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Implementation, Verification, and Detection

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Abstract

We investigate implementation of social choice functions, where we make severe restrictions on mechanisms such as detail-freeness, boundedness, only tiny transfers permitted, and uniqueness of iteratively undominated strategy profile in the ex-post term. After the determination of allocation, some partial information about the state becomes verifiable. The central planner can make the transfers contingent on this information. By demonstrating a sufficient condition for implementation, namely full detection, we show that a wide variety of social choice functions are uniquely implementable even if the range of players' lies that the verified information can directly detect is quite narrow. With full detection, we can detect all possible lies, not by the verified information alone, but by processing a chain of detection triggered by this information. The designed mechanism is sequential in that each player makes announcements twice at two distinct stages. This paper does not assume expected utility, quasi-linearity, and risk neutrality.

Keywords: Unique Implementation, Ex-Post Verification, Iterative Dominance, Detail-Freeness, Boundedness, Full Detection, Sequential Mechanisms, Tiny Transfers.

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1. Introduction

This paper investigates unique implementation of a social choice function (SCF), where the central planner attempts to achieve the allocation implied by the SCF that is contingent on the state. The central planner, however, cannot observe the state before he (or she) determines an allocation. Hence, the central planner designs a mechanism to induce informed players to reveal their knowledge about the state. In this case, the mechanism must incentivize these players to make the truthful announcements as unique equilibrium behavior. The requirement of uniqueness is a quite substantial restriction in the implementation problem. The basic problem is therefore to clarify whether the central planner can design such a well-behaved mechanism.[3]

The main departure from the previous works on implementation is that in our model, some partial information about the state becomes public and verifiable after the central planner determines an allocation. The central planner can utilize this verified information as the clue to detecting players' lying. By making the monetary transfers contingent on this verified information as well as their announcements, the central planner attempts to design a mechanism that can effectively penalize any detected liar, making players willing to tell the truth. This paper clarifies the extent to which a wider variety of SCFs are uniquely implementable with partial verification than without it. This paper is the first attempt to consider the role of such ex-post verification in the unique or full implementation literature.

Except for the allowance of verification, this paper makes a number of very severe restrictions on mechanism design as follows. Firstly, we select *iterative dominance* as the equilibrium concept, which is defined as the set of all strategy profiles that survive through the iterative removal of strategies that are dominated in the ex-post term, i.e., weakly dominated at every state and strictly dominated at some states. This set implies the set of all strategy profiles that survive through the iterative removal of strategies that are strictly dominated irrespective of the specification of full-support prior distribution. We then require the uniqueness of iteratively undominated strategy profile. Since this

[3] For the surveys on implementation theory, see Moore (1992), Palfrey (1992), Osborne and Rubinstein (1994, Chapter 10), Jackson (2001), and Maskin and Sjöström (2002), for instance.

iterative dominance notion is a very weak equilibrium concept, our uniqueness requirement should be a very severe restriction.

Secondly, from the viewpoints of robustness in terms of higher-order belief perturbations and Knightian uncertainty, we require a mechanism to be ‘detail-free’ in terms of prior distribution on the state space. Since our definition of iterative dominance is on the ex-post term, the mechanism must inevitably be made irrelevant to the specification of prior distribution, i.e., detail-free.

Thirdly, from the viewpoint of plausibility in mechanism design, we require a mechanism to be ‘bounded’ in that it is not incorporated with any construction that has no equilibrium, such as the integer game. Because of this boundedness requirement, we focus on a class of mechanisms in which the message space is finite for each player.

Fourthly, we permit only tiny transfers because of players’ limited liability. To be precise, we require any transfer close to zero off the equilibrium path and no transfers on the equilibrium path.

The above requirements, i.e., uniqueness of iteratively undominated strategy profile, detail-freeness, boundedness, and tiny transfers, will make our implementation problem quite substantial to be solved even if we permit partial verification. Hence, the main purpose of this paper is to demonstrate a sufficient condition on the state space, under which, a SCF is uniquely implementable in iterative dominance with partial verification, where we design a mechanism that is bounded and detail-free, and utilizes only tiny transfers.

In order to design a well-behaved bounded mechanism with tiny transfers, we will apply the basic concept of mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994), where the central planner requires each player to make multiple announcements at once, randomly selects one announcement profile from their announcements, and fines the first deviants from some ‘reference point’. Once we can establish the reference point truthfully, the mechanism design a la Abreu-Matsushima well motivates all players to make the truthful announcements, successfully implementing the SCF. The remaining problem is therefore to clarify the manner in which we can establish such a truthful reference point.

This remaining problem appears difficult to be solved, because the relevance of the verified information to the state is limited and the prior distribution is unspecified in our model. In fact, the range of players' lies that the verified information can directly detect is quite narrow, and we cannot even apply any device of proper scoring rule to incentivize players to reveal the distributions, i.e., their private information, truthfully.

We, however, can overcome this difficulty by finding the following 'chain of detection'. The verified information detects a limited but non-empty class of some players' lies, which motivates these players to reveal their respective aspects of the state truthfully; this truthful revelation along with the verified information detects another class of lies, which motivates the relevant players to reveal other aspects of the state truthfully; and so on.

We demonstrate a condition on the state space, namely full detection, implying that there exists a chain of detection, throughout which, we can iteratively detect all possible lies. Hence, with full detection, truth-telling is the only announcement for each player that survives through the iterative removal of detected lies.

With full detection, by penalizing detected liars in an appropriate manner, we can make all players reveal their private information truthfully. This implies that we can establish the truthful reference point as the combination of the verified information and their truthfully announced private information.

Based on these observations, we show as the main theorem of this paper that full detection is a sufficient condition, under which, a SCF is uniquely implementable in iterative dominance with partial verification, where the designed mechanism is bounded, detail-free, and permits only tiny transfers.

Full detection appears to be an involved condition. In fact, in order to detect a player's lie, we have to find out a state at which the other players never announce any message profile that they may possibly announce if his lie is true. However, by demonstrating a tractable sufficient condition for full detection, we show that, despite of this complexity, full detection covers a wide range of state space formulations. This is in contrast to the case without verification, where any non-trivial deterministic SCF is never implementable in the exact term.

In order to detect all possible lies, the central planner must design, not a single-stage mechanism, but a sequential mechanism, in which, each player is required to make announcements twice at two distinct stages, i.e., at the first stage and at the second stage. The central planner regards their first announcements along with the verified information as the reference point. He utilizes only their second announcements for the determination of allocation.

At the first stage, each player is informed of his (or her) private information that the central planner asks him to reveal, but is less informed of the other information than at the second stage. By requiring players to make announcements when they are less informed, the central planner can prevent them from finding a mean of escape from detection, making the truthful reference point easier to be established.

For instance, let us consider the case with $n \geq 2$ players, where a state is described by $\omega = (\omega_0, \omega_1, ..., \omega_n)$. Assume that ω_i is either 1, 2, or 3 for each $i \in \{0,1,...,n\}$, and ω_0 is ex-post verifiable. The state space Ω is defined as a subset of $\{1,2,3\}^{n+1}$. At the first stage, each player $i \in \{0,1,...,n\}$ observes ω_i as his private information and the central planner requires him to announce about it. It must be noted that if $\Omega = \{1,2,3\}^{n+1}$, it is impossible to detect any lie.

Let us suppose that Ω is a proper subset of $\{1,2,3\}^{n+1}$, and it is 'moth-eaten' in the sense that each player's observation is different from his neighbor's observation at every state, i.e., $\omega \in \{1,2,3\}^{n+1}$ belongs to Ω if and only if

$$\omega_i \neq \omega_{i-1} \text{ for all } i \in \{1,...,n\}.$$

Because of this moth-eaten nature, the verified information ω_0 can directly detect any lie about ω_1, because player 1 cannot exclude the possibility that his lie $\omega_1' \neq \omega_1$ is equivalent to ω_0, i.e., $\omega_1' = \omega_0$. This motivates player 1 to tell the truth about ω_1.

In the same manner, the truthful announcement about ω_1 can detect any lie about ω_2. This motivates player 2 to tell the truth about ω_2. Recursively, any player $i \in \{1,...,n\}$ is well motivated to tell the truth about ω_i, implying full detection.

In the process of such iterative removal of detected lies, it is crucial to assume that each player i is not informed of ω_{i-1} at the first stage. Otherwise, he can find a way to escape from detection by announcing $\tilde{\omega}_i \notin \{\omega_i, \omega_{i-1}\}$.

This paper further investigates the case in which full detection does not hold, i.e., the case of partial detection. By replacing uniqueness of strategy by uniqueness of outcome, we define full implementation in iterative dominance. We define a concept of measurability of a SCF with respect to partial detection. We then show that with partial detection, this measurability is sufficient for the SCF to be fully implementable in iterative dominance.

Throughout this paper, we mostly assume complete information at the second stage. Without any substantial modification, however, we can replace this complete information by incomplete information at the second stage. By requiring a version of ex-post incentive compatibility, we show that the same arguments hold even in the incomplete information environment at the second stage.

We should refer to another departure from the previous works; we do not assume expected utility, quasi-linearity, and risk-neutrality. This paper makes only basic assumptions on preferences such that each player's utility function is continuous in lottery over allocations and is continuous and increasing in monetary transfer.

The organization of this paper is as follows. Section 2 explains related literatures. Section 3 shows the basic model. Section 4 investigates the case of full verification. Section 5 investigates the case of partial verification. Section 6 introduces the concepts of detection and full detection, and demonstrates the main theorem of this paper. Section 7 investigates the case of partial detection. Section 8 investigates incomplete information at the second stage. Section 9 discusses about the generalization of detection and about the impact of boundedness on implementation with partial verification. Section 10 concludes.

2. Literatures

The basic framework for the implementation problem was explored by Hurwicz (1972) and Maskin (1999). Makin showed that monotonicity is a necessary condition for a social choice correspondence to be fully implementable in Nash equilibrium. This result should be regarded as being negative, because monotonicity is a quite demanding condition for a deterministic SCF. In fact, with some additional restrictions, any deterministic SCF that is fully implementable in Nash equilibrium must be dictatorial.

The purpose of this paper is therefore to show permissive results for full, or unique, implementation.

Matsushima (1988) and Abreu and Sen (1991) showed a permissive result that any full-support stochastic SCF is monotonic, and therefore, fully implementable in Nash equilibrium. Hence, if we permit the tiny probability to select unwanted allocations even on the equilibrium path, even any deterministic SCF becomes fully implementable in Nash equilibrium, not in the exact sense, but in the virtual sense. In contrast, this paper sticks to exact implementation.

