CIRJE-F-996

Effects of Jumps and Small Noise in High-Frequency Financial Econometrics

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November 2015; Revised in October 2016 and April 2017

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September 29, 2016
February 10, 2017 (final version)

Abstract

Several new statistical procedures for high-frequency financial data analysis have been developed to estimate risk quantities and test the presence of jumps in the underlying continuous-time financial processes. Although the role of micro-market noise is important in high-frequency financial data, there are some basic questions on the effects of presence of noise and jump in the underlying stochastic processes. When there can be jumps and (micro-market) noise at the same time, it is not obvious whether the existing statistical methods are reliable for applications in actual data analysis. We investigate the misspecification effects of jumps and noise on some basic statistics and the testing procedures for jumps proposed by Ait-Sahalia and Jacod (2009, 2010) as an illustration. We find that their first test (testing the presence of jumps as a null-hypothesis) is asymptotically robust in the small-noise asymptotic sense against possible misspecifications while their second test (testing no-jumps as a null-hypothesis) is quite sensitive to the presence of noise.

Keywords

High-frequency Financial Data, Continuous-time Processes, Jumps, Micro-market Noise, Small-noise Asymptotics, Asymptotic Robustness of Jump-Test.

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*2017-2-10. This is a revision of Discussion Paper CIRJE-F-996. We thank Katsumi Shimotsu for useful comments to our work. This work was supported by Grant-in-Aid for Scientific Research No.25245033.

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1. Introduction

In the past decades there has been considerable interest in the problem of statistical estimation and testing on some properties of the underlying stochastic processes for high-frequency financial data analysis (see Ait-Sahalia and Jacod (2014) for an extensive list of the related literature; also see Hayashi and Yoshida (2008) for other related problems). Several new statistical procedures for high-frequency financial data analysis have been developed to estimate risk quantities such as volatility and test the presence of jumps in the underlying continuous-time financial processes by using high-frequency financial data. The role of micro-market noise is important in high-frequency financial data; however, there are basic questions on the effects of noise and jumps in the underlying stochastic processes. When there can be jumps and micro-market noise at the same time, it is not certain whether the existing methods are reliable for the applications in actual data analysis. We will investigate the misspecification effects of jumps and noise on the realized volatility and some test statistics for jumps. In particular we examine the test statistics for jumps developed by Jacod (2009) and Ait-sahalia and Jacod (2009, 2010) as an illustration.

The main purpose of this paper is to develop a small-noise asymptotic method to assess the effects of the possible existence of jumps and noises on statistical procedures and to examine their validity in realistic situations. We first investigate the effects of jumps and noise by using the realized volatility to estimate the quadratic variation for risk management. We then investigate some higher-order functionals and the limiting distributions of some test procedures of the jumps proposed by Ait-Sahalia and Jacod (2009) which are included as an illustration within our analysis. As seen in later sections the small-noise asymptotic analysis sheds some new insights on the role of noise and jumps in high-frequency econometric analysis.

Several recent studies have been related to the main topic of this paper such as Ait-Sahalia, Jacod and Li (2012), Li and Mykland (2015) and many others on the pre-averaging method and the rounding-error problem, respectively. We shall mention these studies briefly in the course of our related discussions.

In Section 2 we introduce the statistical setting of the underlying It$\hat{o}$ semimartingale and noise and then investigate the asymptotic property of the realized quadratic variation (QV) when we have noise and jumps in the *small-noise* asymptotic sense. In Section 3 we investigate the asymptotic property of the higher order realized variation in the *small-noise* asymptotic sense, and in Section 4 we apply our method to investigate the asymptotic distributions of test statistics for detecting jumps. In Section 5, we report our simulation results including some results on the modified tests based on the pre-averaging method by Ait-Sahalia, Jacod and Li (2012). We give concluding remarks in Section 6. The proofs and the derivations of our results are included in Appendix A, and some figures are presented in Appendix B.

2. Effects of Noise on QV in a Simple Case

We consider the continuous-time financial market in a fixed terminal interval $[0, T]$ and set $T = 1$ as the end of a market day without loss of generality. The underlying security price is a continuous time semimartingale and we first consider the case when it is a diffusion stochastic process without the drift term. It is possible to investigate the essential aspects of the effects of micro-market noise on the underlying (continuous time) stochastic processes in the simple setting. We observe the financial price process in the high frequency, but we have micro-market noise and assume the additive model as the simple case.

2.1 A Formulation of Small Micro-market Noise

Let the first-filtered probability space be $(\mathbf{\Omega}^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, P^{(0)})$ on which the Itô semimartingale X_t $(0 \leq t \leq 1)$ is well-defined. Let also the second-filtered probability space be $(\mathbf{\Omega}^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\geq 0}, P^{(1)})$ on which the micro-market noise terms $v(t_i^n)$ $(i = 1, \cdots, n)$ are well-defined with $0 \leq t_i^n \leq 1$. Then we construct the filtered probability space and the probability measure as $(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathbf{\Omega} = \mathbf{\Omega}^{(0)} \times \mathbf{\Omega}^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)}$ with $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^{(0)} \otimes \mathcal{F}_s^{(1)}$ $(0 \leq t \leq s \leq 1)$ and $P = P^{(0)} \times P^{(1)}$.

We first consider the additive model for the observed (log-)price at $t_i^n \in [0, 1]$ as

$$Y(t_i^n) = X(t_i^n) + \epsilon_n v(t_i^n) \quad (i = 1, \cdots, n), \tag{2.1}$$

where $X(t)$ is the continuous-time Brownian martingale as the simplest case with

$$X(t) = X(0) + \int_0^t \sigma_s dB_s \ (0 \leq s \leq 1) \ , \tag{2.2}$$

where B_s is the standard Brownian motion and σ_s is the (instantaneous) volatility function, which are predictable (and progressively measurable) with respect to $(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ (see Ikeda and Watanabe (1989) for the general reference). We use the notations $X_i = X(t_i^n)$

and $Y_i = Y(t_i^n)$ in the following analysis.

In this paper we investigate the situation in which the micro-market noise terms $v(t_i^n)$ $(= v_i)$ are a sequence of random variables with $\mathcal{E}[v_i] = 0, \mathcal{E}[v_i^2] = 1$, and ϵ_n (≥ 0) is a (non-negative) sequence of parameters depending on n, which goes to 0 as $n \longrightarrow \infty$. We call this situation the small-noise case.

The standard models of micro-market noise have been the case when $\epsilon_n = \epsilon$ $(> 0$, a constant) while $n \longrightarrow \infty$. The classical high frequency models correspond to the case when $\epsilon_n = 0$. Thus we can fill the gaps between the classical high frequency models and the micro-market noise models because the two situations represent as the extreme cases. In the constant noise case as $n \longrightarrow \infty$, the market noise dominates the hidden intrinsic price movements as the limit eventually, which may not be reasonable in the real financial markets.

In regard to the micro-market noise, the additive model we are considering in (2.1) can be regarded as an approximation to the possible non-linear models, such as

$$(2.3) \qquad Y(t_i^n) = f_n(X(t_i^n), \Delta X(t_i^n), v(t_i^n)) \quad (i = 1, \cdots, n),$$

where $f_n(\cdot)$ is a sequence of measurable functions depending on n and $\Delta X(t_i^n) = X(t_i^n) - X(t_{i-1}^n)$. One example of (2.3) without the second term $\Delta X(t_i^n)$ would be the small rounding error models investigated by Li, Zhang and Li (2015) as well as other studies. As a second example of (2.3), we can consider the situations of $Y(t_i^n) = f(X(t_i^n), \epsilon_{ni} v(t_i^n)) \quad (i = 1, \cdots, n)$ when the conditional variance is given by $[\epsilon_{ni}]^2 = \mathcal{E}[(\Delta X(t_i^n))^2 | \mathcal{F}(t_{i-1}^n)] = O_p(n^{-1})$. Then we have the linearized small-noise model which is represented by $Y(t_i^n) = X(t_i^n) + \epsilon_{ni} v(t_i^n)$ with $\mathcal{E}[v(t_i^n)] = 0$.

When there are many observations of traded prices in a fixed interval, meaning that the financial market is quite active, we can expect that the micro-market noise plays a less important role. Therefore, the additive model (2.1) may be a reasonable description of some financial markets. At this point, we would like to stress that the general case is beyond the scope of this paper and our investigation is the first step toward understanding the effects of micro-market noise in more general situations. We expect that our analyses would be extended to several important cases without much difficulty.

In the following analysis we consider (2.1) when ϵ_n is a sequence of constants depending on n; and $v(t_i^n)$ $(= v_i)$ is a sequence of random variables independent of $X(t)$ with $\mathcal{E}[v_i] = 0$,$\mathcal{E}[v_i^2] = 1$ and the higher moment conditions as $\mathcal{E}[v_i^8] < \infty$. We denote $\mathcal{E}[v_i^4] = 3 + \kappa_4 < \infty$ and

$\kappa_4 = 0$ for the Gaussian noise random variable.

2.2 Effects of Small Noise

We consider the situation in which $\epsilon_n \to 0$ $(n \to \infty)$ and we call this case *the small noise* asymptotic sequence. We set the observed times as $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ with $t_i^n - t_{i-1}^n = 1/n$ $(= \Delta_n)$, and we consider the asymptotic behavior of the basic statistics in the small noise situation when $n \longrightarrow \infty$. This situation gives useful information on the relation of the no-noise case and the noise case in high-frequency econometric analysis.

In this setting we first investigate the estimation problem of the quadratic variation (QV) by using the realized volatility

$$V_n(2) = \sum_{i=1}^{n} (Y_i - Y_{i-1})^2 , \tag{2.4}$$

which can be decomposed as

$$\begin{aligned} V_n(2) &= \sum_{i=1}^{n} (X_i - X_{i-1})^2 + 2\epsilon_n \sum_{i=1}^{n} (X_i - X_{i-1})(v_i - v_{i-1}) \\ &\qquad + \epsilon_n^2 \sum_{i=1}^{n} (v_i - v_{i-1})^2 \\ &= (I) + (II) + (III) , \end{aligned}$$

where we denote v_0 $(= v(t_0^n)) = 0$ as a convention in this paper and write each term as (I), (II) and (III), which are the components of the above decomposition.

When ϵ_n is a constant (which is independent of n), the effect of the third term in the decomposition dominates the other terms in $V_n(2)$ as $n \to \infty$. To make our analysis meaningful, we consider the *small-noise case* when

$$n\epsilon_n^2 = c + o(1) , \tag{2.5}$$

where c is a nonnegative constant and $o(1)$ denotes a smaller order than constants.

Then we write (III) as

$$2c + \frac{c}{\sqrt{n}} \times \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^{n} (v_i - v_{i-1})^2 - 2 \right] . \tag{2.6}$$

For the stochastic integration of the Brownian motion, we approximate that for $i = 1, \cdots, n$

$$\begin{aligned} X_i - X_{i-1} &= \int_{t_{i-1}^n}^{t_i^n} \sigma_s dB_s \\ &\sim \sigma(t_{i-1}^n)[B(t_i^n) - B(t_{i-1}^n)] = \sigma_{i-1}^n (\frac{1}{\sqrt{n}}) Z_i , \end{aligned} \tag{2.7}$$

where $\sigma_{i-1} = \sigma(t_{i-1}^n)$ (we set σ_0 as a constant) and Z_i $(i = 1, \cdots, n)$ is a sequence of i.i.d. $N(0,1)$ random variables. Then we can write (I) as

$$(I) = \frac{1}{n} \sum_{i=1}^{n} \sigma_{i-1}^2 + \frac{1}{\sqrt{n}} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_{i-1}^2 (Z_i^2 - 1) \right] + o_p(\frac{1}{\sqrt{n}})$$

and (II) as

$$(II) = \frac{1}{\sqrt{n}} \left[\frac{2\sqrt{c}}{\sqrt{n}} \sum_{i=1}^{n} \sigma_{i-1} Z_i (v_i - v_{i-1}) \right] + o_p(\frac{1}{\sqrt{n}}) \ ,$$

where $o_p(\frac{1}{\sqrt{n}})$ means a smaller order than $1/\sqrt{n}$ in probability.

(We will also use the standard notation $O_p(n^{-1})$ as the stochastic order of n^{-1} in the following analysis.)

