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On Robust Properties of the SIML Estimation of Volatility under Micro-market noise and Random Sampling

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Abstract

For estimating the integrated volatility and covariance by using high frequency data, Kunitomo and Sato (2008, 2011) have proposed the Separating Information Maximum Likelihood (SIML) method when there are micro-market noises. The SIML estimator has reasonable finite sample properties and asymptotic properties when the sample size is large under general conditions with *non-Gaussian processes* or *volatility models*. We shall show that the SIML estimator has the asymptotic robustness property in the sense that it is consistent and has the stable convergence (i.e. the asymptotic normality in the deterministic case) when there are micro-market noises and the observed high-frequency data are sampled randomly with the underlying (continuous time) stochastic process. The SIML estimation has also reasonable finite sample properties with these effects.

Key Words

Integrated Volatility with Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Random Sampling, Asymptotic Robustness.

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1. Introduction

Recently a considerable interest has been paid on the estimation problem of the integrated volatility by using high-frequency data of financial prices. Although the earlier studies often had ignored the presence of micro-market noises in financial markets, there have been arguments that the micro-market noises are important in high-frequency financial data, and then several new statistical estimation methods have been developed. See Ait-Sahalia, Mykland and Zhang (2005), Zhang, Mykland and Ait-Sahalia (2005), Bandorff-Nielsen, Hansen, Lund and Shephard (2008), and Malliavin and Mancino (2009) as recent literatures on the related topics among many. In this respect Kunitomo and Sato (2008, 2011) have proposed a new statistical method called the Separating Information Maximum Likelihood (SIML) method for estimating the integrated volatility and the integrated covariance by using high frequency data under the presence of micro-market noises. The SIML estimator has reasonable asymptotic properties as well as finite sample properties ; it is consistent and it has the asymptotic normality (or the *stable convergence* in the more general case) when the sample size is large and the data frequency interval is small under a set of regularity conditions for the *non-Gaussian* stochastic processes and *volatility models*. Kunitomo and Sato (2011) have also shown that the SIML estimator has the asymptotic robustness properties, that is, it is consistent and asymptotically normal even when the noise terms are autocorrelated and/or there are endogenous correlations between the market-noise terms and the (underlying) efficient market price process. These aspects have important implications because there have been recent finance literatures which focused on the micro-market aspects in high frequency financial data including Engle and Sun (2007) as an example.

In this paper we shall investigate the properties of the SIML estimation when we have the micro-market noises and random sampling data. In actual high frequency financial data they are recorded irregularly which can be regarded as the observations at random times and the effects of the randomness could be significant when we have micro-market noises. When the effects of measurement errors are present, it seems that the standard statistical methods measuring the integrated volatility and covariance have some problem. They could handle the problem of our interest, but often they need some special consideration on the underlying mechanism of price process. We shall show that the SIML estimator is asymptotically robust in these situations; that is, it is consistent and asymptotically normal as the sample size increases under a reasonable set of assumptions. The asymptotic robustness of the SIML method on the integrated volatility and covariance has desirable properties over other estimation methods for the underlying continuous stochastic process with micro-market noise. Because the SIML estimation is a simple method, it can be practically used for analyzing the multivariate (high frequency) financial time series.

The problem of random sampling on the covariance estimation by high frequency financial data has been investigated by Hayashi and Yoshida (2005, 2008) for the problem of covariance estimation by using randomly sampled data. In this paper we shall focus on the estimation of integrated volatility, but subsequently in Kunitomo and Misaki (2012) we shall investigate the problem of covariance estimation and hedging coefficients.

In Section 2 we introduce the high frequency micro-market models with micro-market noise and the estimation methods including the Separating Information Maximum Likelihood (SIML) method. Then in Section 3 we give some numerical results on the SIML estimation in the basic simulation framework. In Section 4 we shall give the asymptotic properties of the SIML estimator and in Section 5 we shall report the simulation results under a variety of assumptions on the alternative micro-market models. Finally, in Section 6 some brief remarks will be given. Some mathematical details of the proofs of theorems shall be given in Appendix.

2. Micro-market noise models and the estimation of the integrated volatility

2.1 A General Formulation

Let $y(t_i^n)$ be the $i-$th observation of the (log-) price at t_i^n for $0 = t_0^n < t_1^n < \cdots < t_n^{n^*} \leq t_n^n = 1$ and $t_{n^*}^n = \max_{t_i^n \leq 1} t_i^n$. (We shall use n and n^* as indices; n^* is a stochastic index while n is a constant index.) We set $\mathbf{y}_n = (\mathbf{y}(t_j^n))$ be a vector of n^* observations. We consider the situation that the high-frequency data are observed at random times t_i^n and for the simplicity we assume some conditions on random sampling.

Assumption 2.1 : There exists a positive constant c such that

$$t_{n^*}^n \xrightarrow{p} 1 \ , \ \frac{n^*}{n} \xrightarrow{p} c \tag{2.1}$$

for $0 = t_0^n < t_1^n < \cdots < t_{n^*}^n \leq t_n^n = 1$ and

$$\mathcal{E}\left[|t_i^n - t_{i-1}^n|\right] = O(n^{-1}) \tag{2.2}$$

as $n \to \infty$.

For the ease of our analysis, we take $c = 1$ and we make a further assumption on the independence of $X(t)$ and t_i^n. Let $\tau_i^n = t_i^n - t_{i-1}^n$ be the random duration for price change intervals. The above conditions imply that n^{-1} corresponds to the average length of duration intervals in $[0, 1]$ when n is relatively large.

A typical example is the Poisson Process Sampling with the intensity function $\lambda_n = nc$. In this case the sequence of random variables τ_i^n are exponentially distributed with $\mathcal{E}(\tau_i^n) = 1/n$.

Assumption 2.2 : The underlying stochastic process $X(t)$ $(0 \leq t \leq 1)$ is independent of the sequence t_i^n $(i \geq 1)$.

We assume that the underlying continuous process $X(t)$ $(0 \leq t \leq 1)$ is not necessarily the same as the observed (log-)price at $t_i^n (i = 1, \cdots)$ and

$$X(t) = X(0) + \int_0^t \sigma_x(s) dB(s) \ \ (0 \leq t \leq 1), \tag{2.3}$$

where $B(s)$ is the standard Brownian motion, $\sigma_x(s)$ is the instantaneous volatility function adapted to the $\sigma-$field $\mathcal{F}(\mathbf{x}_r, \mathbf{B}(r), r \leq s)$. The main statistical objective is to estimate the integrated volatility

$$\sigma_x^2 = \int_0^1 \sigma_x^2(s) ds \tag{2.4}$$

of the underlying continuous process $X(t)$ $(0 \leq t \leq 1)$ from the set of discretely observed prices $y(t_i^n)$ with the condition that $\sigma_x(s)$ is a progressively measurable function and $\sup_{0 \leq s \leq 1} \sigma_x^2(s) < \infty$ $(a.s.)$.
Then we consider the situation when the observed (log-)price $y(t_i^n)$ is a sequence of discrete stochastic process generated by

$$y(t_i^n) = h\left(X(t_i^n), y(t_{i-1}^n), u(t_i^n)\right) \ , \tag{2.5}$$

where $h(\cdot)$ is a measurable function, the (unobservable) continuous martingale process $X(t)$ $(0 \leq t \leq 1)$ is defined by (2.3) and $u(t_i^n)$ is the micro-market noise process. Although in this paper we assume that $u(t_i^n)$ are a sequence of independently and identically distributed random variables with $\mathcal{E}(u(t_i^n)) = 0$ and $\mathcal{E}(u(t_i^n)^2) = \sigma_u^2$, it is straightforward to extend our analysis to the cases when we have weakly dependent random sequences.

There are special cases of (2.3) and (2.5), which reflect the important aspects on modeling financial markets and the high frequency financial data for market applications. The simple (high-frequency) financial model with micro market noise can be represented by

$$y(t_i^n) = X(t_i^n) + u(t_i^n) \ , \tag{2.6}$$

where the underlying process $X(t)$ is given by (2.3).

The most important statistical aspect of (2.6) is the fact that it is an additive (signal-plus-noise) measurement error model. However, there are economic reasons why the standard situation as (2.6) is not necessarily enough for applications. The high-frequency financial models for micro-market price adjustments and the round-off-errors models for financial prices are not necessarily represented as the form of

(2.6), but they can be represented as special cases of (2.3) and (2.5). Sato and Kunitomo (2012) have discussed several important examples of (2.5).

2.2 The SIML Method

We summarize the derivation of the separating information maximum likelihood (SIML) estimation, which was originally proposed by Kunitomo and Sato (2008, 2011) with the equidistance observations. Let $y(t_i^n)$ be the $i-$th observation of the (log-) price at t_i^n $(0 = t_0^n < t_1^n < \cdots < t_n^n = 1)$, t_i^n are the sequence of observed times (n is the non-random number of observations) and we set $\mathbf{y}_n = (y(t_i^n))$ be an $n \times 1$ vector of observations. The underlying (hidden) continuous process $X(t)$ $(0 \leq t \leq 1)$, which is not necessarily the same as the observed (log-)price at t_i^n $(i = 1, \cdots, n)$ and $u(t_i^n)$ is the micro-market noise at t_i^n.

We consider the basic situation of (2.3) and (2.5) with fixed observation intervals when we have $y(t_i^n) = X(t_i^n) + u(t_i^n)$ and $h_n = t_i^n - t_{i-1}^n = 1/n$, where $X(t)$ $(0 \leq t \leq 1)$ and $u(t_i^n)$ $(i = 1, \cdots, n)$ are independent of $\sigma_x^2(s) = \sigma_x^2$ (constant). Under the assumption that $u(t_i^n)$ are independently, identically and normally distributed as $N(0, \sigma_u^2)$, then given the initial condition y_0 and n we have

$$\mathbf{y}_n \sim N_n\left(y(0)\mathbf{1}_n, \sigma_v^2 \mathbf{I}_n + h_n \sigma_x^2 \mathbf{C}_n \mathbf{C}_n^{'}\right) \ , \tag{2.7}$$

where an $n \times n$ matrix

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}^{-1} \tag{2.8}$$

and a $1 \times n$ vector $\mathbf{1}_n^{'} = (1, \cdots, 1)$.

When the observations are randomly sampled, the durations $\tau_i^n = t_i^n - t_{i-1}^n$ are a sequence of random variables, we take n^* instead of n in (2.7) and (2.8). By transforming $\mathbf{y}_{n^*}$ to $\mathbf{z}_{n^*}$ $(= (z_k))$ by

$$\mathbf{z}_{n^*} = \sqrt{n^*}\mathbf{P}_{n^*}\mathbf{C}_{n^*}^{-1}\left(\mathbf{y}_{n^*} - \bar{\mathbf{y}}_0\right) \tag{2.9}$$

where $\bar{\mathbf{y}}_0 = y(0)\mathbf{1}_{n^*}$, $\mathbf{P}_{n^*} = (p_{jk})$ and for $j, k = 1.\cdots, n^*$,

$$p_{jk} = \sqrt{\frac{2}{n^* + \frac{1}{2}}} \cos\left[\frac{2\pi}{2n^* + 1}(j - \frac{1}{2})(k - \frac{1}{2})\right] . \tag{2.10}$$

Then the transformed variables z_k $(k = 1, \cdots, n^*)$ given n^* are mutually independent and $z_k \sim N(0, \sigma_x^2 + a_{k,n^*}\sigma_v^2)$, where

$$a_{k,n^*} = 4n^* \sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n^*+1}\right)\right] \;(k = 1, \cdots, n^*) . \tag{2.11}$$

Because the ML estimator of unknown parameters is a rather complicated function of all observations and each a_{kn^*} terms depend on k as well as n^*, one way to have a simple solution is to approximate the likelihood function under the Gaussian distributions.