Moore and Repullo (1988) and Palfrey and Srivastava (1991) replaced Nash equilibrium with their respective refinements such as subgame perfect equilibrium and undominated Nash equilibrium. Abreu and Matsushima (1994) replaced Nash equilibrium with weak iterative dominance, or according to the terminology of Moulin (1979), dominance solvability, and then showed a permissive result for unique implementation by permitting just tiny monetary transfers off the equilibrium path. Chen, Kunimoto, and Chung (2015) extended this result to the Bayesian environment. Because of the use of refinement, these works commonly permitted the existence of Nash equilibria that fail to achieve the value of the SCF. In contrast, this paper will replace Nash equilibrium even with a weaker equilibrium concept namely iterative dominance in the ex-post term, eliminating all unwanted Bayesian Nash equilibria irrespective of the specification of full-support prior distribution.

Many previous works in the implementation literature have constructed mechanisms that have ‘implausible’ features such that the mechanisms are incorporated with constructions that have no equilibrium, such as the integer games. In order to exclude such constructions that are implausible, or according to the terminology of Jackson (1992), are unbounded, Abreu and Matsushima (1992a) innovated a new method, namely AM mechanism design, that makes the mechanism bounded by permitting only a finite strategy space for each player. Abreu and Matsushima then showed a very permissive result for unique virtual implementation in iterative dominance. Abreu and Matsushima (1992b) extend this result to the Bayesian environment. Abreu and Matsushima (1994) and Kunimoto (2015) also utilized the AM mechanism design for exact implementation, by replacing iterative dominance with weak iterative dominance. The present paper will apply the AM bounded mechanism design for exact implementation without replacing iterative dominance with any stronger equilibrium concept such as weak iterative dominance.

In the Bayesian framework, the designed mechanisms generally depend on the fine details of a fixed prior distribution. From the viewpoint of robustness in higher-order belief perturbations, Bergemann and Morris (2009) emphasized the importance of detail-free mechanism design and the usage of ex-post equilibrium concepts. Alternatively, from the viewpoint of Knightian uncertainty, where each player possesses multiple prior distributions, Lopomo, Rigotti, and Shannon (2009) investigated detail-free mechanism design. This paper will define the iterative dominance notion on the ex-post term and design mechanisms that are detail-free in terms of prior distribution.

Based on these backgrounds, we should regard this paper as the first work to show the permissive result for exact implementation of SCFs by using only detail-free bounded mechanisms with tiny transfers.

The construction in this paper is divided into two parts, i.e., the application of the AM mechanism design, and the establishment of the truthful reference point. The technical contribution of this paper is mainly devoted to the latter part.

In order to establish the reference point truthfully, the pioneering works such as Abreu and Matsushima (1992a, 1992b, 1994) have utilized the incentive devices of ‘virtualness’. Alternatively, Matsushima (2008a, 2008b) assumed the presence of tiny psychological cost of dishonesty for a player, and then incentivized him to make the truthful announcements for the reference point. In contrast, this paper will not utilize either the incentive device of virtualness or the psychological cost of dishonesty.

In order to establish the truthful reference point, this paper demonstrates an alternative approach by assuming that some partial information about the state is ex-post verifiable. There exist many previous works, such as Hansen (1985), Mezzetti (2004), DeMarzo, Kremer, and Skrzypaz (2005), Mylovanov and Zapechelnyuk (2014), Deb and Mishra (2014), and Carroll (2015), that incorporated such ex-post verification into the problems of mechanism design. These works commonly showed that the presence of ex-post verification makes incentive compatibility easier to be satisfied.

In contrast, this paper’s concern is about the impact of ex-post verification, not on incentive compatibility, but on uniqueness of equilibrium. In this respect, this paper should be regarded as the first attempt to incorporate ex-post verification into the unique or full implementation theory.

The literature of persuasion games is also related to this paper, where players voluntarily reveal verifiable information, i.e., hard evidences, which can partially prove their announcements to be correct, encouraging the correct public decision making. See Grossman (1981) and Kamenica and Gentzkov (2011), for instance. Kartik and Tercieux (2012) and Ben-Porath and Lipmann (2012) investigated full implementation with such hard evidences, stating that the great degree to which hard evidences directly prove players' announcements to be correct is crucial in implementing a wide variety of SCFs. In contract, this paper emphasizes that a wide variety of SCFs are implementable even if the verifiable information is quite limited.

In order to show such permissive results even with tiny verification, this paper assumes that the state space is common knowledge and the central planner can make the mechanism dependent on the state space. We then demonstrate a condition concerning the shape of state space, namely full detection, which guarantees any SCF to be implementable.

Full detection assumes that there exists a rare event, the occurrence of which, each player assigns probability zero. As pointed out by the authors in behavioral economics, such as Camerer and Kunreuther (1989), real people tend to assign a rare event with probability zero because of their psychological biases such as the optimistic bias. This justifies the relevancy of full detection.

In the literature on models of knowledge, it has been discussed as 'the puzzle of the hats', for instance, that a tiny information release gives a big influence on players' reasoning. In contrast, this paper focuses on the influence of tiny information release, i.e., verified information, on player's incentives. This information should be hidden from players when they make announcements for the purpose of establishing the reference point.

The method of AM mechanism design has long been criticized without any formal analysis, because of the conjecture that this method crucially depends on the expected utility assumption. This paper will prove that this criticism is groundless. The functioning of AM mechanism relies just on the local linearity of preferences, implying the irrelevance of global linearity such as expected utility and quasi-linearity. Hence, this paper will be expected to promote more popularity of this essentially powerful method.

3. The Model

We consider a situation in which the central planner determines an allocation and makes monetary transfers. Let $N \equiv \{1,...,n\}$ denote the finite set of all players, where we assume $n \geq 3$ except for Section 8. Let A denote the finite set of all allocations. Let Δ denote the set of all lotteries over allocations[4]. Let Ω denote the finite set of all states, i.e., the finite state space. A social choice function, shortly a SCF, is defined as $f : \Omega \rightarrow \Delta$.

We define the state-contingent utility function for each player $i \in N$ as

$$u_i : \Delta \times R \times \Omega \rightarrow R.$$

where $u_i(\alpha, t_i, \omega)$ implies the utility for player i when he (or she) expects the state ω to occur, the central planner to determine an allocation according to the lottery $\alpha \in \Delta$, and make a monetary transfer $t_i \in R$ to player i. Let $u \equiv (u_i)_{i \in N}$.

We assume that $u_i(\alpha, t_i, \omega)$ is *continuous* with respect to $\alpha \in \Delta$ and $t_i \in R$, and that $u_i(\alpha, t_i, \omega)$ is *increasing* in t_i. Importantly, this paper does not assume expected utility, quasi-linearity, and risk-neutrality. Let U_i denote the set of all utility functions for player i.

4. Full Verification

As a benchmark for this paper's analysis, this section will assume *full verification* as follows. After the central planner determines an allocation, but before he makes monetary transfers, the state becomes public and verifiable to the court. The central planer can make the monetary transfers contingent on the state as well as the players' announcements, while he cannot make the allocation choice contingent on the state.

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[4] We denote $\alpha \in \Delta$. We write $\alpha = a$ if $\alpha(a) = 1$.

We define a *mechanism* as $G \equiv (M, g, x)$, where $M \equiv \underset{i \in N}{\times} M_i$, M_i denotes the set of all *messages* of player i, $g : M \to \Delta$ denotes the *allocation rule*, $x \equiv (x_i)_{i \in N}$ denotes the *transfer rule*, and $x_i : M \times \Omega \to R$ denotes the transfer rule for player i. We confine our attention to mechanisms such that M_i is *finite* for all $i \in N$, i.e., we focus on a class of mechanisms that are so-called bounded.[5]

This section assumes complete information in that each player each player observes the state $\omega \in \Omega$, while the central planer cannot observe it, before his allocation choice. Each player $i \in N$ announces a message $m_i \in M_i$ that is contingent on the state ω. The central planner then determines an allocation according to the lottery $g(m) \in \Delta$, where $m \equiv (m_i)_{i \in N} \in M$ denotes the message profile. After the state ω becomes verifiable, the central planner receives the monetary transfer $x_i(m, \omega) \in R$ from each player i.

A *strategy for each player* i *in a mechanism* G is defined as $s_i : \Omega \to M_i$. Player i announces the message $s_i(\omega) \in M_i$ when he observes ω. Let S_i denote the set of all strategies for player i. Let $S \equiv \underset{i \in N}{\times} S_i$ and $s \equiv (s_i)_{i \in N} \in S$.

4.1. Iterative Dominance

We introduce the equilibrium concept, namely *iterative dominance*, which is defined as the survival of iterative removals of messages that are dominated with strict inequality in the ex-post term, in the following manner. For every $i \in N$ and $\omega \in \Omega$, let

$$M_i(0, \omega) \equiv M_i .$$

Recursively, for each $h \geq 1$, we define a subset of player i's messages $M_i(h, \omega) \subset M_i$ in the manner that $m_i \in M_i(h, \omega)$ if and only if there exists no $m_i' \in M_i(h-1, \omega)$ such that for every $m_{-i} \in M_{-i}(h-1, \omega)$,

$$u_i(g(m), -x_i(m, \omega), \omega) < u_i(g(m_i', m_{-i}), -x_i(m_i', m_{-i}, \omega), \omega),$$

[5] For a further discussion about boundedness, see Subsection 9.2.

where $M_{-i}(h-1,\omega) \equiv \underset{j\in N\setminus\{i\}}{\times} M_j(h-1,\omega)$. Here, we require each player i to prefer m_i' to m_i irrespective of $m_{-i} \in M_{-i}(h-1,\omega)$. We require strict inequalities for the iterative steps of eliminating dominated messages.

Definition 1: A strategy $s_i \in S_i$ for player i is said to be *iteratively undominated in* G *with full verification* if

$$s_i(\omega) \in \bigcap_{h=0}^{\infty} M_i(h,\omega) \quad \text{for all} \quad \omega \in \Omega .$$

Because of the requirement of strict inequalities, the order of elimination does not matter in the definition of iterative dominance. By requiring the uniqueness of iteratively undominated strategy profile, we define unique implementation in iterative dominance with full verification as follows.

Definition 2: A mechanism G is said to *uniquely implement a SCF* f *in iterative dominance with full verification* if there exists the unique iteratively undominated strategy profile $s \in S$ in G, i.e.,

$$\bigcap_{h=0}^{\infty} M_i(h,\omega) = \{s_i(\omega)\} \quad \text{for all} \quad \omega \in \Omega \quad \text{and} \quad i \in N ,$$

and it induces the value of the SCF, i.e.,

$$g(s(\omega)) = f(\omega) \quad \text{for all} \quad \omega \in \Omega .$$

The definition of iterative dominance is irrelevant to the specification of prior distribution on Ω. That is, the mechanism that uniquely implements a SCF in iterative dominance with full verification is “detail-free” with respect to prior distribution on Ω.