By adding the three terms, we have

$$\begin{aligned} V_n(2) \ &= \left[\frac{1}{n} \sum_{i=1}^{n} \sigma_{i-1}^2 + 2c \right] \\ &+ \frac{1}{\sqrt{n}} \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_{i-1}^2 (Z_i^2 - 1) \right] \\ &+ \frac{1}{\sqrt{n}} \left[\frac{2\sqrt{c}}{\sqrt{n}} \sum_{i=1}^{n} \sigma_{i-1} Z_i (v_i - v_{i-1}) \right] \\ &+ \frac{c}{\sqrt{n}} \times \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^{n} (v_i - v_{i-1})^2 - 2 \right] \\ &+ O_p(\frac{1}{n}) \ . \end{aligned} \tag{2.8}$$

When $n \longrightarrow \infty$, the first term converges in probability to

$$V_c = \int_0^1 \sigma_s^2 ds + 2c \tag{2.9}$$

and the discretization error in the leading term is

$$D_n(0) = \frac{1}{n} \sum_{i=1}^{n} \sigma(t_{i-1}^n)^2 - \int_0^1 \sigma_s^2 ds \ . \tag{2.10}$$

We also need to evaluate the discretization error

$$D_n(1) = (X_i - X_{i-1}) - \sigma(t_{i-1}^n)(\frac{1}{\sqrt{n}}) Z_i \ \ (i = 2, \cdots, n). \tag{2.11}$$

To investigate the present problem in detail, we make a simple assumption about the (stochastic) volatility function, which is a solution of the stochastic differential equation (SDE) given by

$$\sigma_t = \sigma_0 + \int_0^t \mu_s^\sigma ds + \int_0^t \omega_s^\sigma dB_s^\sigma \ \ (0 \le t \le 1) \ , \tag{2.12}$$

where B_s^σ is the second Brownian motion (which can be correlated with $B(s)$), and μ_s^σ and ω_s^σ (which are the drift term and the diffusion term of the volatility), are predictable and progressively measurable with respect to $(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$. We assume that they are bounded and Lipschitz continuous such that the volatility process of (2.12) is smooth and has higher order moments for the resulting simplicity (see Chaper 5 of Ikeda and Watanabe (1989)).

Then we have the following result. For the sake of completeness, we give the proof in Appendix A. It can be also regarded as a simple consequence of Theorem 6.12 of Jacod and Protter (2012).

Lemma 1 : Assume (2.2) and (2.12). For $i = 1, \cdots, n$ and $t_{i-1}^n < t \le t_i^n$,

$$\sigma_t^2 - \sigma_{i-1}^2 = \int_{t_{i-1}^n}^t [2\sigma_s \mu_s^\sigma + (\omega_s^\sigma)^2] ds + \int_{t_{i-1}^n}^t 2\sigma_s \omega_s^\sigma dB_s^\sigma \tag{2.13}$$

and

$$\int_0^1 \sigma_s^2 ds - \frac{1}{n} \sum_{i=1}^n \sigma_{i-1}^2 = O_p(\frac{1}{n}) \ . \tag{2.14}$$

We also need to evaluate the discretization errors. As an example we have the following relation (the proof is given in Appendix A).

Lemma 2 : Assume (2.2) and (2.12). For $i = 1, \cdots, n$ we have

$$(2.15) \qquad (X_i - X_{i-1}) - \sigma(t_{i-1}^n)\frac{1}{\sqrt{n}}Z_i = O_p(\frac{1}{n})$$

and

$$(2.16) \sum_{i=1}^{n}(X_i - X_{i-1})^2 - \left[\int_0^1 \sigma_s^2 ds\right] = 2\sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n}[X_s - X_{i-1}]dX_s = O_p(\frac{1}{\sqrt{n}}) \ ,$$

By investigating the weak convergence of the three terms in the decomposition of $V_n(2)$, we obtain the next result (the proof is given in Appendix A).

Theorem 1 : Assume (2.1), (2.2), and (2.12) with $\mathcal{E}[v_i^3] = 0$ and $\mathcal{E}[v_i^4] = 3 + \kappa_4 < \infty$. Furthermore, assume that v_0 is a random variable v_i $(i = 1, \cdots, n)$ as the initial condition. Define a sequence of random variables as

$$(2.17) \qquad U_{0n} = \frac{1}{n}\sum_{i=1}^{n}\sigma_{i-1}^2 + 2c \,,$$

$$(2.18) \qquad U_{1n} = \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sigma_{i-1}^2(Z_i^2-1)\right] \,,$$

$$(2.19) \qquad U_{2n} = \frac{1}{\sqrt{n}}\left[\frac{2\sqrt{c}}{\sqrt{n}}\sum_{i=1}^{n}\sigma_{i-1}Z_i(v_i-v_{i-1})\right] \,,$$

$$(2.20) \qquad U_{3n} = \frac{c}{\sqrt{n}} \times \sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}(v_i-v_{i-1})^2 - 2\right] \,,$$

where $\sigma_{i-1} = \sigma(t_{i-1}^n)$ and $Z_i = \sqrt{n}[B(t_i^n) - B(t_{i-1}^n)]$ $(i = 1, \cdots, n)$. We set the limiting random variable of U_{0n} as $U_0 = \int_0^1 \sigma_s^2 ds + 2c \ (= V_c)$ as the leading term of $V_n(2)$. Then as $n \longrightarrow \infty$ with (2.4), we have the stable convergence as

$$(2.21) \qquad \sqrt{n}\,[V_n(2) - U_0] \xrightarrow{S-c} U = U_1 + U_2 + U_3 \,,$$

where U_i $(i = 1, 2, 3)$ are $\mathcal{F}-$ conditional and thus mutually independent Gaussian random variables with zero means and the (conditional) asymptotic variances are given by $\mathcal{E}[U_1^2|\mathcal{F}] = 2\int_0^1 \sigma_s^4 ds$, $\mathcal{E}[U_2^2|\mathcal{F}] = 8c\int_0^1 \sigma_s^2 ds$, and $\mathcal{E}[U_3^2|\mathcal{F}] = [12 + 4\kappa_4]c^2$.

In the above expression, we freely use the stable convergence arguments and $\mathcal{F}-$conditional Gaussianity, which were developed and explained in some detail by Jacod (2008) and Jacod and Protter (2012).

When σ_s^2 is a deterministic function, U and U_i $(i=1,2)$ in Theorem 1 are Gaussian random variables. In the general case, however, we need to extend the original probability space such that the quantities $\int_0^1 \sigma_s^2 ds$ and $\int_0^1 \sigma_s^4 ds$ are mathematically meaningful as the limits of the sequence of random variables. Thus we need to extend the probability space $(\boldsymbol{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ to $(\tilde{\boldsymbol{\Omega}}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$ so that these quantities are well-defined random variables. As an illustration, we take U_{1n}. In this case $U_{1n} \xrightarrow{S-c} U_1$ means the CLT (central limit theorem) such that

$$\frac{U_{1n}}{\sqrt{2\int_0^1 \sigma_s^4 ds}} \xrightarrow{\mathcal{L}} N(0,1) \ , \tag{2.22}$$

which is the convergence in law. We do not discuss the details of this procedure, but as a general reference on stable convergence, we refer to Hausler and Luschgy (2015), Jacod and Protter (2012).

2.3 Effects of Jumps and Noise

When there can be jumps in the underlying stochastic process, it is natural to assume that the underlying stochastic process is an Itô semimartingale (continuous-time) process

$$\begin{aligned} X(t) &= X(0) + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_{|x|<1} \delta(s,x)(\mu-\nu)(ds,dx) \\ &+ \int_0^t \int_{|x|\geq 1} \delta(s,x)\mu(ds,dx) \ , \end{aligned} \tag{2.23}$$

where μ_s (drift parameter) and σ_s (diffusion parameter) are predictable and progressively measurable, $\delta(s,x)$ is a predictable process, $\mu(\cdot)$ is a jump measure and $\nu(\cdot)$ is the compensator of $1_A * \mu$ for $1 * \nu(\omega)_t = \nu(\omega : [0,t) \times A)$ (we have used the notation of Jacod and Protter (2012, Section 2), and Jacod (2009)). We also assume that the jump sizes $|\Delta X_s|$ are bounded in the following anlysis.

We generally do not need any condition on jump sizes at the end. However, in order to show the CLT and the stable convergence to be held, we need to develop the similar mathematical arguments as Section 5 (or Section 5.3.2 in particular) of Jacod and Protter (2012), which has given the related derivations in detail already. We do not pursue the resulting arguments for the proofs of our results in the general cases although they hold.

The filtered probability space $(\mathbf{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ can be constructed as in Section 2.1 except for the fact that the first probability space is constructed by the general It$\hat{o}$ semimartingale process X_t $(0 \leq t \leq 1)$, which is well-defined in the extended probability space.

For the general case of (2.23) we still have the same representation of (2.21) in Theorem 1, but U_i $(i = 0, 1)$ should be replaced by

$$U_0 = \int_0^1 \sigma_s^2 ds + \sum_{0 \le s \le 1} (\Delta X_s)^2 + 2c \,, \tag{2.24}$$

and

$$\mathcal{E}[U_1^2|\mathcal{F}] = 2\int_0^1 \sigma_s^4 ds + 4 \sum_{0 \le s \le 1} \sigma_s^2 (\Delta X_s)^2 \,, \tag{2.25}$$

where $\Delta X_s = X_s - X_{s-}$ is the jump at s, and U_1 is an $\mathcal{F}-$conditionally Gaussian process with zero mean. Since we can apply the martingale CLT to U_{2n}, U_2 should be replaced by an $\mathcal{F}-$conditionally Gaussian process with zero mean and

$$\mathcal{E}[U_2^2|\mathcal{F}] = 8c \left[\int_0^1 \sigma_s^2 ds + \sum_{0 \le s \le 1} (\Delta X_s)^2 \right] \tag{2.26}$$

because the limiting random variable of the sum of $(X_i - X_{i-1})(v_i - v_{i-1})$ is conditionally Gaussian random variable.

The outline of our derivations in (2.24)-(2.26) for the jump case is given in Appendix A. It is a protptype of our evaluation methods in Sections 3 and 4. We notice that without the presence of jumps the above three terms can be reduced to the expressions in Theorem 1. When $c = 0$, the result reduces to the standard situation without the noise term.

3. Effects of Noise on $V_n(4)$

By using the small-noise asymptotics, it is possible to investigate the asymptotic properties of $V_n(p)$ and $V_n(4)$ in particular, where

$$V_n(p) = \sum_{i=1}^{n}(Y_i - Y_{i-1})^p \ . \tag{3.1}$$

For $V_n(p)$ $(p \geq 2)$, we need to evaluate the discretization errors and then utilize $\text{It}\hat{o}'s$ Lemma in the general semimartingales with jumps (see Protter (2003)). As an illustration we give a simple consequence of $\text{It}\hat{o}'s$ Lemma as the next proposition (the proof is given in Appendix A).

Lemma 3 : Assume (2.23) for X_t $(0 \leq t \leq 1)$. For any positive integer p (≥ 2), we have

$$\begin{aligned}
&\sum_{i=1}^{n}(X_i - X_{i-1})^p - \left[\sum_{0\leq s\leq 1}(\Delta X_s)^p\right] \\
= &\sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n} p(X_{s-} - X_{i-1})^{p-1}dX_s \\
&+\sum_{i=1}^{n}\int_{t_{i-1}^n}^{t_i^n} \frac{p(p-1)}{2}(X_{s-} - X_{i-1})^{p-2}d[X_s, X_s]^c \\
&+\sum_{i=1}^{n}\sum_{t_{i-1}^n\leq s<t_i^n}\sum_{j=2}^{p-1} {}_pC_j(X_{s-} - X_{i-1})^{p-j}(\Delta X_s)^j \ ,
\end{aligned} \tag{3.2}$$

where ${}_pC_j = p!/[(p-j)!j!]$, $[X_s, X_s]^c$ is the continuous part of the quadratic variation of X_s and we use the convention that the last term in (3.2) is zero when $p = 2$. When $p \geq 3$, (3.2) is $O_p(n^{-1/2})$.