Let m and l be positive integers, which are dependent on n^* and we write m_{n^*} and l_{n^*}. Then we define the SIML estimator of $\hat{\sigma}_x^2$ by

$$\hat{\sigma}_x^2 = \frac{1}{m_{n^*}} \sum_{k=1}^{m_{n^*}} z_k^2 \tag{2.12}$$

and the SIML estimator of $\hat{\sigma}_v^2$ by

$$\hat{\sigma}_v^2 = \frac{1}{l_n} \sum_{k=n^*+1-l_{n^*}}^{n^*} a_{k,n^*}^{-1} z_k^2 . \tag{2.13}$$

The numbers of terms m_{n^*} and l_{n^*} are dependent on n^* such that $m_{n^*}, l_{n^*} \to \infty$ as $n \to \infty$. We impose the order requirements that $m_{n^*} = O_p(n^\alpha)$ $(0 < \alpha < \frac{1}{2})$ and $l_{n^*} = O(n^\beta)$ $(0 < \beta < 1)$ for σ_x^2 and σ_u^2, respectively.

3. Basic Simulation

We have investigated the properties of the SIML estimator for the integrated volatility based on a set of simulations and the number of replications is 1000. We have taken the sample size as $n \sim 1,800$ and $n \sim 18,000$, and we have chosen $\alpha = 0.4$ and $\beta = 0.8$. The details of the simulation procedure are similar to the corresponding ones reported by Kunitomo and Sato (2008, 2011). In Tables *Raw* means the case when we apply the SIML estimation as if the high frequency data are equally observed. An alternative way of the SIML estimation is to fix the length of intervals and use the last observation in each interval as if it was observed at a fixed time. This method is called the grid estimation method. We have used 1 second, 10 second, 30 second, 60 second and 300 seconds as grids of duration times.

In our basic simulations we consider two cases when the observations are the sum of signal and micro-market noise, which correspond to the models with (2.3) and (2.6). We use the Poisson Sampling as the basic stochastic sampling with the parameter $\lambda = n$. In the first example the signal is the Brownian motion with the instantaneous volatility function

$$\sigma_x^2(s) = \sigma(0)^2 \left[a_0 + a_1 s + a_2 s^2 \right], \tag{3.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we need restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. In this case the integrated volatility is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{3.2}$$

In this example we have taken several intra-day instantaneous volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) volatility movements and the U-shaped volatility movements.

In the second example we take that the instantaneous volatility follows the stochastic volatility model

$$dh(t) = \gamma \, h(t) dt + c \, dW(t) \tag{3.3}$$

and

$$\sigma_x(t)^2 = \sigma^2 e^{h(t)} ,$$

where $W(t)$ is a Brownian Motion which is independent of $B(t)$ and $v(t)$ $(0 \le t \le 1)$ and σ^2 is a constant. For the stationary processes we have set the condition $-2 < \gamma < 0$ in our experiments. Then we set $\sigma_x^2 = \frac{1}{n^*} \sum_{i=1}^{n^*} \sigma_x(t_i^n)^2$.

We summarize our estimation results of the first example in Tables 3.1 (Case 1), 3.2 (Case 2) and the second example in Table 3.3 (Case 3), respectively. In each table we have calculated the value of the historical volatility as HI for comparison. When there are micro-market noise components with the martingale signal part, the value of HI often differs from the true integrated volatility of the signal part significantly. However, we have found that it is possible to estimate the integrated volatility and the noise variance when we have the signal-noise ratio as $10^{-2} \sim 10^{-6}$ at least by the SIML estimation method. Although we have omitted the details of the second example, the estimation results are similar in the stochastic volatility model as Table 3.3 has suggested.

While the noise variance is relatively small, the HI estimate gives reasonable value, its bias becomes large when the noise variance is relatively large in all cases. The bias depends on the relative magnitude of the signal variance and the noise variance as well as the sampling scheme. This finding corresponds to the common observation on the use of the HI method. The SIML estimate, however, gives reasonable value in most cases and it does not depend on the relative magnitude of the signal variance and the noise variance as well as the sampling schemes. As the sampling scheme is sparse, the variances of the HI estimator and the SIML estimator become large.

By our basic simulations we can conclude that we can estimate both the integrated volatility of the hidden martingale part and the market noise part reasonably in all cases we have examined by the SIML estimation even if we have the Poisson Random Sampling scheme.

Table 3.1 : Estimation of volatility :

Case 1 ($a_0 = 1, a_1 = a_2 = 0$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.06E-04	2.22E-04	2.07E-04	2.05E-04	2.04E-04	2.05E-04
		6.54E-05	4.52E-05	6.44E-05	8.34E-05	9.48E-05	1.31E-04
$\hat{\sigma}_v^2$	2.00E-06	2.03E-06	1.04E-07	9.82E-07	1.92E-06	2.18E-06	3.00E-06
		1.43E-07	6.00E-09	8.43E-08	2.19E-07	3.19E-07	8.30E-07
HI		7.39E-03	7.04E-03	4.74E-03	2.48E-03	1.40E-03	4.40E-04
		3.45E-04	3.35E-04	2.61E-04	1.79E-04	1.34E-04	8.46E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.02E-04	2.01E-04	2.00E-04	1.99E-04	1.99E-04
		6.39E-05	4.06E-05	6.27E-05	8.12E-05	9.25E-05	1.27E-04
$\hat{\sigma}_v^2$	2.00E-07	2.29E-07	1.30E-08	1.24E-07	2.72E-07	3.82E-07	1.20E-06
		1.61E-08	6.80E-10	1.01E-08	3.04E-08	5.60E-08	3.30E-07
HI		9.19E-04	8.84E-04	6.54E-04	4.28E-04	3.20E-04	2.24E-04
		3.98E-05	3.89E-05	3.27E-05	2.75E-05	2.72E-05	4.02E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.00E-04	1.99E-04	1.98E-04	1.99E-04
		6.38E-05	4.01E-05	6.26E-05	8.10E-05	9.23E-05	1.26E-04
$\hat{\sigma}_v^2$	2.00E-08	4.90E-08	3.84E-09	3.86E-08	1.08E-07	2.02E-07	1.02E-06
		3.68E-09	1.79E-10	2.88E-09	1.17E-08	2.96E-08	2.82E-07
HI		2.72E-04	2.68E-04	2.45E-04	2.23E-04	2.12E-04	2.02E-04
		1.15E-05	1.15E-05	1.17E-05	1.39E-05	1.76E-05	3.64E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.00E-04	1.99E-04	1.98E-04	1.99E-04
		6.38E-05	4.00E-05	6.26E-05	8.10E-05	9.23E-05	1.26E-04
$\hat{\sigma}_v^2$	2.00E-09	3.10E-08	2.92E-09	3.00E-08	9.11E-08	1.84E-07	1.00E-06
		2.52E-09	1.42E-10	2.23E-09	9.95E-09	2.69E-08	2.78E-07
HI		2.07E-04	2.07E-04	2.05E-04	2.02E-04	2.01E-04	2.00E-04
		9.57E-06	9.57E-06	1.01E-05	1.27E-05	1.67E-05	3.61E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.06E-04	2.07E-04	2.07E-04	2.07E-04	2.07E-04	2.06E-04
		4.12E-05	4.14E-05	6.55E-05	8.45E-05	9.66E-05	1.31E-04
$\hat{\sigma}_v^2$	2.00E-06	2.00E-06	9.40E-07	2.03E-06	2.09E-06	2.18E-06	3.00E-06
		5.54E-08	3.08E-08	1.41E-07	2.33E-07	3.19E-07	8.38E-07
HI		7.22E-02	4.57E-02	7.41E-03	2.60E-03	1.40E-03	4.40E-04
		1.06E-03	7.80E-04	2.96E-04	1.83E-04	1.35E-04	8.63E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.02E-04	2.01E-04	2.01E-04
		4.00E-05	4.00E-05	6.33E-05	8.15E-05	9.36E-05	1.27E-04
$\hat{\sigma}_v^2$	2.00E-07	2.03E-07	9.65E-08	2.29E-07	2.89E-07	3.82E-07	1.20E-06
		5.60E-09	3.15E-09	1.60E-08	3.20E-08	5.60E-08	3.33E-07
HI		7.40E-03	4.75E-03	9.20E-04	4.40E-04	3.20E-04	2.24E-04
		1.07E-04	7.99E-05	3.45E-05	2.76E-05	2.79E-05	4.10E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.01E-04	2.01E-04	2.00E-04
		3.99E-05	3.99E-05	6.31E-05	8.12E-05	9.32E-05	1.26E-04
$\hat{\sigma}_v^2$	2.00E-08	2.28E-08	1.22E-08	4.90E-08	1.09E-07	2.02E-07	1.01E-06
		6.28E-10	3.82E-10	3.44E-09	1.21E-08	2.95E-08	2.83E-07
HI		9.20E-04	6.55E-04	2.72E-04	2.24E-04	2.12E-04	2.02E-04
		1.23E-05	1.01E-05	9.28E-06	1.32E-05	1.78E-05	3.70E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.00E-04
		3.99E-05	3.99E-05	6.31E-05	8.11E-05	9.32E-05	1.26E-04
$\hat{\sigma}_v^2$	2.00E-09	4.82E-09	3.76E-09	3.10E-08	9.13E-08	1.84E-07	9.97E-07
		1.37E-10	1.08E-10	2.17E-09	1.02E-08	2.68E-08	2.77E-07
HI		2.72E-04	2.45E-04	2.07E-04	2.02E-04	2.01E-04	2.00E-04
		3.66E-06	3.77E-06	7.05E-06	1.18E-05	1.66E-05	3.65E-05

Table 3.2 : Estimation of volatility :