4.2. Construction of Mechanisms

Fix arbitrary real numbers $\eta_1 > 0$ and $\eta_2 > 0$. Fix an arbitrary integer $K > 0$. We construct a mechanism $G^* = G^*(f, \eta_1, \eta_2, K) = (M, g, x)$ in the following manner. For every $i \in N$, let

$$M_i = \underset{k=1}{\overset{K}{\times}} M_i^k ,$$

and

$$M_i^k = \Omega \quad \text{for all } k \in \{1, ..., K\}.$$

Player $i \in N$ announces K *sub-messages* $m_i^k \in M_i^k$ at once.

For each $k \in \{1, ..., K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } m_i^k = \omega \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \omega,$$

where $a^* \in A$ is an arbitrary allocation, which is regarded as the status quo allocation. Let

$$g(m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K}.$$

The central planner randomly selects an integer k from $\{1, ..., K\}$, and determines an allocation according to the lottery $g^k(m^k) \in \Delta$. The central planner selects an allocation according to the value of the SCF, i.e., $f(\omega)$, if at least $n-1$ players i announce $m_i^k = \omega$. Otherwise, he selects the status quo allocation a^*.

Let

$$x_i(m, \omega) = \eta_1 + \frac{r_i}{K}\eta_2 \qquad \text{if there exists } k \in \{1, ..., K\} \text{ such that}$$

$$m_i^k \neq \omega, \text{ and}$$

$$m_j^{k'} = \omega \text{ for all } k' < k \text{ and } i \in N,$$

and

$$x_i(m,\omega) = \frac{r_i}{K}\eta_2 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i \in \{0,...,K\}$ denotes the number of integers $k \in \{1,...,K\}$ such that $m_i^k \neq \omega$.

If a player is one of the first deviants from ω, i.e., one of the players who tell lies as the youngest sub-message among all liars, he is fined the monetary amount η_1. Any player $i \in N$ is fined the monetary amount $\frac{r_i}{K}\eta_2$.

Since

$$0 \leq x_i(m,\omega) \leq \eta_1 + \eta_2,$$

by selecting $\eta_1 + \eta_2$ close to zero, we can make the monetary transfer $x_i(m,\omega)$ as close to zero as possible.

We denote a strategy $s_i = (s_i^k)_{k=1}^K$, where $s_i^k : \Omega \to M_i^k$. We define the *honest* strategy for player i, $s_i^* = (s_i^{*k})_{k=1}^K$, as

$$s_i^{*k}(\omega) = \omega \text{ for all } k \in \{1,...,K\} \text{ and } \omega \in \Omega.$$

The honest strategy profile $s^* \equiv (s_i^*)_{i \in N}$ induces the value of the SCF f in G^*, i.e.,

$$g(s^*(\omega)) = f(\omega) \text{ for all } \omega \in \Omega,$$

and no monetary transfers, i.e.,

$$x_i(s^*(\omega),\omega) = 0 \text{ for all } i \in N \text{ and } \omega \in \Omega.$$

The construction of the mechanism G^* is based on the bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994). Abreu and Matsushima demonstrated the basic concepts relevant to G^*, such that each player announces multiple sub-messages at once, the central planner randomly selects one sub-message profile, and he fines the first deviants.

There is a substantial difference between this paper and Abreu and Matsushima in that we do not utilize any incentive device of ‘virtualness’ that originates in Matsushima (1988) and Abreu and Sen (1991). Virtualness permits the selections of undesirable allocation even on the equilibrium path. In contrast, this paper does not permit such selections at all, i.e., requires a mechanism to achieve, not virtually, but exactly, the value of the SCF.

4.3. Possibility Theorem

Since $u_i(\alpha, t_i, \omega)$ is continuous in (α, t_i) and increasing in t_i, we can select a sufficient K such that whenever

$$\max_{a \in A} |\alpha(a) - \alpha'(a)| \leq \frac{1}{K},$$

then

(1) $$u_i(\alpha, -t_i, \omega) > u_i(\alpha', -t_i - \eta_1, \omega) \text{ for all } t_i \in [0, \eta_2] \text{ and } \omega \in \Omega.$$

The inequalities (1) imply that a first-deviant's loss from the monetary fine η_1 is always greater than his gain from the change of allocation caused by his lying.

The following theorem shows that G^* uniquely implements f in iterative dominance with full verification. Since G^* is well-defined, we can conclude that *with full verification, any SCF is uniquely implementable in iterative dominance, where we need almost no monetary transfers off the equilibrium path, and need no monetary transfers on the equilibrium path.*

Theorem 1: *The honest strategy profile s^* is the unique iteratively undominated strategy profile in G^* with full verification.*

Proof: We can show that each player $i \in N$ prefers $m_i^1 = \omega$. Suppose that there exists a player $j \in N \setminus \{i\}$ who announces $m_j^1 \neq \omega$. Then, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined η_1 or even more. From (1), the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation.

Next, suppose that there exists no player $j \in N \setminus \{i\}$ who announces $m_j^1 \neq \omega$. Then, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined η_2 or even more. (Even if he announces $m_i^1 = \omega$, he may still be one of the first deviants, and therefore, he may not save the fine η_1 in this case.) From the specification of g, there is no resultant change of allocation. These observations imply that he prefers $m_i^1 = \omega$ regardless of the other players' announcements.

Fix an arbitrary integer $h \in \{2,...,K\}$. Suppose that each player $i \in N$ announces $m_i^{h'} = \omega$ for all $h' \in \{1,...,h-1\}$. According to the same manner as above, we can show that he prefers $m_i^h = \omega$. Suppose that there exists a player $j \in N \setminus \{i\}$ who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined η_1 or even more. From (1), the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation. Next, suppose that there exists no player $j \in N \setminus \{i\}$ who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined η_2 or even more. The specification of g implies that there is no resultant change of allocation in this case. Hence, he prefers $m_i^h = \omega$.

Q.E.D.

5. Partial Verification

From this section, let us describe a state as

$$\omega = (\omega_0, \omega_1, ..., \omega_n).$$

For each $i \in N \bigcup \{0\}$, Ω_i denotes the set of possible ω_i. Let $\Omega_{-i} \equiv \underset{j \in N \bigcup \{0\} \setminus \{i\}}{\times} \Omega_j$, $\omega_{-i} \equiv (\omega_j)_{j \in N \bigcup \{0\} \setminus \{i\}} \in \Omega_{-i}$, $\Omega_{-i-j} \equiv \underset{l \in N \bigcup \{0\} \setminus \{i, j\}}{\times} \Omega_l$, and $\omega_{-i-j} \equiv (\omega_l)_{l \in N \bigcup \{0\} \setminus \{i, j\}} \in \Omega_{-i-j}$.

We assume that Ω is a proper subset of $\underset{i \in N \bigcup \{0\}}{\times} \Omega_i$. Each player $i \in N$ regards the set difference $\underset{i \in N \bigcup \{0\}}{\times} \Omega_i \setminus \Omega \neq \phi$ as the rare event the occurrence of which is ignorable. Let $\Omega_{-i}(\omega_i) \subset \Omega_{-i}$ denote the set of possible ω_{-i} such that $(\omega_i, \omega_{-i}) \in \Omega$. We assume that for every $\omega_i \in \Omega_i$, $\Omega_{-i}(\omega_i)$ is nonempty.

We assume *partial verification* as follows. After the central planner determines an allocation, but before he determines monetary transfers, only ω_0 becomes public and verifiable to the court.

5.1. Sequential Mechanisms

We assume that each player $i \in N$ observes ω_i at the first stage, i.e., observes ω_i earlier than $\omega_{-i} = (\omega_j)_{j \in N \cup \{0\} \setminus \{i\}}$. The central planner requires each player i to announce about ω_i as his first announcement at the first stage, i.e., announce before he observes ω_{-i}.

This section assumes that each player $i \in N$ observes ω_{-i} at the second stage, i.e., just before the central planner selects an allocation. The central planner requires each player i to announce about the state ω as his second announcement at the second stage, i.e., just after he observes ω_{-i}. In other world, this section assumes incomplete information at the first stage, while complete information at the second stage. In Section 8, we will replace this complete information assumption with the more general incomplete information at the second stage.

Based on the above-mentioned twice requirements of announcement, we define a *sequential mechanism* as $\Gamma \equiv (M^0, M, g, x)$, where $M^0 \equiv \underset{i \in N}{\times} M_i^0$, M_i^0 denotes the set of possible first announcements by player i, $M \equiv \underset{i \in N}{\times} M_i$, M_i denotes the set of possible second announcements by player i, $g : M^0 \times M \to \Delta$ denotes the allocation rule, $x \equiv (x_i)_{i \in N}$ denotes the transfer rule, and $x_i : M^0 \times M \times \Omega_0 \to R$ denotes the transfer rule for player i. We assume that both M_i^0 and M_i are finite sets for each $i \in N$, i.e., we focus on a set of bounded sequential mechanisms.

After observing ω_i, but before observing ω_{-i}, i.e., at the first stage, each player i makes his first announcement $m_i^0 \in M_i^0$. After observing ω_{-i}, i.e., at the second stage, each player i makes his second announcement $m_i \in M_i$. The central planner then selects an allocation according to $g(m_0, m) \in \Delta$. After ω_0 becomes verifiable, the central planner receives $x_i(m^0, m, \omega_0) \in R$ from each player i.

We will assume imperfect information in that each player cannot observe the other players' first and second announcements.

A *strategy for player* i *in a sequential mechanism* Γ is defined as $\psi_i \equiv (s_i^0, s_i)$, where $s_i^0 : \Omega_i \to M_i^0$ and $s_i : \Omega \to M_i$. He announces $s_i^0(\omega_i) \in M_i^0$ as his first announcement. He announces $s_i(\omega) \in M_i$ as his second announcement. Because of the imperfect information assumption, his second announcement does not depend on the other players' announcements. Let S_i^0 denote the set of possible s_i^0. Let $\Psi_i \equiv S_i^0 \times S_i$ denote the set of all strategies for player i in Γ. Denote $S^0 \equiv \underset{i\in N}{\times} S_i^0$, $\Psi \equiv \underset{i\in N}{\times} \Psi_i$, and $\psi \equiv (\psi_i)_{i\in N} \in \Psi$.