For applications of testing jumps in the underlying stochastic processes, we need to investigate the asymptotic behavior of $V_n(4) = \sum_{i=1}^{n}(Y_i - Y_{i-1})^4$. We first decompose $V_n(4)$ into five terms

$$\begin{aligned}V_n(4) &= \sum_{i=1}^{n}(X_i - X_{i-1})^4 \\ &\quad +4\epsilon_n \sum_{i=1}^{n}(X_i - X_{i-1})^3(v_i - v_{i-1}) + 6\epsilon_n^2 \sum_{i=1}^{n}(X_i - X_{i-1})^2(v_i - v_{i-1})^2 \\ &\quad +4\epsilon_n^3 \sum_{i=1}^{n}(X_i - X_{i-1})(v_i - v_{i-1})^3 + \epsilon_n^4 \sum_{i=1}^{n}(v_i - v_{i-1})^4 \\ &= (I) + (II) + (III) + (IV) + (V) \ ,\end{aligned}$$

where each of the terms are defined as the corresponding terms in the decomposition.

When we have the condition $n\epsilon_n^2 = c + o(1)$ in (2.5) on the noise term, we represent the leading term of $V_n(4)$ as

$$U_0 = \sum_{0 \le s \le 1} (\Delta X_s)^4 \ . \tag{3.3}$$

Then we need to evaluate the asymptotic property (i.e., the asymptotic distribution in particular) of the several quantities in the order of $o_p(1)$. Let the sequence of random variables be

$$U_{1n} = \sum_{i=1}^{n}(X_i - X_{i-1})^4 - \left[\sum_{0 \le s \le 1} (\Delta X_s)^4\right] \ , \tag{3.4}$$

$$U_{2n} = 4\epsilon_n \sum_{i=1}^{n}(X_i - X_{i-1})^3(v_i - v_{i-1}) \ , \tag{3.5}$$

(3.6) $$U_{3n} = 6\epsilon_n^2 \sum_{i=1}^{n} (X_i - X_{i-1})^2 (v_i - v_{i-1})^2 \ ,$$

(3.7) $$U_{4n} = 4\epsilon_n^3 \sum_{i=1}^{n} (X_i - X_{i-1})(v_i - v_{i-1})^3 \ ,$$

and

(3.8) $$U_{5n} = \epsilon_n^4 n \left[\frac{1}{n} \sum_{i=1}^{n} (v_i - v_{i-1})^4 \right] \ .$$

When we have the condition $n\epsilon_n^2 = c + o(1)$ and after lengthy evaluations of each term outlined in Appendix A, we find that $U_{1n} = O_p(n^{-1/2})$, $U_{2n} = O_p(n^{-1/2})$, $U_{3n} = O_p(n^{-1})$, $U_{4n} = O_p(n^{-3/2})$, and $U_{5n} = O_p(1/n)$. Thus we have the next result and the proof is given in Appendix A.

Theorem 2 : Assume (2.1), (2.12) and (2.23) with $\mathcal{E}[v_i^8] < \infty$. Let the leading term of $V_n(4)$ be $U_0 = \sum_{0 \le s \le 1} (\Delta X_s)^4$. Then as $n \longrightarrow \infty$ with (2.5) we have the stable convergence

(3.9) $$\sqrt{n} \left[V_n(4) - U_0 \right] \stackrel{S-c}{\longrightarrow} U = U_1 + U_2 \ ,$$

where U_1 is an $\mathcal{F}-$ conditionally Gaussian random variable with zero means and the asymptotic variance

(3.10) $$\mathcal{E}[U_1^2 | \mathcal{F}] = 16 \sum_{0 \le s < 1} \sigma_s^2 (\Delta X_s)^6 \ .$$

In addition,

$$\mathcal{E}[U_2^2|\mathcal{F}] = 32c \sum_{0\leq s<1} (\Delta X_s)^6 \tag{3.11}$$

and U_2 is an $\mathcal{F}-$conditionally (and U_i $(i = 1,2)$ are mutually independent) Gaussian random variable with zero mean if we assume Gaussianity for v_i $(i = 1, \cdots, n)$.

There can be other possibilities in the approximations, which are different from the condition in (2.5) as the small-noise asymptotics. For instance, if we have

$$n\epsilon_n^4 = c^* + o(1) \ , \tag{3.12}$$

where c is a non-negative constant. The first term should be

$$U_0 = \sum_{0\leq s\leq 1} (\Delta X_s)^4 + c^*\sigma^4(\Delta v) \tag{3.13}$$

and

$$\sigma^4(\Delta v) = 8 + 2\mathcal{E}[(v^2-1)^2] \ . \tag{3.14}$$

Similarly, we can develop the corresponding decomposition for each sequence of ϵ_n. However, the small disturbance asymptotics under the condition (2.5) often give useful approximations, as we discuss in Sections 4 and 5.

4. An Application to the Testing Procedures for Jumps

One interesting problem in high-frequency econometrics is whether the underlying stochastic process for asset prices has some jump component or whether the continuous diffusion process is appropriate. Several testing procedures have been proposed to detect whether there is a jump compnent. In this section we investigate two testing procedures developed by Ait-Sahalia and Jacod (2009) as an important application of our analysis. We consider the asymptotic properties of their test statistics. For this purpose, we use their notations

$$
\begin{aligned}
W_n(p,k) &= \sqrt{n}\left[(\hat{B}(p,k\Delta_n)_1 - B(p)_1) - (\hat{B}(p,\Delta_n)_1 - B(p)_1)\right] \\
&= \sqrt{n}\left[\sum_{i=1}^{[n/k]} |\Delta_i^n Y(k)|^p - \sum_{i=1}^{n} |\Delta_i^n Y(1)|^p\right] , \qquad (4.1)
\end{aligned}
$$

where

$$
\hat{B}(p,k\Delta_n)_t = \sum_{i=1}^{[t/k\Delta_n]} |\Delta_i^n Y(k)|^p \ , \ B(p)_t = \sum_{0<s\leq t} |\Delta X_s|^p \ , \qquad (4.2)
$$

$$
\Delta_i^n Y(k) = (X_{ik\Delta_n} - X_{(i-1)k\Delta_n}) + \epsilon_n (v(ik\Delta_n) - v((i-1)k\Delta_n)) \qquad (4.3)
$$

and $v(ik\Delta_n)$ $(= v_{ik})$ are i.i.d. noise terms and $\Delta_n = 1/n$.

Then we have the next result when $p = 4$ and $k = 2$ (the proof is given in Appendix A).

Theorem 3 : Assume (2.1), (2.12), (2.23), and $\mathcal{E}[v_i^8] < \infty$.

(i) We assume an Itô semimartingale (2.23) for the underlying process X, and v_i $(i = 1, \cdots, n)$ is the sequence of Gaussian random variables. Then for $W_n(p,k)$ with $p = 4$ and $k = 2$, as $n \longrightarrow \infty$ with (2.5), we have the following stable convergence

$$W_n(p,k) \xrightarrow{S-c} U = U_1 + U_2 \ , \tag{4.4}$$

where U_i $(i = 1, 2)$ are $\mathcal{F}-$ conditional (and mutually independent) Gaussian random variables with zero means and

$$\mathcal{E}[U_1^2|\mathcal{F}] = p^2(k-1) \sum_{0 \le s < 1} \sigma_s^2 (\Delta X_s)^{2(p-1)} \ , \tag{4.5}$$

and

$$\mathcal{E}[U_2^2|\mathcal{F}] = p^2(k-1) \times 2c \sum_{0 \le s < 1} (\Delta X_s)^{2(p-1)} \ . \tag{4.6}$$

(ii) When X_t is a Brownian Itô semimartingale (2.23) without jump components, let the fourth-order realized variation be

$$V_n(4,k) = \sum_{i=1}^{[n/k]} |\Delta_i^n Y(k)|^4 \ . \tag{4.7}$$

Then as $n \longrightarrow \infty$ with (2.5), $nV_n(4,k)$ converges in probability to

$$U_0^*(4,k) = \frac{m_4}{k} \int_0^1 \sigma_s^4 ds + \frac{12c}{k} \int_0^1 \sigma_s^2 ds + \frac{c^2}{k} \mathcal{E}[(\Delta v)^4] \ , \tag{4.8}$$

($\mathcal{E}[(\Delta v)^4] = \mathcal{E}[(v_i - v_{i-1})^4]$) and we have the stable convergence

$$\begin{aligned} &(4.9)\ \sqrt{n}\left[nV_n(4,k) - U_0^*(4,k)\right] \\ &\xrightarrow{S-c}\ U^*(4,p) = U_1^*(4,k) + U_2^*(4,k) + U_3^*(4,k) + U_4^*(4,k) + U_5^*(4,k) \ , \end{aligned}$$

where $U_i^*(4,k)$ $(i = 1,2,3,4,5)$ are (mutually independent) $\mathcal{F}-$conditionally Gaussian random variables with zero means and the (conditional) asymptotic variances are given by

$$\mathcal{E}[(U_1^*(4,k))^2|\mathcal{F}] = k^3(m_8 - m_4^2)\int_0^1 \sigma_s^8 ds \ , \tag{4.10}$$

$$\mathcal{E}[(U_2^*(4,k))^2|\mathcal{F}] = 32ck^2\int_0^1 \sigma_s^6 ds \ , \tag{4.11}$$

(4.12)

$$\mathcal{E}[(U_3^*(4,k))^2|\mathcal{F}] = 36c^2k[(m_4 - m_2^2)\mathbf{Var}(\Delta v)^2 + (8m_4 - 4)]\int_0^1 \sigma_s^4 ds \ ,$$

$$\mathcal{E}[(U_4^*(4,k))^2|\mathcal{F}] = 16c^3\mathbf{Var}[(\Delta v)^3]\int_0^1 \sigma_s^2 ds \tag{4.13}$$

and

$$\mathcal{E}[(U_5^*(4,k))^2|\mathcal{F}] = \frac{c^4}{k}\mathbf{Var}[(\Delta v)^4] \ , \tag{4.14}$$

and m_p is the $p-th$ moment of v_1. When v_i are Gaussian random variables, $m_2 = 1, m_4 = 3, m_6 = 15$ and $m_8 = 105$. Then $\mathbf{Var}[(\Delta v)]$ $(= \mathcal{E}[(v_i - v_{i-1})^2]) = 2$ $(i = 2,\cdots,n)$, $\mathcal{E}[(\Delta v)^4] = 12$, $\mathcal{E}[(\Delta v)^6] = 120$ and $\mathcal{E}[(\Delta v)^8] = 1680$.

We notice that the above results correspond to Theorem 2 of Ait-Sahalia and Jacod (2009) when $c = 0$. It is straightforward to extend this approach to more general cases with k and p.

Let

(4.15) $$\hat{S}(p,k,\Delta_n) = \frac{\hat{B}(p,k\Delta_n)_1}{\hat{B}(p,\Delta_n)_1}$$

be a test statistic for the composite hypothesis that there is a jump in the underlying stochastic process for the first test of Ait-Sahalia and Jacod (2009). We call their first test as the null hypothesis of the presence of jump while we call their second test as the null-hypothesis of the non-existence of jump terms. The first test can be constructed based on the first part of Theorem 3 because its asymptotic distribution and the critical region can be constructed by $C_{n,t} = \{\hat{S}(p,k,\Delta_n) > c_n\}$, where c_n is a constant. The second test can be constructed from the second part of Theorem 3 because its asymptotic distribution and the critical region can be constructed by $C_{n,t} = \{\hat{S}(p,k,\Delta_n) < c_n^c\}$, where c_n^c is a constant (we can construct $\hat{V}_{n,1}^j$ and $\hat{V}_{n,1}^c$ in their notations).

In addition, let

(4.16)
$$\hat{D}(6,\Delta)_t = \frac{1}{k_n\Delta_n}\sum_{i=1}^{[1/\Delta_n]} |\Delta_i^n X|^p \sum_{j\in I_{n,1}(j)} (\Delta_j^n X)^2 I\{|\Delta_j^n X| \le \alpha\Delta_n^{\omega}\}\ ,$$

where $\alpha > 0$, $\omega \in (0,1/2)$ and $I_{n,1}(i) = \{j \in N : j \neq i, 1 \le j \le [1/\Delta_n], |i-j| \le k_n\}$. We need to take $k_n\Delta \to 0$ and $k_n \to \infty$ as $n \to \infty$.

Since

$$\hat{D}(6,\Delta)_1 \xrightarrow{p} \sum_{0\le s<1} (\Delta X_s)^6 \times \left[2\sigma_s^2 + 4c\right] \tag{4.17}$$

when $n\epsilon_n^2 = c + o(1)$, it is possible to investigate the asymptotic properties of the jump test statistics.