Case 2 ($a_0 = 1, a_1 = -1, a_2 = 1$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	1.67E-04	1.72E-04	1.89E-04	1.74E-04	1.71E-04	1.71E-04	1.72E-04
		5.52E-05	3.88E-05	5.44E-05	7.04E-05	8.01E-05	1.10E-04
$\hat{\sigma}_v^2$	2.00E-06	2.02E-06	1.04E-07	9.78E-07	1.90E-06	2.14E-06	2.83E-06
		1.43E-07	5.99E-09	8.40E-08	2.18E-07	3.15E-07	7.85E-07
HI		7.36E-03	7.01E-03	4.71E-03	2.44E-03	1.36E-03	4.06E-04
		3.44E-04	3.34E-04	2.60E-04	1.77E-04	1.32E-04	7.92E-05
$\hat{\sigma}_x^2$	1.67E-04	1.67E-04	1.69E-04	1.67E-04	1.66E-04	1.66E-04	1.66E-04
		5.35E-05	3.41E-05	5.24E-05	6.80E-05	7.76E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-07	2.24E-07	1.25E-08	1.20E-07	2.57E-07	3.51E-07	1.03E-06
		1.58E-08	6.64E-10	9.79E-09	2.88E-08	5.16E-08	2.85E-07
HI		8.85E-04	8.51E-04	6.21E-04	3.94E-04	2.86E-04	1.91E-04
		3.87E-05	3.79E-05	3.14E-05	2.56E-05	2.46E-05	3.44E-05
$\hat{\sigma}_x^2$	1.67E-04	1.67E-04	1.67E-04	1.67E-04	1.66E-04	1.65E-04	1.66E-04
		5.34E-05	3.36E-05	5.23E-05	6.78E-05	7.74E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-08	4.41E-08	3.37E-09	3.37E-08	9.26E-08	1.72E-07	8.52E-07
		3.28E-09	1.57E-10	2.53E-09	1.02E-08	2.52E-08	2.37E-07
HI		2.39E-04	2.35E-04	2.12E-04	1.89E-04	1.79E-04	1.69E-04
		1.00E-05	9.97E-06	1.01E-05	1.18E-05	1.49E-05	3.06E-05
$\hat{\sigma}_x^2$	1.67E-04	1.67E-04	1.66E-04	1.67E-04	1.66E-04	1.65E-04	1.66E-04
		5.34E-05	3.35E-05	5.23E-05	6.78E-05	7.74E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-09	2.62E-08	2.45E-09	2.51E-08	7.62E-08	1.54E-07	8.34E-07
		2.12E-09	1.19E-10	1.87E-09	8.36E-09	2.26E-08	2.33E-07
HI		1.74E-04	1.73E-04	1.71E-04	1.69E-04	1.68E-04	1.67E-04
		8.03E-06	8.03E-06	8.50E-06	1.07E-05	1.40E-05	3.02E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	1.67E-04	1.72E-04	1.73E-04	1.74E-04	1.74E-04	1.73E-04	1.73E-04
		3.50E-05	3.52E-05	5.56E-05	7.15E-05	8.16E-05	1.11E-04
$\hat{\sigma}_v^2$	2.00E-06	2.00E-06	9.39E-07	2.03E-06	2.08E-06	2.15E-06	2.84E-06
		5.54E-08	3.08E-08	1.41E-07	2.32E-07	3.14E-07	7.92E-07
HI		7.22E-02	4.57E-02	7.37E-03	2.57E-03	1.36E-03	4.07E-04
		1.06E-03	7.80E-04	2.95E-04	1.81E-04	1.33E-04	8.06E-05
$\hat{\sigma}_x^2$	1.67E-04	1.68E-04	1.68E-04	1.68E-04	1.68E-04	1.68E-04	1.67E-04
		3.36E-05	3.35E-05	5.32E-05	6.83E-05	7.83E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-07	2.02E-07	9.60E-08	2.24E-07	2.75E-07	3.52E-07	1.03E-06
		5.59E-09	3.14E-09	1.56E-08	3.04E-08	5.17E-08	2.87E-07
HI		7.37E-03	4.72E-03	8.87E-04	4.07E-04	2.86E-04	1.91E-04
		1.07E-04	7.95E-05	3.36E-05	2.58E-05	2.52E-05	3.51E-05
$\hat{\sigma}_x^2$	1.67E-04	1.67E-04	1.67E-04	1.68E-04	1.68E-04	1.68E-04	1.67E-04
		3.35E-05	3.34E-05	5.30E-05	6.79E-05	7.79E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-08	2.24E-08	1.17E-08	4.42E-08	9.44E-08	1.71E-07	8.47E-07
		6.16E-10	3.69E-10	3.10E-09	1.05E-08	2.53E-08	2.37E-07
HI		8.87E-04	6.22E-04	2.39E-04	1.91E-04	1.78E-04	1.69E-04
		1.20E-05	9.71E-06	8.18E-06	1.13E-05	1.51E-05	3.10E-05
$\hat{\sigma}_x^2$	1.67E-04	1.67E-04	1.67E-04	1.68E-04	1.68E-04	1.67E-04	1.67E-04
		3.35E-05	3.34E-05	5.29E-05	6.78E-05	7.79E-05	1.06E-04
$\hat{\sigma}_v^2$	2.00E-09	4.35E-09	3.29E-09	2.62E-08	7.64E-08	1.53E-07	8.30E-07
		1.23E-10	9.53E-11	1.84E-09	8.52E-09	2.25E-08	2.31E-07
HI		2.39E-04	2.12E-04	1.74E-04	1.69E-04	1.68E-04	1.67E-04
		3.18E-06	3.25E-06	5.95E-06	9.93E-06	1.39E-05	3.06E-05

Table 3.3 : Estimation of volatility :

Case 3 (Stochastic Volatility)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.08E-04	2.23E-04	2.09E-04	2.07E-04	2.08E-04	2.11E-04
		6.94E-05	4.65E-05	6.78E-05	8.91E-05	1.07E-04	1.43E-04
$\hat{\sigma}_v^2$	2.00E-06	2.02E-06	1.04E-07	9.81E-07	1.93E-06	2.16E-06	3.03E-06
		1.42E-07	5.91E-09	8.50E-08	2.19E-07	3.19E-07	8.44E-07
HI		7.39E-03	7.04E-03	4.74E-03	2.48E-03	1.39E-03	4.45E-04
		3.49E-04	3.37E-04	2.52E-04	1.77E-04	1.33E-04	8.72E-05
$\hat{\sigma}_x^2$	2.00E-04	2.03E-04	2.03E-04	2.02E-04	2.02E-04	2.02E-04	2.03E-04
		6.70E-05	4.21E-05	6.55E-05	8.69E-05	1.05E-04	1.38E-04
$\hat{\sigma}_v^2$	2.00E-07	2.29E-07	1.30E-08	1.24E-07	2.73E-07	3.80E-07	1.21E-06
		1.62E-08	6.82E-10	9.90E-09	3.14E-08	5.40E-08	3.43E-07
HI		9.19E-04	8.85E-04	6.54E-04	4.28E-04	3.19E-04	2.26E-04
		4.07E-05	3.97E-05	3.19E-05	2.74E-05	2.66E-05	4.20E-05
$\hat{\sigma}_x^2$	2.00E-04	2.02E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04	2.03E-04
		6.67E-05	4.18E-05	6.53E-05	8.67E-05	1.04E-04	1.37E-04
$\hat{\sigma}_v^2$	2.00E-08	4.90E-08	3.84E-09	3.86E-08	1.08E-07	2.02E-07	1.02E-06
		3.66E-09	1.83E-10	2.80E-09	1.26E-08	2.87E-08	2.90E-07
HI		2.72E-04	2.69E-04	2.45E-04	2.23E-04	2.12E-04	2.04E-04
		1.18E-05	1.18E-05	1.19E-05	1.40E-05	1.73E-05	3.77E-05
$\hat{\sigma}_x^2$	2.00E-04	2.02E-04	2.00E-04	2.02E-04	2.01E-04	2.01E-04	2.02E-04
		6.66E-05	4.17E-05	6.52E-05	8.67E-05	1.04E-04	1.37E-04
$\hat{\sigma}_v^2$	2.00E-09	3.10E-08	2.93E-09	3.01E-08	9.11E-08	1.84E-07	1.00E-06
		2.44E-09	1.42E-10	2.20E-09	1.05E-08	2.66E-08	2.83E-07
HI		2.07E-04	2.07E-04	2.05E-04	2.02E-04	2.01E-04	2.02E-04
		9.61E-06	9.63E-06	1.03E-05	1.28E-05	1.65E-05	3.72E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.06E-04	2.07E-04	2.08E-04	2.07E-04	2.08E-04	2.05E-04
		4.09E-05	4.10E-05	6.36E-05	8.17E-05	9.63E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-06	2.00E-06	9.40E-07	2.02E-06	2.10E-06	2.19E-06	2.95E-06
		5.53E-08	3.04E-08	1.41E-07	2.27E-07	3.31E-07	8.17E-07
HI		7.22E-02	4.57E-02	7.40E-03	2.60E-03	1.40E-03	4.36E-04
		1.08E-03	7.96E-04	2.93E-04	1.79E-04	1.35E-04	8.61E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.02E-04	2.03E-04	2.01E-04	2.02E-04	2.00E-04
		3.98E-05	3.98E-05	6.08E-05	7.92E-05	9.40E-05	1.25E-04
$\hat{\sigma}_v^2$	2.00E-07	2.03E-07	9.65E-08	2.28E-07	2.90E-07	3.82E-07	1.18E-06
		5.62E-09	3.12E-09	1.59E-08	3.21E-08	5.89E-08	3.34E-07
HI		7.40E-03	4.75E-03	9.19E-04	4.40E-04	3.19E-04	2.23E-04
		1.10E-04	8.17E-05	3.42E-05	2.70E-05	2.68E-05	4.03E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04	1.99E-04
		3.98E-05	3.98E-05	6.05E-05	7.88E-05	9.38E-05	1.25E-04
$\hat{\sigma}_v^2$	2.00E-08	2.28E-08	1.22E-08	4.88E-08	1.09E-07	2.01E-07	1.01E-06
		6.41E-10	3.85E-10	3.42E-09	1.22E-08	2.93E-08	2.83E-07
HI		9.20E-04	6.55E-04	2.71E-04	2.23E-04	2.11E-04	2.02E-04
		1.28E-05	1.05E-05	9.15E-06	1.28E-05	1.68E-05	3.60E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04	1.99E-04
		3.98E-05	3.98E-05	6.04E-05	7.88E-05	9.37E-05	1.25E-04
$\hat{\sigma}_v^2$	2.00E-09	4.83E-09	3.76E-09	3.09E-08	9.06E-08	1.83E-07	9.90E-07
		1.39E-10	1.11E-10	2.16E-09	9.97E-09	2.62E-08	2.77E-07
HI		2.72E-04	2.45E-04	2.07E-04	2.02E-04	2.00E-04	2.00E-04
		3.73E-06	3.86E-06	6.99E-06	1.15E-05	1.60E-05	3.56E-05

4. Asymptotic Properties of the SIML Estimation

It is important to investigate the asymptotic properties of the SIML estimator when the instantaneous volatility function $\sigma_x^2(s)$ is not constant over time. When the integrated volatility is a positive (deterministic) constant a.s. (i.e. σ_x^2 is not stochastic) while the instantaneous volatility function is time varying, we have the consistency and the asymptotic normality of the SIML estimator as $n \to \infty$. For the deterministic time varying volatility case with the random sampling, the asymptotic properties of the SIML estimator can be summarized as the next proposition and the proof shall be given briefly in Appendix A, which is similar to the one by Kunitomo and Sato (2008, 2011).

Theorem 4.1 : We assume that $X(t_i^n)$ and $u(t_i^n)$ $(i = 1, \cdots, n^*)$ in (2.3) and (2.6) are independent with Assumptions 2.1 and 2.2, $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds$ is a positive constant (or deterministic), and $\mathcal{E}[u(t_i^n)^4] < \infty$. Define the SIML estimator $\hat{\sigma}_x^2$ of σ_x^2 by (2.12).

(i) For $m_n = n^\alpha$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$ (and $m_{n^*}/m_n \xrightarrow{p} 1$),

$$\hat{\sigma}_x^2 - \sigma_x^2 \xrightarrow{p} 0 . \tag{4.1}$$

(ii) For $m_n = n^\alpha$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] \xrightarrow{d} N\left[0, V\right] , \tag{4.2}$$

where

$$V = 2\int_0^1 \left[\sigma_x^2(s)\right]^2 ds . \tag{4.3}$$

When the instantaneous covariance $\sigma_x^2(s)$ $(= \sigma_x^2)$ is constant, then

$$V = 2\sigma_x^4 . \tag{4.4}$$

When σ_x^2 is a random variable, we need the concept of *stable convergence in law* because the limiting distribution of the SIML estimator is the mixed-Gaussian distribution. In order to discuss the stable convergence in law we extend the probability space $(\Omega, \mathcal{F}, P)$ to the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as explained by Chapter VIII of Jacod and Syriyaev (2003) or Jacod (2007).

We say that a sequence of random variables Z_n with an index n stably converges in law if

$$\mathbf{E}[Yf(Z_n)] \longrightarrow \tilde{\mathbf{E}}[Yf(Z)] \tag{4.5}$$

for all bounded continuous functions $f(\cdot)$ and all bounded random variables Y, and $\tilde{\mathbf{E}}[\cdot]$ is the expectation operator with respect to the extended probability space. We write this convergence as

$$Z_n \stackrel{\mathcal{L}-s}{\longrightarrow} Z \ . \tag{4.6}$$

(See Jacod and Shriyaev (2003) and Jacod (2007) for the details.)