5.2. Iterative Dominance

With partial verification, we define *iterative dominance* as follows. For every $i \in N$, let

$$\hat{M}_i^0(0,\omega_i) \equiv M_i^0 \text{ for all } \omega_i \in \Omega_i ,$$

and

$$\hat{M}_i(0,\omega) \equiv M_i \text{ for all } \omega \in \Omega .$$

Let $\hat{M}^0(0,\omega_{-0}) \equiv \underset{i\in N}{\times} \hat{M}_i^0(0,\omega_i)$, and $\hat{M}_{-i}^0(0,\omega_{-i-0}) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j^0(0,\omega_j)$. Recursively, for each $h \geq 1$, we define a subset of player $i's$ first messages $\hat{M}_i^0(h,\omega_i) \subset \hat{M}_i^0$ in the manner that $m_i^0 \in \hat{M}_i^0(h,\omega_i)$ if and only if there exists no $\tilde{m}_i^0 \in \hat{M}_i^0(h-1,\omega_i)$ such that for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$, $m \in \hat{M}(h-1,\omega)$, and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$\leq u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega) ,$$

and there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that for every $m \in \hat{M}(h-1,\omega)$ and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega) .$$

We define a subset of player $i's$ second messages $\hat{M}_i(h,\omega) \subset M_i$ in the manner that $m_i \in \hat{M}_i(h,\omega)$ if and only if there exists no $m_i' \in \hat{M}_i(h-1,\omega)$ such that for every $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$ and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m^0,m_i',m_{-i}),-x_i(m^0,m_i',m_{-i},\omega_0),\omega).$$

Definition 3: A strategy $\psi_i = (s_i^0, s_i) \in \Psi_i$ for player i is said to be *iteratively undominated in* Γ *with partial verification* if

$$s_i^0(\omega_i) \in \bigcap_{h=0}^{\infty} \hat{M}_i^0(h,\omega_i) \text{ for all } \omega_i \in \Omega_i,$$

and

$$s_i(\omega) \in \bigcap_{h=0}^{\infty} \hat{M}_i(h,\omega) \text{ for all } \omega \in \Omega.$$

Suppose that $\Omega_i = \Omega_0$ for all $i \in N$, and

$$\omega_i = \omega_0 \text{ for all } i \in N \text{ if and only if } \omega \in \Omega.$$

This supposition corresponds to the full verification case studied in Section 4, in which, $\Omega_{-i}(\omega_i)$ is a singleton for all $i \in N$ and $\omega_i \in \Omega_i$. Clearly, the definition of iterative dominance in this section is equivalent to that of iterative dominance with full verification.

The order of elimination does not matter in the definition of iterative dominance with partial verification. Even if we change the order of eliminating strategies, the set of eventually survived strategies is unchanged. The reason for this irrelevance is that for every $\omega_i \in \Omega_i$, there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ for which the strict inequalities hold for players' incentive irrespective of the other players' announcements.

Definition 4: A sequential mechanism Γ is said to *uniquely implement a SCF* f *in iterative dominance with partial verification* if there exists the unique iteratively undominated strategy profile $\psi \in \Psi$ in Γ with partial verification, and this profile induces the value of the SCF, i.e.,

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

The definition of iterative dominance is irrelevant to the specification of prior distribution on Ω. That is, the sequential mechanism that uniquely implements a SCF in iterative dominance with partial verification is "detail-free" with respect to prior distribution on Ω.

It is implicit to assume in this paper that at the first stage, each player $i \in N$ prefers a (ω_i) -contingent choice $(f^{\omega_i}, t_i^{\omega_i})$ to another (ω_i) -contingent choice $(f'^{\omega_i}, t_i'^{\omega_i})$ if $(f^{\omega_i}, t_i^{\omega_i})$ makes a more preferable choice of allocation and transfer for player i than $(f'^{\omega_i}, t_i'^{\omega_i})$ irrespective of ω_{-i}, i.e., if for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$,

$$u_i(f_i^{\omega_i}(\omega_{-i}), t_i^{\omega_i}(\omega_{-i}), \omega) > u_i(f_i'^{\omega_i}(\omega_{-i}), t_i'^{\omega_i}(\omega_{-i}), \omega),$$

where $f_i^{\omega_i} : \Omega_{-i}(\omega_i) \to \Delta$ and $t_i^{\omega_i} : \Omega_{-i}(\omega_i) \to R$. With this implicit assumption, we can safely say that any message that is eliminated through the iterative procedure is regarded as a message that is dominated with strict inequality irrespective of the specification of full-support prior distribution on Ω.

In the definition of iterative dominance with partial verification, we required strict inequalities for, not all, but some, states. This implies that any eliminated message is weakly dominated for all non-full-support distributions, while it is strictly dominated for all full-support distributions.

6. Full Detection

This section demonstrates a sufficient condition under which any SCF is uniquely implementable in iterative dominance with partial verification. This section is the main part of this paper.

6.1. Definitions

For each $i \in N$, let us denote $\chi_{-i} : \Omega_{-i} \to 2^{\Omega_{-i}}$, where

$$\omega_{-i} \in \chi_{-i}(\omega_{-i}),$$

and

$$\tilde{\omega}_0 = \omega_0 \text{ for all } \tilde{\omega}_{-i} \in \chi_{-i}(\omega_{-i}).$$

We regard the function χ_{-i} as describing the pattern of the announcements made by all players other than player i. That is, they announce a profile that belongs to $\chi_{-i}(\omega_{-i}) \subset \Omega_{-i}$ when ω_{-i} occurs. Here, we regard player 0 as the dummy player who always announce about ω_0 truthfully.

We introduce a notion on χ_{-i}, namely *detection*, as follows.

Definition 5: A function χ_{-i} is said to *detect player* i *for* ω_i *against* ω_i' if there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

(2) $$\chi_{-i}(\tilde{\omega}_{-i}) \cap \chi_{-i}(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(\omega_i').$$

Suppose that ω_i is correct, but player i announces $\omega_i' \neq \omega_i$ incorrectly. Suppose that for every ω_{-i}, the other players announce according to $\chi_{-i}(\omega_{-i}) \subset \Omega_{-i}$, i.e., they announce a profile ω_{-i-0}' that satisfies

$$(\omega_0, \omega_{-i-0}') \in \chi_{-i}(\omega_{-i}).$$

Note that if player $i's$ announcement ω_i' is correct, the other players announce according to $\chi_{-i}(\tilde{\omega}_{-i})$ for some $\tilde{\omega}_{-i} \in \Omega_{-i}(\omega_{-i}')$.

Suppose that player i expects $\omega_{-i} = (\omega_0, \omega_{-i-0}) \in \Omega_{-i}(\omega_i)$ to occur. Then, player i expects the other players to announce according to $\chi_{-i}(\omega_{-i})$. However, this along with (2) implies that the other players never announce according to $\chi_{-i}(\omega_{-i})$, which contradicts the expectation of player i that ω_{-i} occurs. Hence, in this case, we can recognize that player $i's$ announcement ω_i' is incorrect, i.e., χ_{-i} detects player i for ω_i ag*ainst* ω_i'.

Based on the detection notion, we define *full detection* as follows. For every $h \in \{0,1,...\}$ and $i \in N \bigcup \{0\}$, we specify $\chi_i(h) : \Omega_i \to 2^{\Omega_i}$ and $\chi_{-i}(h) : \Omega_{-i} \to 2^{\Omega_{-i}}$ in the following manner. Let

$$\chi_0(h)(\omega_0) = \{\omega_0\} \text{ for all } \omega_0 \in \Omega_0 \text{ and } h \in \{0,1,...\},$$

and

$$\chi_i(0)(\omega_i) = \Omega_i \text{ and } \chi_{-i}(0)(\omega_{-i}) = \{\omega_0\} \times \Omega_{-i} \text{ for all } i \in N \text{ and } \omega \in \Omega.$$

Recursively, for each $h \in \{1,2,...\}$, we define $\chi_i(h)(\omega_i) \subset \chi_i(h-1)(\omega_i)$ and $\chi_{-i}(h)(\omega_{-i}) \subset \chi_{-i}(h-1)(\omega_{-i})$ in the manner that for every $\omega_i' \in \chi_i(h-1)(\omega_i)$,

$$\omega_i' \in \chi_i(h)(\omega_i) \quad \text{if and only if } \chi_{-i}(h-1) \text{ fails to detect player } i$$
$$\text{for } \omega_i \text{ against } \omega_i', \text{ i.e.,}$$
$$\{\bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(\omega_{-i}')} \chi_{-i}(\tilde{\omega}_{-i})\} \bigcap \chi_{-i}(\omega_{-i}) \neq \phi \text{ for all}$$
$$\omega_{-i} \in \Omega_{-i}(\omega_i),$$

and for every $\omega_{-i} \in \chi_{-i}(h-1)(\omega_{-i})$,

$$\omega_{-i} \in \chi_{-i}(h)(\omega_{-i}) \quad \text{if and only if } \omega_j \in \chi_j(h)(\tilde{\omega}_j) \text{ for all}$$
$$j \in N \bigcup \{0\} \setminus \{i\}.$$

Here, $\chi_{-i}(h)$ implies the set of all announcements that can survive through the h -round iterative removal of detected lies.

Full Detection: For every $i \in N$ and $\omega_i \in \Omega_i$,

$$\bigcap_{h \to \infty} \chi_i(h)(\omega_i) = \{\omega_i\}.$$

The sequence $((\chi_i(h), \chi_{-i}(h-1))_{i \in N})_{h=0}^{\infty}$ describes the iterative removal of detected lies. Full detection implies that the iterative removal of detected lies eventually eliminates all lies. Truth-telling is therefore the only announcement that survives through such removal procedure. Since Ω is finite, there exists a positive integer h^* such that

$$\chi_i(h)(\omega_i) = \{\omega_i\} \text{ for all } h \geq h^*.$$

We demonstrate the following tractable sufficient condition for full detection. Let us describe a state as

$$\omega = (\theta_0, ..., \theta_L),$$

where L is a positive integer. We assume that there exists a function $\tau : \{0,...,L\} \to 2^{N \cup \{0\}}$ such that for every $i \in N \bigcup \{0\}$

$$\omega_i = \theta_{L(i)},$$

where we denote

$$L(i) \equiv \{l \in \{0,...,L\} \,|\, i \in \tau(l)\},$$

and

$$\theta_C \equiv (\theta_l)_{l \in C} \text{ for each } C \subset \{0,...,L\}.$$

We assume that

$$L(0) = \{0\}, \text{ i.e., } \omega_0 = \theta_0.$$

Here, $\tau(l) \subset N$ implies the set of all players who observe θ_l at the first stage, and $L(i) \subset \{1,...,L\}$ implies the set of all components of the state that player i observes at the first stage. Let Ξ_l denote the set of possible θ_l, i.e., $\Omega \subset \underset{l \in \{0,...,L\}}{\times} \Xi_l$.

