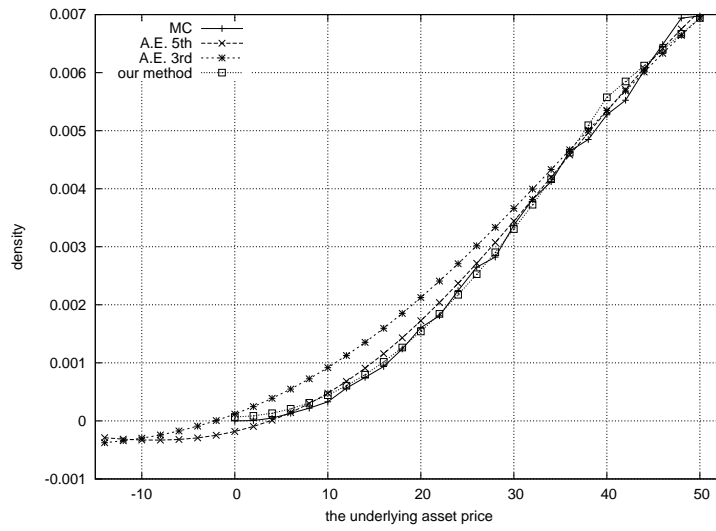


Figure 6: Densities under SABR model with  $c = 1$ (right tail)

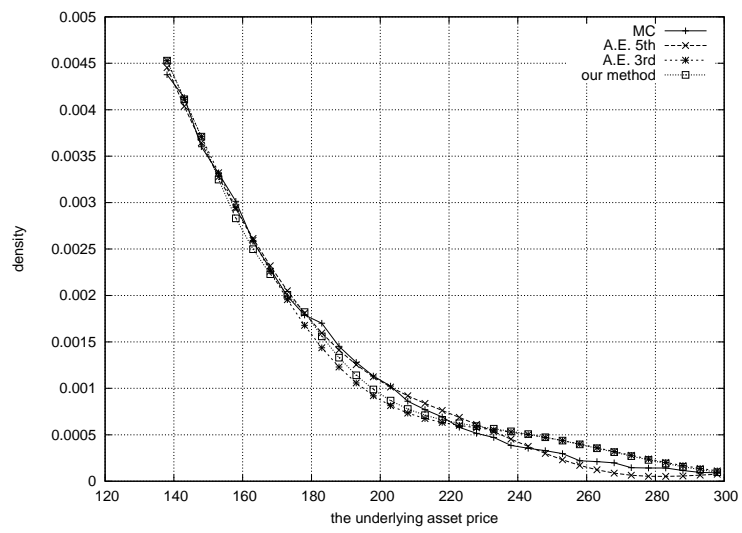


Table 4: Option prices under SABR model with  $c = \frac{1}{2}$

strike	(a)	(b)	(c)	(d)	(b)-(a)	(c)-(a)	(d)-(a)
10	0.13	0.15	0.28	0.15	-0.02	-0.15	-0.02
20	0.43	0.46	0.61	0.46	-0.02	-0.17	-0.03
30	0.98	1.00	1.16	1.00	-0.03	-0.18	-0.03
40	1.86	1.89	2.04	1.88	-0.03	-0.18	-0.02
50	3.19	3.21	3.35	3.21	-0.03	-0.16	-0.02
60	5.06	5.09	5.20	5.08	-0.02	-0.14	-0.01
70	7.58	7.60	7.69	7.60	-0.02	-0.11	-0.01
80	10.82	10.84	10.91	10.84	-0.02	-0.08	-0.02
90	14.84	14.86	14.90	14.87	-0.01	-0.06	-0.03
100	19.66	19.68	19.70	19.70	-0.01	-0.03	-0.03
110	15.30	15.29	15.29	15.27	0.02	0.01	0.03
120	11.67	11.66	11.64	11.64	0.01	0.03	0.03
130	8.73	8.72	8.69	8.71	0.01	0.04	0.02
140	6.42	6.41	6.36	6.40	0.01	0.06	0.02
150	4.64	4.62	4.56	4.62	0.01	0.08	0.01
160	3.29	3.28	3.21	3.28	0.01	0.09	0.01
170	2.30	2.29	2.21	2.27	0.01	0.09	0.03
180	1.59	1.57	1.49	1.54	0.01	0.09	0.04
190	1.08	1.07	0.99	1.03	0.01	0.09	0.05
200	0.72	0.72	0.65	0.67	0.01	0.08	0.05

Figure 7: Densities under SABR model with  $c = \frac{1}{2}$ (total)

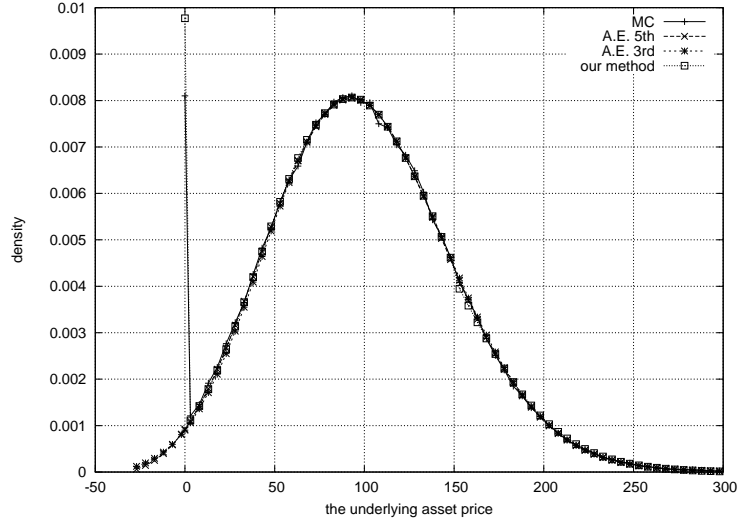


Figure 8: Densities under SABR model with  $c = \frac{1}{2}$ (left tail)

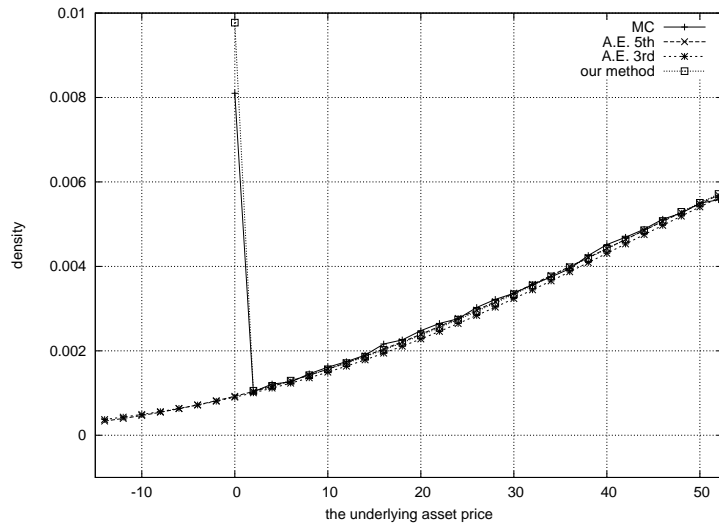


Figure 9: Densities under SABR model with  $c = \frac{1}{2}$  (right tail)