For the first test of Ait-Sahalia and Jacod (2009), we have the next result when $p = 4$ and $k = 2$.

Corollary 4 : We assume an Itô semimartingale (2.23) for the underlying process X and micro-market noise satisfying the condition (2.5). For $\hat{S}(p,k)$ with $p = 4, k = 2$, as $n \longrightarrow \infty$ with (2.5), we have the stable convergence

$$\Delta_n^{-1/2}\left[\hat{S}(p,k) - 1\right] \xrightarrow{S-c} S = \frac{U_1 + U_2}{U_0}\ , \tag{4.18}$$

where U_i $(i = 0,1,2)$ are defined in Theorem 3.

Some Remarks :

(i) It is straightforward to derive the asymptotic distribution of the test statistic when we have the Brownian Itô semimartingale as the underlying stochastic process for the second test of Ait-Sahalia and Jacod (2009). The result becomes rather complicated and for $\hat{S}(p,k)$ with $p = 4$ and $k = 2$ and $U_0^*(4,k)$, as $n \longrightarrow \infty$ with (2.5) we have the stable convergence

$$
\begin{aligned}
& \Delta_n^{-1/2}\left[\hat{S}(p,k) - \frac{U_0^*(4,k)}{U_0^*(4,1)}\right] \\
\xrightarrow{S-c}\ & S = \frac{1}{U_0^*(4,1)}\left[(U_1^*(4,k) + U_2^*(4,k) + U_3^*(4,k) + U_4^*(4,k) + U_5^*(4,k))\right. \\
& \left. - \frac{U_0^*(4,k)}{U_0^*(4,1)}(U_1^*(4,1) + U_2^*(4,1) + U_3^*(4,1) + U_4^*(4,1) + U_5^*(4,1))\right],
\end{aligned}
$$

where $U_i^*(4,k)$ $(i = 0,1,2,3,4,5)$ are defined in Theorem 3.

(ii) We can obtain the local power of the testing procedure by using the limiting distributions of the test statistic, which are equivalent to those in Ait-Sahalia and Jacod (2009) when $c = 0$,

(iii) It is possible to use the limiting distributions of test statistic when $c > 0$ by using $\hat{c}$, which can be constructed by estimating the noise variance. It would be interesting to improve the testing procedure by Ait-Sahalia and Jacod (2009) when $c > 0$. We could subsequently apply the resulting procedure even when the noise terms are not necessarily small.

5. Simulation

In our simulations the data generating process can be written as

$$
dX_t = bdt + \sigma_t dB_t + dJ_t\ , \tag{5.1}
$$

where b is a (constant) drift term, B_t is the standard Brownian motion,

σ_t is the volatility parameter at t, J_t is a compound Poisson process. We have adopted the simulation procedure for jumps developed by Cont and Tankov (2004, Section 6). As a stochastic volatility model with the leverage effect, we use

$$d\sigma_t^2 = \kappa(\beta - \sigma_t^2)dt + \omega\sigma_t d\tilde{B}_t^\sigma \ , \tag{5.2}$$

where $\kappa = 5, \beta = 0.2, \omega = 0.5$, and $\rho = \mathcal{E}[dB_t d\tilde{B}_t^\sigma] = -0.5$.
For the micro-market noise we have adopted the Gaussian noise although it is straightforward to use other distributions for the resulting simplicity.

In Appendix B, we have given several figures on finite sample distributions of normalized statistics to illustrate the effects of CLT in Theorem 1, Theorem 2, and Corollary 4. We standardized the empirical density and distribution because they give more information than the original parameters with their own scales. Figure 1 gives the standardized empirical density (n=1,000) with the bias-variance correction implied by Theorem 1 and Theorem 2, respectively (because we have additional terms in U_0 when there are noise term). Theorem 1 gives a reasonable approximation to the limiting Gaussian distribution while we need finer data (such as $n = 5,000$) in Figure 2 mainly because our jump-related terms in Theorem 2. Figure 2 gives the standardized empirical density (n=5,000) with the bias-variance correction implied by

Theorem 2 and Corollary 4. First, we find that even a small amount of noise contamination has significant effects on the distributions of the limiting random variables when $c \neq 0$. However, we have good approximations for the finite sample distributions of the statistics if we correct for the effects of noise by using the small-noise asymptotics.

We also include several figures to examine the finite sample and asymptotic distributions of the (standardized) test statistics proposed by Ait-Sahalia and Jacod (2009) under the null-hypothesis and the alternative hypotheses in Figure 3 and Figure 4. For this purpose we consider four cases.

(i) X_0, which stands for the diffusion + Poisson jump model versus X_1, which stands for the diffusion model.

(ii) Y_0, which stands for the diffusion + Poisson jump + large noise model versus Y_1, which stands for the diffusion + large noise model.

(iii) V_0, which stands for the diffusion + Poisson jump + type-1 small noise model versus V_1, which stands for the diffusion model + type-1 small noise model.

(iv) W_0, which stands for the diffusion + Poisson jump + type-2 small noise model versus X_1, which stands for the diffusion + type-2 small noise model.

Here the type-1 small noise corresponds to $\epsilon_n v_i \sim (c\Delta_n)^{1/2} N(0,1)$, while the type-2 small noise corresponds to $\epsilon_n v_i \sim (c\Delta_n^{1/2})^{1/2} N(0,1)$, where $N(0,1)$ is the standard normal random variable. The *large noise* corresponds to the case when $\epsilon_n = c$ (a constant). For the Poisson jumps we have set λ (intensity) to 10, the jump size $N(0,5)$, and the simulation size is 1,000. We note that it is sometimes difficult to see the figures drawn according to the same scale because their distributions are also drawn using the same scale when there are noise terms.

We simulated many cases, but we only report those that provide essential findings in our simulations. First for the diffusion case we have good approximations of the finite distributions with their limiting distributions, even when $n = 1,000$. We find that we need finer data to have good approximations when there are jumps and noise at the same time. This is reasonable because when there are jump terms, we need finer data to distinguish between the effects of jumps and noise.

Second, we investigated the rejection and acceptance probabilities by using the empirical distributions of two statistics proposed by Ait-Sahalia and Jacod (2009) for detecting possible jumps. Then in Figure 3 and Figure 4 we compared the limiting distributions of the statistics in the small-noise asymptotics we have developed. The red curves correspond to the standard normal distribution, which gives often reasonable approximations of the finite sample distributions in the small-noise asymptotics. In two cases, W_0 and Y_0, the normalizations by the small-noise asymptotics are not appropriate. The scales of the empirical distributions and the approximate Gaussian distributions are quite different in these cases.

When we have large noise in comparison with the underlying stochastic process, the so-called large noise cases, we cannot obtain good approximations because the effects of noise dominate the finite distributions of the statistics. However, when we have small noise, i.e., the (2.5) condition, the formulas we derived in Theorem 1 and Theorem 2 give good approximations. Among the two test procedures, the limiting distributions of the second test are more complicated in Theorem 3 and it seems that we need finer data to have good approximations of the finite sample distributions. However, for the first test statistics, the approximations derived in Theorem 3 are often good approximations. In sum, we find that the first test statistic is quite robust to the presence of small noise, while the second test statistic is quite sensitive to the presence of noise (the details of our simulation results on the power properties of the test statistics for jumps will be reported on another occasion). The distributions of the test statistics can be approximated quite well by the small-noise asymptotics.

In their jump tests under micro-market noise, Ait-Sahalia, Jacod and Li (2012) developed several modified testing procedure based on the pre-averaging method. As an application of our simulations we investigated the finite sample distribution of the test statistics with their notations defined by

$$S(Y, g, h, 4) = \frac{\bar{V}(Y, g, 4)^n}{2\bar{V}(Y, h, 4)^n}\ , \tag{5.3}$$

where we take the simplest weight functions $g(x) = (0.5 - |x - 0.5|)^+$ and $h(x) = g(2x)$. In Figure 5, we plot the empirical density of the normalized statistics $N^X = n^{1/4}(S(X_0, g, h, 4) - 1)/\sqrt{\Sigma}$, N^Y, N^V and N^W, implied by Theorem 3 in Ait-Sahalia, Jacod and Li (2012) and use their notations such that

$$\Sigma = \frac{D(g, g) - 2D(g, h) + 4D(h, h)}{(0.0125 \times \sum_{0 \le s \le 1} |\Delta X_s|^4)^2},$$

$$\begin{aligned} D(g, h) = {} & 16\,[\psi_-(g, h) + \psi_+(g, h)] \left(\sum_{0 \le s \le 1} \sigma_s^2 |\Delta X_s|^6 \right) \\ & + 16c^2\,[\psi_-(g, h) + \psi_+(g, h)] \left(\sum_{0 \le s \le 1} |\Delta X_s|^6 \right), \end{aligned}$$

and $\psi_{\pm}(g, h)$ are variables defined in their paper. In this simulation, the simulation size is $1,000$, and we set $\Delta_n = 1/10,000$, the averaging window is $k_n = 120$; the parameter of the noise term is $c = 10$. As we expected, the normal approximation is not often appropriate mainly because the construction of their test statistics are rather complicated when based on our settings. Kurisu (2016) gave the more details of the simulation method we have used and the resulting simulation results in a systematic way.

6. Concluding Remarks

In this paper, we investigated the effects of jumps and noise in financial high-frequency data problems. For this purpose, we developed a small noise asymptotic analysis when the size of the micro-market noise depends on the sample size. By using this approach we identified the effects of jumps and noise on our volatility estimation and on some jump test procedures. We found that the first test by Ait-Sahalia and Jacod (2009) (testing the presence of jumps as a null-hypothesis) is asymptotically robust in the small-noise asymptotic sense against possible misspecifications, while the second test (testing no jumps as a null hypothesis) is quite sensitive to the presence of noise.

We conducted a number of simulations and found that the asymptotic distributions obtained in the small noise asymptotics gave good approximations for the distributions of the estimators and test statistics in the standard high-frequency asymptotics.

In addition to these results, it is possible to investigate the sample size and noise size needed to justify the estimation and testing procedures proposed in earier studies. In this paper we have developed small-noise asymptotics, that allow to analyze the asymptotic theory without jumps and noise as well as the finite sample properties of the statistics with jumps and noise. We show that the approximations based on the small-noise asymptotics often give quite accurate distributions.

Furthermore, it is interesting to note that even when $n = 1,000$ and $c = 100$ the asymptotic distributions we derived often gave useful approximations for the limiting distributions of the sequence of random variables. We illustrate this finding in Figure 6, Appendix B where we show the empirical densities of the normalized limiting random variables with $c = 0$ (the uncorrected random variable) and $c = 100$ (the corrected random variable). (In Figure 6 the small-noise approximated distributions are on the right-hand side while the original simulated distributions are on the left-hand side.)

In addition, if (2.1) is a reasonable approximation of the real situation, it is possible to estimate the parameter c by using high-frequency data with noise. Let the estimator of the integrated volatility be $\hat{\sigma}_n^2$. One possibility is to use the SIML estimator proposed by Kunitomo and Sato (2013), and another possibility is to use the spectral method by Bibinger and Reiss (2014) (there are also other ways to estimate

c). We can then construct

$$\hat{c} = \frac{1}{2}[\sum_{i=1}^{n}(\Delta_i^n Y_i)^2 - \hat{\sigma}_n^2] \ , \tag{6.1}$$

where we denote $\Delta_i^n Y = Y_i - Y_{i-1}$ and $Y_i = Y(t_i^n)$.

By using an estimate of c, we can use the asymptotic distributions derived in the previous sections and Kurisu (2016) investigated the related problems in details. In Section 2.1, we have indicated that it is possible to extend our investigation to more general non-linear situations. We are currently iunvestigating this problem as well as the asymptotic behavior of other statistics and statistical procedures when there are jumps and noise in underlying continuous processes.

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APPENDIX A : Mathematical Derivations

In this Appendix A we give some details of the proofs of mathgematical results omitted in previous sections. Since the derivations are often slight modifications of Jacod and Protter (2012), and Ait-Sahalia and Jacod (2009), we refer to their book and some related papers for some mathematical details on the underlying arguments. We need some additional arguments because of the presence of noise in the following arguments.