For every $i \in N \bigcup \{0\}$ and $l \in L(i)$, let

$$L(l,i) \equiv \{\tilde{l} \in \{0,...,l-1\} \,|\, i \notin \tau(\tilde{l})\},$$

which implies the set of all components of the state that are younger than l and player i cannot observe at the first stage. For every $i \in N \bigcup \{0\}$, $l \in L(i)$, and $\omega_i \in \Omega_i$, let

$$\theta_{L(l,i)} \in \Xi_{L(l,i)}(\omega_i) \text{ if and only if there exists } \tilde{\omega} \in \Omega \text{ such that}$$

$$\tilde{\omega}_i = \omega_i \text{ and } \tilde{\theta}_{L(l,i)} = \theta_{L(l,i)}.$$

Here, $\Xi_{L(l,i)}(\omega_i)$ implies the set of all $\theta_{L(l,i)}$ that are consistent with player i's observation at the first stage.

Proposition 2: *Suppose that for every $l \in \{1,...,L\}$, there exist $i \in \tau(l)$ such that for every $\omega_i \in \Omega_i$ and $\tilde{\omega}_i \in \Omega_i$,*

$$\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i) \text{ if } \theta_l \neq \tilde{\theta}_l \text{ and } \theta_{l'} = \tilde{\theta}_{l'} \text{ for all } l' \in L(i) \setminus \{l\}.$$

Then, Ω *satisfies full detection.*

Proof: Consider $l=1$. In this case, $\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i)$ implies that θ_0 detects player $i=\tau(1)$ for ω_i against $\tilde{\omega}_i$ whenever $\theta_1 \neq \tilde{\theta}_1$, where we must note that the dummy player 0 always tell the truth about θ_0 because of its verification.

Fix an arbitrary $h \in \{2,...,L\}$. Suppose that for every $h' \in \{0,...,h-1\}$, player $\tau(l')$ tells the truth about $\theta_{l'}$. In this case, $\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i)$ implies that $\theta_{L(l,i)}$ detects player $i=\tau(l)$ for ω_i against $\tilde{\omega}_i$ whenever $\theta_l \neq \tilde{\theta}_l$.

Q.E.D.

A special case of the sufficient condition in Proposition 2 is introduced as follows. Suppose that

$$\Xi_l = \Xi_0 \text{ for all } l \in \{1,...,L\},$$

and each component of the state is always different from its neighbors, i.e., for every $\omega \in \underset{l\in\{0,...,L\}}{\times} \Xi_l$,

$$\omega \in \Omega \text{ if and only if } \theta_{l-1} \neq \theta_l \text{ for all } l \in \{1,...,L\}.$$

Moreover, suppose that for every $l \in \{1,...,L\}$, there exists $i=\iota(l) \in N$ such that

$$l \in L(i),$$

$$\{l-2, l-1\} \not\in L(i) \text{ if } l \geq 2,$$

and

$$0 \not\in L(i) \text{ if } l=1.$$

Hence, player $i=\iota(l)$ cannot receive any information about θ_{l-1} at the first stage.

In this special case, it is clear that Ω satisfies the sufficient condition in Proposition 2, i.e., Ω satisfies full detection. A player's lie about θ_l is detected through the observation of θ_{l-1}. Since θ_0 is verifiable, we can eliminate all lies about

θ_1 through the observation of θ_0. Recursively, for every $l \in \{2,...,L\}$, we can eliminate all lies about θ_l through the truthful announcement about θ_{l-1}[6].

6.2. The Theorem

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 \equiv (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. In order to uniquely implement a SCF f in iterative dominance with partial verification, we construct a sequential mechanism $\Gamma^* = \Gamma^*(f, \eta_1, \eta_2, \eta_3, K) = (M^0, M, g, x)$ as follows. Let

$$M_i^0 = \Omega_i ,$$

and

$$M_i = \prod_{k=1}^{K} M_i^k \quad \text{and} \quad M_i^k = \Omega \quad \text{for all} \quad k \in \{1,...,K\} .$$

For each $k \in \{2,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } m_i^k = \omega \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \omega.$$

Let

$$g(m^0, m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K} .$$

The allocation choice does not depend on the first announcements m^0.

Let

$$x_i(m^0, m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0, \omega_0) + z_i(m^0, m, \omega_0) ,$$

where

$$x_i^h(m^0, \omega_0) = \eta_1(h) \qquad \text{if } (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}) ,$$

[6] We will show some generalization of this special case in Subsection 9.1.

$$x_i^h(m^0,\omega_0)=0 \qquad \text{if } (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}),$$

$z_i(m^0,m,\omega_0)=\eta_2+\dfrac{r_i}{K}\eta_3$ if there exists $k\in\{1,...,K\}$ such that

$m_i^k \neq (m^0,\omega_0)$, and

$m_j^{k'}=(m^0,\omega_0)$ for all $k'<k$ and

$j\in N\setminus\{i\}$,

and

$z_i(m^0,m,\omega_0)=\dfrac{r_i}{K}\eta_3$ if there exists no such $k\in\{1,...,K\}$,

where $r_i\in\{0,...,K\}$ implies the number of $k\in\{2,...,K\}$ such that $m_i^k\neq(m^0,\omega_0)$.

We select (η_1,η_2,η_3) such that

(3) $$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1}\eta_1(h)+\eta_2+\eta_3 \text{ for all } \tilde{h}\in\{1,...,h^*\}.$$

Since

$$0\leq x_i(m^0,m,\omega_0)\leq\sum_{h=1}^{h^*}\eta_1(h)+\eta_2+\eta_3 \text{ for all } i\in N \text{ and } (m^0,m,\omega_0),$$

by choosing $\sum_{h=1}^{h^*}\eta_1(h)+\eta_2+\eta_3$ close to zero, we can make $x_i(m^0,m,\omega_0)$ as close to zero as possible.

We define the *honest* strategy for player i in Γ^*, $\psi_i^*=(s_i^{0*},s_i^*)$, as

$$s_i^{0*}(\omega_i)=\omega_i \text{ for all } \omega_i\in\Omega_i,$$

and

$$s_i^{*k}(\omega)=\omega \text{ for all } k\in\{1,...,K\} \text{ and } \omega\in\Omega.$$

The honest strategy profile $\psi^*\equiv(\psi_i^*)_{i\in N}$ induces the value of the SCF f, i.e.,

$$g(\psi^*(\omega))=f(\omega) \text{ for all } \omega\in\Omega,$$

and no monetary transfers, i.e.,

$$x_i(\psi^*(\omega),\omega)=0 \text{ for all } i\in N \text{ and } \omega\in\Omega.$$

Because of the continuity assumption, we can select a sufficient K such that whenever $\max_{a\in A}\left|\alpha(a)-\alpha'(a)\right|\leq\frac{1}{K}$, then

(4) $$u_i(\alpha,-t_i,\omega_i)>u_i(\alpha',-t_i-\eta_2,\omega_i) \text{ for all } t_i\in[0,\sum_{h=1}^{h^*}\eta_1(h)+\eta_3] \text{ and } \omega_i\in\Omega_i.$$

The inequalities (4) imply that η_2 is close to zero but is sufficient compared with the change of allocation within the $\frac{1}{K}-$limit.

According to x_i^h, any player $i\in N$ is fined the monetary amount $\eta_1(h)$ if he makes a first announcement that is detected by $\chi_{-i}(h-1)$ which describes the profiles of the other players' announcements that are survived through the $(h-1)-$round iterative removals of detected messages. This along with (3) and full detection implies that each player is willing to announce an undetected message, i.e., the honest message, as his first announcement.

According to z_i, any first deviant from the combination of the profile of first announcements and the verified information (ω^0,m_0) in the second announcement stage is fined the monetary amount η_2. Any player is additionally fined $\frac{\eta_3}{K}$ whenever he deviates from (ω^0,m_0). We apply the bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994), showing that by setting the first announcement and the verified information as the reference, any player is willing to make the truthful second announcement.

We can see this paper's main technical contribution in the arguments about the players' incentive at the first stage. It is crucial in incentives that each player i makes the first announcement before he observes ω_{-i}. The informational restriction to ω_i at the first stage serves to prohibit each player from seeking a mean of escape from detection.

Based on these observations, we can demonstrate the following theorem, which states that under full detection, the sequential mechanism Γ^* uniquely implements f in iterative dominance with partial verification. Since Γ^* is well-defined, we can conclude that *with full detection, any SCF is uniquely implementable in iterative dominance with partial verification, where we need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path*.

Theorem 3: *Under full detection, the honest strategy profile* ψ^* *is the unique iteratively undominated strategy profile in* Γ^*. *That is,* Γ^* *uniquely implements* f *in iterative dominance with partial verification.*

Proof: Suppose that player i observes ω_i and announces $m_i^0 \notin \chi_i(1)(\omega_i)$ as his first announcement. In this case, $\chi_{-i}(0)$ detects him for ω_i against m_i^0; there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

$$\chi_{-i}(0)(\tilde{\omega}_{-i}) \cap \chi_{-i}(0)(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0).$$

Since $\chi_{-i}(0)(\omega_{-i}) = \{\omega_0\} \times M_{-0-i}$, the announcement by any other player $j \in N \setminus \{i\}$ belongs to $\chi_j(0)(\omega_j)$. This implies that, by announcing m_i^0, he is fined $\eta_1(1)$. In contrast, he can save this fine by announcing ω_i truthfully. Since the announcement of m_i^0 is irrelevant to the allocation choice and $\eta_1(1)$ is large enough to satisfy (3), it follows that player i never announces any element that does not belong to $\chi_i(1)(\omega_i)$.

Consider an arbitrary $h \in \{2, ..., h^*\}$. Suppose that any player $i \in N$ announces the message for the first announcement that belongs to $\chi_i(h-1)(\omega_i)$. Suppose that player i observes ω_i and announces $m_i^0 \notin \chi_i(h)(\omega_i)$. In this case, $\chi_{-i}(h-1)$ detects him for ω_i against m_i^0. That is, there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

$$\chi_{-i}(h-1)(\tilde{\omega}_{-i}) \cap \chi_{-i}(h-1)(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0).$$

Since the announcement by any other player $j \in N \setminus \{i\}$ belongs to $\chi_j(h-1)(\omega_j)$, he is fined $\eta_1(h)$. In contrast, he can save this fine by announcing ω_i truthfully. Since the announcement of m_i^0 is irrelevant to the allocation choice and $\eta_1(h)$ is large enough to satisfy (3), it follows that player i never announces any element that does not belong to $\chi_i(h)(\omega_i)$.