For the stochastic volatility case in the continuous time, we further assume that the instantaneous volatility $\sigma_x(t)$ is a strong solution of the stochastic differential equation (SDE)

$$\begin{aligned} \sigma_x(t) &= \sigma_x(0) + \int_0^t \mu_\sigma(s, \sigma_x(s))ds + + \int_0^t \gamma_\sigma(s, \sigma_x(s))dB(s) \\ &\quad + \int_0^t \gamma_\sigma^*(s, \sigma_x(s))dB^*(s) \ , \end{aligned} \tag{4.7}$$

where the coefficients $\mu_\sigma(s), \gamma_\sigma(s)$ and $\gamma_\sigma^*(s)$ are in the class of $\mathcal{A}^1$ (extensively measurable, continuous and bounded), and $B^*(s)$ is a Brownian motion which is orthogonal to $B(s)$. Here we set $\mathbf{B}(t) = (B(t), B(t)^*)'$ as the vector of Brownian motions.

Then there exists a strong solution such that $\sup_{0 \leq s \leq 1} \mathcal{E}[\sigma_x^4(s)] < \infty$. (There can be weaker conditions on the coefficients which give the existence of a strong solution and the moment conditions. See Chapter III of Ikeda and Watanabe (1989) for the notations and their details.)

The asymptotic properties of the SIML estimator in the stochastic volatility cases can be summarized as Theorem 4.2.

Theorem 4.2 : We assume that $X(t)$ and u_i $(i = 1, \cdots, n^*)$ in (2.3) and (2.6) are independent with Assumptions 2.1 and 2.2 and $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds > 0$ (positive) is finite (a.s.). We assume that $\mathcal{E}[u(t_i^n)^4] < \infty$. Define the SIML estimator $\hat{\sigma}_x^2$ of σ_x^2 by

(2.12).

(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$ (and $m_{n^*}/m_n \xrightarrow{p} 1$),

$$\hat{\sigma}_x^2 - \sigma_x^2 \xrightarrow{p} 0 \ . \tag{4.8}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$ we have the weak convergence

$$Z_{n^*} = \sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] \xrightarrow{w} Z^* \ , \tag{4.9}$$

where the characteristic function $g_n(t) = \mathcal{E}[\exp(itZ_{n^*})]$ of Z_{n^*} converges to the characteristic function of Z^*, which is written as

$$g(t) = \mathcal{E}[e^{-\frac{Vt^2}{2}}] \ , \tag{4.10}$$

where

$$V = 2\int_0^1 \left[\sigma_x^2(s)\right]^2 ds \ . \tag{4.11}$$

It would be possible to derive the corresponding results on the asymptotic distribution of the SIML estimator when we have the non-linear transformations of (2.5). Some results have been reported in Sato and Kunitomo (2011) for the case of fixed observation intervals under the additional conditions.

5. Further Simulations

By extending the basic simulation framework reported in Section 3, we have conducted a large number of simulations. As we have explained in Section 3, we have considered two situations when the volatility function ($\sigma_x^2(s)$) is given by $\sigma_x^2(s) = \sigma(0)^2\left[a_0 + a_1 s + a_2 s^2\right]$ (a_i ($i = 0,1,2$) are constants) and $\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2\left[a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right]$.

In this section we have adopted the similar situations investigated by Sato and Kunitomo (2011) and among many Monte-Carlo simulations we summarize main results as Tables in Appendix B.

As the fourth case (Case 4) we use the EACD(1,1) model, which was proposed by Engle and Russel (1997). Financial econometricians have been interested in the autoregressive conditional duration (ACD) models because the assumption of Poisson sampling leads to the sequence of i.i.d. random variables for durations while we sometimes observe the dependent structure on the duration process. Although there can be many types of duration dependence models, we shall use the EACD(1,1) model as a representative one. Let $\tau_i^n = t_i^n - t_{i-1}^n$ and $\tau_i^n = \psi_i^n \epsilon_i^n$, where

$$\psi_i^n = \omega + a\tau_i^n + b\psi_{i-1}^n \tag{5.1}$$

and ϵ_i^n are a sequence of i.i.d. exponential random variables with $a > 0, b > 0$ and $\omega > 0$. In order to make a fair comparison with the Poisson sampling and other possible cases, we have set $\mathcal{E}[\tau_i^n] = 1/n$.

In our simulations we use several non-linear transformation models in the form of (2.5). Among them Case 5 is the linear price adjustment model such that $y(t_i^n) = P(t_i^n)$ and

$$P(t_i^n) - P(t_{i-1}^n) = g\left[X(t_i^n) - P(t_{i-1}^n)\right] + u(t_i^n) \ , \tag{5.2}$$

where $u(t_i^n)$ is an i.i.d. sequence of mnicro-market noises with $\mathcal{E}[u(t_i^n)] = 0, \mathcal{E}[u(t_i^n)^2] = \sigma_u^2$ and g is a constant adjustment coefficient. When $0 < g < 2$, this model in the general case corresponds to the linear adjustment model with the micro-market noise.

Case 6 is the micro-market models with the round-off errors. We set

$$P(t_i^n) - P(t_{i-1}^n) = g_\eta\left[X(t_i^n) - P(t_{i-1}^n) + u(t_i^n)\right] \ , \tag{5.3}$$

where $u(t_i^n)$ is an i.i.d. sequence of micro-market noises with $\mathcal{E}[u(t_i^n)] = 0, \mathcal{E}[u(t_i^n)^2] = \sigma_u^2$ and

$$g_\eta(x) = \eta\left[\frac{x}{\eta}\right] \tag{5.4}$$

is the round-off part of x, $[x]$ is the largest integer being less than x and η is a (small) positive constant. This formulation corresponds to the micro-market model with the restriction of the minimum price change and η is the parameter of minimum price change. Hence Case 6 is the basic round-off model while Case 7 is its variant when $P(t_i^n) - P(t_{i-1}^n) = g_\eta[X(t_i^n) - P(t_{i-1}^n)] + u(t_i^n)$.

Case 8 is the nonlinear price adjustment model with micro-market noise when the nonlinear transformation is given by

$$g(x) = g_1 x \mathrm{I}(x \geq 0) + g_2 x \mathrm{I}(x < 0) \ , \tag{5.5}$$

where g_i $(i = 1, 2)$ are some constants and $\mathrm{I}(\cdot)$ is the indicator function. This has been called the SSAR (simultaneous switching autoregressive) model, which has been investigated by Sato and Kunitomo (1996) and Kunitomo and Sato (1999). Sato and Kunitomo (2011) have discussed the economic meaning of models in more details.

For Cases 5-8, the estimates obtained by historical-volatility (HI) are badly-biased, which have been known in the analysis of high frequency financia data. The estimates of HI are often badly-biased except the cases when the noise variance is quite small in comparison with the signal variance and we take relatively large grids of observations. The SIML method, on the other hand, gives reasonable estimate and the variance of the SIML estimator is within a reasonable range for practical purposes.

For Case 4, the effects of the duration dependence on the integrated volatility estimation are of the small size. This aspect is common for Cases 4, 5 and 8 and the SIML estimation of the integrated volatility is reliable for the linear and non-linear price adjustments models.

For Cases 6 and 7, the SIML estimator has some bias when the threshold parameter η is quite small. However, as η gets small, it becomes negligible in its effects. This corresponds to the fact that for Nikkei-225-futures the minimim price change was 10 yen while the average spot index price has been 10,000 yen, which corresponds to the level $\eta = 0.001$.

By examining these results of our simulations we can conclude that we can estimate the integrated volatility of the hidden martingale part reasonably by the SIML estimation method. It may be surprising to find that the SIML method gives reasonable estimates even when we have nonlinear transformations of the original unobservable security (intrinsic) values. We have conducted a number of further simulations, but the results are quite similar as we have reported in this section.

Table 5.1 : Estimation of integrated volatility :

Case 4 (Autoregressive Conditional Duration)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.03E-04	2.23E-04	2.06E-04	2.04E-04	2.03E-04	2.05E-04
		6.75E-05	4.50E-05	6.62E-05	8.41E-05	9.70E-05	1.33E-04
$\hat{\sigma}_v^2$	2.00E-06	2.03E-06	1.04E-07	9.75E-07	1.89E-06	2.17E-06	3.01E-06
		1.44E-07	8.05E-09	9.60E-08	2.18E-07	3.12E-07	8.29E-07
HI		7.41E-03	7.05E-03	4.70E-03	2.46E-03	1.39E-03	4.41E-04
		5.22E-04	4.85E-04	3.01E-04	1.78E-04	1.32E-04	8.47E-05
$\hat{\sigma}_x^2$	2.00E-04	1.99E-04	2.02E-04	1.99E-04	1.98E-04	1.98E-04	1.99E-04
		6.61E-05	4.06E-05	6.39E-05	8.18E-05	9.46E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-07	2.29E-07	1.30E-08	1.23E-07	2.69E-07	3.81E-07	1.20E-06
		1.66E-08	8.62E-10	1.11E-08	3.03E-08	5.52E-08	3.33E-07
HI		9.21E-04	8.85E-04	6.50E-04	4.26E-04	3.19E-04	2.24E-04
		5.58E-05	5.23E-05	3.62E-05	2.77E-05	2.75E-05	4.07E-05
$\hat{\sigma}_x^2$	2.00E-04	1.98E-04	2.00E-04	1.99E-04	1.98E-04	1.97E-04	1.99E-04
		6.60E-05	4.02E-05	6.37E-05	8.16E-05	9.44E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-08	4.90E-08	3.84E-09	3.84E-08	1.07E-07	2.02E-07	1.02E-06
		4.13E-09	1.87E-10	2.91E-09	1.19E-08	2.96E-08	2.85E-07
HI		2.72E-04	2.68E-04	2.45E-04	2.23E-04	2.12E-04	2.02E-04
		1.23E-05	1.21E-05	1.20E-05	1.42E-05	1.80E-05	3.71E-05
$\hat{\sigma}_x^2$	2.00E-04	1.98E-04	2.00E-04	1.99E-04	1.97E-04	1.97E-04	1.99E-04
		6.60E-05	4.01E-05	6.37E-05	8.16E-05	9.44E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-09	3.10E-08	2.92E-09	2.99E-08	9.09E-08	1.85E-07	1.01E-06
		3.07E-09	1.44E-10	2.23E-09	1.01E-08	2.71E-08	2.81E-07
HI		2.07E-04	2.07E-04	2.04E-04	2.02E-04	2.01E-04	2.00E-04
		9.74E-06	9.73E-06	1.03E-05	1.30E-05	1.71E-05	3.68E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.05E-04	2.06E-04	2.05E-04	2.04E-04	2.05E-04	2.07E-04
		4.14E-05	4.12E-05	6.49E-05	8.31E-05	9.73E-05	1.33E-04
$\hat{\sigma}_v^2$	2.00E-06	2.00E-06	9.31E-07	2.03E-06	2.10E-06	2.19E-06	3.00E-06
		5.56E-08	3.35E-08	1.41E-07	2.28E-07	3.18E-07	8.39E-07
HI		7.22E-02	4.52E-02	7.40E-03	2.61E-03	1.40E-03	4.40E-04
		1.67E-03	9.32E-04	2.95E-04	1.78E-04	1.34E-04	8.67E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.00E-04	1.99E-04	2.00E-04	2.01E-04
		4.02E-05	3.96E-05	6.28E-05	8.11E-05	9.49E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-07	2.03E-07	9.57E-08	2.29E-07	2.90E-07	3.83E-07	1.20E-06
		5.64E-09	3.41E-09	1.59E-08	3.16E-08	5.58E-08	3.38E-07
HI		7.40E-03	4.70E-03	9.20E-04	4.41E-04	3.21E-04	2.24E-04
		1.68E-04	9.49E-05	3.45E-05	2.70E-05	2.69E-05	4.10E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.00E-04	1.99E-04	2.00E-04	2.00E-04
		4.02E-05	3.95E-05	6.26E-05	8.09E-05	9.47E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-08	2.28E-08	1.21E-08	4.90E-08	1.09E-07	2.02E-07	1.02E-06
		6.44E-10	4.01E-10	3.38E-09	1.20E-08	2.94E-08	2.86E-07
HI		9.20E-04	6.50E-04	2.72E-04	2.24E-04	2.12E-04	2.03E-04
		1.80E-05	1.15E-05	9.12E-06	1.28E-05	1.73E-05	3.68E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.00E-04	1.99E-04	1.99E-04	2.00E-04
		4.02E-05	3.95E-05	6.26E-05	8.10E-05	9.47E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-09	4.82E-09	3.75E-09	3.10E-08	9.13E-08	1.84E-07	1.00E-06
		1.51E-10	1.09E-10	2.16E-09	1.01E-08	2.67E-08	2.80E-07
HI		2.72E-04	2.45E-04	2.07E-04	2.03E-04	2.01E-04	2.01E-04
		3.95E-06	3.85E-06	6.87E-06	1.15E-05	1.64E-05	3.64E-05