Proof of Lemma 1 : From (2.12) we represent that for $t_{i-1}^n < t \le t_i^n$ and $t_i^n - t_{i-1}^n = 1/n$ $(i = 1, \cdots, n)$,

$$\sigma_t = \sigma(t_{i-1}^n) + \int_{t_{i-1}^n}^t \mu_s^\sigma ds + \int_{t_{i-1}^n}^t \omega_s^\sigma dB_s^\sigma \ . \tag{A.2}$$

Then we have

$$\int_{t_{i-1}^n}^t \sigma_u du = \frac{1}{n}\sigma(t_{i-1}^n) + O_p(\frac{1}{n\sqrt{n}}) \ . \tag{A.3}$$

It is because

$$\int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^u \mu_s^\sigma ds du = O_p(\frac{1}{n^2}) \tag{A.4}$$

due to the assumption that μ_s^σ is bounded and

$$\int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^u \omega_s^\sigma dB_s^\sigma du = \int_{t_{i-1}^n}^{t_i^n} (\int_s^{t_i^n} du) \omega_s^\sigma dB_s^\sigma = O_p(\frac{1}{n\sqrt{n}}) \ . \tag{A.5}$$

Then by using Itô's Lemma

$$\text{(A.6)} \quad \int_{t_{i-1}^n}^{t_i^n} \sigma_u^2 du - (t_i^n - t_{i-1}^n)\sigma^2(t_{i-1}^n)$$
$$= \int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{u} [2\sigma_u \mu_s^\sigma + (\omega_s^\sigma)^2 ds] du + \int_{t_{i-1}^n}^{t_i^n} [\int_{t_{i-1}^n}^{u} 2\sigma_u \omega_s^\sigma dB_s^\sigma] du \ .$$

Again by interchanging the integrations, using (A.3)-(A.5) and evaluating two terms separately, we find the result. **Q.E.D**.

Proof of Lemma 2 : We use the notation that $\omega^\sigma(t_i^n) = \omega_s^\sigma$ at $s = t_i^n$.

(i) From (2.12) we find that for $t_{i-1}^n < t \le t_i^n$ and $t_i^n - t_{i-1}^n = 1/n$ $(i = 1, \cdots, n)$,

$$\text{(A.7)} \quad \sigma_t = \sigma(t_{i-1}^n) + \omega^\sigma(t_{i-1}^n)[B^\sigma(t) - B^\sigma(t_{i-1}^n)] + o_p(\frac{1}{\sqrt{n}})$$

because

$$\text{(A.8)} \quad \mathcal{E}[\int_{t_{i-1}^n}^{t} (\omega^\sigma(s) - \omega^\sigma(t_{i-1}^n))^2 ds] = o(\frac{1}{n}) \ .$$

Then we write

$$X_i - X_{i-1} = \int_{t_{i-1}^n}^{t_i^n} \sigma_s dB_s$$
$$= \int_{t_{i-1}^n}^{t_i^n} [\sigma(t_{i-1}^n) + \omega^\sigma(t_{i-1}^n)(B^\sigma(s) - B^\sigma(t_{i-1}^n))] dB_s + o_p(\frac{1}{n}) \ ,$$

where we denote $X_i = X(t_i^n)$ $(i = 1, \cdots, n)$. Here we decompose for $s \ge t_{i-1}^n$

$$B^\sigma(s) - B^\sigma(t_{i-1}^n) = \rho[B_s - B(t_{i-1}^n)] + \sqrt{1-\rho^2}[B_s^* - B^*(t_{i-1}^n)]$$

to make B_s^* and B_s being independent (ρ is the correlation coefficient of B_s and B_s^*), and we use the fact that

$$\text{(A.9)} \quad \int_{t_{i-1}^n}^{t_i^n} [B_s - B(t_{i-1}^n)] dB_s = \frac{1}{2} \left[(B_s - B(t_{i-1}^n))^2 - (s - t_{i-1}^n)^2 \right]_{t_{i-1}^n}^{t_i^n} ,$$

which is $O_p(n^{-1})$. Hence we have (2.15).

(ii) We write $(X_i - X_{i-1})^2 = X_i^2 - X_{i-1}^2 - 2X_{i-1}(X_i - X_{i-1})$ and apply Ito's Lemma to X_i^2. Then we have

$$\begin{aligned} (X_i - X_{i-1})^2 &= \int_{t_{i-1}^n}^{t_i^n} 2X_s dX_s + \int_{t_{i-1}^n}^{t_i^n} d[X_s, X_s] - 2X_{i-1}(X_i - X_{i-1}) \\ &= \int_{t_{i-1}^n}^{t_i^n} 2(X_s - X_{i-1}) dX_s + [X_i, X_i] - [X_{i-1}, X_{i-1}] , \end{aligned}$$

where $[X_i, X_i]$ is the quadratic variation of X_i $(i = 1, \cdots, n)$ with the notation $[X_0, X_0] = 0$. Since

$$\text{(A.10)} \qquad [X_i, X_i] - [X_{i-1}, X_{i-1}] = \int_{t_{i-1}^n}^{t_i^n} \sigma_s^2 ds ,$$

we have

$$\text{(A.11)} \quad \sum_{i=1}^n (X_i - X_{i-1})^2 - \int_{t_{i-1}^n}^{t_i^n} \sigma_s^2 ds = \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} 2(X_s - X_{i-1}) dX_s .$$

Because the right-hand side is a martingale,

$$[\int_{t_{i-1}^n}^{t_i^n} (X_s - X_{i-1}) dX_s, \int_{t_{i-1}^n}^{t_i^n} (X_s - X_{i-1}) dX_s] = \int_{t_{i-1}^n}^{t_i^n} (X_s - X_{i-1})^2 \sigma_s^2 ds ,$$

and $(X_s - X_{i-1})^2 = O_p(n^{-1})$, we have the result of (2.16). **Q.E.D.**

Proof of Theorem 1 : We use the fact that the limiting random variables of U_{in} $(i = 1,2,3)$ follow the ($\mathcal{F}-$conditionally) Gaussian distributions, which are mutually independent, by using the central limit theorem. Because we can apply the arguments in Jacod and Protter (2012), we have the stable convergence of the underlying random variables and then U_i $(i = 1,2,3)$ are $\mathcal{F}-$conditionally independent. Since $\mathbf{Var}[Z_i^2 - 1] = 2$, we find that

$$\text{(A.12)} \qquad \mathbf{Var}[U_{1n}] \sim 2\int_0^1 \sigma_s^4 ds \ ,$$

$$\text{(A.13)} \qquad \mathbf{Var}[U_{2n}] \sim 8c\int_0^1 \sigma_s^2 ds \ ,$$

Also the asymptotic variance of the third term U_{3n} is approximately equal to $(1/n)\mathcal{E}[4\sum_{i=2}^n (v_i^2-1)^2 + 4\sum_{i=2}^n v_i^2 v_{i-1}^2]$. It is because we use a simple relation

$$\begin{aligned} [\sum_{i=1}^n (v_i - v_{i-1})^2 - 2n]^2 &\sim [2\sum_{i=1}^n (v_i^2-1) - 2\sum_{i=1}^n v_i v_{i-1}]^2 \\ &= 4[\sum_{i=1}^n (v_i^2-1)^2 + \sum_{i=1}^n v_i^2 v_{i-1}^2 - 2\sum_{i=1}^n (v_i^2-1)\sum_{i=1}^n v_i v_{i-1}] \end{aligned}$$

and $4[\mathcal{E}(v_i^4) - (\mathcal{E}(v_i^2))^2)] + 4(\mathcal{E}(v_i^2))^2) = 4[2+\kappa_4] + 4$. Thus we find that

$$\text{(A.14)} \qquad \mathbf{Var}[U_{3n}] \sim [12 + 4\kappa_4]c^2 \ .$$

Since U_i $(i = 1,2,3)$ are $\mathcal{F}-$conditionally uncorrelated, they are $\mathcal{F}-$conditionally independent. By applying the stable convergence theorem and the central limit theorem (CLT), we have the desired result. **Q.E.D**.

Derivations of (2.24)-(2.26) : We use the relation that

$$\sum_{i=1}^{n}(X_i - X_{i-1})^2 = \sum_{i=1}^{n}\left[\int_{t_{i-1}^n}^{t_i^n}\mu_s ds + \int_{t_{i-1}^n}^{t_i^n}\sigma_s dB_s + \sum_{t_{i-1}^n < s \le t_i^n}\Delta X_s\right]^2 .$$

We note that the last term does make sense because we have assumed the boundedness of jumps and $t_i^n - t_{i-1}^n = 1/n$ $(i = 1, \cdots, n)$. Since the effect of the drift term is $o_p(1)$, which is stochastically negligible, by using the standard result on semi-martingales we have

$$\text{(A.15)} \qquad \sum_{i=1}^{n}(X_i - X_{i-1})^2 \xrightarrow{p} \int_0^1 \sigma_s^2 ds + \sum_{0<s\le 1}(\Delta X_s)^2 .$$

Then we have (2.24) by using LLN (the law of large numbers) to $(1/n)\sum_{i=1}^{n}(v_i - v_{i-1})^2$.

Let

$$\text{(A.16)} \qquad Z_n = \sqrt{n}\left[\sum_{i=1}^{n}(X_i - X_{i-1})^2 - (\int_0^1 \sigma_s^2 ds + \sum_{0<s\le 1}(\Delta X_s)^2)\right] ,$$

which is approximately equal to

$$\text{(A.17)} \qquad \sqrt{n}\left(\sum_{i=1}^{n}\left[\sigma(t_{i-1}^n)(B(t_i^n) - B(t_{i-1}^n)) + \sum_{t_{i-1}^n < s \le t_i^n}\Delta X_s\right]^2 \right.$$
$$\left. -[\int_0^1 \sigma_s^2 ds + \sum_{0<s\le 1}(\Delta X_s)^2]\right) .$$

Then the additional term of the constant order in probability from (A.16) is given by

$$
\text{(A.18)} \qquad 2\sum_{i=1}^{n}\left[\sum_{t_{i-1}^{n}<s\le t_{i}^{n}}\Delta X_s\right]\sqrt{n}\sigma(t_{i-1}^{n})(B(t_{i}^{n})-B(t_{i-1}^{n}))\ .
$$

The additional cross product of jump term and the noise term from the second term of $V_n(2)$, i.e. $2\epsilon_n\sum_{i=2}^{n}(X_i-X_{i-1})(v_i-v_{i-1})$ is given by

$$
\text{(A.19)} \qquad \sqrt{n}\times\frac{2\sqrt{c}}{\sqrt{n}}\left[\sum_{i=1}^{n}\sum_{t_{i-1}^{n}<s\le t_{i}^{n}}\Delta X_s\right](v_i-v_{i-1})\ .
$$

If we assume the Gaussianity on v_i $(i=1,\cdots,n)$, we have the $\mathcal{F}-$conditional Gaussianity.

The remaining arguments of the stable convergence are followed by the corresponding ones of Jacod and Protter (2012). **Q.E.D.**

Proof of Lemma 3 : Let $Y(t)=[X(t)-X(t_{i-1}^{n})]^p$ for $p\ge 2$ and we apply $\text{It}\hat{o}'$s lemma for the general Ito semi-martingale to $Y(t)$. (Theorem 32 of Protter (2003), for instance.) For $p\ge 3$, (3.2) is $o_p(1)$.