From the above arguments, we have proved that if ψ_i is strictly iteratively undominated, then,

$$s_i^0(\omega_i) \in \bigcap_{h\to\infty} \chi_i(h)(\omega_i) \text{ for all } \omega_i \in \Omega_i ,$$

which, along with full detection, implies that

$$s_i^0(\omega_i) = \omega_i \text{ for all } \omega_i \in \Omega_i .$$

Since all players tell the truth for their first announcements, i.e., any player $i \in N$ announces $m_i^0 = \omega_i$, we can prove in the same manner as in Theorem 1 that for each $i \in N$, if ψ_i is strictly iteratively undominated, then,

$$s_i^k(\omega) = \omega \text{ for all } \omega \in \Omega \text{ and } k \in \{1,...,K\} ,$$

where we utilized the inequality (4) for deriving this statement.

From these observations, we have proved that ψ^* is the unique quasi-strict iteratively undominated strategy profile in Γ^*.

Q.E.D.

In the proof, we utilize the basic concept of bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994) such that the central planner requires each player to make multiple announcements at one time, selects one profile from their announcements, and fines the first deviants from the reference point. Once we can establish the truthful reference point, the mechanism a la Abreu-Matsushima can successfully implement the SCF in iterative dominance.

Hence, the remaining problem is to show how the central planner can establish such truthful reference point. This problem becomes quite substantial once we require the mechanism to be detail-free in terms of prior distribution. In fact, this is an easy problem to solve if we permit a particular prior distribution $p:\Omega \to [0,1]$ to be common knowledge. For instance, let us denote by $p_i(\cdot|\omega_i):\Omega_0 \to [0,1]$ the $(\omega_i)-$ conditional distribution on Ω_0 induced by p. Assume that for each $i \in N$,

$$p_i(\cdot|\omega_i) \neq p_i(\cdot|\omega_i') \text{ whenever } \omega_i \neq \omega_i' .$$

In this case, by introducing a device of proper scoring rule, the central planner can incentivize each player i to reveal ω_i truthfully, establishing the truthful reference point.

Since this paper assumes no such common knowledge, we need to utilize the 'moth-eaten' nature of the state space, which is expressed by full detection, in the more complicated manner than the differences in conditional distribution.

7. Partial Detection

This section considers the case in which Ω does not satisfy full detection. We weaken implementation by replacing the uniqueness of iteratively undominated strategy profile with the uniqueness of outcome induced by iteratively undominated strategy profiles. By doing this manner, we define full implementation in iterative dominance with partial verification as follows.

Definition 6: A sequential mechanism Γ is said to *fully implement a SCF* f *in iterative dominance with partial verification* if every iteratively undominated strategy profile $\psi \in \Psi$ in Γ induces the value of the SCF, i.e.,

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

Full implementation permits the multiplicity of iteratively undominated strategies. However, it require any profile of iteratively undominated strategies to correctly achieve the value of the SCF.

A *partition on* Ω_i is defined as $\Phi_i : \Omega_i \to 2^{\Omega_i} \setminus \{\phi\}$, where for every $\omega_i \in \Omega_i$ and $\omega_i' \in \Omega_i$,

$$\text{either } \Phi_i(\omega_i) = \Phi_i(\omega_i') \text{ or } \Phi_i(\omega_i) \cap \Phi_i(\omega_i') = \phi .$$

We can regard a partition Φ_i as the set of subsets $\phi_i \subset \Omega_i$ such that

$$\phi_i \in \Phi_i \text{ if and only if } \phi_i = \Phi_i(\omega_i) \text{ for some } \omega_i \in \Omega_i .$$

Let $\Phi \equiv (\Phi_i)_{i \in N \cup \{0\}}$. Denote $\Phi(\omega) = (\Phi_i(\omega_i))_{i \in N \cup \{0\}}$.

We specify Φ_i^* as the finest partition on Ω_i satisfying that

$$\bigcap_{h\to\infty} \chi_i(h)(\omega_i) \subset \Phi_i^*(\omega_i) \quad \text{for all} \quad \omega_i \in \Omega_i\,,$$

where $\chi_i(h)$ was introduced in the definition of full detection. Note that full detection holds if and only if for every $i \in N \bigcup \{0\}$, Φ_i^* is the full partition, i.e., $\Phi_i^*(\omega_i) = \{\omega_i\}$ for all $\omega_i \in \Omega_i$.

Definition 7: A SCF f is said to be *measurable* if for every $\omega \in \Omega$ and $\omega' \in \Omega$,

$$f(\omega) = f(\omega') \quad \text{whenever} \quad \Phi_i^*(\omega_i) = \Phi_i^*(\omega_i') \quad \text{for all} \quad i \in N \bigcup \{0\}\,.$$

Measurability implies that the value of f is the same between ω_i and ω_i' whenever both belong to the same cell of Φ^*. The following theorem shows that the measurability is a sufficient condition for a SCF to be fully implementable in iterative dominance with partial verification.

Theorem 4: *Suppose that a SCF f is measurable. Then, it is fully implementable in iterative dominance with partial verification.*

Proof: See Appendix A.

8. Incomplete Information at the Second Stage

Throughout the previous sections, we made the complete information assumption at the second stage. This section eliminates this assumption, and instead assumes *incomplete information at the second stage* as follows.

For each $i \in N$, let us fix an arbitrary set $C(i) \subset N \bigcup \{0\}$, where we assume $i \in C(i)$. Each player i observes $\omega_{C(i)} = (\omega_j)_{j \in C(i)}$ at the second stage. He cannot observe $\omega_{N \bigcup \{0\} \setminus C(i)}$. Hence, he makes his second announcement contingent only on $\omega_{C(i)}$.

We define $\Omega_C(\omega_{N\cup\{0\}\backslash C}) \subset \underset{i\in C}{\times}\Omega_i$ in the manner that

$$\omega_C \in \Omega_C(\omega_{N\cup\{0\}\backslash C}) \text{ if and only if } (\omega_C, \omega_{N\cup\{0\}\backslash C}) \in \Omega.$$

We redefine a strategy $\psi_i = (s_i^0, s_i)$ for each player i by replacing $s_i : \Omega \to M_i$ with $s_i : \Omega_{C(i)} \to M_i$.

We redefine iterative dominance as follows. For every $i \in N$, let

$$\hat{M}_i^0(0,\omega_i) \equiv M_i^0 \text{ for all } \omega_i \in \Omega_i,$$

and

$$\hat{M}_i(0,\omega_{C(i)}) \equiv M_i \text{ for all } \omega \in \Omega.$$

Let $\hat{M}(0,\omega) \equiv \underset{j\in N}{\times}\hat{M}_j(0,\omega_{C(j)})$ and $\hat{M}_{-i}(0,\omega) \equiv \underset{j\in N\backslash\{i\}}{\times}\hat{M}_j(0,\omega_{C(j)})$. Recursively, for each $h \geq 1$, we define $\hat{M}_i^0(h,\omega_i) \subset \hat{M}_i^0$ in the manner that $m_i^0 \in \hat{M}_i^0(h,\omega_i)$ if and only if there exists no $\tilde{m}_i^0 \in \hat{M}_i^0(h-1,\omega_i)$ such that for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$, $m \in \hat{M}(h-1,\omega)$, and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$\begin{aligned}&u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)\\&\leq u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega),\end{aligned}$$

and there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that for every $m \in \hat{M}(h-1,\omega)$ and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$\begin{aligned}&u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)\\&< u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega).\end{aligned}$$

We define $\hat{M}_i(h,\omega_{C(i)}) \subset M_i$ in the manner that $m_i \in \hat{M}_i(h,\omega)$ if and only if there exists no $m_i' \in \hat{M}_i(h-1,\omega_{C(i)})$ such that for every $\omega_{N\cup\{0\}\backslash C(i)} \in \Omega_{N\cup\{0\}\backslash C(i)}(\omega_{C(i)})$, $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$, and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$\begin{aligned}&u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)\\&\leq u_i(g(m^0,m_i',m_{-i}),-x_i(m^0,m_i',m_{-i},\omega_0),\omega),\end{aligned}$$

and there exists $\omega_{N\cup\{0\}\backslash C(i)} \in \Omega_{N\cup\{0\}\backslash C(i)}(\omega_{C(i)})$ such that for every $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$ and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$u_i(g(m^0, m), -x_i(m^0, m, \omega_0), \omega)$$
$$< u_i(g(m^0, m_i', m_{-i}), -x_i(m^0, m_i', m_{-i}, \omega_0), \omega) .$$

A strategy $\psi_i = (s_i^0, s_i) \in \Psi_i$ for player i is said to be *iteratively undominated in* Γ *with partial verification under incomplete information at the second stage* if

$$s_i^0(\omega_i) \in \bigcap_{h=0}^{\infty} \hat{M}_i^0(h, \omega_i) \text{ for all } \omega_i \in \Omega_i ,$$

and

$$s_i(\omega) \in \bigcap_{h=0}^{\infty} \hat{M}_i(h, \omega) \text{ for all } \omega \in \Omega .$$

Definition 8: A SCF f is said to be *strictly incentive compatible* if there exist a positive real number $\xi > 0$ and a function $\tilde{f} : \times_{i \in N} \Omega_{C(i)} \to \Delta$ such that for every $\omega \in \Omega$,

$$\tilde{f}((\omega_{C(i)})_{i \in N}) = f(\tilde{\omega}) \text{ whenever } \omega_{C(i)} = \tilde{\omega}_{C(i)} \text{ for all } i \in N \text{, and}$$
$$\bigcup_{i \in N} C(i) = N \bigcup \{0\}$$

and for every $i \in N$, $\omega'_{C(j)} \in \Omega_{C(i)}$, and $t_i \in [0, \xi]$,

$$u_i(\tilde{f}((\omega_{C(j)})_{j \in N}), -t_i, \omega_i) \geq u_i(\tilde{f}((\omega_{C(j)})_{j \in N \setminus \{i\}}, \omega'_{C(i)}), -t_i, \omega_i) .$$

Strict incentive compatibility implies that irrespective of the constant transfer $-t_i$ within the $\xi-$limit, truth-telling is a weakly dominated strategy for each player i in the direct mechanism given by $\tilde{f}$. Clearly, strict incentive compatibility automatically holds under complete information.

From the continuity assumption, it follows that a SCF f is strictly incentive compatible if there exists $\tilde{f}$ such that for every $\omega \in \Omega$,

$$\tilde{f}((\omega_{C(i)})_{i \in N}) = f(\tilde{\omega}) \text{ whenever } \omega_{C(i)} = \tilde{\omega}_{C(i)} \text{ for all } i \in N \text{, and}$$
$$\bigcup_{i \in N} C(i) = N \bigcup \{0\}$$

and for every $i \in N$ and $\omega'_{C(j)} \in \Omega_{C(i)}$,

$$u_i(\tilde{f}((\omega_{C(j)})_{j\in N}),0,\omega) > u_i(\tilde{f}((\omega_{C(j)})_{j\in N\setminus\{i\}},\omega'_{C(i)}),0,\omega) \text{ if}$$

$$\tilde{f}((\omega_{C(j)})_{j\in N}) \neq \tilde{f}((\omega_{C(j)})_{j\in N\setminus\{i\}},\omega'_{C(i)}) .$$

The following theorem states that with full detection and strict incentive compatibility, we can construct a sequential mechanism that uniquely implements the SCF f in iterative dominance with partial verification under incomplete information at the second stage. Here, we need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path.

