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A Robust Estimation of Integrated Volatility under Round-off Errors, Micro-market Price Adjustments and Noises

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Abstract

For estimating the integrated volatility by using high frequency data, Kunitomo and Sato (2008, 2011, 2013) have proposed the Separating Information Maximum Likelihood (SIML) method when there are micro-market noises. The SIML estimator has reasonable finite sample properties and asymptotic properties when the sample size is large under reasonable conditions. We show that the SIML estimator has the robustness properties in the sense that it is consistent and has the stable convergence (i.e. the asymptotic normality in the deterministic case) when there are round-off errors and micro-market price adjustments and noises for the underlying (continuous time) stochastic process. The SIML estimation has also reasonable finite sample properties with these effects and dominate the existing methods such as the realized kernel method and the pre-averaging method in some situations.

Key Words

Integrated Volatility with Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Round-off errors, Price adjustments, Micro-market noises, Asymptotic Robustness

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1. Introduction

Recently a considerable interest has been paid on the estimation problem of the integrated volatility by using high-frequency data of financial prices. Although the earlier studies often had ignored the presence of micro-market noises in financial markets, there have been arguments that the micro-market noises are important in high-frequency financial data, and then several new statistical estimation methods have been developed. Among many studies in recent literature on the related topics three methods have attracted some attention : the linear combination of realised volatilities constructed with subsampling by Zhang, Mykland and Ait-Sahalia (2005), the realised kernel method with autocovariances and kernels by Bandorff-Nielsen, Hansen, Lund and Shephard (2008, 2011), and the pre-averaging method by Jacod, Li, Mykland, Podolskij and Vetter (2009). See Jacod and Protter (2012) and Ait-Sahalia and Jacod (2014) for the recent developments on the related issues and their references. As an alternative method Kunitomo and Sato (2008, 2011, 2013) have proposed an alternative statistical method called the Separating Information Maximum Likelihood (SIML) method for estimating the integrated volatility and the integrated covariance by using high frequency data under the presence of micro-market noises. The SIML estimator has reasonable asymptotic properties as well as finite sample properties. (There is a related spectral estimation method, which is a rather recent study, by Bibinger and Reiβ (2014).)

In this paper we shall investigate the robustness property of the SIML estimation when we have the round-off errors and the micro-market adjustment mechanisms in the process of forming the observed transaction prices. We investigate the round-off error model as a non-linear transformation of the underlying financial price process with micro-market noises. The statistical problem of round-off error model of continuous stochastic processes has been previously investigated by Delattre and Jacod (1997), Rosenbaum (2009), Li and Myckland (2012). But our formulation has a new aspect and our motivation is the empirical observation that we have the tick-size effects (the minimum price change size and the minimum order size) and we often observe bid-ask spreads on securities in actual financial markets.

The micro-market models including the price adjustments have been discussed in *micro-market literatures* (Engle and Sun (2007), and Hansbrouck (2007) for instance). Among possible micro-market statistical models, we first take the (linear) adjustment model proposed by Amihud and Mendelson (1987) as a benchmark case in our investigation. Then we shall investigate the linear and nonlinear price adjustment models in which a continuous martingale is the hidden intrinsic value on the underlying security. A new statistical feature of our approach is to utilize the nonlinear (discrete) transformations of continuous time diffusion process with discrete time noise. Examples of relevant non-linear time series models are the Simultaneous Switching Autoregressive (SSAR) model developed by Kunitomo and Sato (1999) and the threshold autoregressive (TAR) type models by Tong (1990).

The main theme of this study is the fact that the observed price can be different from the underlying intrinsic value of the security and we can interpret this discrepancy as a nonlinear transformation from the intrinsic value to the observed price. We can represent the present situation as the nonlinear statistical model of an unobservable (continuous-time) state process and the observed (discrete-time) stochastic process with measurement error. When the effects of measurement errors are present, the SIML estimator is robust, that is, it is consistent and asymptotically normal (or the mixed normal in the *stable convergence* sense) as the sample size increases under a set of assumptions. The required condition on the threshold parameter for the round-off errors in this paper is quite weak, for instance. The asymptotic robustness of the SIML method on the integrated volatility and covariance has desirable properties over other estimation methods from a large number of data for the underlying continuous stochastic process with micro-market noise. Because the SIML estimation is a simple method, it can be practically used for analyzing the multivariate (high frequency) financial time series. As a companion paper, Kunitomo, Misaki and Sato (2015) are investigating the multivariate problem when we need to estimate the covariance and the hedging coefficient.

In Section 2 we introduce the round-off error model and the micro-market adjustment models. In Section 3 we explain the SIML method and the MSIML (modified SIML) method to improve the convergence rate. Then we give the asymptotic properties of the SIML estimator when the integrated volatility is stochastically time dependent there are the round-off errors and/or there are micro-market adjustments. In Section 4 we shall report the finite sample properties of the SIML estimator based on a set of simulations. Finally, in Section 5 some brief remarks will be given. Some mathematical details of the proofs of theorems are given in Appendix A. Tables and figures based on simulations are given in Appendix B.

2. Round-off error Models and Micro-market adjustment Models

2.1 A General Formulation

Let $y(t_i^n)$ be the $i-$th observation of the (log-) price at t_i^n for $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$. We consider the situation when the underlying continuous-time stochastic process $X(t)$ $(0 \le t \le 1)$ is not necessarily the same as the observed (log-)price at t_i^n $(i = 1, \cdots, n)$ and it is a Brownian semi-martingale as

$$X(t) = X(0) + \int_0^t \mu_x(s)ds + \int_0^t \sigma_x(s)dB(s) \quad (0 \le t \le 1), \tag{2.1}$$

where $\mu_x(s)$ is a predictable locally bounded drift term, $\sigma_x(s)$ is an adapted continuous (and bounded) volatility process, and $B(s)$ is the standard Brownian motion. The solution of stochastic differential equation has often the representation as (2.1) although we exclude the jumps in the following analysis mainly because we have many jump-noises. See Ikeda and Watanabe (1989) or Jacod and Protter (2012) for the details.

The main statistical objective is to estimate the integrated volatility (or the quadratic variation)

$$\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds \tag{2.2}$$

of the underlying continuous process $X(t)$ $(0 \le t \le 1)$ from the set of observations on $y(t_i^n)$.

In this paper we consider the situation that the observed (log-)price $y(t_i^n)$ is not necessarily Brownian semi-martingale, but it is generated by

$$y(t_i^n) = h\left(X(t_i^n), y(t_{i-1}^n), u(t_i^n)\right) , \tag{2.3}$$

where $h(\,\cdot\,)$ is a measurable function, the (unobservable) continuous Brownian semi-martingale $X(t)$ $(0 \le t \le 1)$ is defined by (2.1), and $u(t_i^n)$ is the micro-market noise process. For the simplicity we assume that $u(t_i^n)$ are a sequence of independently and identically distributed random variables with $\mathbf{E}(u(t_i^n)) = 0$ and $\mathbf{E}(u(t_i^n)^2) = \sigma_u^2$ $(0 = t_0^n < t_1^n < \cdots < t_n^n = 1;\ t_i^n - t_{i-1}^n = 1/n, i = 1, \cdots, n)$.

There are special cases of (2.1) and (2.3), which can describe important aspects on modeling financial markets and the high frequency financial data for practical market applications. The simple (high-frequency) financial model with micro market noise can be represented by

$$y(t_i^n) = X(t_i^n) + u(t_i^n) \ , \tag{2.4}$$

where the underlying process $X(t)$ is given by (2.1) and $u(t_i^n)$ is a sequence of (mutually) independent random variables.

The most important statistical aspect of (2.4) is the fact that it is an additive (signal-plus-noise) measurement error model and the signal is a continuous process. However, there are economic reasons why the standard situation such as (2.4) is not enough for applications. The round-off-errors models and the high-frequency financial models with micro-market price adjustments financial prices cannot be reduced to (2.4), but they can be represented as special cases of (2.1) and (2.3).

2.2 The basic Round-off-error model

First, we investigate the basic round-off-error model with the micro-market noise. One motivation for this model has been the fact that in actual financial markets transactions occur with the minimum tick size and the observed price data do not have continuous path over time. For instance, the traded price and quantity usually have the minimum size and the Nikkei-225-futures, which have been the most important traded derivatives in Japan (as explained in Kunitomo and Sato (2011)) for instance, has the minimum 10 yen size while the Nikkei-225-stock index is around 9,000 yen in the year of 2011. (See Hansbrouck (2007) for the details of business practice of major stock markets in the U.S..) Thus it is quite interesting and important to see the effects of round-off-errors on the estimates of the integrated volatility.

Let $y_i = P(t_i^n)$ and

$$P(t_i^n) = g_\eta \left(X(t_i^n) + u(t_i^n) \right) \ , \tag{2.5}$$

where the micro-market noise term $u(t_i^n)$ is a sequence of i.i.d. random variables with $\mathcal{E}[u(t_i^n)] = 0$, $\mathcal{E}[u(t_i^n)^2] = \sigma_u^2$ and the nonlinear function

$$g_\eta(x) = \eta \left[\frac{x}{\eta} \right] \tag{2.6}$$

is the round-off part of x, $[x]$ is the largest integer being equal or less than x, and the threshold parameter η is a (small) positive constant.

This model corresponds to the micro-market model with the restriction of the minimum price change with micro-market noise and η represents the parameter of the level of minimum price change. In Figure 2-3 typical sample paths in the standard round-off model are shown by using simulations. (The true process is in black while the observed process is in red with a pre-determined threshold level.)

2.3 A Micro-market Price Adjustment Model

There have been a large number of micro-market models in the area of financial economics in the past which have tried to explain the role of noise traders, insiders, bid-ask spreads, the transaction prices, the effects of tax and fee, and the associated price adjustment processes. As an illustration we give an underlying typical argument on the financial market mechanism by Figures 2-1 and 2-2 in Appendix B. In the underlying financial market we denote that P and Q are the price and the quantity (in demand, supply and traded) of a security [1] . When the demand curve and supply curve for a security do not meet, there is no transaction occurring at the moment. The minimum (desired) supply price level $\bar{P}$ is higher than the maximum (desired) demand price level $\underline{P}$, and then there is a (bid-ask) spread. On the one hand, when there were some information in the supply side indicating that the intrinsic value of a security X_t at t is less than the latest observed price $P_{t-\Delta t}$ at $t-\Delta t$ (i.e. $X_t - P_{t-\Delta t} < 0$, $\Delta t > 0$), the supply schedule would be shifted down-ward. On the other hand, when there were some information in the demand side indicating that the intrinsic value of a security at t is higher than the latest observed price (i.e. $X_t - P_{t-\Delta t} > 0$), the demand schedule would be shifted up-ward. In these situations while the trade of a security would occur at the price P^* and the quantity Q^* as in Figure 2.2, the financial market would be under pressure for price changes.