Table 5.2 : Estimation of integrated volatility :

Case 5 ($g = 0.2$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.35E-04	4.22E-04	2.40E-04	2.16E-04	2.10E-04	2.08E-04
		7.42E-05	9.28E-05	7.44E-05	8.85E-05	9.76E-05	1.35E-04
$\hat{\sigma}_v^2$	2.00E-06	6.42E-07	5.67E-08	5.78E-07	1.73E-06	3.21E-06	6.31E-06
		4.58E-08	2.79E-09	4.75E-08	2.03E-07	4.79E-07	1.75E-06
HI		4.01E-03	3.98E-03	3.66E-03	3.07E-03	2.42E-03	8.33E-04
		1.66E-04	1.69E-04	1.89E-04	2.13E-04	2.26E-04	1.71E-04
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.12E-04	2.01E-04	1.99E-04	1.98E-04	1.98E-04
		6.32E-05	4.27E-05	6.20E-05	8.20E-05	9.29E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-07	6.46E-08	5.95E-09	6.08E-08	1.84E-07	3.58E-07	1.32E-06
		4.60E-09	2.86E-10	4.92E-09	2.14E-08	5.35E-08	3.61E-07
HI		4.21E-04	4.19E-04	4.00E-04	3.67E-04	3.29E-04	2.37E-04
		1.69E-05	1.74E-05	2.02E-05	2.42E-05	2.82E-05	4.24E-05
$\hat{\sigma}_x^2$	2.00E-04	1.98E-04	1.91E-04	1.97E-04	1.98E-04	1.97E-04	1.97E-04
		6.24E-05	3.85E-05	6.12E-05	8.14E-05	9.26E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-08	6.79E-09	8.77E-10	9.07E-09	2.98E-08	7.28E-08	8.15E-07
		4.84E-10	4.32E-11	7.45E-10	3.46E-09	1.09E-08	2.29E-07
HI		6.21E-05	6.35E-05	7.49E-05	9.62E-05	1.20E-04	1.77E-04
		2.56E-06	2.76E-06	4.25E-06	7.16E-06	1.10E-05	3.19E-05
$\hat{\sigma}_x^2$	2.00E-04	1.97E-04	1.89E-04	1.97E-04	1.98E-04	1.97E-04	1.97E-04
		6.24E-05	3.81E-05	6.12E-05	8.14E-05	9.26E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-09	1.01E-09	3.70E-10	3.89E-09	1.44E-08	4.43E-08	7.64E-07
		7.40E-11	2.84E-11	4.27E-10	1.87E-09	6.88E-09	2.18E-07
HI		2.62E-05	2.79E-05	4.24E-05	6.91E-05	9.85E-05	1.71E-04
		1.74E-06	1.92E-06	3.32E-06	6.13E-06	9.97E-06	3.11E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.27E-04	2.30E-04	2.10E-04	2.11E-04	2.10E-04	2.13E-04
		4.57E-05	4.60E-05	6.63E-05	8.60E-05	9.84E-05	1.35E-04
$\hat{\sigma}_v^2$	2.00E-06	6.28E-07	5.63E-07	4.32E-06	5.58E-06	5.72E-06	6.56E-06
		1.73E-08	1.74E-08	3.08E-07	6.24E-07	8.39E-07	1.83E-06
HI		4.00E-02	3.63E-02	1.74E-02	6.82E-03	3.51E-03	8.59E-04
		5.21E-04	5.86E-04	7.00E-04	4.85E-04	3.44E-04	1.79E-04
$\hat{\sigma}_x^2$	2.00E-04	2.03E-04	2.03E-04	2.02E-04	2.02E-04	2.02E-04	2.02E-04
		3.99E-05	3.99E-05	6.30E-05	8.19E-05	9.37E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-07	6.28E-08	5.66E-08	4.41E-07	6.16E-07	7.12E-07	1.53E-06
		1.73E-09	1.75E-09	3.14E-08	6.87E-08	1.05E-07	4.26E-07
HI		4.02E-03	3.66E-03	1.85E-03	8.35E-04	5.17E-04	2.63E-04
		5.22E-05	5.90E-05	7.27E-05	5.56E-05	4.66E-05	4.85E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.01E-04
		3.94E-05	3.93E-05	6.26E-05	8.14E-05	9.32E-05	1.27E-04
$\hat{\sigma}_v^2$	2.00E-08	6.31E-09	5.95E-09	5.29E-08	1.20E-07	2.12E-07	1.02E-06
		1.74E-10	1.82E-10	3.79E-09	1.32E-08	3.13E-08	2.84E-07
HI		4.22E-04	4.01E-04	2.96E-04	2.36E-04	2.17E-04	2.03E-04
		5.33E-06	6.35E-06	1.05E-05	1.40E-05	1.85E-05	3.66E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.01E-04
		3.94E-05	3.93E-05	6.26E-05	8.13E-05	9.31E-05	1.27E-04
$\hat{\sigma}_v^2$	2.00E-09	6.63E-10	8.85E-10	1.42E-08	7.00E-08	1.62E-07	9.71E-07
		1.83E-11	2.77E-11	1.01E-09	7.83E-09	2.38E-08	2.69E-07
HI		6.22E-05	7.51E-05	1.40E-04	1.77E-04	1.88E-04	1.97E-04
		8.34E-07	1.43E-06	5.19E-06	1.05E-05	1.58E-05	3.54E-05

Table 5.3 : Estimation of integrated volatility :

Case 6 ($\eta = 0.001$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.05E-04	2.22E-04	2.06E-04	2.05E-04	2.04E-04	2.05E-04
		6.46E-05	4.55E-05	6.36E-05	8.45E-05	9.54E-05	1.34E-04
$\hat{\sigma}_v^2$	2.00E-06	2.11E-06	1.08E-07	1.02E-06	1.99E-06	2.25E-06	3.09E-06
		1.51E-07	6.26E-09	8.79E-08	2.29E-07	3.27E-07	8.52E-07
HI		7.68E-03	7.32E-03	4.93E-03	2.57E-03	1.44E-03	4.50E-04
		3.58E-04	3.47E-04	2.70E-04	1.86E-04	1.37E-04	8.62E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.03E-04	2.00E-04	2.00E-04	1.99E-04	1.99E-04
		6.33E-05	4.10E-05	6.20E-05	8.25E-05	9.36E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-07	3.12E-07	1.72E-08	1.64E-07	3.48E-07	4.66E-07	1.30E-06
		2.17E-08	9.22E-10	1.35E-08	3.88E-08	6.76E-08	3.53E-07
HI		1.22E-03	1.17E-03	8.43E-04	5.23E-04	3.70E-04	2.35E-04
		5.32E-05	5.20E-05	4.24E-05	3.38E-05	3.15E-05	4.18E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.01E-04	2.00E-04	2.00E-04	1.98E-04	1.99E-04
		6.33E-05	4.06E-05	6.20E-05	8.23E-05	9.35E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-08	1.22E-07	7.72E-09	7.65E-08	1.83E-07	2.86E-07	1.11E-06
		9.80E-09	4.17E-10	6.13E-09	2.01E-08	4.05E-08	3.06E-07
HI		5.46E-04	5.32E-04	4.30E-04	3.18E-04	2.62E-04	2.13E-04
		2.58E-05	2.51E-05	2.08E-05	1.92E-05	2.15E-05	3.84E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.01E-04	2.00E-04	2.00E-04	1.98E-04	1.99E-04
		6.33E-05	4.05E-05	6.21E-05	8.23E-05	9.35E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-09	9.08E-08	6.37E-09	6.56E-08	1.66E-07	2.68E-07	1.09E-06
		7.57E-09	3.45E-10	5.29E-09	1.81E-08	3.83E-08	3.00E-07
HI		4.51E-04	4.46E-04	3.83E-04	2.97E-04	2.51E-04	2.11E-04
		2.19E-05	2.15E-05	1.87E-05	1.79E-05	2.05E-05	3.81E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.06E-04	2.07E-04	2.07E-04	2.08E-04	2.07E-04	2.08E-04
		4.08E-05	4.10E-05	6.53E-05	8.48E-05	9.73E-05	1.33E-04
$\hat{\sigma}_v^2$	2.00E-06	2.09E-06	9.78E-07	2.11E-06	2.17E-06	2.26E-06	3.10E-06
		5.76E-08	3.20E-08	1.47E-07	2.41E-07	3.29E-07	8.58E-07
HI		7.52E-02	4.76E-02	7.71E-03	2.70E-03	1.45E-03	4.51E-04
		1.10E-03	8.10E-04	3.06E-04	1.90E-04	1.41E-04	8.84E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.02E-04	2.02E-04	2.02E-04
		3.96E-05	3.95E-05	6.30E-05	8.19E-05	9.37E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-07	2.86E-07	1.35E-07	3.13E-07	3.73E-07	4.63E-07	1.28E-06
		8.07E-09	4.34E-09	2.18E-08	4.03E-08	6.89E-08	3.51E-07
HI		1.04E-02	6.65E-03	1.22E-03	5.40E-04	3.68E-04	2.34E-04
		1.51E-04	1.09E-04	4.59E-05	3.38E-05	3.26E-05	4.27E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04
		3.96E-05	3.95E-05	6.29E-05	8.17E-05	9.35E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-08	8.51E-08	4.25E-08	1.28E-07	1.93E-07	2.83E-07	1.09E-06
		3.70E-09	1.98E-09	9.64E-09	2.11E-08	4.07E-08	3.03E-07
HI		3.23E-03	2.17E-03	5.62E-04	3.23E-04	2.61E-04	2.12E-04
		9.78E-05	6.81E-05	2.24E-05	1.92E-05	2.16E-05	3.85E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04
		3.96E-05	3.95E-05	6.29E-05	8.17E-05	9.35E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-09	3.79E-08	2.29E-08	1.05E-07	1.75E-07	2.65E-07	1.07E-06
		2.23E-09	1.33E-09	8.08E-09	1.90E-08	3.86E-08	3.01E-07
HI		1.67E-03	1.29E-03	4.87E-04	3.02E-04	2.50E-04	2.10E-04
		6.74E-05	5.15E-05	2.02E-05	1.75E-05	2.07E-05	3.80E-05

Table 5.4 : Estimation of integrated volatility :