Theorem 5: *Assume incomplete information at the second stage, full detection, and strict incentive compatibility. Then, the SCF* f *is uniquely implementable in iterative dominance with partial verification under incomplete information at the second stage. That is, there exists a sequential mechanism* Γ *that has the unique iteratively undominated strategy profile* ψ *, and*

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega\in\Omega .$$

We need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path.

Proof: See Appendix B.

We further introduce a condition on a SCF as a combination of measurability and ex-post incentive compatibility as follows.

Definition 9: A SCF f is said to be *strict measurable incentive compatible* if there exist a positive real number $\xi > 0$ and a function $\tilde{f} : \underset{i\in N}{\times} \Omega_i \to \Delta$ such that for every $\omega\in\Omega$,

$$\tilde{f}(\omega_{-0}) = f(\omega_{-0}) ,$$

and for every $i\in N$, $\tilde{\omega}_i \in \Omega_i$, and $t_i \in [0,\xi]$,

$$u_i(\tilde{f}(\omega_{-0}),-t_i,\omega_i) \geq u_i(\tilde{f}(\tilde{\omega}_i,\omega_{-i-0}),-t_i,\omega_i) ,$$

where $\hat{f}$ is measurable in that for every $\omega \in \underset{i \in N}{\times} \Omega_i$ and $\omega' \in \underset{i \in N}{\times} \Omega_i$,

$$f(\omega) = f(\omega') \text{ whenever } \Phi_i^*(\omega_i) = \Phi_i^*(\omega_i') \text{ for all } i \in N \bigcup \{0\}.$$

From the continuity assumption, it follows that a SCF f is strict measurable incentive compatible if there exists $\tilde{f} : \underset{i \in N}{\times} \Omega_i \to \Delta$ such that for every $\omega \in \Omega$,

$$\tilde{f}(\omega_{-0}) = f(\omega_{-0}),$$

and for every $i \in N$ and $\tilde{\omega}_i \in \Omega_i$,

$$u_i(\tilde{f}(\omega_{-0}), 0, \omega_i) > u_i(\tilde{f}(\tilde{\omega}_i, \omega_{-i-0}), 0, \omega_i) \text{ whenever } \tilde{f}(\omega_{-0}) \neq \tilde{f}(\tilde{\omega}_i, \omega_{-i-0}),$$

where $\hat{f}$ is measurable.

The following theorem shows that the strict measurable incentive compatibility is a sufficient condition for a SCF to be fully implementable in iterative dominance with partial verification under incomplete information at the second stage.

Theorem 6: *Assume incomplete information at the second stage and strict measurable incentive compatibility. Then, the SCF f is fully implementable in iterative dominance with partial verification under incomplete information at the second stage.*

Proof: See Appendix C.

9. Discussion

9.1. Multi-Round Announcements

Throughout this paper, we assumed that the central planner requires each player $i \in N$ to make the first announcement about ω_i at the first stage. This subsection assumes that the central planner can require each player i to make announcements even before the first stage, i.e., even when he observes only partial information about

ω_i. We show a weaker sufficient condition than full detection for guaranteeing unique implementation in iterative dominance.

We consider a modification of the special case introduced in Subsection 6.1 as follows. Before the first stage, there exist T rounds, i.e., round 1, round 2, …, and round T, where round T corresponds to the first stage, and each player i can observe all components of $\omega_i = (\theta_l)_{l\in L(i)}$ by round T.

For each $i \in N$ and each $l \in L(i)$, player i observes θ_l at round $t = t(i,l) \in \{1,...,T\}$. Let us denote $t(i,l) = \infty$ for each $l \notin L(i)$.

We assume that there exists a mapping $\iota : \{1,...,L\} \to N$ such that

$$t(\iota(1),1) < t(\iota(1),0),$$

and for every $l \in \{2,...,L\}$,

$$\iota(l) \neq \iota(l-1),$$

$$t(\iota(l),l) < t(\iota(l),l-1), \text{ and } t(\iota(l),l) < t(\iota(l),l-2).$$

Based on this assumption, we consider the T-round procedure in which for every $l \in \{1,...,L\}$, the central planner requires player $i = \iota(l)$ to make an announcement about θ_l at round $t(i,l)$. Since $\theta_l \neq \theta_{l-1}$, it is clear that player $\iota(l-1)'s$ announcement about θ_{l-1} detects player $\iota(l)'s$ lies about θ_l. This can make unique implementation in iterative dominance possible to achieve by constructing some modification of sequential mechanism.

The above-mentioned assumption is substantially weaker than the sufficient condition in Subsection 6.1. In fact, we can permit that for every $l \in \{1,...,L\}$ and $i \in N$,

$l-2 \in L(i)$ even if $l \in L(i)$ and player i is the only person who observes θ_{l-2} at the first stage.

For instance, let us consider the case in which

$$L = 2n,$$

for every $i \in N$,

$$L(i) = \{2i-1, 2i+1\} \text{ if } i \text{ is odd,}$$

and

$$L(i) = \{2(i-1), 2i\} \text{ if } i \text{ is even.}$$

Note that for every $l \in \{1,...,L\}$ and $i \in N$,

$$l-2 \in L(i) \text{ whenever } l \in L(i),$$

which violates the sufficient condition in Subsection 6.1.

Let us further suppose that $T = 2$, and each player i observes

$$\theta_{2i+1} \text{ at round 1 and } \theta_{2i-1} \text{ at round 2 if } i \text{ is odd,}$$

and

$$\theta_{2i} \text{ at round 1 and } \theta_{2(i-1)} \text{ at round 2 if } i \text{ is even.}$$

Then, this case satisfies the assumption of this subsection, and therefore, we can make unique implementation in iterative dominance possible to achieve.

9.2. Unbounded Mechanisms

Throughout this paper, we confined our attention to bounded mechanisms, in which each player has a finite set of messages. This confinement is crucial in making the unique implementation problem non-trivial to solve.

Let us reconsider a sequential mechanism, in which, each player i makes an announcement m_i^0 about ω_i at the first stage. In order to incentivize each player to make the honest announcement at the first stage, the previous sections assumed that Ω satisfies full detection.

We however show below that once we permit a variant of sequential mechanism to be unbounded, i.e., permit an infinite set of messages for each player, we can easily eliminate all dishonest announcements at the first stage under a much weaker condition than full detection.

Let us denote $\pi_i : \Omega_i \to \Omega_i$ and $\pi \equiv (\pi_i)_{i \in N} : \Omega_{-0} \to \Omega_{-0}$. Assume on Ω that for every π, if it is not truth-telling, i.e.,

$$(\omega_0, \pi(\omega_{-0})) \neq \omega \text{ for some } \omega \in \Omega,$$

then there exists $\omega \in \Omega$ such that

$$(\omega_0, \pi(\omega_{-0})) \notin \Omega.$$

This implies that whenever there exists a player who tells a lie, then there exists a state at which the resultant announcement profile, along with the verified ω_0, belongs to the rare event. It is clear that this assumption is weaker than full detection, and it is not sufficient for a bounded sequential mechanism to eliminate all dishonest announcements at the first stage.

However, by constructing an unbounded mechanism, we can eliminate all dishonest announcements at the first stage. Consider the integer game, in which, each player simultaneously announces a positive integer, and the player who is the only person that announces the biggest integer among all players wins a positive monetary amount. Note that the integer game has no iteratively undominated strategy profile.

We then construct an unbounded mechanism in the manner that the players play the sequential mechanism, and they additionally play the integer game if and only if the resultant announcements at the first stage, $m^0 = (m_i^0)_{i\in N}$, and the verified ω_0 satisfy

$$(\omega_0, m^0) \notin \Omega .$$

Note that if a player tell a lie, then there exists a state at which the resultant combination (ω_0, m^0) belong to the rare event, i.e., $(\omega_0, m^0) \notin \Omega$. In this case, the players fail to play any iteratively undominated strategy profile, because of the emptiness of iteratively undominated strategy profile in the integer game. Hence, we can eliminate all dishonest announcements.

10. Conclusion

We investigated unique implementation of a SCF, where we required a mechanism to be detail-free and bounded, utilize only tiny transfers, and satisfy uniqueness of iteratively undominated strategy profile. We defined the iterative dominance notion on the ex-post terms, and required the strict inequalities for all full-support distributions as the incentive constraints.

The central concern of this paper was to clarify the impact of partial verification on implementation. We demonstrated a condition on the state space, namely full detection, under which, any SCF was uniquely implementable with partial verification. In contrast to the case with no such verification, a wide variety of SCFs were uniquely implementable with partial verification. This permissive result held even if the range of players' lies that the verified information can directly detect was quite narrow.

It was crucial for the central planner to design, not a static, but a sequential, mechanism, in which, each player announced two times at two distinct stages. Each player was assumed to be less informed at the first stage than at the second stage. By doing this sequential manner, the central planner could establish the reference point truthfully. With the establishment of truthful reference point, we could successfully apply the bounded mechanism design that originated in Abreu and Matsushima (1992).

This was the first paper to demonstrate permissive results in exact implementation with uniqueness of Nash equilibrium. This was also the first paper to investigate bounded mechanism design with unique mixed Nash equilibrium without the expected utility hypothesis.

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Appendix A: Proof of Theorem 4

Fix arbitrary real numbers $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 \equiv (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a sequential mechanism $\Gamma^{**} = \Gamma^{**}(f, \eta_1, \eta_2, \eta_3, K) = (M^0, M, g, x)$ as follows. Let

$$M_i^0 = \Omega_i \, ,$$

and

$$M_i = \prod_{k=1}^{K} M_i^k \quad \text{and} \quad M_i^k = \Phi^* \quad \text{for all} \quad k \in \{1,...,K\} .$$

In contrast with the sequential mechanism Γ^* specified in Subsection 6.2, each player i announces an element of the partition Φ_i^* as a sub-message in the second announcement stage instead of announcing an element of Ω_i.