Let $y(t_i^n) = P(t_i^n)$ $(i = 1, \cdots, n)$ and consider the (linear) micro-market price adjustment model given by

$$P(t_i^n) - P(t_{i-1}^n) = g\left(X(t_i^n) - P(t_{i-1}^n)\right) + u(t_i^n) \; , \tag{2.7}$$

where $X(t)$ (the intrinsic value of a security at t) and $P(t_i^n)$ (the observed log-price at t_i^n) are measured in logarithm, the adjustment (constant) coefficient g $(0 < g < 2)$, and $u(t_i^n)$ is an i.i.d. sequence of noises with $\mathbf{E}[u(t_i^n)] = 0$ and $\mathbf{E}[u(t_i^n)^2] = \sigma_u^2$. The specific model (2.7), which was originally proposed by Amihud and Mendelson (1987), is a typical example because it has been one of well-known micro-market models involving transaction costs and interactions among different type of market participants in financial markets. We depart our discussion from the Amihud-Mendelson model, however, because we focus on the integrated volatility estimation while their main purpose was to investigate the micro-market mechanisms by using daily (open-to-open and close-to-close) data. Also while Amihud and Mendelson (1987) used that $X(t_i^n)$ follows a (discrete) random walk process in the discrete time series framework, we consider the case when $X(t)$ is a general continuous-time mar-

[1] This is a typical illustration for the expository purpose, which may be analogous to the current market practice for the periodic call option of Tokyo Stock Exchange (TSE). We thank Wataru Ohta for pointing out this practice.

tingale given by (2.1), and the integrated volatility of the intrisic value is defined by $0 < \int_0^t \sigma_s^2 ds < \infty$ (a.s.).

2.4 Round-off Errors and Nonlinear Price Adjustment models

We can generalize the round-off errors model and the linear price adjustment model we have introduced. We discuss two variants along this direction for an illustration although there can be other directions. As a nonlinear model we represent the micro-market price adjustment models with round-off error effects. One motivation has been the fact that in actual financial markets transactions occur with the minimum tick size, the observed price data do not have continuous path over time and we also have some price adjustment mechanism. Again we can illustrate the underlying typical argument on the movements of prices in financial markets by using Figure 2-2 in Appendix B. When the demand curve and supply curve do meet at a point as Figure 2-2, the quantity Q^* is traded at the price P^*. Still there would be excess demand which could not be traded at the particular moment because of the positive tick-size ($\eta > 0$) and the minimum order size effects, i.e. the number of orders should be an integer above the specified fixed size in financial markets.

Let y_i $(= y(t_i^n)) = P(t_i^n)$ and

$$P(t_i^n) - P(t_{i-1}^n) = g_\eta \left(X(t_i^n) - P(t_{i-1}^n) + u(t_i^n) \right) , \tag{2.8}$$

where $u(t_i^n)$ is a sequence of i.i.d. noises with $\mathbf{E}[u(t_i^n)] = 0$, $\mathbf{E}[u(t_i^n)^2] = \sigma_u^2$ and $g_\eta(x)$is defined by (2.6). Then from (2.8) the difference between the observed price and the underlying intrisic value can be represented as

$$\begin{aligned} (2.9)\quad & P(t_i^n) - X(t_i^n) \\ = \; & g_\eta \left(-(P(t_{i-1}^n) - X(t_{i-1}^n)) + \Delta X(t_i^n) + u(t_i^n) \right) + (P(t_{i-1}^n) - X(t_{i-1}^n) - \Delta X(t_i^n)) \\ = \; & g_\eta^* \left(P(t_{i-1}^n) - X(t_{i-1}^n), \Delta X(t_i^n), u(t_i^n) \right) , \end{aligned}$$

where

$$\Delta X(t_i^n) = \int_{t_{i-1}^n}^{t_i^n} \mu_x(s) ds + \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s) dB_s \tag{2.10}$$

is a sequence of differences of $X(t_i^n)$ and g_η^* is defined by (2.9) implicitly. The round-off model of (2.8) and (2.9) can be represented as a nonlinear adjustment model with noise components.

As the price adjustment mechanism, there have been discussions on asymmetrical movements of financial price processes, which are related to the financial risk management problems. It may be natural to consider the situation that there are different mechanisms in the up-ward phase of financial prices (that is $y(t_i^n) \geq y(t_{i-1}^n)$) and in the down-ward phase of financial prices (that is, $y(t_i^n) \leq y(t_{i-1}^n)$). In the context of micro-market models, some economists have tried to find econometric models involving transaction costs and micro-market structures such as market practices and regulations on the maximum limits of down-ward price movements within a day. While one possible approach would be to build statistical models with asymmetrical movements of the instantaneous volatility function, our approach is slightly different from them because we are trying to consider the micro-market price adjustment process directly. As an example of the discrete time series modeling of the nonlinear price adjustment model of the security price, we consider a non-linear extension of (2.7) with

$$(2.11) P(t_i^n) - P(t_{i-1}^n) = g_1 \mathrm{I}(X(t_i^n) \geq P(t_{i-1}^n)) + g_2 \mathrm{I}(X(t_i^n) < P(t_{i-1}^n)) + u(t_i^n),$$

where g_i $(i = 1, 2)$ are some constants and $\mathrm{I}(\cdot)$ is the indicator function.
This model is the SSAR (simultaneous switching autoregressive) model investigated by Kunitomo and Sato (1996, 1999), which is similar to the threshold autoregressive (TAR) models developed by Tong (1990) having the delayed parameters. A set of sufficient conditions for the geometric ergodicity of the SSAR process is given by $g_1 > 0, g_2 > 0$ and $(1 - g_1)(1 - g_2) < 1$ with some moment condition on $u(t_i)$. If we set $g_1 = g_2 = g$, then we have the linear adjustment case as (2.7) and the geometrically ergodicity condition is given by $0 < g < 2$. The non-linear time series models such as the class of TAR models and the exponential AR models have been known in the non-linear time series analysis whose statistical properties have been extensively discussed by Tong (1990).

In order to consider the general nonlinar price adjustment models, which are different from the round-off error type models, we take $y_i = P(t_i^n)$ and

$$P(t_i^n) - P(t_{i-1}^n) = g\left(X(t_i^n) - P(t_{i-1}^n)\right) + u(t_i^n) \ , \tag{2.12}$$

where $g(\ \cdot\ )$ is a non-linear measurable function, and $u(t_i^n)$ is a sequence of i.i.d. micro-market noises with $\mathbf{E}[u(t_i^n)] = 0$ and $\mathbf{E}[u(t_i^n)^2] = \sigma_u^2$.

This representation includes the linear and non-linear price adjustment modes as special cases.

3. The SIML Estimation and Asymptotic Robustness Properties

3.1 The SIML Method

We summarize the derivation of the separating information maximum likelihood (SIML) estimation. The method was originally proposed by Kunitomo and Sato (2008, 2011) based on the basic model of (2.1) and (2.4) with a constant volatility under the Gaussian noise, but they are not needed to have the asymptotic properties of the SIML estimator.

Let $y(t_i^n)$ be the $i-$th observation of the (log-) price at t_i^n $(0 = t_0^n < t_1^n < \cdots < t_n^n = 1)$ and we set $\mathbf{y}_n = (y(t_i^n))$ be an $n \times 1$ vector of observations. The underlying (hidden) continuous-time stochastic process $X(t)$ $(0 \leq t \leq 1)$, which is not necessarily the same as the observed (log-)price at t_i^n $(i = 1, \cdots, n)$ and we set $u(t_i^n)$ being the micro-market noise at t_i^n. We consider the basic additive model of (2.1) and (2.4) when we have $y(t_i^n) = X(t_i^n) + u(t_i^n)$, where $X(t)$ $(0 \leq t \leq 1)$ and $u(t_i^n)$ $(i = 1, \cdots, n)$ are independent with $\sigma_x^2(s) = \sigma_x^2$ (time-invariant). Under the assumption that $u(t_i^n)$ are independently, identically and normally distributed as $N(0, \sigma_u^2)$ given the initial condition y_0, we have

$$\mathbf{y}_n \sim N_n\left(y(0)\mathbf{1}_n, \sigma_u^2\mathbf{I}_n + h_n\sigma_x^2\mathbf{C}_n\mathbf{C}_n'\right) \ , \tag{3.1}$$

where

$$
\text{(3.2)} \qquad \mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}^{-1} ,
$$

$\mathbf{1}_n' = (1, \cdots, 1)$ and $h_n = 1/n$ $(= t_i^n - t_{i-1}^n)$.
By transforming $\mathbf{y}_n$ to $\mathbf{z}_n$ $(= (z_k))$ by

$$
\text{(3.3)} \qquad \mathbf{z}_n = h_n^{-1/2} \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{y}_n - \bar{\mathbf{y}}_0)
$$

where $\bar{\mathbf{y}}_0 = y(0)\mathbf{1}_n$, $\mathbf{1}_n = (1, \cdots, 1)'$, $\mathbf{P}_n = (p_{jk})$ and for $j, k = 1. \cdots, n$,

$$
\text{(3.4)} \qquad p_{jk} = \sqrt{\frac{2}{n + \frac{1}{2}}} \cos\left[\frac{2\pi}{2n+1} (j - \frac{1}{2})(k - \frac{1}{2}) \right] .
$$

The transformed variables z_k $(k = 1, \cdots, n)$ are mutually independent and $z_k \sim N(0, \sigma_x^2 + a_{kn}\sigma_u^2)$. Then the log-likelihood function is given by

$$
\text{(3.5)} \qquad L_n = -\frac{1}{2} \sum_{k=1}^{n} \log[\sigma_x^2 + a_{kn}\sigma_u^2] - \frac{1}{2} \sum_{k=1}^{n} \frac{z_k^2}{\sigma_x^2 + a_{kn}\sigma_u^2} ,
$$

where

$$
\text{(3.6)} \qquad a_{kn} = 4n \sin^2 \left[\frac{\pi}{2} \left(\frac{2k-1}{2n+1} \right) \right] \ (k = 1, \cdots, n) .
$$

Because the ML estimator of unknown parameters is a complicated function of all observations and each a_{kn} terms depend on k as well as n, one way to have a simple solution is to approximate the likelihood function because a_{kn} are small for $k = 1, \cdots, m$ while a_{kn} dominate each terms for $k = n - l + 1, \cdots, n$ when m and l are in the smaller order than n.
Let m and l be positive integers, which are dependent on n and we write m_n and l_n. If we ignore the effects of a_{kn} $(k = 1, \cdots, m_n)$, then for the basic additive model we define the SIML estimator of $\hat{\sigma}_x^2$ by

$$
\text{(3.7)} \qquad \hat{\sigma}_x^2 = \frac{1}{m_n} \sum_{k=1}^{m_n} z_k^2 .
$$

By the similar reasoning, the SIML estimator of $\hat{\sigma}_u^2$ by

$$\hat{\sigma}_u^2 = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} z_k^2 \ . \tag{3.8}$$

In this estimation method we separate the information on σ_x^2 and σ_u^2 and the numbers of terms m_n and l_n are dependent on n such that $m_n, l_n \to \infty$ as $n \to \infty$. We impose the order requirement such that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for σ_x^2 and σ_u^2, respectively.

Furthermore, we can modify the SIML method such that the asymptotic bias can be removed. The modified SIML (MSIML) estimator of σ_x^2 is given by

$$\hat{\sigma}_{x,m}^2 = \frac{1}{m_n} \sum_{k=1}^{m_n} [z_k]^2 - \left[\frac{1}{m_n} \sum_{k=1}^{m_n} a_{kn} \right] \hat{\sigma}_u^2 \ , \tag{3.9}$$

where $\hat{\sigma}_u^2$ is given by (3.8).

This modification may be interpreted as approximation to the one proposed by Cai, T., A. Munk and J. Schmidt-Hieber (2010), which has an optimal property when the volatility function is constant and the noise terms follow the simple Gaussian process.