Then we have

$$
\begin{aligned}
& (X_i - X_{i-1})^p - \left[\sum_{t_{i-1}^n \le s \le t_i^n} (\Delta X_s)^p \right] \\
= & \int_{t_{i-1}^n}^{t_i^n} p(X_{s-} - X_{i-1})^{p-1} dX_s + \int_{t_{i-1}^n}^{t_i^n} \frac{p(p-1)}{2} (X_{s-} - X_{i-1})^{p-2} d[X_s, X_s]^c \\
& + \sum_{t_{i-1}^n \le s < t_i^n} \left[(X_s - X_{i-1})^p - (X_{s-} - X_{i-1})^p - p(X_{s-} - X_{i-1})^{p-1} \Delta X_s \right] \\
& - \sum_{t_{i-1}^n \le s < t_i^n} [(X_s - X_{i-1}) - (X_{s-} - X_{i-1})]^p \quad .
\end{aligned}
$$

In the above derivation we use the relation $(X_s - X_{i-1})^p - (X_{s-} - X_{i-1})^p = [\Delta + (X_{s-} - X_{i-1})]^p - (X_{s-} - X_{i-1})^p$, which equal to $\sum_{j=2}^{p-1} {}_pC_j (X_{s-} - X_{i-1})^{p-j} (\Delta X_s)^j$. Then by taking the summation with respect to $i = 1, \cdots, n$, we have the result. **Q.E.D.**

Proof of Theorem 2 : By using Lemma 3, we have

$$
\text{(A.20)} \qquad \sum_{i=1}^{n} (X_i - X_{i-1})^4 \xrightarrow{p} \sum_{0 \le s < 1} (\Delta X_s)^4 \ .
$$

By using the similar arguments as (A.15) and (A.16), we can express

$$
\begin{aligned}
\text{(A.21)} \qquad & \sqrt{n} \left[\sum_{i=1}^{n} (X_i - X_{i-1})^4 - \sum_{0 \le s < 1} (\Delta X_s)^4 \right] \\
& = \frac{4}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t_{i-1}^n \le s < t_i^n} \sigma(\tau_{i-1}^n) Z(\tau_i^n) (\Delta X_s)^3 + o_p(\frac{1}{\sqrt{n}}) \ ,
\end{aligned}
$$

where τ_i^n $(i = 1, 2, \cdots)$ correspond to the stopping times for jumps of the Itô semimartingale (and if there were no jumps in $t_{i-1}^n \le s < t_i^n$, they do not appear). The remaining term in the decomposition we need to evaluate is

$$\text{(A.22)}\quad U_{2n} = \frac{4\sqrt{c}}{\sqrt{n}}\sum_{i=1}^{n}(X_i - X_{i-1})^3(v_i - v_{i-1})$$
$$= \frac{4\sqrt{c}}{\sqrt{n}}\sum_{0\le s\le 1}(\Delta X_s)^3[v(\tau_i^n) - v(\tau_{i-1}^n)] + O_p(\frac{1}{\sqrt{n}})\ ,$$

where we have used the notation $\Delta_n = 1/n$, $\Delta_i^n X = X_i - X_{i-1}$ $(i = 2, \cdots, n)$. We set

$$U_{2n}^{'} = \sqrt{\epsilon_n}\left[\sum_{i=1}^{n}(\Delta_i^n X)^3 v_i - \sum_{i=1}^{n}(\Delta_i^n X)^3 v_{i-1}\right]$$
$$= \sqrt{\epsilon_n}\left[\sum_{i=1}^{n-1}[(\Delta_i^n X)^3 - (\Delta_{i+1}^n X)^3]v_i + (\Delta_n^n X)^3 v_n - (\Delta_1^n X)^3 v_0\right]\ .$$

Then by using the Cauchy-Swartz inequality, we evaluate the conditional expectation $\mathcal{E}[(U_{2n}^{'})^2|X]$, which is given by

$$\mathcal{E}[(U_{2n}^{'})^2|X] = \epsilon_n\left[\sum_{i=1}^{n-1}[(\Delta_i^n X)^3 - (\Delta_{i+1}^n X)^3]^2 + (\Delta_n^n X)^6(\Delta_1^n X)^6\right]$$
$$= 2\epsilon_n\left[\sum_{i=1}^{n}(\Delta_i^n X)^6 - \sum_{i=1}^{n-1}(\Delta_i^n X)^3(\Delta_{i+1}^n X)^3\right]$$
$$\le 4\epsilon_n\sum_{i=1}^{n}(\Delta_i^n X)^6\ .$$

We notice that $\sum_{i=1}^{n}(\Delta_i^n X)^6 = O_p(1)$, $\sum_{i=1}^{n-1}(\Delta_i^n X)^3(\Delta_{i+1}^n X)^3 = o_p(1)$, and then we have $\mathcal{E}[(U_{2n}^{'})^2] = O(n^{-1})$. In these relations the most important step is to show that $\sum_{i=1}^{n-1}(\Delta_i^n X)^3(\Delta_{i+1}^n X)^3 = o_p(1)$, which can be proven by applying Theorem 8.2.1 of Jacod and Protter (2012). We set $F(x_1, x_2) = x_1^3 x_2^3, f_1(x) = F(x_1, 0), f_2(x) = F(0, x_2)$ and $f_1 *$

$\mu_1 + f_2 * \mu_1 = 0$ (here μ_1 is the associated jump measure) in their notation.

Then we have

$$\text{(A.23)} \qquad \mathcal{E}[(\Delta_n^{-1/2} U'_{2n})^2 | X] \xrightarrow{p} 2c\mathcal{E}[v_1^2] \sum_{0<s\leq 1} (\Delta X_s)^6 \, .$$

Finally, since U_{1n} and U_{2n} are $\mathcal{F}-$conditionally uncorrelated, they are $\mathcal{F}-$conditionally independent. By using the stable convergence arguments in Jacod and Protter (2012), we have the desired result. **Q.E.D.**

Proof of Theorem 3 : We follow Ait-Sahalia and Jacod (2009) for basic method of their proof, but we need some additional arguments because we have the effects of noise as well as jumps in the underlying processes.

[Part 1] : We apply Lemma 3 with a general k($k \geq 1$) and $p \geq 4$ to

$[\Delta_i^n Y(k)]^p$, and then we can express that

$$\begin{aligned}
& \sum_{i=1}^{[n/k]}[Y(ik\Delta_n) - Y((i-1)k\Delta_n)]^p \\
= & \sum_{i=1}^{[n/k]}[X(ik\Delta_n) - X((i-1)k\Delta_n)]^p \\
& +p\epsilon_n \sum_{i=1}^{[n/2]}[X(ik\Delta_n) - X((i-1)k\Delta_n)]^{p-1}[v(ik\Delta_n) - v((i-1)k\Delta_n)] \\
& +o_p(\frac{1}{\sqrt{n}}) \\
= & \sum_{0\le s<1}(\Delta X_s)^p + p\sqrt{\frac{k}{n}}[\sum_{i=1}^{[n/k]} \sum_{t_{(i-1)k}^n \le s < t_{ik}^n} \sigma(t_{(i-1)k}^n) Z_i^n(k)(\Delta X_s)^{p-1}] \\
& +\frac{p\sqrt{c}}{\sqrt{n}}[\sum_{i=1}^{[n/k]} \sum_{t_{(i-1)k}^n \le s < t_{ik}^n} (\Delta X_s)^{p-1}[v(ik\Delta_n) - v((i-1)k\Delta_n)]] + o_p(\frac{1}{\sqrt{n}}) ,
\end{aligned}$$

where we denote that $Z_i^n(k) = \sqrt{n/k}[B(t_{ik}^n) - B(t_{(i-1)k}^n)]$ and then $Z_i^n = Z_i^n(1)$ in (2.7).

For the ease of exposition we shall use the notations that $\Delta_i^n X(k) = X(ik\Delta_n) - X((i-1)k\Delta_n)$, $\Delta_i^n B(k) = B(ik\Delta_n) - B((i-1)k\Delta_n)$, and $\Delta_i^n v(k) = v(ik\Delta_n) - v((i-1)k\Delta_n) = v_{ik} - v_{(i-1)k}$ in the following analysis.

By taking the $p-$the realized variation with k ($k \geq 2$) and the $p-$the realized variation with k ($k = 1$) for $p = 4$, we decompose

(A.24)
$$\sqrt{n}\left[\sum_{i=1}^{[n/k]}[Y(ik\Delta_n) - Y((i-1)k\Delta_n)]^4 - \sum_{i=1}^{n}[Y(i\Delta_n) - Y((i-1)\Delta_n)]^4\right]$$

into four terms except other negligible terms asymptotically, which are given by

$$\begin{aligned} W_{1n} &= 4\sum_{i=1}^{[n/k]}\sigma(t^n_{(i-1)k})[\sqrt{k}Z_i^n(k)\sum_{\tau^n_{(i-1)k}\le s<\tau^n_{ik}}(\Delta X_s)^3]\ , \\ W_{2n} &= -4\sum_{i=1}^{n}\sigma(t^n_{(i-1)})[Z_i^n(1)\sum_{\tau^n_{(i-1)}\le s<\tau^n_{i}}(\Delta X_s)^3]\ , \\ W_{3n} &= 4\sqrt{c}\sum_{i=1}^{[n/k]}\sum_{t^n_{(i-1)k}\le s<t^n_{ik}}(X(ik\Delta_n) - X((i-1)k\Delta_n)]^3\Delta_i^n v(k) \end{aligned}$$

and

$$\text{(A.25)}\quad W_{4n} = -4\sqrt{c}\sum_{i=1}^{n}\sum_{t^n_{(i-1)}\le s<t^n_{i}}[X(i\Delta_n) - X((i-1)\Delta_n)]^3\Delta_i^n v(k)\ ,$$

respectively, where $Z_i^n(k)$ ($i = 1, \cdots, n$) are Gaussian random variables with $N(0,1)$.

In order to obtain the asymptotic distribution of (A.24), we need to evaluate the asymptotic variances of $W_{1n}+W_{2n}$ and $W_{3n}+W_{4n}$, respectively. Since the asymptotic covariance of $W_{1n} + W_{2n}$ and $W_{3n} + W_{4n}$ are asymptotically negligible, we need to evaluate the variance of $W_{1n} + W_{2n}$ (and that of $W_{3n} + W_{4n}$). For this purpose we decompose $W_{1n} + W_{2n} = 4\sum_{i=1}^{[n/k]} U_{i,k}^n$, where we further decompose $U_{i,k}^{(n)} =$

$U^n_{i,k}(1) - U^n_{i,k}(2)$ as

$$U^n_{i,k}(1) = \sigma(t^n_{(i-1)k}) \sum_{t^n_{(i-1)k} \leq s < t^n_{ik}} (\Delta X_s)^{p-1} \sum_{l=0}^{k-1} [B(t^n_{(i-l)k}) - B(t_{(i-1)k-1})],$$

$$U^n_{i,k}(2) = \sum_{l=0}^{k-1} \sigma(t^n_{(i-1))k+l}) \sum_{t^n_{(i-1)k+l} \leq s < t^n_{(i-1)k+l+1}} (\Delta X_s)^{p-1} [B(t^n_{(i-1)k+l+1}) - B(t^n_{(i-l)k+l})].$$

Then it is straightforward to evaluate that for $p = 4$

$$\mathbf{Var}[U^n_{i,k}(1)|\mathcal{F}_{(i-1)k}] \sim \frac{k}{n}[\sigma(t^n_{(i-1)k})]^2 [\sum_{t^n_{(i-1)k} \leq s < t^n_{ik}} (\Delta X_s)^6],$$

$$\mathbf{Var}[U^n_{i,k}(2)|\mathcal{F}_{(i-1)k}] \sim \frac{1}{n} \sum_{l=0}^{k-1} [\sigma(t^n_{(i-1)k+l})]^2 [\sum_{t^n_{(i-1)k+l} \leq s < t^n_{(i-1)k+l+1}} (\Delta X_s)^6],$$

$$\mathbf{Cov}[U^n_{i,k}(1) U^n_{i,k}(2)|\mathcal{F}_{(i-1)k}] \sim \frac{1}{n} \sum_{l=0}^{k-1} \sigma(t^n_{(i-1)k}) \sigma(t^n_{(i-1)k+l}) \sum_{t^n_{(i-1)k+l} \leq s < t^n_{(i-1)k+l+1}} (\Delta X_s)^6,$$

where we have used the notation that $\mathcal{F}_{(i-1)k}$ is the $\sigma-$field given at $t = (i-1)k\Delta_n$ in the discretization of the underlying continuous time processes.

Because we have assumed the volatility process as (2.12), we find that

$$\begin{aligned} \mathbf{Var}[U^n_{i,k}|\mathcal{F}_{(i-1)k}] &= \mathbf{Var}[U^n_{i,k}(1)|\mathcal{F}_{(i-1)k}] + \mathbf{Var}[U^n_{i,k}(2)|\mathcal{F}_{(i-1)k}] \\ &\quad -2\mathbf{Cov}[U^n_{i,k}(1), U^n_{i,k}(2)|\mathcal{F}_{(i-1)k}] \end{aligned}$$

is approximately

$$\text{(A.26)} \quad \mathbf{Var}[U^{n,*}_{i,k}|\mathcal{F}_{(i-1)k}] = [\frac{k-1}{n}][\sigma(t^n_{(i-1)k})]^2 [\sum_{t^n_{(i-1)k} \leq s < t^n_{ik}} (\Delta X_s)^6] \; .$$

By taking the summation with respect to $i = 1, \cdots, [n/k]$, we can obtain (4.5). By applying the similar arguments to $W_{3n} + W_{4n}$, we also obtain (4.6). Since the remaining arguments are similar to the proof of Theorem 2 as we have done in the derivations of Theorem 2, we have the first part of Theorem 3.