For each $k \in \{2,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } \; m_i^k = \Phi^*(\omega) \; \text{ for at least } \; n-1 \; \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \; \omega .$$

Let

$$g(m^0, m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K} .$$

Let

$$x_i(m^0, m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0, \omega_0) + z_i(m^0, m, \omega_0) \, ,$$

where

$$x_i^h(m^0, \omega_0) = \eta_1(h) \qquad \text{if } \; (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}) \, ,$$

$$x_i^h(m^0, \omega_0) = 0 \qquad \text{if } \; (\omega^0, m_{-i}^0) \in \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}) \, ,$$

$$z_i(m^0, m, \omega_0) = \eta_2 + \frac{r_i}{K}\eta_3 \text{ if there exists } k \in \{1,...,K\} \text{ such that}$$

$(m^0, \omega_0) \notin m_i^k$, and

$(m^0, \omega_0) \in m_j^{k'}$ for all $k' < k$ and

$j \in N \setminus \{i\}$,

and

$$z_i(m^0, m, \omega_0) = \frac{r_i}{K}\eta_3 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i \in \{0,...,K\}$ implies the number of $k \in \{2,...,K\}$ such that $(m^0, \omega_0) \notin m_i^k$. According to z_i, any first player i who reports an element of Φ_i^* as his sub-message in the second announcement stage that does not include the combination of the profile of first announcements and the verified information (ω^0, m_0) is fined the tiny amount given by η_2.

We select (η_1, η_2, η_3) such that

$$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1} \eta_1(h) + \eta_2 + \eta_3 \text{ for all } \tilde{h} \in \{1,...,h^*\}.$$

With this, in the same manner as the proof of Theorem 3, we can prove that for every $i \in N$, if ψ_i is quasi-strictly iteratively undominated, then

$$s_i^0(\omega_i) \in \Phi_i^*(\omega_i) \text{ for all } \omega_i \in \Omega_i,$$

and

$$s_i^k(\omega) = \Phi^*(\omega) \text{ for all } k \in \{1,...,K\} \text{ and } \omega \in \Omega.$$

This along with the measurability implies that if ψ is a quasi-strictly iteratively undominated strategy profile, then

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega,$$

and

$$x_i(\psi(\omega), \omega) = 0 \text{ for all } i \in N \text{ and } \omega \in \Omega.$$

Appendix B: Proof of Theorem 5

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 = (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a sequential mechanism, denoted by $\hat{\Gamma}^* = \hat{\Gamma}^*(f,\eta_1,\eta_2,\eta_3,K) = (M^0, M, g, x)$, as follows. Let

$$M_i^0 = \Omega_i\,,\quad M_i = \prod_{k=1}^{K} M_i^k\,,\text{ and }\quad M_i^k = \Omega_{C(i)}\quad\text{for all}\quad k \in \{1,...,K\}.$$

Let

$$g(m) = \frac{\sum_{k=2}^{K} \tilde{f}(m^k)}{K-1}\quad\text{for all}\quad m \in M\,,$$

which $\tilde{f}$ is the function introduced in Definition 6. Let

$$x_i(m,\omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0,\omega_0) + z_i(m^0,m,\omega_0)\,,$$

where

$$x_i^h(m^0,\omega_0) = \eta_1(h)\qquad\text{if}\quad (m_{-i}^0,\omega_0) \notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i})\,,$$

$$x_i^h(m^0,\omega_0) = 0\qquad\text{if}\quad (m_{-i}^0,\omega_0) \notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i})\,,$$

$$z_i(m^0,m,\omega_0) = \eta_2 + \frac{r_i}{K}\eta_3\quad\text{if there exists}\quad k \in \{1,...,K\}\quad\text{such that}$$

$$m_i^k \neq m_{C(i)}^0\,,\text{ and}$$

$$m_j^{k'} = m_{C(j)}^0\quad\text{for all}\quad k' < k\quad\text{and}\quad j \in N \setminus \{i\}\,,$$

and

$$z_i(m^0,m,\omega_0) = \frac{r_i}{K}\eta_3\qquad\text{if there exists no such}\quad k \in \{1,...,K\}\,,$$

where we denote $m_0^0 = \omega_0$, and $r_i \in \{0,...,K\}$ implies the number of $k \in \{1,...,K\}$ satisfying $m_i^k \neq m_{C(i)}^0$. We select (η_1,η_2,η_3) such that

$$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1} \eta_1(h) + \eta_2 + \eta_3\quad\text{for all}\quad \tilde{h} \in \{1,...,h^*\}\,.$$

Note that

$$0 \leq x_i(m,\omega_0) \leq \sum_{h=1}^{h^*} \eta_1(h) + \eta_2 + \eta_3\quad\text{for all}\quad i \in N\quad\text{and}\quad (m,\omega_0)\,.$$

Hence, by choosing $\sum_{h=1}^{h^*}\eta_1(h)+\eta_2+\eta_3$ close to zero, we can make the monetary transfer $x_i(m,\omega_0)$ close to zero, i.e., lesser than ξ, where ξ was the real number in Definition 6, which was selected close to zero.

We define the *honest* strategy for player i, denoted by $\hat{\psi}_i^* = (\hat{s}_i^{0*}, (\hat{s}_i^{*k})_{k=1}^K)$, as

$$\hat{s}_i^{*0}(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i,$$

and

$$\hat{s}_i^{*k}(\omega_{C(i)}) = \omega_{C(i)} \quad \text{for all} \quad k \in \{1,...,K\} \quad \text{and} \quad \omega_i \in \Omega_i .$$

The honest strategy profile $\hat{\psi}^* = (\hat{\psi}_i^*)_{i\in N}$ always induces the value of the SCF f and no monetary transfers.

Because of the continuity assumption, we can select a sufficiently large K such that whenever $\max_{a\in A}|\alpha(a)-\alpha'(a)| \leq \frac{1}{K}$, then

(B-1) $$u_i(\alpha,-t_i,\omega_i) > u_i(\alpha',-t_i-\eta_2,\omega_i) \quad \text{for all} \quad t_i \in [0,\sum_{h=1}^{h^*}\eta_1(h)+\eta_3] \quad \text{and} \quad \omega_i \in \Omega_i .$$

In the same manner as in Theorem 3, we can prove that if s_i is strictly iteratively undominated in $\hat{G}^*$, then, $s_i(\omega_i) \in \bigcap_{h\to\infty} \chi_i(h)(\omega_i)$, that is,

$$s_i^1(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i .$$

Suppose $m_j^1 = \omega_j$ for all $j \in N$. We can show that if s_i is strictly iteratively undominated, then,

$$s_i^2(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i .$$

Suppose that there exists a player $j \in N\setminus\{i\}$ who announces $m_j^2 \neq m_j^1$, i.e., $m_j^2 \neq \omega_j$. Then, by announcing $m_i^2 \neq \omega_i$ instead of ω_i, player i is fined the monetary amount given by η_2 or more. From (B-1), the impact of the monetary fine η_2 on his welfare is greater than the impact of the resultant change of allocation. Next, suppose that there exists no player $j \in N\setminus\{i\}$ who announces $m_j^2 \neq \omega_j$. Then, by announcing $m_i^2 \neq \omega_i$ instead of ω_i, player i is fined the monetary amount given by η_3. Because of strict incentive compatibility, the resultant change of allocation never improves his welfare. Hence, player i prefers $m_i^2 = \omega_i$.

Fix an arbitrary $h \geq 3$. Suppose that $m_j^{h'} = \omega_j$ for all $j \in N$ and $h' < h$. In the same manner as above, we can show that each player i prefers $m_i^h = \omega_i$. These observations imply that if s_i is strictly iteratively undominated, then,

$$s_i^k(\omega_i) = \omega_i \text{ for all } \omega_i \in \Omega_i \text{ and all } k \in \{1, ..., K\},$$

that is, $s_i = \hat{s}_i^*$.

Appendix C: Proof of Theorem 6

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1, ..., h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 = (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a mechanism denoted by $\hat{G}^{**} = \hat{G}^{**}(f, \eta_1, \eta_2, \eta_3, K) = (M, g, x)$ as follows. Let

$$M_i = \prod_{k=1}^{K} M_i^k,$$

$$M_i^k = \Omega_i,$$

and

$$M_i^k = \Phi_i^* \text{ for all } k \in \{2, ..., K\}.$$

In contrast with $\hat{G}^*$, each player i announces about not ω_i but $\phi_i \in \Phi_i^*$ for all sub-messages except the first sub-message m_i^1. Let

$$g(m) = \frac{\sum_{k=2}^{K} \tilde{f}(m^k)}{K-1} \text{ for all } m \in M,$$

which $\tilde{f}$ is the function introduced in Definition 9. We will write $\tilde{f}(\phi) = \tilde{f}(\omega)$ if $\omega_i \in \phi_i$ for all $i \in N$, where we denote $\phi = (\phi_i)_{i \in N}$. Let

$$x_i(m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^1, \omega_0) + z_i(m),$$

where

$$x_i^h(m^1,\omega_0)=\eta_1(h) \qquad \text{if } (m_{-i}^1,\omega_0)\notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i}),$$

$$x_i^h(m^1,\omega_0)=0 \qquad \text{if } (m_{-i}^1,\omega_0)\notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i}),$$

$$z_i(m)=\eta_2+\frac{r_i}{K-1}\eta_3 \qquad \text{if there exists } k\in\{2,...,K\} \text{ such that}$$

$$(m_i^1,\omega_0)\notin m_i^k, \text{ and}$$

$$(m_j^1,\omega_0)\in m_j^{k'} \text{ for all } k'<k \text{ and } j\in N\setminus\{i\},$$

and

$$z_i(m)=\frac{r_i}{K-1}\eta_3 \qquad \text{if there exists no such } k\in\{2,...,K\},$$

where $r_i\in\{0,...,K-1\}$ implies the number of $k\in\{2,...,K\}$ satisfying $m_i^k\neq m_i^1$. We select (η_1,η_2,η_3) such that

$$\eta_1(\tilde{h})>\sum_{h=1}^{\tilde{h}-1}\eta_1(h)+\eta_2+\eta_3 \quad \text{for all } \tilde{h}\in\{1,...,h^*\}.$$

In the same manner as the proof of Theorem 5, we can prove that for every $i\in N$, if ψ_i is quasi-strictly iteratively undominated, then

$$s_i^1(\omega_i)\in\Phi_i^*(\omega_i) \quad \text{for all } \omega_i\in\Omega_i,$$

and

$$s_i^k(\omega)=\Phi^*(\omega) \quad \text{for all } k\in\{2,...,K\} \text{ and } \omega\in\Omega.$$

This joint with the measurability implies that if ψ is a quasi-strictly iteratively undominated strategy profile, then

$$g(\psi(\omega))=f(\omega) \quad \text{for all } \omega\in\Omega,$$

and

$$x_i(\psi(\omega),\omega)=0 \quad \text{for all } i\in N \text{ and } \omega\in\Omega.$$