3.2 On Asymptotic Properties of the SIML estimator under the basic additive Model

It is important to investigate the asymptotic properties of the SIML estimator when the instantaneous volatility function $\sigma_x^2(s)$ is not constant over time. When the integrated volatility is a positive (deterministic) constant a.s. (i.e. σ_x^2 is not stochastic) while the instantaneous volatility function is time varying, we have the consistency and the asymptotic normality of the SIML estimator as $n \to \infty$. For the expository purpose, the asymptotic properties of the SIML estimator deterministic integrated volatility case, can be summarized as Proposition 3.1 whose proof has been given in Kunitomo and Sato (2008, 2013).

Proposition 3.1 : We assume that $X(t_i^n)$ and $u(t_i^n)$ $(i = 1, \cdots, n)$ in (2.1) and (2.4) are independent, $|\mu_s|$ is bunded, $\sigma_x(s)$ is a deterministic and $\sigma_x^2 = \int_0^1 \sigma_x^2(s) ds$,

$\sup_{0\leq s\leq 1}\sigma_x^4(s) < \infty$ and $\mathbf{E}[u(t_i^n)^4] < \infty$. Define the SIML estimator $\hat{\sigma}_x^2$ of σ_x^2 by (3.7).

(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\sigma}_x^2 - \sigma_x^2 \xrightarrow{p} 0 \ . \tag{3.10}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] \xrightarrow{d} N\left[0, V\right] \ , \tag{3.11}$$

where

$$V = 2\int_0^1 \sigma_x^4(s)ds \ . \tag{3.12}$$

When the instantaneous volatility $\sigma_x^2(s)$ $(= \sigma_x^2)$ is constant, then

$$V = 2\sigma_x^4 \ . \tag{3.13}$$

When σ_x^2 is a random variable, it may be convenient to assume that the process $\sigma_x(t)$ is a Brownian semi-martingale. In this case we need to use the concept of *stable convergence in law* and the limiting distribution of the SIML estimator follows the mixed-Gaussian distribution under a set of reasonable assumptions. For the stochastic volatility case in the continuous time, we assume that $\sigma_x(t)$ follows

$$\begin{aligned} \sigma_x(t) &= \sigma_x(0) + \int_0^t \mu_\sigma(s)ds + \int_0^t \gamma_\sigma(s)dB(s) \\ &\quad + \int_0^t \gamma_\sigma^*(s)dB^*(s) \ , \end{aligned} \tag{3.14}$$

where the coefficients $\mu_\sigma(s), \gamma_\sigma(s)$ and $\gamma_\sigma^*(s)$ are extensively measurable, continuous and bounded, and $B^*(s)$ is a Brownian motion which is orthogonal to $B(s)$.
We extend the probability space $(\Omega, \mathcal{F}, P)$ to the extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as explained by Chapter VIII of Jacod and Shyiyaev (2003) or Jacod and Protter (2012). Then we say that a sequence of random variables Z_n with an index n stably converges in law if $\mathbf{E}[Yf(Z_n)] \longrightarrow \tilde{\mathbf{E}}[Yf(Z)]$ for all bounded continuous functions

$f(\cdot)$ and all bounded random variables Y, and $\tilde{\mathbf{E}}[\cdot]$ is the expectation operator with respect to the extended probability space. We write this convergence as

$$Z_n \xrightarrow{\mathcal{L}-s} Z \ . \tag{3.15}$$

The asymptotic properties of the SIML estimator in the stochastic volatility cases can be summarized as Theorem 3.2 and the proof will be given in Appendix-A.

Theorem 3.2 : We assume that $X(t)$ and u_i $(i = 1, \cdots, n)$ in (2.1) and (2.4) are independent, $|\mu_x(s)|$ is bounded, $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds > 0$ is positive (a.s.) with $\sup_{0 \le s \le 1} \mathbf{E}[\sigma_x^4(s)] < \infty$, and $\mathbf{E}[u(t_i^n)^4] < \infty$. Define the SIML estimator $\hat{\sigma}_x^2$ of σ_x^2 by (3.7).
(i) For $m_n = n^\alpha$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\sigma}_x^2 - \sigma_x^2 \xrightarrow{p} 0 \ . \tag{3.16}$$

(ii) For $m_n = n^\alpha$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$ we have the weak convergence

$$Z_n = \sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] \xrightarrow{\mathcal{L}-s} Z^* \ , \tag{3.17}$$

where Z^* is a continuous process defined on a very good filtered extension of $(\Omega, \mathcal{F}, P)$, which conditionally on the $\sigma-$field $\mathcal{F}$ is a Gaussian proces with independent increments satisfying

$$V_t = \tilde{\mathbf{E}}[(Z_t^*)|\mathcal{F}] = 2\int_0^t \sigma_x^4(s)ds \tag{3.18}$$

for $0 < t \le 1$.

In the present situation it has been known that the asymptotic bound of the convergence rate is $n^{0.5}$. Since the convergence rate of the SIML estimator is $n^{0.4}$, it is sub-optimal. However, it is possible to improve the convergence rate and we have the following result on the MSIML estimation.

Corollary 3.3 : Define the MSIML estimator $\hat{\sigma}_{x,m}^2$ of σ_x^2 by (3.9). Under Assumptions of Proposition 3.1 or Theorem 3.2, for $m_n = n^\alpha$ and $0 < \alpha < 0.5$, as

$n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{x,m}^2 - \sigma_x^2\right] \xrightarrow{\mathcal{L}-s} Z^* \ , \tag{3.19}$$

where Z^* is defined as the same as in Theorem 3.2.

3.3 On Asymptotic Robustness of the SIML estimator for the round-off error and micro-market price adjustment models

We investigate the asymptotic properties of the SIML estimation under the round-off error models and the micro-market adjustment models. First, we investigate the situation of the basic round-off model in Section 2.2 when we have a sequence of discrete observations $P(t_i^n)$ with $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ and we estimate the integrated volatility of the underlying security $\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds$.

Let $[x]$ and $\{x\}$ be the integer part and the fractional part of a real number x, respectively. If there were no micro-market noise term in (2.5), we can decompose

$$X(t_i^n) = \eta[\frac{X(t_i^n)}{\eta}] + \eta\{\frac{X(t_i^n)}{\eta}\} \tag{3.20}$$

for $i = 1, \cdots, n$. Then we have $y(t_i^n) = \eta[\frac{X(t_i^n)}{\eta}]$ and

$$\begin{aligned} \sum_{i=1}^n (\Delta y(t_i^n))^2 &= \sum_{i=1}^n (\Delta X(t_i^n))^2 + \eta^2 \sum_{i=1}^n (\{\frac{X(t_i^n)}{\eta}\} - \{\frac{X(t_{i-1}^n)}{\eta}\})^2 \\ &\quad -2\eta \sum_{i=1}^n (\Delta X(t_i^n))(\{\frac{X(t_i^n)}{\eta}\} - \{\frac{X(t_{i-1}^n)}{\eta}\}) \ . \end{aligned} \tag{3.21}$$

We set the threshold parameter $\eta = \eta_n$, which is dependent on n. If it satisfies the condition

$$\eta_n \sqrt{n} = o(1) \ , \tag{3.22}$$

then the first term of (3.23) converges to the integrated volatility σ_x^2 as $n \to \infty$ since the realized volatility is a consistent estimator of the integrated volatility when we do not have micro-market noise. However, if this condition is not satisfied and also there is a micro-market noise term at the same time, it is not obvious how to estimate the integrated volatility. Since n is finite in practice, we need to search a weak condition on the threshold parameter η_n and in this respect we have the next result.

Theorem 3.4 : We assume that in (2.1) and (2.2), $\mu_x(s)$ and $\sigma_x(s)$ is bounded and i.e., there exist μ^*, σ_* and σ^* such that $|\mu_x(s)| < \mu^*, 0 < \sigma_* < \sigma_x(s) < \sigma^*$. Also we assume that in (2.5) and (2.6) there exists γ ($\gamma > 0$) such that the threshold parameter $\eta = \eta_n$ satisfies

$$\eta_n n^{\gamma} = O(1) \ . \tag{3.23}$$

Define the SIML estimator of the integrated volatility of $X(t)$ with $m_n = n^{\alpha}$ ($0 < \alpha < 0.4$) by (3.7). Then the limiting distribution of the normalized estimator $\sqrt{m_n}\,[\hat{\sigma}_x^2 - \sigma_x^2]$ is asymptotically ($m_n, n \to \infty$) equivalent to the limiting distributions given by Proposition 3.1 and Theorem 3.2 under their assumptions.

As the second case we consider the linear price-adjustment model (2.7) and the observed price at t_i^n ($i = 1, \cdots, n$) can be expressed as

$$\begin{aligned} P(t_i^n) &= (1-g)P(t_{i-1}^n) + gX(t_i^n) + u(t_i^n) \\ &= g\sum_{j=0}^{i-1}(1-g)^j X(t_{i-j}^n) + \sum_{j=0}^{i-1}(1-g)^j u(t_{i-j}^n) + (1-g)^i P(t_0^n) \ , \end{aligned} \tag{3.24}$$

which is a weighted linear combination of past intrsic values and past noise terms. In this case, however, from (2.7) we notice that

$$\begin{aligned} P(t_i^n) - u(t_i^n) - X(t_i^n) &= (1-g)\left[P(t_{i-1}^n) - X(t_i^n)\right] \\ &= (1-g)\left[P(t_{i-1}^n) - X(t_{i-1}^n) - u(t_{i-1}^n)\right] \\ &\quad +(1-g)\left[u(t_{i-1}^n) - \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s)dB_s\right] \ . \end{aligned} \tag{3.25}$$

Then if we define a sequence of random variables $U_a(t_i^n) = P(t_i^n) - u(t_i^n) - X(t_i^n)$ and $W^*(t_i^n) = (1-g)[u(t_{i-1}^n) - \int_{t_{i-1}^n}^{t_i^n} \mu_x(s)ds - \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s)dB_s]$, then we can represent the present model as

$$U_a(t_i^n) = (1-g)U_a(t_{i-1}^n) + W^*(t_i^n) \ . \tag{3.26}$$

Although $W^*(t_i^n)$ are correlated with $U_a(t_{i-1}^n)$, the linear price-adjustment model can be regarded as an extension of the basic model of (2.1) and (2.4). Then we have the next result on the limiting distribution of the SIML estimator whose the proof will be given in Appendix A.

Theorem 3.5 : We assume that $X(t)$ and u_i $(i = 1, \cdots, n)$ in (2.1) and (2.7) are independent, and g $(0 < g < 2)$ in (2.7) is a constant. Define the SIML estimator of the integrated volatility of $X(t)$ with $m_n = n^\alpha$ $(0 < \alpha < 0.4)$ by (3.7). Then the asymptotic distribution of $\sqrt{m_n}\,[\hat{\sigma}_x^2 - \sigma_x^2]$ is asymptotically $(m_n, n \to \infty)$ equivalent to the limiting distributions given by Proposition 3.1 and Theorem 3.2 under their assumptions.

We notice that the present micro-market (linear) adjustment model is quite similar to the structure of the micro-market model with autocorrelated micro-market noises, which was discussed in Kunitomo and Sato (2011).