[Part 2] : We re-write $V_n(4,k)$, which can be decomposed into five terms as

$$\begin{aligned}
V_n(4,k) &= \sum_{i=1}^{[n/k]} (Y(ik\Delta_n) - Y((i-1)k\Delta_n))^4 \\
&= \sum_{i=1}^{[n/k]} (X(ik\Delta_n) - X((i-1)k\Delta_n))^4 \\
&\quad +4\epsilon_n \sum_{i=1}^{[n/k]} (X(ik\Delta_n) - X((i-1)k\Delta_n))^3 (v_{ik} - v_{(i-1)k}) \\
&\quad +6\epsilon_n^2 \sum_{i=1}^{[n/k]} (X(ik\Delta_n) - X((i-1)k\Delta_n))^2 (v_{ik} - v_{(i-1)k})^2 \\
&\quad +4\epsilon_n^3 \sum_{i=1}^{[n/k]} (X(ik\Delta_n) - X((i-1)k\Delta_n)) (v_{ik} - v_{(i-1)k})^3 \\
&\quad +\epsilon_n^4 \sum_{i=1}^{[n/k]} (v_{ik} - v_{(i-1)k})^4 \\
&= (I) + (II) + (III) + (IV) + (V) \ \ \text{(say)}.
\end{aligned}$$

Then we evaluate the stochastic order of each terms and we find that $(I) = O_p(n^{-1})$, $(II) = (F_{2n}, say) = O_p(n^{-3/2})$, $(III) = O_p(n^{-1})$, $(IV) = (F_{4n}, say) = O_p(n^{-3/2})$ and $(V) = O_p(n^{-1})$. Because

$n \times (I) \sim k(k/n) \sum_{i=1}^{[n/k]} [\Delta_i^n X(k)/\sqrt{k\Delta_n}]^4$,

$n \times (III) \sim n \times 6(c/n) \sum_{i=1}^{[n/k]} [\Delta_i^n X(k)/\sqrt{k\Delta_n}]^2 [\Delta_i v(k)]^2$

and

$n \times (V) \sim n(c/n)^2 \sum_{i=1}^{[n/k]} [\Delta_i v(k)]^2$,

we can obtain the limiting random variable as (4.8).

Then we set

$$\text{(A.27)} \qquad F_{1n} = n \times (I) - km_4 \int_0^1 \sigma_s^4 ds \ ,$$

$$\text{(A.28)} \qquad F_{3n} = n \times (III) - 6c \times 2 \int_0^1 \sigma_s^2 ds \ ,$$

and

$$\text{(A.29)} \qquad F_{5n} = n \times (V) - \frac{c^2}{k} \times \mathcal{E}[(\Delta_i v(k))^4] \ ,$$

which are $O_p(n^{-1/2})$.

For the ease of expositions we shall use the notations that U_{in} correspond to F_{in} for $i = 2, 3, 4$ except constant terms. Then we need to evaluate the limiting random variables of U_{in} $(i = 2, \cdots, 4)$ and the limiting random variables of $\sqrt{n} \times F_{1n}$, $n\sqrt{n} \times (II)$, $\sqrt{n} \times F_{3n}$, $n\sqrt{n} \times (IV)$ and $\sqrt{n} \times F_{5n}$, separately.

The explicit evaluations of these terms are straightforward, but they are a little bit tedious especially for F_{3n}. Since careful calculations at several places are needed, we give some details of those points.

First, it is straightford to find that the limiting random variable of

$\mathbf{Var}[\sqrt{n}F_{1n}]$ as $\mathcal{E}[(U_1^*(4,k))^4|\mathcal{F}] \sim k^3(m_8 - m_4^2)\int_0^1 \sigma_s^8 ds$ in (4.10).

Second, let

$$
\text{(A.30)} \qquad \begin{aligned} U_{2n} &= k^{3/2}\Delta_n^2 \sum_{i=1}^{[n/k]} [\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}}]^3 [\Delta_i v(k)] \\ &= k^{3/2}\Delta_n^2 \sum_{i=1}^{[n/k]} [(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})^3 - (\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})^3][\Delta_i v(k)] \\ &\quad + k^{3/2}\Delta_n^2 [(\frac{\Delta_n^n X(k)}{\sqrt{k\Delta_n}})^3 v_n - (\frac{\Delta_1^n X(k)}{\sqrt{k\Delta_n}})^3 v_0], \end{aligned}
$$

Then we have the conditional expectation given X as

$$
\begin{aligned} & \Delta_n^{-3}\mathcal{E}[(U_{2n})^2|X] \\ = \; & k^3\Delta_n \sum_{i=1}^{[n/k]} [(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})^3 - (\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})^3]^2 \\ & + k^3\Delta_n [(\frac{\Delta_{[n/k]} X(k)}{\sqrt{k\Delta_n}})^6 + (\frac{\Delta_1^n X(k)}{\sqrt{k\Delta_n}})^6] \\ = \; & 2k^2 \left[k\Delta_n \sum_{i=1}^{[n/k]} (\frac{\Delta_{n/k} X(k)}{\sqrt{k\Delta_n}})^6 - k\Delta_n \sum_{i=1}^{[n/k]} (\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})^3 (\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})^3 \right], \end{aligned}
$$

which converges in probability to $2k^2\int_0^1 \sigma_s^6 ds$ because the second term is stochastically negligible. By multiplying 4^3c to $\mathbf{Var}[U_{2n}]$, we have (4.11) as the second term.

Third, we set

$$
\text{(A.31)} \qquad U_{3n} = k\Delta_n^2 \sum_{i=1}^{[n/k]} [\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}}]^2 [\Delta_i v(k)]^2 \,,
$$

which is the order $O_p(\Delta_n)$ ($= O_p(n^{-1})$). (The explicit calculation of the limiting random variables of U_{3n} involves some complications because of the evaluation of the associated discretization errors and auto-correlation structures.) For this evaluation we define a sequence of random variables $W_n = nU_{3n} - 2\mathcal{E}[v_1^2]m_2 \int_0^1 \sigma_s^2 d;$, which can be re-written as

$$
\begin{aligned}
W_n &= k\Delta_n \left[\sum_{i=1}^{[n/k]-1} \Big((\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 + (\Delta_{i+1}^n X(k)/\sqrt{k\Delta_n})^2 \Big) v_{ik}^2 \right] \\
&- 2\mathcal{E}[v_1^2]m_2 \int_0^1 \sigma_s^2 ds \\
&- 2k\Delta_n \sum_{i=1}^{[n/k]-1} (\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 v_{ik} v_{(i-1)k} + o_p(\frac{1}{\sqrt{n}}) \ ,
\end{aligned}
$$

where we have used the relation

$$
\int_0^1 \sigma_s^2 ds = \frac{k}{n} \sum_{i=1}^{[n/k]} \sigma_{t_{(i-1)k}}^2 + O_p(\frac{1}{n}) \ .
$$

We further decompose W_n as

$$
\begin{aligned}
W_n &= \frac{k}{n} \left[\sum_{i=1}^{[n/k]-1} (\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 (v_{ik}^2 - \mathcal{E}[v_{ik}^2]) \right] \\
&+ \frac{k}{n} \left[\sum_{i=1}^{[n/k]-1} (\Delta_{i+1}^n X(k)/\sqrt{k\Delta_n})^2 (v_{ik}^2 - \mathcal{E}[v_{ik}^2]) \right] \\
&+ \mathcal{E}[v_1^2] \frac{k}{n} \left[\sum_{i=1}^{[n/k]-1} \Big((\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 + (\Delta_{i+1}^n X(k)/\sqrt{k\Delta_n})^2 - 2\sigma_{t_{k(i-1)}}^2 \Big) \right] \\
&\quad -2\frac{k}{n} \left[\sum_{i=1}^{[n/k]-1} (\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 v_{ik} v_{(i-1)k} \right] + o_p(\frac{1}{\sqrt{n}}) \\
&= W_I + W_{II} + W_{III} + W_{IV} + o_p(\frac{1}{\sqrt{n}}) \quad (say).
\end{aligned}
$$

Then we shall evaluate the asymptotic variances and covariances of each terms in the following analysis.

(i) **Evaluation of** W_I

We set

$$\xi_i^{n,\mathbf{I}} = \frac{k}{n}\left(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}}\right)^2 [v_{ik}^2 - \mathcal{E}(v_{ik}^2)] ,$$

and

$$\widetilde{\xi}_i^{n,\mathbf{I}} = \frac{k}{n}\sigma_{(i-1)k}^2 (Z_i^n(k))^2 [v_{ik}^2 - \mathcal{E}(v_{ik}^2)] .$$

Since $(\Delta_i^n X(k)/\sqrt{k\Delta_n})^2 = \sigma_{(i-1)k}^2 (Z_i^n(k))^2 + O_p(\frac{1}{\sqrt{n}})$ by the result of Lemma 2, we find that $n\sum_{i=1}^{[n/k]-1}((\xi_i^{n,\mathbf{I}})^2 - (\widetilde{\xi}_i^{n,\mathbf{I}})^2)$ is asymptotically negligible. Therefore we can replace $\xi_i^{n,\mathbf{I}}$ with $\widetilde{\xi}_i^{n,\mathbf{I}}$ in W_n. Moreover,

$$\begin{aligned} k \times \frac{1}{(\frac{k}{n})}\sum_{i=1}^{[n/k]-1} \mathcal{E}[(\widetilde{\xi}_i^{n,\mathbf{I}})^2 | \mathcal{F}_{k(i-1)}] &= k\mathbf{Var}[v_1^2] m_4 \left(k\Delta_n \sum_{i=1}^{[n/k]-1} \sigma_{(i-1)k}^4\right) \\ &\xrightarrow{p} k\mathbf{Var}[v_1^2] m_4 \int_0^1 \sigma_s^4 ds \;(\equiv V_{\mathbf{I}} = V_{\mathbf{II}}) . \end{aligned}$$

Hence we have the stable convergence as

$$\text{(A.32)} \qquad \sqrt{n}\times W_I \xrightarrow{\mathcal{L}-s} \mathbf{N}\left(0, k\mathbf{Var}[v_1^2] m_4 \int_0^1 \sigma_s^4 ds\right) .$$

(ii) **Evaluation of** W_{II}

We set

$$\xi_i^{n,\mathbf{II}} = \frac{k}{n}\left(\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}}\right)^2 [v_{ik}^2 - \mathcal{E}(v_{ik}^2)],$$

and

$$\widetilde{\xi}_i^{n,\mathbf{II}} = \frac{k}{n}\sigma_{(i-1)k}^2 (Z_{i+1}^n(k))^2 (v_{ik}^2 - \mathcal{E}[v_{ik}^2]) .$$

By using Lemma 1, we can evaluate **II** in the same way as W_I.

(iii) **Evaluation of** W_{III}

We use the fact that $\sqrt{n}W_{III}$ is approximately equivalent to

$$2\sqrt{k} \times \frac{1}{\sqrt{\frac{n}{k}}} \sum_{i=1}^{[\frac{n}{k}]} \left[\left(\frac{\Delta_i^n X(k)}{k\Delta_n} \right)^2 - \sigma_{t_{k(i-1)}}^2 \right] .$$

Then by applying CLT, we have that $\sqrt{n} \times W_{III} \xrightarrow{\mathcal{L}-s} \mathbf{N}(0, V_{\mathbf{III}})$, where $V_{\mathbf{III}} = 4k(m_4 - m_2^2) \int_0^1 \sigma_s^4 ds$.