Case 7 ($\eta = 0.001$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.05E-04	2.23E-04	2.07E-04	2.05E-04	2.04E-04	2.05E-04
		6.46E-05	4.58E-05	6.36E-05	8.47E-05	9.57E-05	1.34E-04
$\hat{\sigma}_v^2$	2.00E-06	2.11E-06	1.09E-07	1.02E-06	1.99E-06	2.26E-06	3.09E-06
		1.48E-07	6.22E-09	8.75E-08	2.30E-07	3.30E-07	8.48E-07
HI		7.69E-03	7.33E-03	4.93E-03	2.57E-03	1.44E-03	4.50E-04
		3.53E-04	3.43E-04	2.69E-04	1.86E-04	1.38E-04	8.60E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.03E-04	2.00E-04	2.00E-04	1.99E-04	2.00E-04
		6.33E-05	4.10E-05	6.20E-05	8.24E-05	9.36E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-07	3.11E-07	1.72E-08	1.64E-07	3.48E-07	4.64E-07	1.29E-06
		2.24E-08	9.25E-10	1.36E-08	3.90E-08	6.66E-08	3.51E-07
HI		1.22E-03	1.17E-03	8.43E-04	5.22E-04	3.69E-04	2.35E-04
		5.34E-05	5.21E-05	4.27E-05	3.32E-05	3.11E-05	4.20E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.01E-04	2.00E-04	2.00E-04	1.99E-04	2.00E-04
		6.33E-05	4.06E-05	6.19E-05	8.22E-05	9.35E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-08	1.15E-07	7.52E-09	7.56E-08	1.83E-07	2.85E-07	1.11E-06
		9.38E-09	4.12E-10	6.13E-09	2.02E-08	4.13E-08	3.12E-07
HI		5.32E-04	5.21E-04	4.28E-04	3.18E-04	2.61E-04	2.13E-04
		2.54E-05	2.49E-05	2.12E-05	1.96E-05	2.18E-05	3.87E-05
$\hat{\sigma}_x^2$	2.00E-04	2.00E-04	2.01E-04	2.00E-04	2.00E-04	1.99E-04	1.99E-04
		6.32E-05	4.06E-05	6.20E-05	8.22E-05	9.36E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-09	8.94E-08	6.33E-09	6.56E-08	1.66E-07	2.67E-07	1.10E-06
		7.64E-09	3.46E-10	5.34E-09	1.81E-08	3.83E-08	2.99E-07
HI		4.48E-04	4.43E-04	3.83E-04	2.97E-04	2.51E-04	2.11E-04
		2.22E-05	2.19E-05	1.95E-05	1.84E-05	2.05E-05	3.78E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.06E-04	2.07E-04	2.07E-04	2.08E-04	2.07E-04	2.08E-04
		4.08E-05	4.10E-05	6.55E-05	8.51E-05	9.75E-05	1.33E-04
$\hat{\sigma}_v^2$	2.00E-06	2.09E-06	9.78E-07	2.11E-06	2.17E-06	2.26E-06	3.09E-06
		5.73E-08	3.20E-08	1.47E-07	2.38E-07	3.30E-07	8.53E-07
HI		7.52E-02	4.76E-02	7.71E-03	2.70E-03	1.45E-03	4.51E-04
		1.10E-03	8.09E-04	3.07E-04	1.89E-04	1.41E-04	8.79E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.02E-04	2.03E-04	2.02E-04	2.02E-04
		3.96E-05	3.96E-05	6.33E-05	8.22E-05	9.41E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-07	2.85E-07	1.35E-07	3.13E-07	3.73E-07	4.63E-07	1.28E-06
		7.87E-09	4.45E-09	2.21E-08	4.19E-08	6.74E-08	3.51E-07
HI		1.04E-02	6.64E-03	1.22E-03	5.40E-04	3.69E-04	2.34E-04
		1.53E-04	1.13E-04	4.66E-05	3.53E-05	3.26E-05	4.23E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.02E-04
		3.96E-05	3.95E-05	6.30E-05	8.19E-05	9.36E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-08	6.77E-08	3.86E-08	1.32E-07	1.92E-07	2.84E-07	1.09E-06
		2.70E-09	1.54E-09	9.08E-09	2.11E-08	3.99E-08	3.02E-07
HI		2.86E-03	2.07E-03	5.71E-04	3.23E-04	2.61E-04	2.12E-04
		6.85E-05	4.96E-05	1.99E-05	1.90E-05	2.17E-05	3.86E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04
		3.96E-05	3.95E-05	6.30E-05	8.17E-05	9.35E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-09	3.28E-08	2.19E-08	1.07E-07	1.75E-07	2.67E-07	1.08E-06
		1.81E-09	1.16E-09	7.97E-09	1.88E-08	3.79E-08	2.98E-07
HI		1.56E-03	1.27E-03	4.93E-04	3.02E-04	2.51E-04	2.10E-04
		5.85E-05	4.62E-05	1.93E-05	1.75E-05	2.05E-05	3.77E-05

Table 5.5 : Estimation of integrated volatility :

Case 8 ($g_1 = 0.2, g_2 = 5$)

1800	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.28E-04	3.12E-04	2.32E-04	2.23E-04	2.22E-04	2.25E-04
		7.27E-05	6.87E-05	7.29E-05	9.22E-05	1.05E-04	1.46E-04
$\hat{\sigma}_v^2$	2.00E-06	2.46E-06	1.70E-07	1.69E-06	4.29E-06	5.93E-06	7.08E-06
		2.11E-07	1.46E-08	1.79E-07	5.65E-07	9.76E-07	2.26E-06
HI		1.21E-02	1.17E-02	9.32E-03	6.06E-03	3.72E-03	9.28E-04
		9.54E-04	9.43E-04	8.27E-04	6.15E-04	4.61E-04	2.36E-04
$\hat{\sigma}_x^2$	2.00E-04	2.04E-04	2.17E-04	2.04E-04	2.02E-04	2.01E-04	2.02E-04
		6.43E-05	4.40E-05	6.30E-05	8.30E-05	9.46E-05	1.31E-04
$\hat{\sigma}_v^2$	2.00E-07	3.17E-07	2.36E-08	2.38E-07	6.49E-07	9.94E-07	1.88E-06
		3.01E-08	2.19E-09	2.62E-08	9.13E-08	1.77E-07	5.75E-07
HI		1.67E-03	1.64E-03	1.39E-03	1.00E-03	6.99E-04	3.05E-04
		1.48E-04	1.47E-04	1.35E-04	1.15E-04	9.21E-05	6.51E-05
$\hat{\sigma}_x^2$	2.00E-04	2.02E-04	2.10E-04	2.02E-04	2.01E-04	2.00E-04	2.01E-04
		6.35E-05	4.23E-05	6.22E-05	8.24E-05	9.38E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-08	1.16E-07	9.94E-09	1.02E-07	3.07E-07	5.53E-07	1.43E-06
		1.57E-08	1.31E-09	1.47E-08	5.09E-08	1.14E-07	4.50E-07
HI		7.05E-04	6.99E-04	6.46E-04	5.44E-04	4.36E-04	2.51E-04
		9.12E-05	9.05E-05	8.57E-05	7.66E-05	6.69E-05	5.32E-05
$\hat{\sigma}_x^2$	2.00E-04	2.02E-04	2.09E-04	2.02E-04	2.01E-04	1.99E-04	2.01E-04
		6.35E-05	4.22E-05	6.23E-05	8.22E-05	9.36E-05	1.30E-04
$\hat{\sigma}_v^2$	2.00E-09	9.83E-08	8.81E-09	9.02E-08	2.77E-07	5.19E-07	1.40E-06
		1.50E-08	1.30E-09	1.42E-08	4.78E-08	1.08E-07	4.49E-07
HI		6.24E-04	6.20E-04	5.82E-04	5.06E-04	4.17E-04	2.48E-04
		8.94E-05	8.88E-05	8.35E-05	7.39E-05	6.50E-05	5.26E-05
18000	True	Raw	1 sec.	10 sec.	30 sec.	60 sec.	300 sec.
$\hat{\sigma}_x^2$	2.00E-04	2.24E-04	2.27E-04	2.23E-04	2.25E-04	2.23E-04	2.27E-04
		4.62E-05	4.69E-05	7.21E-05	9.28E-05	1.05E-04	1.45E-04
$\hat{\sigma}_v^2$	2.00E-06	2.35E-06	1.60E-06	5.85E-06	5.88E-06	5.97E-06	6.82E-06
		7.32E-08	6.13E-08	4.46E-07	7.24E-07	9.81E-07	2.14E-06
HI		1.16E-01	8.90E-02	2.11E-02	7.15E-03	3.67E-03	8.98E-04
		3.01E-03	2.55E-03	1.09E-03	6.43E-04	4.56E-04	2.21E-04
$\hat{\sigma}_x^2$	2.00E-04	2.03E-04	2.03E-04	2.03E-04	2.04E-04	2.03E-04	2.04E-04
		3.99E-05	4.00E-05	6.38E-05	8.28E-05	9.45E-05	1.29E-04
$\hat{\sigma}_v^2$	2.00E-07	2.42E-07	1.66E-07	6.30E-07	6.88E-07	7.78E-07	1.59E-06
		7.55E-09	6.32E-09	4.79E-08	8.40E-08	1.22E-07	4.48E-07
HI		1.21E-02	9.32E-03	2.36E-03	9.18E-04	5.57E-04	2.71E-04
		3.09E-04	2.63E-04	1.18E-04	7.49E-05	5.83E-05	5.11E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.02E-04	2.01E-04	2.01E-04
		3.95E-05	3.94E-05	6.29E-05	8.16E-05	9.33E-05	1.28E-04
$\hat{\sigma}_v^2$	2.00E-08	3.11E-08	2.32E-08	1.15E-07	1.75E-07	2.65E-07	1.07E-06
		1.05E-09	8.96E-10	9.14E-09	2.14E-08	4.03E-08	2.99E-07
HI		1.67E-03	1.38E-03	5.08E-04	3.02E-04	2.50E-04	2.10E-04
		4.74E-05	4.31E-05	2.47E-05	2.11E-05	2.22E-05	3.78E-05
$\hat{\sigma}_x^2$	2.00E-04	2.01E-04	2.00E-04	2.01E-04	2.01E-04	2.01E-04	2.01E-04
		3.95E-05	3.94E-05	6.28E-05	8.15E-05	9.32E-05	1.27E-04
$\hat{\sigma}_v^2$	2.00E-09	1.13E-08	9.94E-09	7.09E-08	1.31E-07	2.23E-07	1.03E-06
		5.00E-10	4.84E-10	6.14E-09	1.59E-08	3.37E-08	2.88E-07
HI		7.04E-04	6.45E-04	3.51E-04	2.50E-04	2.24E-04	2.05E-04
		2.91E-05	2.74E-05	1.85E-05	1.72E-05	1.96E-05	3.69E-05

6. Conclusions

In this paper, we have shown that the Separating Information Maximum Likelihood (SIML) estimator has the asymptotic robustness in the sense that it is consistent and it has the asymptotic normality under a set of fairly general conditions when the high frequency financial data are randomly sampled. They include not only the cases when we have the micro-market noises but also the cases when the micro-market structure has the nonlinear adjustments and the round-off errors under a set of reasonable assumptions. The micro-market factors in actual financial markets are common in the sense that we have the minimum price change and the minimum order size rules and also we often observe the bid-ask differences in stock markets. Therefore the robustness of the estimation methods of the integrated volatility is quite important. By conducting a large number of simulations, we have confirmed that the SIML estimator has reasonable robust properties in finite samples even in the non-standard situations.

As a concluding remark, we should stress on the fact that the SIML estimator is very simple and it can be practically used not only for the integrated volatility but also the integrated covariance and the hedging coefficients from the multivariate high frequency financial series. Some results on these problems will be given in Kunitomo and Misaki (2012).

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We first prepare useful formulas. The derivations are the results of elementary use of trigonometric functions, which are straightforward. Most of them have been given in Kunitomo and Sato (2008, 2011). When we apply the results with a fixed number ofobservations n, we shall often ignore the differences caused by a random number of observations n^* under Assumption 2.1 with $c = 1$.