Third, we investigate the situation when we have a sequence of price adjustments with the round-off error effect introduced in Section 2.4. Define

$$U_a(t_i^n) = P(t_i^n) - X(t_i^n) - u(t_i^n) \ . \tag{3.27}$$

When $|P(t_{i-1}^n) - X(t_i^n) - u(t_i^n)| > \eta$, then by using the fact that $x + \eta > x \geq g_\eta(x) > x - \eta$, we have $|U_a(t_i^n)| \leq \eta$. On the other hand, when $|P(t_{i-1}^n) - X(t_i^n) - u(t_i^n)| \leq \eta$, then $P(t_i^n) = P(t_{i-1}^n)$ and $|U_a(t_i^n)| \leq \eta$.
Define $y_i = P(t_i^n)$ and $v_i = u(t_i^n) + U_a(t_i^n)$ $(i = 1, \cdots, n)$, we have $y_i = x_i + v_i$ and

$$|U_a(t_i^n)| \leq \eta \ . \tag{3.28}$$

By using the similar arguments to the results reported as Theorems 3.1 and 3.2 on the limiting distribution of the integrated volatility estimator we have the next result and the proof will be given in Appendix A.

Theorem 3.6 : We assume that $X(t)$ and u_i $(i = 1, \cdots, n)$ in (2.1) and (2.8) are independent. The threshold parameter $\eta = \eta_n$ depends on n satisfying

$$\eta_n \sqrt{n} = O(1) \ . \tag{3.29}$$

Define the SIML estimator of the integrated volatility of $X(t)$ with $m_n = n^{\alpha}$ ($0 < \alpha < 0.4$) by (3.7). Then the limiting distribution of the normalized estimator $\sqrt{m_n}\,[\hat{\sigma}_x^2 - \sigma_x^2]$ is asymptotically ($m_n, n \to \infty$) equivalent to the limiting distributions given by Proposition 3.1 and Theorem 3.2 under their assumptions.

In the above theorem we have imposed the condition (3.30) on η, which is weaker than (3.22), but stronger than (3.23). This condition could be relaxed because our simulations have suggested that the asymptotic result does not essentially depend upon the this condition.

Finally, we shall investigate the situation when we have a sequence of discrete observations under the non-linear adjustment model given by (2.12). We set $y(t_i^n) = P(t_i^n)$ and use a sequence of differences of $X(t_i^n)$ as $\Delta X(t_i^n) = X(t_i^n) - X(t_{i-1}^n) = \int_{t_{i-1}^n}^{t_i^n} \mu_x(s)ds + \int_{t_{i-1}^n}^{t_i^n} \sigma_x(s)dB_s$.
Also we let

$$U_a(t_i^n) = P(t_i^n) - [X(t_i^n) + u(t_i^n)] \tag{3.30}$$

and

$$w(t_i^n) = -\Delta X(t_i^n) + u(t_{i-1}^n) \ . \tag{3.31}$$

The adjustment process $U_a(t_i^n)$ is the difference between the observed price and the true process with noise, which describes the additional price adjustment mechanism because $P(t_i^n) - X(t_i^n) = u(t_i^n) + U_a(t_i^n)$.
If there is no drift term (i.e. $\mu_x(s) = 0$), $w(t_i^n)$ are a sequence of uncorrelated random variables when $\Delta X(t_i^n)$ and $u(t_i^n)$ are independent. The difference between the observed price and the underlying intrisic value plus noise can be represented as

$$\begin{aligned} U_a(t_i^n) &= U_a(t_{i-1}^n) + w(t_i^n) + g\left[-U_a(t_{i-1}^n) - w(t_i^n)\right] \\ &= g^*\left[U_a(t_{i-1}^n) + w(t_i^n)\right] \ , \end{aligned} \tag{3.32}$$

where $g^*(z) = z + g(-z)$, $\mathbf{E}[w(t_i^n)] = 0$ and $\mathbf{E}[w(t_i^n)^2] < \infty$.
When we have an (ergodic) stationary solution for $U_a(t_i^n)$, we can reduce (2.13) and (2.14) as the signal-plus-noise stationary process such that $y_i = x_i + v_i$ $(i = 1, \cdots, n)$, where we set $y_i = P(t_i^n)$, $x_i = X(t_i^n)$ and $v_i = U_a(t_i^n) + u(t_i^n)$. But by our construction we have the situation when the signal term x_i and the noise term v_i are mutually correlated and v_i are autocorrelated over time. Because the discrete time series $U_a(t_i^n)$ satisfies the stochastic difference equation, it has a Markovian property. In order to investigate the limiting behavior of the volatility estimation, we need a set of sufficient conditions, which are some type of ergodic condition. We summarize our results under some additional conditions with the nonlinear price adjustments and the proof will be given in Appendix A.

Theorem 3.7 : We assume that $X(t)$ and u_i $(i = 1, \cdots, n)$ in (2.1) and (2.12) are independent. For $U_a(t_i^n)$ satisfying (3.32) and $\mathbf{E}[U_a(t_i^n)^4] < \infty$, we further assume that there exist functions $\rho_1(\,\cdot\,)$ and $\rho_2(\,\cdot, \cdot\,)$ such that

$$\text{(3.33)} \qquad \mathrm{Cov}[U_a(t_i^n), U_a(t_j^n)] = c_1 \rho_1(|i - j|) \ ,$$

where c_1 is a (positive) constant and $\sum_{s=0}^{\infty} \rho_1(s) < \infty$ and

$$\text{(3.34)} \mathrm{Cov}\left[U_a(t_i^n) U_a(t_{i'}^n), U_a(t_j^n) U_a(t_{j'}^n))\right] = c_2 \rho_2(i - i^{'}, j - j^{'}) \text{ for } j > j^{'} > i > i^{'} ,$$

where c_2 is a (positive) constant and $\sum_{s,s^{'}=0}^{\infty} \rho_2(s, s^{'}) < \infty$.
Define the SIML estimator of the realized volatility of $P(t_i^n)$ with $m_n = n^{\alpha}$ $(0 < \alpha < 0.4)$ by (3.7). Then the asymptotic distribution of $\sqrt{m_n}\,[\hat{\sigma}_x^2 - \sigma_x^2]$ is asymptotically (as $m_n, n \to \infty$) equivalent to the limiting distributions given by Proposition 3.1 and Theorem 3.2 under their assumptions.

In the above theorem we impose a set of sufficient conditions for (3.34), which is a strong-mixing condition. If we can find positive constants ρ $(|\rho| < 1)$ and c_3 such that the conditional expectation

$$\text{(3.35)} \qquad |\mathbf{E}[U_a(t_j^n) U_a(t_{j'}^n) | U_a(t_i^n), U_a(t_{i'}^n)]| < c_3 \rho^{|j - j^{'}|}$$

for any $j > j^{'} > i > i^{'}$, then we have (3.34) and (3.35).

There are many discrete time series models in statistics satisfying the ergodicity conditions. A simple example is the linear case when $g(x) = c\, x$ (c is a constant with $0 < c < 2$ and $U_a(t_i^n)$ are weakly dependent process. It is straightforward to have the above conditions in this case based on the arguments which are similar to the derivations in Chapter 8 of Anderson (1971). The second example is the SSAR(1) model with (2.11). If we impose the strong condition such as $0 < g_1, g_2 < 1$, it is also straightforward to use the arguments for linear processes. According to our simulations, however, it is often too strong to have the desired results and it may be an interesting future topic. There can be a large number of non-linear price adjustment models and non-linear discrete time series models for $X(t_i^n)$ and $P(t_i^n)$.

4. Simulations

We have investigated the robustness properties of the SIML estimator for the integrated volatility based on a set of simulations and the number of replications is 1,000. We have taken the sample size $n = 20,000$, and we have chosen $\alpha = 0.4$(or 0.45) and $\beta = 0.8$ in all cases. The details of the simulation procedure are similar to the corresponding ones reported by Kunitomo and Sato (2008, 2011).

In our simulations we consider several cases when the observations are generated by (2.1) and (2.3). For the simplicity, we set $\mu_x(s) = 0$ and the volatility function $(\sigma_x^2(s))$ is given by

$$\sigma_x^2(s) = \sigma(0)^2 \left[a_0 + a_1 s + a_2 s^2 \right], \tag{4.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. It is a typical time varying (but deterministic) case and the integrated volatility σ_x^2 is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{4.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the

U-shaped movements. We have omitted the stochastic integrated volatility case in this paper. But as Kunitomo and Sato (2011) have shown, we have found that the results reported are essentially not changed even when the instantaneous volatility is a stochastic process.

Among many Monte-Carlo simulations, we summarize our main results as Tables in Appendix B. We have used several models in the form of (2.1) and (2.3); each model corresponds to the typical cases when we take $h(\cdot,\cdot,\cdot)$ in (2.3) as

$$
\begin{aligned}
&\text{Model 1} && h_1(x,y,u) = g_\eta(x+u) \ \ (g_\eta(\cdot) \text{ is } (2.6))\ , \\
&\text{Model 2} && h_2(x,y,u) = y + g(x-y) + u \ (g : \text{a constant})\ , \\
&\text{Model 3} && h_3(x,y,u) = y + g_\eta(x-y+u) \ \ (g_\eta(\cdot) \text{ is } (2.6))\ , \\
&\text{Model 4} && h_4(x,y,u) = y + g_\eta(x-y) + u \ \ (g_\eta(\cdot) \text{ is } (2.6))\ , \\
&\text{Model 5} && h_5(x,y,u) = y + u + \begin{cases} g_1(x-y) & \text{if } y \geq 0 \ (g_1 : \text{a constant}) \\ g_2(x-y) & \text{if } y < 0 \ (g_2 : \text{a constant}) \end{cases}\ , \\
&\text{Model 6} && h_6(x,y,u) = y + \left[g_1 + g_2 \exp(-\gamma|x-y|^2)\right](x-y) \\
& && \qquad\qquad (g_1, g_2 : \text{constants})\ , \\
&\text{Model 7} && h_7(x,y,u) = y + h_2 \ o \ h_4 \ o \ h_1(x,y,u)\ ,
\end{aligned}
$$

respectively.

Model 1 is the basic round-off error model in Section 2.2. Model 2 corresponds to the linear price adjustment model with the micro-market noise when the adjustment coefficient g is a constant. When $0 < g < 2$, Model 2 corresponds to the stationary linear price adjustment model with the micro-market noise. Model 1, Model 3 and Model 4 are the micro-market models with the round-off errors. Model 1 is the basic round-off error model while Model 3 and Model 4 are the round-off errors models with price adjustment mechanisms. Model 5 and Model 6 are the SSAR model and the exponential AR model, which have been known as nonlinear (discrete) time series models. Model 7 is a combination of three nonlinear models with micro-market noise, the round-off errors and the non-linear adjustments at the same time, which corresponds to the most complicated nonlinear model among our examples. As an illustration on our simulatons we show two typical simulation (sample) paths by using the basic round-off model in Figure 2-3 among many examples.

For a comparison we have calculated and compared the historical integrated volatility (HI) estimate and the SIML estimate in each table. Overall the estimates of the SIML method are quite stable and robust against the possible values of the variance ratio even in the nonlinear transformations we have considered. For Model-1, the estimates obtained by historical-volatility (H-vol) are badly-biased, which have been known in the analysis of high frequency financial data. Actually, the values of H-vol are badly-biased in all cases of our simulations while the SIML method gives reasonable estimates in all cases of Model-1 to Model 7. (See Tables B-1$\sim$B-12.) We give some representative cases as Table 3 to Table 12 and in these tables the estimates of the SIML method are quite stable and robust against the possible values of the variance ratio even in the nonlinear transformations we considered. The bias of the SIML estimator is often small and the variance of the SIML estomator is often stable.