Lemma 1 : For any integer l and m $(1 \leq l, m \leq n)$

$$\text{(A.1)} \qquad \sum_{k=1}^{m} \left[\cos \pi \frac{2k-1}{2n+1} l \right] = \frac{1}{2} \frac{\sin 2\pi m \frac{l}{2n+1}}{\sin \pi \frac{l}{2n+1}}$$

and

$$\text{(A.2)} \qquad \sum_{k=1}^{m} \left[\cos \pi \frac{2k-1}{2n+1} l \right]^2 = \frac{m}{2} + \frac{1}{4} \frac{\sin 4\pi m \frac{l}{2n+1}}{\sin 2\pi \frac{l}{2n+1}} \ .$$

Lemma 2 : Let $c_{ij} = (2/m) \sum_{k=1}^{m} s_{ik} s_{jk}$ $\ (i, j = 1, \cdots, n; k = 1, \cdots, m)$ and

$$\text{(A.3)} \qquad s_{jk} = \cos \left[\frac{2\pi}{2n+1} (j - \frac{1}{2})(k - \frac{1}{2}) \right] \ .$$

Then (i) for any positive integers j, k we have

$$\text{(A.4)} \qquad \sum_{i=1}^{n} c_{ij} c_{ik} = \frac{1}{m} (\frac{n}{2} + \frac{1}{4}) c_{jk}$$

and

$$\text{(A.5)} \qquad \sum_{i,j=1}^{n} c_{ij}^2 = \frac{4}{m} \left[\frac{n}{2} + \frac{1}{4} \right]^2 \ .$$

(ii) As $n \to \infty$,

$$\text{(A.6)} \qquad \frac{1}{n} \sum_{i=1}^{n} (c_{ii} - 1) \to 0$$

and

$$\text{(A.7)} \qquad \frac{1}{n} \sum_{i=1}^{n} (c_{ii} - 1)^2 \to 0 \ .$$

In the sequel we shall give the proofs of Theorem 4.1 and Theorem 4.2. Under

Assumptions 2.1 and 2.2 we can consider as if there were n terms when n is a fixed integer because the effects of terms of r_i ($n^* < i \leq n$) are negligible in probability. With the transformations of (2.9) and (2.10) we set $z_k = z_k^{(1)} + z_k^{(2)}$, where $z_k^{(1)}$ and $z_k^{(2)}$ correspond to the $k-$elements of $\mathbf{z}_{n^*}^{(1)} = \sqrt{n^*}\mathbf{P}_{n^*}\mathbf{C}_{n^*}^{-1}(\mathbf{z}_{n^*} - \mathbf{Y}_0)$ and $\mathbf{z}_{n^*}^{(2)} = \sqrt{n^*}\mathbf{P}_{n^*}\mathbf{C}_{n^*}^{-1}\mathbf{V}_{n^*}$, respectively. (We use these notations which are similar to Kunitomo and Sato (2008, 2011).) Then we shall investigate the asymptotic properties as if n^* and m_{n^*} are the fixed numbers of n and m_n in the following analysis.

By using Lemma 1, we have $\mathcal{E}[\mathbf{z}_n^{(1)}] = \mathbf{0}$, $\mathcal{E}[\mathbf{z}_n^{(2)}] = \mathbf{0}$ and

$$\text{(A.8)} \qquad \mathcal{E}[\mathbf{z}_n^{(2)}\mathbf{z}_n^{(2)'}] = \sigma_v^2 n^{-1}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n = \sigma_v^2 n^{-1}\mathbf{D}_n \ ,$$

$\mathbf{D}_n$ is a diagonal matrix with the k-th element

$$\text{(A.9)} \qquad d_k = 2\left[1 - \cos(\pi(\frac{2k-1}{2n+1}))\right] \ \ (k = 1, \cdots, n) \ .$$

In our proof of theorems we shall extensively use the decomposition

$$\text{(A.10)} \qquad \begin{aligned} & \hat{\sigma}_x^2 - \sigma_x^2 \\ = & \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^2 - \sigma_x^2\right] \\ = & \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(1)2} - \sigma_x^2\right] + \sigma_v^2\frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn} + \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(2)2} - \sigma_v^2 a_{kn}\right] \\ & +2\frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(1)}z_k^{(2)}\right] \ . \end{aligned}$$

Under Assumptions 2.1 and 2.2, we can ignore the additional terms of r_j ($j = n^* + 1, \cdots, n$) due to the random sampling, (A.10) is asymptotically equivalent to

$$\text{(A.11)} \qquad \begin{aligned} & \hat{\sigma}_x^{*2} - \sigma_x^2 \\ = & \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(1)2} - \sigma_x^2\right] + \sigma_v^2\frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn} + \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(2)2} - \sigma_v^2 a_{kn}\right] \\ & +2\frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_k^{(1)}z_k^{(2)}\right] \ . \end{aligned}$$

(Again we note that we investigate (A.10) and (A.11) as if n^* and m_{n^*} are the fixed numbers of n and m_n.)

Lemma 3 : Assume the assumptions of Theorem 4.1.

(i) For $0 < \alpha < 0.5$,

$$\hat{\sigma}_x^2 - \sigma_x^2 \xrightarrow{p} 0 \tag{A.12}$$

as $n \to \infty$.

(ii) For $0 < \alpha < 0.4$,

$$\begin{aligned} &\sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2 - \sum_{i\geq 1}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] \\ &-\sqrt{m_n}\left[\frac{1}{m}\sum_{k=1}^{m}\left(z_k^{(1)2}\right) - \sigma_x^2 - \sum_{i\geq 1}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] \\ \xrightarrow{p}\ & 0 \end{aligned} \tag{A.13}$$

as $n \to \infty$.

Proof of Lemma 3 : By using the well-known relation $\sin x = x - (1/6)x^3 + (1/120)x^5 + O(x^7)$,

$$\begin{aligned} \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} &= \frac{1}{m_n}2n\sum_{k=1}^{m_n}\left[1-\cos(\pi\frac{2k-1}{2n+1})\right] \\ &= \frac{n}{m_n}\left[2m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}}\right] \\ &\sim \frac{n}{m_n}\left[2m_n - \frac{(\pi\frac{2m_n}{2n+1}) - \frac{1}{6}(\pi\frac{2m_n}{2n+1})^3}{(\frac{\pi}{2n+1}) - \frac{1}{6}(\frac{\pi}{2n+1})^3}\right] \\ &= O(\frac{m_n^2}{n}) \end{aligned} \tag{A.14}$$

and

$$\begin{aligned} \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^2 &= \frac{1}{m_n}4n^2\sum_{k=1}^{m_n}\left[1-2\cos(\pi\frac{2k-1}{2n+1}) + \frac{1}{2}(1+\cos(2\pi\frac{2k-1}{2n+1}))\right] \\ &= \frac{4n^2}{m_n}\left[\frac{3}{2}m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}} + \frac{1}{4}\frac{\sin\pi\frac{4m_n}{2n+1}}{\sin\pi\frac{2}{2n+1}}\right] \\ &= O(\frac{m_n^4}{n^2}) \end{aligned} \tag{A.15}$$

as $n \to \infty$. Then (A.14) and (A.15) are $o(1)$ when we have the condition that $m_n^2/n \to 0$ $(n \to \infty)$. Hence for the first term of (A.11) we need $0 < \alpha < 0.5$ for the consistency and $0 < \alpha < 0.4$ for the asymptotic normality as the mimimum requirements, respectively, because $(1/\sqrt{m_n})\sum_{i=1}^{m_n} a_{kn}$ should be negligible in the latter case. In order to show that these conditions are sufficient, we shall evaluate each terms of $\sqrt{m_n}[\hat{\sigma}_x^2 - \sigma_x^2]$ based on the decomposition (A.11).

For the third term of (A.11), there exists a positive constant K_1

$$
\begin{aligned}
\text{(A.16)} \qquad & \mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n} z_k^{(1)} z_k^{(2)}\right]^2 \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n} \mathcal{E}\left[z_k^{(1)} z_{k'}^{(1)} z_k^{(2)} z_{k'}^{(2)}\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n} \mathcal{E}\left[2\sum_{j,j'\geq 1} s_{jk}s_{j'k'}\mathcal{E}(r_{jg}r_{j'}|\mathcal{F}_{\min(j,j')}) z_k^{(2)} z_{k'}^{(2)}\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n} \mathcal{E}\left[2\sum_{j\geq 1} s_{jk}s_{j,k'}\mathcal{E}(r_j^2|\mathcal{F}_{j-1}) z_k^{(2)} z_{k'}^{(2)}\right] \\
&\leq K_1 \mathcal{E}\left[\sup_{0\leq s\leq 1}\sigma_x^2(s)\right]\mathcal{E}\left[\max_{i\geq 1}|t_i^n - t_{i-1}^n|\right](\frac{n}{2}+\frac{1}{4})\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} \\
&= O(\frac{m_n^2}{n})\ .
\end{aligned}
$$

where we have used the independence of $x_k^{(1)}$ and $x_k^{(2)}$, and the relation

$$
\text{(A.17)} \qquad \sum_{j=1}^{n} s_{jk}^2 = \frac{n}{2} + \frac{1}{4} \quad \text{for any } k \geq 1\ .
$$

For the second term of (A.11), let $\mathbf{b}_k = \mathbf{e}_k^{'}\mathbf{P}_n^{'}\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0,\cdots,1,0,\cdots)$ is an $n \times 1$ vector. Then we can write $x_{kg}^{(2)} = \sum_{j=1}^{n} b_{kj}v_j$ and

$$
\begin{aligned}
\text{(A.18)} \qquad & \mathcal{E}\left[\frac{1}{\sqrt{m_m}}\sum_{k=1}^{m_n}(z_k^{(2)2} - \sigma_v^2 a_{kn})\right]^2 \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[(z_k^{(2)2} - \sigma_v^2 a_{kn})(z_{k'}^{(2)2} - \sigma_v^2 a_{k'n})\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[(\sum_{j\geq 1} b_{kj}v_j)^2(\sum_{j'\geq 1} b_{k'j'}v_{j'})^2 - \sigma_{gg}^{(v)} a_{kn}a_{k'n}\right] \\
\text{(A.19)} \qquad &\leq K_2\frac{1}{m_n}\sum_{k=1}^{m_n}\sum_{j\geq 1} b_{kj}^4 \sim K_2\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^2 = O(\frac{m_n^4}{n^2})\ ,
\end{aligned}
$$

where K_2 is a positive constant.

Hence we have found that the main effect of the sampling errors associated with the SIML estimator of the realized variance is the first term of (A.11). Then we shall show the consistency and the variance formula. We write $r_i = x_i - x_{i-1}$ $(i = 1, \cdots, n)$ and by using the fact that r_i are a sequence of martingale differences,

$$\text{(A.20)} \qquad \mathcal{E}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(z_k^{(1)2} - \sigma_x^2)\right]^2$$

$$\sim \mathcal{E}\left\{\sum_{i,j\geq 1}\left[c_{ij}\, r_i r_j - \delta_{ij}\int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]\right\}^2$$

$$= \mathcal{E}\left\{\sum_{i=j\geq 1}\left[c_{ij} r_i^2 - \int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right]\right\}^2 + \mathcal{E}\left\{\sum_{i\neq j\geq 1} c_{ij} r_i r_j\right\}^2 ,$$

where $\delta_{ij} = 1\ (i = j); \delta_{ij} = 0\ (i \neq j)$. Then we need to evaluate

$$\mathcal{E}\left\{\sum_{i=1}^{n}\left[c_{ii} r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right]\right\}^2 = \mathcal{E}\left[\sum_{i=1}^{n}(r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds) + \sum_{i=1}^{n}(c_{ii}-1)r_{ig}^2\right]^2 ,$$

$$\text{(A.21)} \qquad \mathcal{E}\left\{\left[\sum_{i\neq j=1}^{n} c_{ij} r_{ig} r_{jg}\right]^2\right\} = 2\sum_{i\neq j=1}^{n} c_{ij}^2 \mathcal{E}(r_{ig}^2)\mathcal{E}(r_{jg}^2) ,$$

and $\mathcal{E}(r_{ig}^2) \leq K_3/n$ with K_3 being a positive constant. Hence the first term of (A.19) is approximately equivalent to

$$\text{(A.22)} \qquad \sum_{i=1}^{n}\left[r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds + (c_{ii}-1)r_{ig}^2\right]$$

$$= \sqrt{\frac{1}{n}}\sqrt{n}\sum_{i=1}^{n}\left[r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds\right] + \left[\sum_{i=1}^{n}(c_{ii}-1)(r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)\right]$$

$$+ \left[\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}ds\right] .$$

By using the basic evaluation obtained by Jacod-Protter (1998), we find that

$$\text{(A.23)} \qquad \sqrt{n}\sum_{i=1}^{n}\left[r_{ig}^2 - \int_{t_{i-1}}^{t_i}\sigma_x^2(s)ds\right] = O_p(1) .$$

Also we apply the inequalities

$$|\sum_{i=1}^{n}(c_{ii}-1)\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds|^2 = \left[\sum_{i=1}^{n}(c_{ii}-1)^2\right]\left[\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i}\sigma_{gg}^{(x)}(s)ds)^2\right]$$

$$\leq \ [t_i^n - t_{i-1}]^2 \, n \sum_{i=1}^{n} (c_{ii}-1)^2 \left[\sup_{0\leq s\leq t} \sigma_{gg}^{(x)}(s) \right]^2$$
$$= \ O_p(\frac{1}{m_n})$$

and

$$\mathcal{E}\left[|\sum_{i=1}^{n}(c_{ii}-1)(r_{ig}^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds)|^2 \right] = \left[\sum_{i=1}^{n}(c_{ii}-1)^2 \mathcal{E}(r_{ig}^2 - \int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds)^2 \right]$$
$$= O(\frac{1}{n}) \ .$$

Hence we have (A.13) when $m_n/n \to 0$ as $n \to \infty$.