We also have compared the SIML estimates, the Realized Kernel (RK) estimates and the Pre-Averaging (PA) estimates, which were developed by Bandorff-Nielsen et al. (2008) and Jacod et al. (2009), respectively. In order to make a fair comparison we have tried to follow the recommendation by Bandorff-Nielsen et al. (2008) on the choice of kernel (Tukey-Hanning) and the band width parameter H in the RK method and we took the triangular function $g(x)$, $\theta = 1$ and $K = \sqrt{n}$ in the PA method. One important issue in the RK method has been to choose H, which depends on the noise variance and the instantaneous variance (which are actually unknown) and we can interpret as $H = c\sqrt{\sigma_u^2/[\sigma_x^2/n]}$ because σ_u^2 and σ_x^2 should not be known in advance. In the basic round-off model the biases of the RK method and the PA method can be relatively large while the SIML method does not have much bias in some situations. (See Table B-1 and B-2 inthis respect.) Also we have found that the RK and PA estimation gives reasonable estimates in some basic cases if we had taken the reasonable value of the key parameters H, θ and K in many cases.

By examining these results reported and other simulations we conclude that we can estimate the integrated volatility of the hidden martingale part reasonably by the

SIML estimation method despite of the possible non-linear transformations. It may be surprising to find that the SIML method gives reasonable estimates even when we have nonlinear transformations of the original unobservable security (intrinsic) values even in some cases when the biases of the RK method and the PA method are not negligible. We also have conducted a large number of further simulations, but the results are quite similar as we have reported in this section.

5. Conclusions

In this paper, we have shown that the Separating Information Maximum Likelihood (SIML) estimator has the asymptotic robustness in the sense that it is consistent and it has *the stable convergence* in the stochastic integrated volatility cases and the asymptotic normality in the sense of stable convergence under a fairly general conditions even when the standard additive models were not necessarily true and the underlying conditions are not satisfied. They include not only the cases when the micro-market noises are possibly autocorrelated and they are endogenously correlated with the underlying continuous signal process, but also the cases when the micro-market structure has the round-off errors and the nonlinear adjustments under a set of reasonable assumptions. The condition on the threshold parameter in the round-off error models are rather weak for the SIML estimation. The micro-market factors in actual financial markets are common in the sense that we have the minimum price change and the minimum order size rules and also we often observe the bid-ask differences in financial markets. Therefore the robustness of the estimation methods of the integrated volatility is quite important. By conducting a large number of simulations, we have confirmed that the SIML estimator has reasonable and robust properties in finite samples even in the non-standard situations.

As a concluding remark, we should stress on the fact that the SIML estimator is very simple and it can be practically used not only for the integrated volatility but also the integrated covariance and the hedging coefficients from the multivariate high frequency financial series. (See Kunitomo, Misaki and Sato (2015).) An applications on the analysis of stock-index futures market has been reported in Kunitomo and

Sato (2011).

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APPENDIX A : Mathematical Derivations of Theorems

In Appendix A, we give some details of the proofs of Theorems in Section 3.2. The methods of proofs are based on the modifications of ones given by Kunitomo and Sato (2008, 2013). We use the notation K_i $(i \geq 1)$ as positive constants in the following analysis.

The proof of Theorem 3.2 : The proof consists of three major steps. Some of our arguments are based on those developed in Kunitomo and Sato (2008, 2013), which are referred at the appropriate places but their details are often omitted.
(**Step 1**) We shall investigate the asymptotic properties of the SIML estimator in several steps. The first step is to investigate the conditions that the measurement errors can be stochastically negligible.

We write $y_i = x_i + v_i$ and $v_i = u(t_i^n)$, where $y_i = P(t_i^n)$ and $x_i = X(t_i^n)$ in the basic additive model for the proof of Theorem 3.2. We write the underlying (unobservable) returns in the period $(t_{i-1}, t_i]$ as

$$\text{(A.1)} \qquad r_i^* = x_i - x_{i-1} = \int_{t_{i-1}}^{t_i} \mu_x(s)ds + \int_{t_{i-1}}^{t_i} \sigma_x(s)dB_s \ \ (i = 1, \cdots, n)$$

and the martingalepart as

$$\text{(A.2)} \qquad r_i = \int_{t_{i-1}}^{t_i} \sigma_x(s)dB_s \ \ (i = 1, \cdots, n)$$

with $0 = t_0 \leq t_1 < \cdots < t_n = 1$ and $t_i - t_{i-1} = 1/n$ $(i = 1, \cdots, n)$. We note that the (instantaneous) volatility function $\sigma_x(s)$ $(0 \leq s \leq 1)$ and the integrated volatility $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds$ can be stochastic.
Under the boundedness assumption of the drift term $\mu_x(s)$, there exists a constant K_1 such that

$$\text{(A.3)} \qquad |r_i - r_i^*| = |\int_{t_{i-1}}^{t_i} \mu_x(s)ds| \leq K_1(\frac{1}{n})$$

because $t_i^n - t_{i-1}^n = 1/n$. Then the rest of our arguments remains the same as if we use r_i instead of r_i^* $(i = 1, \cdots, n)$.
(**Step 2**) Let $z_{in}^{(1)}$ and $z_{in}^{(2)}$ $(i = 1, \cdots, n)$ be the i-th elements of $n \times 1$ vectors

$$\text{(A.4)} \qquad \mathbf{z}_n^{(1)} = h_n^{-1/2}\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{x}_n - \bar{\mathbf{y}}_0) \ , \ \mathbf{z}_n^{(2)} = h_n^{-1/2}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{v}_n,$$

respectively, where we denote $\mathbf{x}_n = (x_i)$, $\mathbf{v}_n = (v_i)$, $\mathbf{z}_n = (z_{in})$ are $n \times 1$ vectors with $z_{in} = z_{in}^{(1)} + z_{in}^{(2)}$ and $\mathbf{P}_n$ is defined by (3.4).

Then by following the method developed by Kunitomo and Sato (2008, 2013), we use the arguments for investigating the effects of the (possibly) autocorrelated noise term on the asymptotic distribution of $\hat{\sigma}_x^2 - \sigma_x^2$ and $\sigma_x^2 = \int_0^1 \sigma_x^2(s)ds$. We use the decomposition

$$
\text{(A.5)} \quad \sqrt{m_n}\left[\hat{\sigma}_x^2 - \sigma_x^2\right] = \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n} z_{kn}^2 - \sigma_x^2\right]
$$

$$
= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n} z_{kn}^{(1)2} - \sigma_x^2\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{E}[z_{kn}^{(2)2}]
$$

$$
+\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}]\right] + 2\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[z_{kn}^{(1)} z_{kn}^{(2)}\right] .
$$

Then we shall investigate the conditions that three terms except the first one of (A.5) are $o_p(1)$. If they are satisfied, we could estimate the integrated volatility consistently as if there were no noise terms because other terms can be ignored asymptotically as $n \to \infty$.

Let $\mathbf{b}_k = \mathbf{e}_k^{'}\mathbf{P}_n\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0, \cdots, 1, 0, \cdots)$ be an $1 \times n$ vector. We write $z_{kn}^{(2)} = \sqrt{n}\sum_{j=1}^{n} b_{kj} v_j$ and use the relation

$$
(\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'})_{k,k'} = \delta(k,k^{'})d_k = \delta(k,k^{'})4\sin^2[\frac{\pi}{2n+1}(k-\frac{1}{2})].
$$

Then we have $\sum_{j=1}^{n} b_{kj}b_{k'j} = \delta(k,k^{'})d_k$ and $n\sum_{j=1}^{n} b_{kj}^2 = a_{kn}$ $(k = 1, \cdots, n)$ (see (3.6)). Also we shall set the notation K_i $(i \geq 2)$ as positive constants which are chosen appropriately.

First, it is straightforward to find

$$
\text{(A.6)} \qquad \mathbf{E}[([z_{kn}^{(2)})^2] = n\mathbf{E}[\sum_{i=1}^{n} b_{ki}v_i \sum_{j=1}^{n} b_{kj}v_j] \leq K_2 \times a_{kn} ,
$$

provided that $\mathbf{E}(v_i^2)$ are bounded and we use the notation $b_{kj} = 0$ $(j \leq 0)$. By using (3.6) and the trigonometric relation $\sin x = x - (1/6)x^3 + (1/120)x^5 + o(x^7)$,

$$
\text{(A.7)} \qquad \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} = \frac{1}{m_n}2n\sum_{k=1}^{m_n}\left[1 - \cos(\pi\frac{2k-1}{2n+1})\right]
$$

$$= \frac{n}{m_n}\left[2m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}}\right]$$

$$\sim \frac{n}{m_n}\left[2m_n - \frac{(\pi\frac{2m_n}{2n+1}) - \frac{1}{6}(\pi\frac{2m_n}{2n+1})^3}{(\frac{\pi}{2n+1}) - \frac{1}{6}(\frac{\pi}{2n+1})^3}\right]$$

$$= O(\frac{m_n^2}{n})$$

Then the second term of (A.5) becomes

$$\text{(A.8)} \qquad \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{E}[z_{kn}^{(2)}]^2 \le K_2\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}a_{kn} = O(\frac{m_n^{5/2}}{n})$$

if $0 < \alpha < 0.4$.

For the fourth term of (A.5),

$$\mathbf{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n}z_{kn}^{(1)}z_{kn}^{(2)}\right]^2 = \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathbf{E}\left[z_{kn}^{(1)}z_{k',n}^{(1)}z_{kn}^{(2)}z_{k',n}^{(2)}\right]$$

$$\text{(A.9)} \qquad = \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathbf{E}\left[2\sum_{j,j'=1}^{n}s_{jk}s_{j'k'}\mathbf{E}(r_jr_{j'}|\mathcal{F}_{n,\min(j,j')})z_{kn}^{(2)}z_{k'n}^{(2)}\right]$$

$$= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathbf{E}\left[2\sum_{j=1}^{n}s_{jk}s_{j,k'}\mathbf{E}(r_j^2|\mathbf{F}_{n,j-1})z_{kn}^{(2)}z_{k',n}^{(2)}\right]$$

$$\le K_3\left[\sup_{0\le s\le 1}\mathbf{E}(\sigma_x^2(s))\right]\frac{2}{n}(\frac{n}{2}+\frac{1}{4})\frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn}$$

$$= O(\frac{m_n^2}{n})\ ,$$

where $\mathcal{F}_{n,j-1}$ is the $\sigma-$field generated by the information given at t_{j-1} for a fixed n and

$$s_{jk} = \cos[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})]$$

for $j,k = 1,2,\cdots,n$. (See Lemma 1.3 of Kunitomo and Sato (2008).) In the above evaluation we have used the relation

$$|\sum_{j=1}^{n}s_{jk}s_{j,k'}| \le [\sum_{j=1}^{n}s_{jk}^2] = n/2+1/4 \ \text{ for any } k \ge 1\ .$$

Hence we need the condition $0 < \alpha < 0.5$.

For the third term of (A.5), we need to consider the variance of

$$z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}] = n\sum_{j,j'=1}^{n}b_{kj}b_{k,j'}\left[v_jv_{j'} - \mathbf{E}(v_jv_{j'})\right]$$

and we evaluate the expectation of $\left[z_{kn}^{(2)2}-\mathbf{E}[z_{kn}^{(2)2}]\right]\left[z_{k',n}^{(2)2}-\mathbf{E}[z_{k',n}^{(2)2}]\right]$.
By using the independence assumption on v_i $(i=1,\cdots,n)$, there exists a positive constant K_4 such that

$$\text{(A.10)}\qquad \mathbf{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}(z_{kn}^{(2)2}-\mathbf{E}[z_{kn}^{(2)2}])\right]^2 \le K_4\frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn}^2 = O(\frac{m_n^4}{n^2})\ .$$

since $\sum_{k=1}^{m}a_{kn}=O(m_n^3/n)$.
Thus the third term of (A.5) is negligible if $0<\alpha<0.5$.

(**Step 3**) Let $Z_{t,k}^{*}=\sum_{j=1}^{[nt]}p_{kj}r_j$ for any $0<t\le 1$ and $1\le k\le n$. Then we can represent

$$\text{(A.11)}\qquad Z_{t,k}^{*}=\int_0^t\left[\sum_{j=1}^{[nt]}p_{kj}I(t_{j-1}^n<s\le t_j^n)\right](\mu_s ds+\sigma_s dB)\ .$$

Then by using Ito's Lemma

$$\text{(A.12)}\qquad \begin{aligned} V_{t,m} &= \sum_{k=1}^{m}(Z_{t,k}^{*})^2 \\ &= \int_0^t\left[\sum_{j=1}^{[nt]}p_{kj}I(t_{j-1}^n<s\le t_j^n)\right]^2\sigma_s^2 ds \\ &\quad +2\int_0^t\left[\sum_{j=1}^{[nt]}Z_{s,k}^{*}p_{kj}I(t_{j-1}^n<s\le t_j^n)\right](\mu_s ds+\sigma_s dB)\ . \end{aligned}$$

Hence we have

$$\text{(A.13)}\qquad \begin{aligned} &\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}z_{kn}^{(1)2}-\sigma_x^2\right] \\ &= \sum_{j=1}^{n}\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}(\sqrt{n}p_{kj})^2-1\right]\int_{t_{j-1}^n}^{t_j^n}\sigma_s^2 ds \\ &\quad +2\sqrt{m_n}\sum_{j=1}^{n}\int_{t_{j-1}^n}^{t_j^n}\left[\frac{n}{m_n}\sum_{k=1}^{m_n}Z_{s,k}^{*}p_{kj}\right](\mu_s ds+\sigma_s dB)\ . \end{aligned}$$

Then we need to evaluate each term of the right-hand side. First, since $p_{kj}=\sqrt{[4/(2n+1)]}\cos[(2\pi/(2n+1))(k-1/2)(j-1/2)]$, we find that

$$\text{(A.14)}\qquad \sqrt{m_n}\left[\frac{n}{m_n}\sum_{k=1}^{m_n}p_{kj}^2-1\right]=o(1)\ .$$

(See Lemma 5 of Kunitomo and Sato (2013).)

Second, because of the boundedness assumption of μ_s $(0 < s \leq 1)$, we can show that the effects of the drift term are negligible. Thus the only term which we need to evaluate the asymptotic distribution of the SIML estimator is $U_n = \sum_{j=1}^{n} X_{nj}$ and

$$\text{(A.15)} \qquad X_{nj} = 2\sqrt{m_n} \int_{t_{j-1}^n}^{t_j^n} \left[\frac{n}{m_n} \sum_{k=1}^{m_n} Z_{s,k}^* p_{kj} \right] \sigma_s dB \; .$$

We notice that X_{nj} can be rewritten as $X_{nj} = [2\sum_{i=1}^{j-1} \sqrt{m_n} c_{ij} r_i] r_j$, where $r_j = \int_{t_{j-1}}^{t_j} \sigma_s dB_s$ $(1 \leq j \leq n)$ is a sequence of discrete martingales and

$$\begin{aligned}
c_{ii} &= (\frac{2n}{2n+1}) \left[1 + \frac{1}{m_n} \frac{\sin 2\pi m_n (\frac{i-1/2}{2n+1})}{\sin(\pi \frac{i-1/2}{2n+1})} \right] \; , \\
c_{ij} &= \frac{1}{2m_n} (\frac{2n}{2n+1}) \left[\frac{\sin 2\pi m_n (\frac{i+j-1}{2n+1})}{\sin(\pi \frac{i+j-1}{2n+1})} + \frac{\sin 2\pi m_n (\frac{j-i}{2n+1})}{\sin(\pi \frac{j-i}{2n+1})} \right] \quad (i \neq j) \; .
\end{aligned}$$

Kunitomo and Sato (2008, 2013) have also shown that the variance of the limiting distribution in the simle case is the limit of

$$\text{(A.16)} \qquad V_n = 2 \sum_{i,j=1}^{n} m_n c_{ij}^2 \int_{t_{i-1}}^{t_i} \sigma_x^2(s) ds \int_{t_{j-1}}^{t_j} \sigma_x^2(s) ds \; .$$

Because we have assumed the continuous process for $\sigma_x(t)$ and the derivations are not trivial, the expression becomes simple. (See Kunitomo and Sato (2013). A more general result than **Lemma A-1** has been given in the revised version of Kunitomo and Sato (2013) as **Lemma 7**.)

Lemma A-1 : Let

$$\text{(A.17)} \qquad V_n = 2 \sum_{i,j=1}^{n} m_n c_{ij}^2 \left[\int_{t_{i-1}}^{t_i} \sigma_x^2(s) ds \int_{t_{j-1}}^{t_j} \sigma_x^2(s) ds \right] \; ,$$

where c_{ij} are defined by (A.11). Then as $n \to \infty$

$$\text{(A.18)} \qquad V_n \xrightarrow{p} V = 2 \int_0^1 \left[\sigma_x^4(s) \right] ds \; .$$

(**Step 4**) Finally, we need to give the proof of the stable convergence in law. Define the sequence of $\sigma-$fields $\mathcal{F}_{n,i} = \mathcal{F}(X(r), B(r), B^*(r), U(r)\ r \leq t_i^n)$. As the proof of Theorem 3 of Kunitomo and Sato (2013), we shall use a sequence of random variables U_n, which is a discrete martingale difference. We set $V_n^* = \sum_{j=2}^n \mathcal{E}[X_{nj}^2|\mathcal{F}_{n,j-1}]$. Then in order to prove

(A.19) $$U_n = \sum_{i=1}^n X_{ni} \xrightarrow{\mathcal{L}-s} MN(0,V)$$

we need to consider the conditions

$$\text{(Condition A)} \quad \sum_{i=1}^n \mathbf{E}[X_{ni}^2|\mathcal{F}_{n,i-1}] \xrightarrow{p} V$$

$$\text{(Condition B)} \quad \sum_{i=1}^n \mathbf{E}[X_{ni}^2 I(|X_{ni}| > \epsilon)|\mathcal{F}_{n,i-1}] \xrightarrow{p} 0 \text{ (for any } \epsilon > 0),$$

$$\text{(Condition C)} \quad \sum_{i=1}^n \mathbf{E}[X_{ni}(\mathbf{B}_i - \mathbf{B}_{i-1})|\mathcal{F}_{n,i-1}] \xrightarrow{p} 0\ ,$$

$$\text{(Condition D)} \quad \sum_{i=1}^n \mathbf{E}[X_{ni}(N_i - N_{i-1})|\mathcal{F}_{n,i-1}] \xrightarrow{p} 0$$

for all bounded martingales N with $[N, \mathbf{B}] = 0$ (quadratic covariation) and we take the vector of Brownian motions $\mathbf{B}_i = (B(t_i^n), B^*(t_i^n))$ $(i = 1, \cdots, n)$. (See Theorem IX 7.28 of Jacod and Shriyaev (2003) and also Section 2 of Jacod and Protter (2012).)

In the present situation, Condition (A) and Condition (B) are given in the proof of Theorem 3 in Kunitomo and Sato (2013) already and Condition (D) is satisfied trivially because r_i $(i = 1, \cdots, n)$ are martingales associated with the Brownian motion $B(t)$ on $[0, 1]$.

Then the remaining task is to show Condition (C). We shall investigate Condition (C) and use the relation

$$\begin{aligned}
\sum_{i=2}^n \mathbf{E}[X_{ni}(B_i - B_{i-1})|\mathcal{F}_{n,i-1}] &= \sum_{i=2}^n [2\sum_{j=1}^{i-1} \sqrt{m_n} c_{ij} r_j] \mathbf{E}[\int_{t_{i-1}}^{t_i} \sigma_s dB_s \int_{t_{i-1}}^{t_i} dB_s|\mathcal{F}_{n,i-1}] \\
&= \sum_{i=2}^n [2\sum_{j=1}^{i-1} \sqrt{m_n} c_{ij} r_j \int_{t_{i-1}}^{t_i} \mathbf{E}(\sigma_s|\mathcal{F}_{n,i-1}) ds]
\end{aligned}$$

and

$$\left(\sum_{i=2}^n \mathbf{E}[X_{ni}(B_i - B_{i-1})|\mathcal{F}_{n,i-1}]\right)^2$$

$$= \sum_{j,j'=1}^{n-1} \int_{t_{j-1}}^{t_j} \sigma_u dB(u) \int_{t_{j'-1}}^{t_{j'}} \sigma_v dB(v) [2 \sum_{i=j+1}^{n} \sqrt{m} c_{ij} \int_{t_{i-1}}^{t_i} \mathbf{E}(\sigma_s | \mathcal{F}_{n,i-1}) ds]$$

$$\times [2 \sum_{i'=j'+1}^{n} \sqrt{m} c_{i'j'} \int_{t_{i'-1}}^{t_{i'}} \mathbf{E}(\sigma_s | \mathcal{F}_{n,i'-1}) ds] \ .$$

Then by using the relation $\mathbf{E}[\int_{t_{j-1}}^{t_j} \sigma_u dB(u) \int_{t_{j'-1}}^{t_{j'}} \sigma_v dB(v) | \mathcal{F}_{n,i-1}] = \delta(j,j') \mathbf{E}[\int_{t_{j-1}}^{t_j} \sigma_u^2 du]$, we can find a positive constant K_5 such that

$$\mathbf{E}\left(\sum_{i=2}^{n} \mathbf{E}[X_{ni}(B_i - B_{i-1}) | \mathcal{F}_{n,i-1}] \right)^2$$

$$\leq K_5 \mathbf{E}\left(\sum_{j=1}^{n-1} \sum_{i,i'=j+1}^{n} m c_{ij} c_{i'j} \int_{t_{j-1}}^{t_j} \mathbf{E}(\sigma_s^2 | \mathcal{F}_{n,j-1}) ds \int_{t_{i-1}}^{t_i} \mathbf{E}(\sigma_u | \mathcal{F}_{n,i-1}) du \int_{t_{i'-1}}^{t_{i'}} \mathbf{E}(\sigma_v | \mathcal{F}_{n,i'-1}) dv \right) .$$

Then we can use the assumption that σ_s^2 $(0 \leq s \leq 1)]$ is bounded and $t_i - t_{i-1} = 1/n$ for all $i = 1, \cdots, n$. By applying the proof of Lemma 5 of Kunitomo and Sato (2008, 2013), we devide the summation of $2 \leq i \leq n$ into (i) $2 \leq i \leq n^\delta$ $(0 < \delta < 1)$ and (ii) $n^\delta + 1 < i \leq n$. Then we can find positive constants K_6 and γ (> 0) such that

$$\text{(A.20)} \qquad | \sum_{i=j+1}^{n} \sqrt{m} c_{ij} \int_{t_{i-1}}^{t_i} \mathbf{E}(\sigma_s) ds |$$

$$= | \sum_{i=j+1}^{n} \frac{1}{\sqrt{m}} \frac{2n}{2n+1} \left[\frac{\sin \frac{2\pi m}{2n+1}(i+j-1)}{\sin \frac{2\pi}{2n+1}(i+j-1)} + \frac{\sin \frac{2\pi m}{2n+1}(i-j)}{\sin \frac{2\pi}{2n+1}(i-j)} \right] \int_{t_{i-1}}^{t_i} \mathbf{E}(\sigma_s) ds |$$

$$\leq K_6 (\frac{1}{n})^\gamma \ .$$

Hence we have Condition (C) and this completes the proof of Theorem 3.2.