Q.E.D.

We can simplify the limiting distribution of the SIML estimator by using the following lemma. The proof has been given by Kunitomo and Sato (2008).

Lemma 4 : For $0 < \alpha \leq 0.4$,

(A.24)
$$\sqrt{m_n}\left[\sum_{i\geq 1}(c_{ii}-1) \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds \right] \xrightarrow{p} 0$$

as $n \to \infty$.

We note that in the above proof of Lemma 4 we have intentionally used $O_p(\ \cdot\ )$ instead of $O(\ \cdot\ )$ because of the stochastic case.

Proofs of Theorem 4.1 and Theorem 4.2 :

We use n as if it is n^* because we need additional arguments, but they are not essential in our analysis.

(**Step 1**) : We shall give only the proof of the asymptotic normality in Theorem 4.1 (and the stable convergence in Theorem 4.2) of the realized variance $\sigma_{gg}^{(x)}$ ($g = 1, \cdots, p$). We give some brief comments on the related problem between Lemma 2 and Lemma 3.) We first use the proofs of Lemma 3 and Lemma 4 for the consistency and the asymptotic behavior of the SIML estimator.

We write $\sqrt{m_n}[\hat{\sigma}_x^2 - \sigma_x^2]$ as

(A.25)
$$\sqrt{m}\left[\sum_{i,j\geq 1} c_{ij} r_i r_j - \delta_{ij} \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds \right]$$

$$= 2\sqrt{m}\sum_{i>j} c_{ij} r_i r_j + \sqrt{m}\left[\sum_{i=1}^{n} c_{ii} r_i^2 - \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] ,$$

where $\delta_{ij} = 1$ $(i = j)$; $\delta_{ij} = 0$ $(i \neq j)$ and the second term is equivalent to

$$\text{(A.26)} \quad \sqrt{m}\sum_{i\geq 1}\left[r_i^2 - \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds + (c_{ii}-1)r_i^2\right]$$
$$= \sqrt{\frac{m}{n}}\sqrt{n}\sum_{i=1}^{n}\left[r_i^2 - \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] + \sqrt{m}\left[\sum_{i\geq 1}(c_{ii}-1)(r_i^2 - \int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds)\right]$$
$$+\sqrt{m}\left[\sum_{i\geq 1}(c_{ii}-1)\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] ,$$

which is $o_p(1)$.

From the proofs of Lemma 3 and Lemma 4, we can ignore each terms of (A.25) for the limiting distribution $\sqrt{m_n}[\hat{\sigma}_x^2 - \sigma_x^2]$.

The summation of the conditional covariances associated with the first term of the right-hand side of (A.24) is

$$4\sum_{i<j} m_n c_{ij}^2 \mathcal{E}_{i-1}[r_i^2]\mathcal{E}_{j-1}[r_j^2]$$
$$= 2\sum_{i,j\geq 1} m_n c_{ij}^2 \mathcal{E}_{i-1}[r_i^2]\mathcal{E}_{j-1}[r_j^2] - 2\sum_{i\geq 1} m_n c_{ii}^2 (\mathcal{E}_{i-1}[r_i^2])^2$$
$$= 2\sum_{i,j\geq 1} \mathcal{E}_{i-1}[r_i^2]\mathcal{E}_{j-1}[r_j^2] + 2\sum_{i,j\geq 1}(m_n c_{ij}^2 - 1)\mathcal{E}_{i-1}[r_i^2]\mathcal{E}_{j-1}[r_j^2] - 2\sum_{i\geq 1} m_n c_{ii}^2 (\mathcal{E}_{i-1}[r_i^2])^2 ,$$

where we use the notation $\mathcal{E}_{i-1}[r_i^2] = \mathcal{E}[r_i^2|\mathcal{F}_{n,i-1}]$.

For the third term, we have

$$\text{(A.27)} \quad \sum_{i\geq 1} m_n c_{ii}^2 \left(\mathcal{E}_{i-1}[r_i^2]\right)^2 \leq \left[\sup_{0\leq\leq 1} \sigma_x^2(s)\right]^2 m_n [t_i^n - t_{i-1}^n]^2 \sum_{i=1}^{n} c_{ii}^2 \xrightarrow{p} 0$$

as $m/n \to 0$. Then the main part of the asymptotic variance is the limit of

$$\text{(A.28)} \quad V_{gg.n}$$
$$= 2\left[\sum_{i\geq 1}\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right]^2 + 2\sum_{i,j\geq 1}(m_n c_{ij}^2 - 1)\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds \int_{t_{j-1}}^{t_j} \sigma_x^2(s)ds ,$$

which is given by (4.3) by using the next Lemma 5. The second term is bounded because by using Lemma 2 we have $\frac{1}{n^2}\sum_{i,j\geq 1}|m_n c_{ij}^2 - 1| \leq 1 + \frac{m_n}{n^2}\sum_{i,j\geq 1} c_{ij}^2$. When the volatility function is constant ($\sigma_x^2(s) = \sigma_x^2$), we have $\mathcal{E}\left[\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds\right] = \sigma_x^2 \mathcal{E}[|t_i^n - t_{i-1}^n|]$ and the second term of (A.29) vanishes because Lemma 2 again implies

$$\frac{1}{n^2}\sum_{i,j\geq 1}(m_n c_{ij}^2 - 1) = \frac{1}{n^2}\left[n^2 + n + \frac{1}{4} - n^2\right] \rightarrow 0$$

and then

(A.29) $$V = 2\left[\sigma_x^2\right]^2 \ .$$

Lemma 5 : Let

$$V_n = 2\sum_{i,j\geq 1} m_n c_{ij}^2 \left[\int_{t_{i-1}}^{t_i} \sigma_x^2(s)ds \int_{t_{j-1}}^{t_j} \sigma_x^2(s)ds\right] \ .$$

Then as $n \rightarrow \infty$

(A.30) $$V_n \xrightarrow{p} V = 2\int_0^1 \sigma_x^4(s)ds \ .$$

Proof of Lemma 5 : See Kunitomo and Sato (2008). **Q.E.D.**

(**Step 2**) : Next, we need to show that the normalized SIML estimator converges to the limiting random variable in the stable convergence sense. Since we do not have the condition $\mathcal{F}_{n,i} \subseteq \mathcal{F}_{n+1,i}$, one way is to use Chapter IX of Jacod and Shiryaev (2003) and Jacod (2007) in particular. In order to do this, we need the condition that V_n converges to V and V is positive a.s. (We need that V takes a non-negative value in Theorem 4.1 while it is non-negative in the stochastic case for Theorem 4.2. When the probability limit V is a random variable, we need the stable convergence.)

In our proof we shall use of a sequence of random variables

(A.31) $$U_n = \sum_{j\geq 2}[2\sum_{i=1}^{j-1}\sqrt{m_n}c_{ij}r_i]r_j,$$

which is a martingale. We notice that for the process $\mathbf{x}_t$ and $r_i = x_i - x_{i-1}$ ($i = 1,\cdots,n$) are the (discrete) martingale parts. Then we apply Chapter IX of Jacod and Shiryaev (2003) by setting $X_{nj} = (2\sum_{i=1}^{j-1}\sqrt{m_n}c_{ij}r_i)r_j$, $W_{nj} = r_j$ ($j = 2,\cdots,n$)

and $V_n^* = \sum_{j=2}^n \mathcal{E}[X_{nj}^2|\mathcal{F}_{n,j-1}]$ in their notation. In our situation, however, we do not need any drift terms on X_{nj} and W_{nj}. Then it is sufficient to check Condition (A)

$$\max_{1\le j\le n} \mathcal{E}[X_{nj}^2|\mathcal{F}_{n,j-1}] \xrightarrow{p} 0 \ , \tag{A.32}$$

Condition (B)

$$\sum_{j=1}^n \mathcal{E}[X_{nj}^4] \longrightarrow 0 \tag{A.33}$$

and Condition (C)

$$\mathcal{E}[(V_n^* - V)^2] \longrightarrow 0 \tag{A.34}$$

as $n \longrightarrow \infty$. It is because Condition (B) implies that

$$\sum_{j=1}^n \mathcal{E}[(X_{nj}W_{nj})^2] \xrightarrow{p} 0 \ . \tag{A.35}$$

First we notice that Condition (B) implies Condition (A) in the present formulation because for $Y_{nj} = \mathcal{E}[X_{nj}^2|\mathcal{F}_{n,j-1}]$ and any $\epsilon > 0$

$$\mathrm{P}(\max_{1\le j\le n} Y_{nj} > \epsilon) \le \sum_{j\ge 1} \mathrm{P}(Y_{nj} > \epsilon) \le (\frac{1}{\epsilon})^2 \sum_{j\ge 1} \mathcal{E}[Y_{nj}^2] \ . \tag{A.36}$$

Then Lemma 6 below shows Condition (B).
Second, we have assumed the condition $V_n \xrightarrow{p} V$ in Theorems and V_n and V are bounded. Then we can find a positive K_6 such that for any $\epsilon > 0$

$$\begin{aligned}
\mathcal{E}[(V_n - V)^2] &= \mathcal{E}[(V_n - V)^2 I(|V_n - V| \ge \epsilon)] \\
&\quad + \mathcal{E}[(V_n - V)^2 I(|V_n - V| < \epsilon)] \\
&\le K_6 \mathrm{P}(|V_n - V| \ge \epsilon) + \epsilon^2 \ .
\end{aligned}$$

Hence we only need to show Condition (D)

$$\mathcal{E}[(V_n^* - V_n)^2] \longrightarrow 0 \tag{A.37}$$

as $n \longrightarrow \infty$. Then Lemma 7 below shows Condition (D).
Q.E.D.

Lemma 6 : Under the assumptions in Theorems 4.1 and 4.2, we have Condition

(B).

Proof of Lemma 6 : See Kunitomo and Sato (2008). **Q.E.D.**

We shall give the proof of Condition (D) for the time-varying deterministic case because the arguments we use are rather clear and straightforward. However, it is possible to show the result in the stochastic case with additional arguments illustrated in the proof of Lemma 6.

Lemma 7 : Under the assumptions in Theorems 4.1 and 4.2, we have Condition (D).

Proof of Lemma 7 : See Kunitomo and Sato (2008). **Q.E.D.**