Q.E.D.

The proof of Corollary 3.3 : The main reason that we have the convergence rate in Proposition 3.1 and Theorem 3.2 is the second term of (A.5), which is not negligible for $0 < \alpha < 0.4$ because of (A.8). Thus if we remove the bias term due to the second term of (A.5), then it is straightforward to show that the convergence bound is $\alpha = 0.5$. We have omitted the arguments because we have discussed most evaluations are in the proof of Theorem 3.2. in the other terms in the as we have derived.

Q.E.D.

Next we shall give the proof of Theorem 3.6 and then show the proof of Theorem 3.4. It is because the first parts are similar, but we have some additional arguments for Theorem 3.4.

The proof of Theorem 3.6 : The most parts of the proof are very similar to the corresponding ones in the proof of Theorem 3.2. We write $y_i = x_i + v_i, v_i = u_i + w_i$ $(i = 1, \cdots, n)$, where $|w_i| \le \eta_n$. Then we need to check that the effects of a sequence of random variables w_i $(i = 1, \cdots, n)$ are negligible under the additional assumption (3.30) with the threshold parameter η_n (> 0).

We shall illustrate the underlying arguments. From (A.4) and (A.5), we notice that

$$\text{(A.21)} \quad \left[z_{kn}^{(2)}\right]^2 = n\left[\sum_{i=1}^n b_{ki}(u_i + w_i)\right]^2 = n\left[\sum_{i=1}^n b_{ki}u_i\right]^2 + 2n\left[\sum_{i=1}^n b_{ki}u_i\right]\left[\sum_{i=1}^n b_{ki}w_i\right] + n\left[\sum_{i=1}^n b_{ki}w_i\right]^2 .$$

By using the Cauchy-Schwarz inequality under (3.30) and $n\sum_{j=1}^n b_{kj}^2 = a_{kn}$ $(k = 1, \cdots, n)$, we have

$$\text{(A.22)} \quad \left[\sum_{i=1}^n b_{ki}u_i\right]^2 \le \eta_n^2 a_{kn} .$$

Then we can find a positive constant such that

$$\text{(A.23)} \quad \mathbf{E}\left[z_{kn}^{(2)}\right]^2 = n\mathbf{E}\left[\sum_{i=1}^n b_{ki}(u_i + w_i)\right]^2 \le K_7 a_{kn}\left[1 + \eta_n\sqrt{n}\right]^2 .$$

Hence under (3.30) the threshold effects in (A.22) and (A.23) are stochastically negligible.

We use the similar arguments to other terms in the decomposition of (A.5) as (A.11) and we can apply the same argument as the proof of Theorem 3.2 to the first term of the decomposition of (A.5) for its limiting distribution. Then we have the desired result in Theorem 3.6.

Q.E.D.

The proof of Theorem 3.4 : The first part of the proof is to use the similar arguments as Theorem 3.6, but in the present situation it is possible to evaluate the expected value of (A.21) more precisely.

We use the fact that the Fourier series of x for any $0 < x < 1$ is given by

$$x = \frac{1}{2} - \sum_{s=1}^{\infty} \frac{\sin 2\pi s x}{\pi s} = \frac{1}{2} - \sum_{s=1}^{\infty} [\frac{e^{i2\pi s x} - e^{-i2\pi s x}}{2i\pi s}] \ . \tag{A.24}$$

Then except the countable discontinuity points the fractional part $\{x\}$ of any real number x $(= [x] + \{x\})$ is given by (A.24). For a random variable X, let

$$\{X\}^* = \sum_{s=1}^{\infty} \frac{1}{2i\pi s}[(e^{i2\pi s X} - \mathcal{E}(e^{i2\pi s X})) - (e^{-i2\pi s X} - \mathbf{E}(e^{-i2\pi s X})] \ . \tag{A.25}$$

(i) First, we assume that $u(t_i^n) = 0$ $(i = 1, \cdots, n)$, $X(0) = 0, \mu_x(s) = 0$ and $\sigma_x(s)$ is a deterministic function such that $X_j = \int_0^{t_j^n} \sigma_x(s) dB_s$ is a Gaussian process. Then by using the Gaussianity, we find

$$\mathbf{E}(e^{i2\pi s X_j}) = \mathbf{E}(e^{-i2\pi s X_j}) = e^{-2\pi^2 s^2 \int_0^{t_j^n} \sigma_x(s)^2 ds} \tag{A.26}$$

and for any j, k $(j > k; j, k = 1, \cdots, n)$

$$\begin{aligned} &\text{(A.27)} \ \mathbf{Cov}[\{x_j/\eta_n\}, \{x_k/\eta_n\}] \\ &= \sum_{s,s'=1}^{\infty} \frac{-1}{4\pi^2 s s'} \mathbf{E}\{[(e^{i2\pi s x_j/\eta_n} - \mathbf{E}(e^{i2\pi s x_j/\eta_n})) - (e^{-i2\pi s x_j/\eta_n} - \mathbf{E}(e^{-i2\pi s x_j/\eta_n}))] \\ &\qquad \times [(e^{i2\pi s' x_k/\eta_n} - \mathbf{E}(e^{i2\pi s' x_k/\eta_n})) - (e^{-i2\pi s' x_k/\eta_n} - \mathbf{E}(e^{-i2\pi s' x_k/\eta_n}))]\} \ . \end{aligned}$$

Let $\mathcal{G}_{k,n}$ be the $\sigma-$field generated by the random variables X_l $(= X(t_l^n),\ l = 1, \cdots, k)$. Under the assumption that $0 < \sigma_* \leq \sigma_s \leq \sigma^*$, then the conditional expected value of each terms of (A.27) for $j > k$ becomes

$$\begin{aligned} &\mathbf{E}[(e^{i2\pi s X_j/\eta_n} - \mathbf{E}(e^{i2\pi s X_j/\eta_n})) - (e^{-i2\pi s X_j/\eta_n} - \mathbf{E}(e^{-i2\pi s X_j/\eta_n}))|\mathcal{G}_{k,n}] \\ &\leq \ K_8 \, e^{-2\pi^2 s^2 \sigma_*^2 (j-k)/[n\eta_n^2]} \ , \end{aligned} \tag{A.28}$$

where K_8 is a positive constant. Since $e^{-x} < 1/x$ $(x > 0)$, the sum of (A.28) with respect to s converges to a finite value. Actually under the condition (3.23), for any $|j-k| > n^{\delta_1}$ $(0 < \delta_1 < 1)$ we have $2\pi^2 s^2 \sigma_*^2 (j-k)/[n\eta_n^2] = O(n^{\delta_1 - 1 + 2\gamma})$, which goes to $+\infty$ as $n \to \infty$ if we can rake $2\gamma + \delta_1 > 1$. By setting $\delta_2 = \delta_1 - 1 + 2\gamma$,

$-2\pi^2 s^2 \sigma_*^2 (j-k)/[n\eta_n^2] = O(e^{-n^{\delta_2}})$.

Now we can evaluate

$$\text{(A.29)} \qquad \mathbf{Var}[\eta_n \sum_{j=1}^n b_{kj} \{\frac{X_j}{\eta_n}\}^*] = \eta_n^2 \sum_{j,j'=1}^n b_{kj} b_{k,j'} \mathbf{Cov}[\{\frac{X_j}{\eta_n}\}^*, \{\frac{X_{j'}}{\eta_n}\}^*] .$$

We decompose the above summation (A.29) into two components as (a) $\sum_{|j-j'|<n^{\delta_3}}$, and (b) $\sum_{|j-j'|\geq n^{\delta_3}}$ for $0 < \delta_3 < 1$. Then there are n^{δ_3} terms in (a) and there are $n - n^{\delta_3}$ terms, but which are of order $o(1/n)$ from (A.28). Thus by using the relation (A.28) and Lemma A-2 below, we can take constants K_9, K_{10} and K_{11} such that for $w_j = \eta_n \{X_j/\eta_n\}$ $(j = 1, \cdots, n)$

$$\text{(A.30)} \qquad \begin{aligned} n\mathbf{E}[\sum_{j=1}^n b_{kj} w_i]^2 &\leq K_9 \, [n \sum_{j=1}^n b_{kj}^2] \eta_n^2 [n^{\delta_3} + (n - n^{\delta_2}) \frac{1}{n}] \\ &= K_{10} \, a_{kn} [n^\gamma \eta_n]^2 \\ &\leq K_{11} \, a_{kn} \end{aligned}$$

by taking $\gamma = \delta_3/2$ $(0 < \delta_3 < 1)$ and positive constants K_i $(i = 9, 10, 11)$. (We can take $\delta_1 = \delta_3$ for instance.) Thus for the rest of evaluation, we can apply the arguments of the standard cases.

(ii) Next, we consider the case when we have the micro-market noise term $u_i = u(t_j^n)$ $(j = 1, \cdots, n)$. In (i) we replace $\eta_n \{(X_j + u_j)/\eta_n\}$ $(j = 1, \cdots, n)$ instead of $\eta_n \{X_j/\eta_n\}$ and apply the same arguments. Because of the assumption on $X(t_j^n)$ and $u(t_j^n)$ $(j = 1, \cdots, n)$, we can utilize the relation

$$\text{(A.31)} \qquad \mathbf{E}[e^{i2\pi s (X_j + u_j)/\eta_n}] = \mathbf{E}[e^{i2\pi s X_j/\eta_n}] \mathbf{E}[e^{i2\pi s u_j/\eta_n}] .$$

When we have the drift term, which is deterministic by our assumption, the effects of $|\int_{t_k^n}^{t_j^n} \mu_x(s) ds| \leq K_{12} |t_j^n - t_k^n|$ and our arguments are valid in this case.

Q.E.D.

By using tedious but straightforward calculations, we can evaluate trigimometric relations, which are needed for Theorem 3.4.

Lemma A-2 : For $j, k = 1, \cdots, n$, let $\theta_{jk} = [2\pi/(2n+1)](j - 1/2)(k - 1/2)$ and

$\theta_k = [2\pi/(2n+1)](k-1/2)$. Then

$$b_{kj} = \frac{2}{\sqrt{2n+1}}(\cos\theta_{kj} - \cos\theta_{k,j+1}) \tag{A.32}$$

and for any positive interger l ($l < n$)

$$\sum_{j=l+1}^{n} b_{kj}b_{j,j-l} = 8\sin^2(\frac{\theta_k}{2})\cos(l\theta_k) + o(1)\ . \tag{A.33}$$

The proof of Lemma A-2 : We use the representation

$$b_{kj} = \frac{1}{2n+1}\left[(1-e^{i\theta_k})e^{i\theta_{kj}} + (1-e^{-i\theta_k})e^{-i\theta_{kj}}\right]\ . \tag{A.34}$$

Then we have $\sum_{j=1}^{n} e^{i2\theta_{kj}} = 1/(1-e^{i\theta_k})$ and $\sum_{j=1}^{n}(1-e^{i\theta_k})^2 e^{i2\theta_{kj}} = 1-e^{i\theta_k}$. Then it is straight-forward to show that for any integer l,

$$\begin{aligned}(2n+1)\sum_{j=1}^{n} b_{kj}b_{kj-l} &= \sum_j \Big\{e^{-il\theta_k}(1-e^{i\theta_k})^2 e^{i2\theta_{kj}} + e^{il\theta_k}(1-e^{-i\theta_k})^2 e^{-i2\theta_{kj}} \\ &\quad + ne^{il\theta_k}(1-e^{i\theta_k})(1-e^{-i\theta_k}) + ne^{-il\theta_k}(1-e^{-i\theta_k})^2(1-e^{i\theta_k})\Big\} \\ &\sim 4n\left[\sin^2\frac{\theta_k}{2}\right][2\cos(l\theta_k)]\ .\end{aligned}$$

Q.E.D.

The proof of Theorem 3.5 and 3.7 : The most parts of the proof of Theorem 3.5 are quite similar to the corresponding ones in the proof of Theorem 3.2. We define $v_i = V(t_i^n) + u(t_i^n)$ ($i = 1,\cdots,n$) by (2.14) and we write $y_i = x_i + v_i$, where $y_i = P(t_i^n)$ and $x_i = X(t_i^n)$ for the proof of Theorem 3.5. The essential difference is the presence of $U_a(t_i^n)$ terms and then v_i ($i = 1,\cdots,n$) are auto-correlated in the present situation.

In stead of (3.27), by using the conditions (3.31), (3.32) and the Cauchy-Schwartz inequality, we find a positive constant K_{13} such that

$$\begin{aligned}\mathbf{E}[z_{kn}^{(2)}]^2 &= n\mathbf{E}[\sum_{i=1}^{n} b_{ki}v_i \sum_{j=1}^{n} b_{kj}v_j] \\ &\leq n\sum_{s=0}^{n} c_1\rho_1(s)[\sum_{i=1}^{n} b_{ki}b_{k,i-s}] \\ &\leq K_{13}\times a_{kn}\ ,\end{aligned} \tag{A.35}$$

For the third term of (A.5) as in Theorem 3.2, we evaluate the variance of

$$z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}] = n \sum_{j,j'=1}^{n} b_{kj} b_{k,j'} \left[v_j v_{,j'} - \mathbf{E}(v_j v_{j'}) \right]$$

and the expectations of $\left[z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}] \right] \left[z_{k',n}^{(2)2} - \mathbf{E}[z_{k',n}^{(2)2}] \right]$.
By using the condition imposed by (3.31) and (3.32), we can find a positive constant K_{14} such that

$$n^2 \sum_{i,i'=1}^{n} \sum_{j,j'=1}^{n} b_{ki} b_{k,i'} b_{k',j} b_{k',j'} \rho_2(|i-i'|, |j-j'|) \sim K_{14} \times a_{kn} a_{k',n} \ . \tag{A.36}$$

Then by collecting each terms, we can find a positive constant K_{15} such that

$$\begin{aligned} \mathbf{E}\left[\frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} (z_{kn}^{(2)2} - \mathbf{E}[z_{kn}^{(2)2}]) \right]^2 &\leq \frac{K_{15}}{m_n} \sum_{k,k'=1}^{m_n} a_{kn} a_{k'n} \\ &= O(\frac{1}{m_n} \times (\frac{m_n^3}{n})^2) \\ &= O(\frac{m_n^5}{n^2}) \end{aligned} \tag{A.37}$$

since $\sum_{k=1}^{m} a_{kn} = O(m_n^3/n)$.
Thus the third term of (A.5) as in Theorem 3.2 is negligible if $0 < \alpha < 0.4$.

By taking care of these changes in our derivations, it is straightforward to proceed the proof of Theorem 3.7 as for Theorem 3.2. Hence we omit the details.
Q.E.D.

APPENDIX B : TABLES and FIGURES

In Tables the estimates of the variances (σ_x^2) are calculated by the SIML method while H-vol are calculated by the historical (or realized) volatility estimation. The true-val means the true parameter value in simulations and mean, SD and MSE correspond to the sample mean, the sample standard deviation and the sample mean squared error of each estimator, respectively.

B-1 : Comparison of alternative estimates (Model-1)
($\sigma_u^2 = 5.0\mathrm{E}-05, \eta = 0.25$)

n=20,000	SIML (σ_x^2)	H-vol	RK	PA
true-val	1.000	1.00	1.00	1.00
mean	1.104	38.84	2.283	1.719
SD	0.258	9.354	0.514	0.387
MSE	0.092	1519.6	1.910	0.667

B-2 : Comparison of alternative estimates (Model-1)
($u^2 = 1.0\mathrm{E}-04; \eta = 0.1$)

n=20,000	SIML (σ_x^2)	H-vol	RK	PA
true-val	1.000	1.00	1.00	1.00
mean	0.969	12.14	0.948	0.934
SD	0.150	0.443	0.078	0.095
MSE	0.023	124.3	0.008	0.013

B-3 : Estimation of integrated volatility (Model-2)
($a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}-04, g = 0.2$)

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	1.01E+00	2.33E+00
SD	1.97E-01	2.32E-02
MSE	3.89E-02	1.78E+00

B-4 : Estimation of integrated volatility (Model-2)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 0.00\text{E} + 00, g = 0.2)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	9.96E-01	1.11E-01
SD	1.93E-01	2.35E-03
MSE	3.74E-02	7.90E-01

B-5 : Estimation of integrated volatility (Model-2)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 0.00\text{E} + 00, g = 1.5)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	1.00E+00	3.00E+00
SD	1.94E-01	4.03E-02
MSE	3.78E-02	4.00E+00

B-6 : Estimation of integrated volatility (Model-2)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 05, g = 1.0)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	9.88E-01	1.40E+00
SD	1.99E-01	1.40E-02
MSE	3.97E-02	1.60E-01

B-7 : Estimation of integrated volatility (Model-2)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 06, g = 0.01)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	8.40E-01	2.51E-02
SD	1.66E-01	5.41E-04
MSE	5.31E-02	9.50E-01

B-8 : Estimation of integrated volatility (Model-3)

$(a_0 = 7, a_1 = -12, a_2 = 6; \sigma_u^2 = 2.00\text{E} - 02, \eta = 0.5)$

n=20,000	σ_x^2	H-vol
true-val	4.50E+01	4.50E+01
mean	4.60E+01	1.37E+02
SD	1.05E+01	6.19E+00
MSE	1.11E+02	8.46E+03

B-9 : Estimation of integrated volatility (Model-4)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 0.00\text{E} + 00, \eta = 0.005)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	1.00E+00	6.85E-01
SD	1.94E-01	8.66E-03
MSE	3.77E-02	9.92E-02

B-10 : Estimation of integrated volatility (Model-5)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 0.00\text{E} + 00, g_1 = 0.2, g_2 = 5)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	1.01E+00	2.22E+00
SD	1.93E-01	6.46E-02
MSE	3.71E-02	1.49E+00

B-11 : Estimation of integrated volatility (Model-5)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 03, g_1 = 0.2, g_2 = 5)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	1.02E+00	6.65E+01
SD	1.94E-01	1.66E+00
MSE	3.79E-02	4.30E+03

B-12 : Estimation of integrated volatility (Model-6)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 0.00\text{E} + 00, g_1 = 1.9, g_2 = -1.7, \gamma = 10,000)$

n=20,000	σ_x^2	H-vol
true-val	1.00E+00	1.00E+00
mean	9.99E-01	6.39E+00
SD	1.92E-01	3.66E-01
MSE	3.68E-02	2.91E+01

In Figures 2.1 and 2.2 P and Q stand for the price and the quantity, respectively. D and S are the demand curve and supply curve, respectively. η in Figure 2.2 denotes the minimum tick size and Q^* is the quantity traded in Figure 2.2.
In Figure 2.3 typical sample paths in the the standard round-off error model shown by simulations.

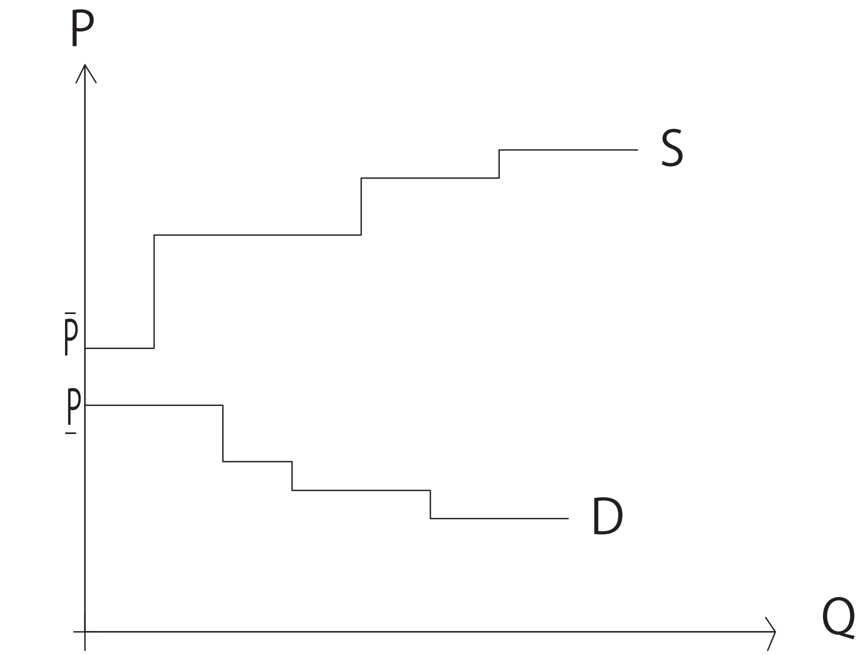

Fig. 2-1

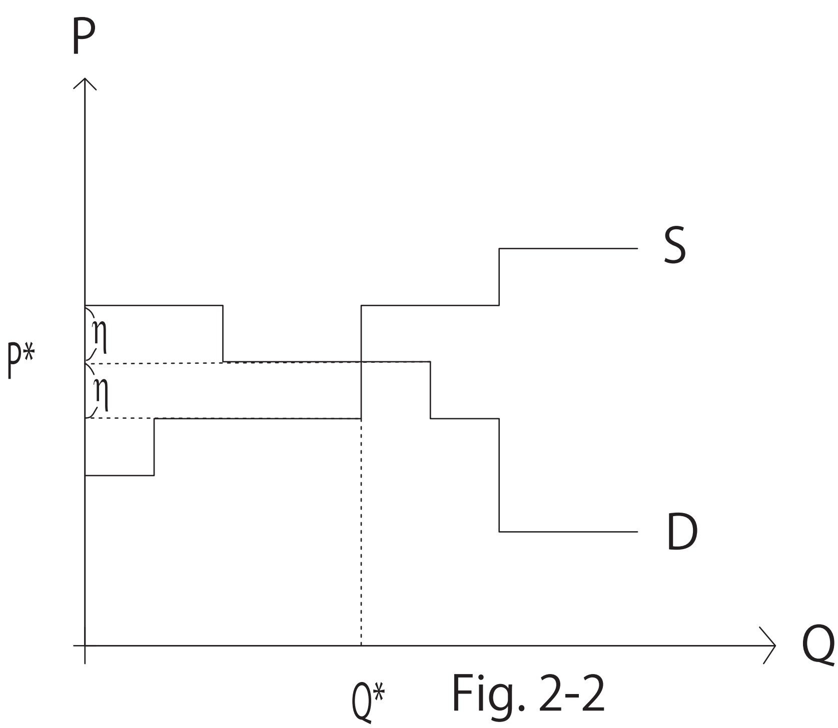

Fig. 2-2

Fig.2-3:Rounding effects