(iv) **Evaluation of** W_{IV}

Let

$$\xi_i^{n,\mathbf{IV}} = \frac{k}{n} \left(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}} \right)^2 v_{ik}v_{(i-1)k},\ \widetilde{\xi}_i^{n,\mathbf{IV}} = \frac{k}{n}\sigma_{(i-1)k}^2 (Z_i^n(k))^2 v_{ik}v_{(i-1)k} .$$

By using the similar argument of the evaluation of W_I, we obtain,

$$\begin{aligned} 4n \sum_{i=1}^{[n/k]-1} \mathcal{E}[(\widetilde{\xi}_i^{n,\mathbf{IV}})^2 | \mathcal{F}_{k(i-1)}] &= 4k\mathcal{E}[v_1^2]^2 m_4 \left(k\Delta_n \sum_{i=1}^{[n/k]-1} \sigma_{(i-1)k}^4 \right) \\ &\xrightarrow{p} 4km_4 \int_0^1 \sigma_s^4 ds \ (\equiv V_{\mathbf{IV}}). \end{aligned}$$

Also we find that the correlations of W_I and W_{III}, W_I and W_{IV}, W_{II} and W_{III}, W_{II} and W_{IV}, W_{III} and W_{IV} is asymptotically negligible.

(v) **Evaluation of the correlation of** W_I **and** W_{II}

From the similar arguments of the evaluation of W_I, we know that

$n\sum_{i=1}^{[n/k]-1}(\xi_i^{n,\mathbf{I}}\xi_i^{n,\mathbf{II}} - \widetilde{\xi}_i^{n,\mathbf{I}}\widetilde{\xi}_i^{n,\mathbf{II}})$ is asymptotically negligible. Moreover,

$$n\sum_{i=1}^{[n/k]-1}\Big(\mathcal{E}[\widetilde{\xi}_i^{n,\mathbf{I}}\widetilde{\xi}_i^{n,\mathbf{II}}|\mathcal{F}_{k(i-1)}] - \mathcal{E}[\widetilde{\xi}_i^{n,\mathbf{I}}|\mathcal{F}_{k(i-1)}]\mathcal{E}[\widetilde{\xi}_i^{n,\mathbf{II}}|\mathcal{F}_{k(i-1)}]\Big)$$
$$= k\mathbf{Var}[v_1^2]m_2^2\left(k\Delta_n\sum_{i=1}^{[n/k]-1}\sigma_{(i-1)k}^4\right)$$
$$\xrightarrow{p} k\mathbf{Var}[v_1^2]m_2^2\int_0^1\sigma_s^4 ds (\equiv V_{\mathbf{I},\mathbf{II}})$$

Then by summarizing (i)-(v), we conclude that $\sqrt{n}W_n \xrightarrow{\mathcal{L}-s} \mathbf{N}\left(0, V_W\right)$, where

$$\begin{aligned} V_W &= V_{\mathbf{I}} + V_{\mathbf{I}} + V_{\mathbf{III}} + V_{\mathbf{IV}} + 2V_{\mathbf{I},\mathbf{II}} \\ &= k\left[\mathbf{Var}(v_1^2)m_4 + \mathbf{Var}(v_1^2)m_4 + 4\mathbf{Var}(v_1^2) + 4m_4 + 2\mathbf{Var}(v_1)\right]\int_0^1 \sigma_s^4 ds\ . \end{aligned}$$

(We have the relation that $\mathbf{Var}[(\Delta v)^2] = 2\mathcal{E}(v_1^4) + 2$ and it is 8 for the Gaussian case.) Then by multiplying 6^2c^2 to V_W, we finally have (4.12) as the third term.

Fourth, we set

$$\begin{aligned} U_{4n} &= k^{1/2}\Delta_n^2\sum_{i=1}^{[n/k]}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})(\Delta v_i(k))^3 \\ &= k^{1/2}\Delta_n^2\sum_{i=1}^{[n/k]}[(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}}) - (\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})]v_{ik}^3 \\ &\quad + k^{1/2}\Delta_n^2[(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})v_{k[n/k]}^3 - (\frac{\Delta_1^n X(k)}{\sqrt{k\Delta_n}})v_0^3] \\ &\quad - 3k^{1/2}\Delta_n^2[(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})v_{ik}^2 v_{(i-1)k} - (\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})v_{ik}v_{(i-1)k}^2]\ . \end{aligned}$$

Then we find that

$$\begin{aligned}
& \Delta_n^{-3}\mathcal{E}[(U_{4n})^2|X] \\
= \quad & 2\mathcal{E}[v_1^6]k\Delta_n\left[\sum_{i=1}^{[n/k]}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})^2 - 2\sum_{i=1}^{[n/k]}(\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})\right] \\
& +2\times 9\mathcal{E}[v_1^4]\mathcal{E}[v_1^2]k\Delta_n\sum_{i=1}^{[n/k]}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})^2 \\
& -2\times 3\mathcal{E}[v_1^4]\mathcal{E}[v_1^2]k\Delta_n\sum_{i=1}^{[n/k]}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}} - \frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}})(\frac{\Delta_{i+1}^n X(k)}{\sqrt{k\Delta_n}}) \\
& +2\times 3\mathcal{E}[v_1^4]\mathcal{E}[v_1^2]k\Delta_n\sum_{i=1}^{[n/k]}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}} - \Delta_{i+1}^n\frac{X}{\sqrt{k\Delta_n}})(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}}) \\
& -2\times 9(\mathcal{E}[v_1^2])^3k\Delta_n\sum_{i=1}^{[n/k]-1}(\frac{\Delta_i^n X(k)}{\sqrt{k\Delta_n}})(\frac{\Delta_{i+1}^n X}{\sqrt{k\Delta_n}}) \\
& +o_p(1) \\
\xrightarrow{p} \quad & [2\mathcal{E}[v_1^6] + 30\mathcal{E}[v_1^4]\mathcal{E}[v_1^2]]\int_0^1 \sigma_s^2 ds\ ,
\end{aligned}$$

which is $\mathbf{Var}([\Delta v])^3]\int_0^1\sigma_s^2 ds$. By multiplying $4^2(c\sqrt{c})^2$, we have (4.13).

Fifth, we use the relation that

$$\epsilon_n^4\sum_{i=1}^{[n/k]}[(\Delta v_i(k))^4 - \mathcal{E}(\Delta v_i(k))^4] \sim \frac{c^2}{n}\sqrt{\frac{n}{k}}\frac{1}{\sqrt{\frac{n}{k}}}\sum_{i=1}^{[n/k]}[(\Delta v_i(k))^4 - \mathcal{E}(\Delta v_i(k))^4],$$

whose asymptotic variance is the limit of $1/n^3 times[c^2\sqrt{k}]\mathbf{Var}(\Delta v)^4$. Then it becomes (4.14) as the limit of $\mathbf{Var}[\sqrt{n}F_{5n}]$.

Finally, because F_{1n} includes the sum of $[Z_i^n(k)]^4$ essentially, F_{2n} includes the sum of $[Z_i^n]^3\Delta_i v(k)$ essentially, F_{3n} includes the sum of

$[Z_i^n]^2[\Delta_i v(k)]^2$ essentially, F_{4n} includes the sum of $[Z_i^n][\Delta_i v(k)]^3$ and F_{5n} includes the sum of $[\Delta_i v(k)]^3$ essentially. Then they are asymptotically and $\mathcal{F}-$conditionally uncorrelated and thus they are $\mathcal{F}-$conditionally independent. By using the stable convergence and summarizing the limiting random variables of each terms, we have the result. **Q.E.D.**

Proof of Corollary 4 : Because of (4.17), we have

$$\text{(A.33)} \qquad \hat{S}(p,k) - 1 = \frac{\hat{B}(p,k\Delta_n)_1 - \hat{B}(p,\Delta_n)_1}{\hat{B}(p,\Delta_n)_1} \ .$$

Then by applying Theorem 3, we have the first part.

For the second part, we consider

$$\begin{aligned} & \frac{\hat{B}(p,k\Delta_n)_1}{\hat{B}(p,\Delta_n)_1} - k^{p/2-1} \\ = & \frac{[\Delta_n^{1-p/2}\hat{B}(p,k\Delta_n)_1 - k^{p/2-1}m_pA(p)] - k^{p/2-1}[\Delta_n^{1-p/2}\hat{B}(p,\Delta_n)_1 - m_pA(p)]}{[\Delta_n^{1-p/2}\hat{B}(p,\Delta_n)_1 - m_pA(p)] + m_pA(p)} \\ & -k^{p/2-1} \ . \end{aligned}$$

Then by applying Theorem 3, we have the second part. **Q.E.D.**

APPENDIX B : Some Figures

In this Appendix B we have given several figures we mentioned to in Sections 5 and 6. We have given the empirical density of the normalized (limiting) random variables for each normalized statistics based on a set of simulations. (The details are explained in Section 5.) For the comparative purpose, the density of the limiting normal density has been drawn by bold (red) curves in each figures.

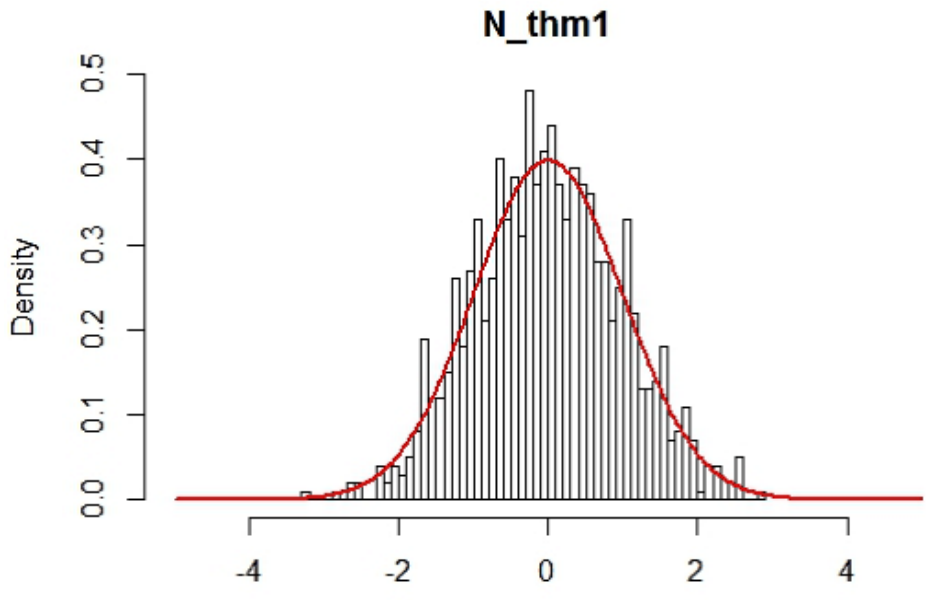

Figure 1 : Effects of Noise for CLT (Theorem 1,$n = 1,000$)

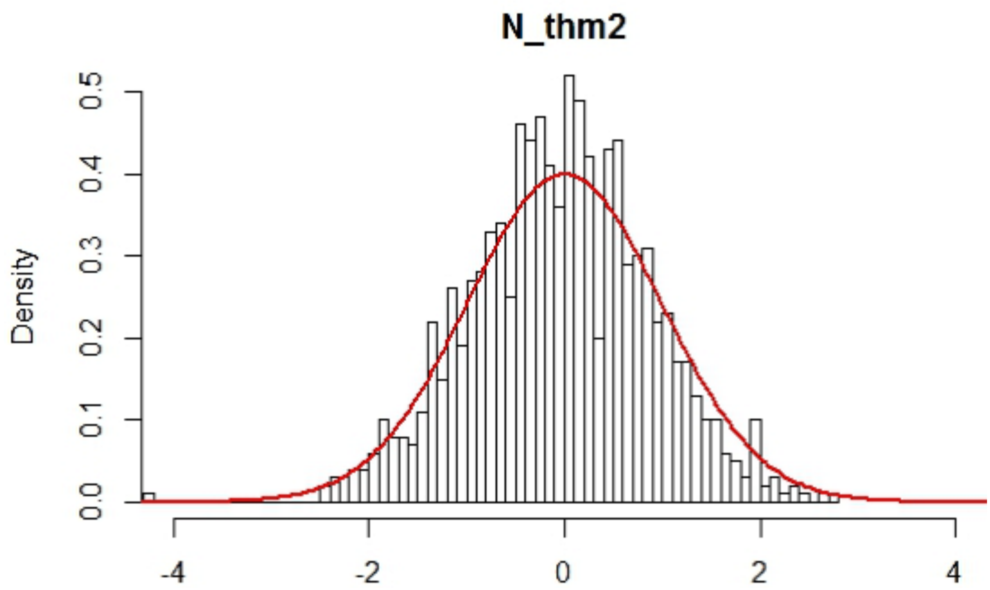

Figure 2 : Effects of Noise for CLT (Theorem 2,$n = 5,000$)

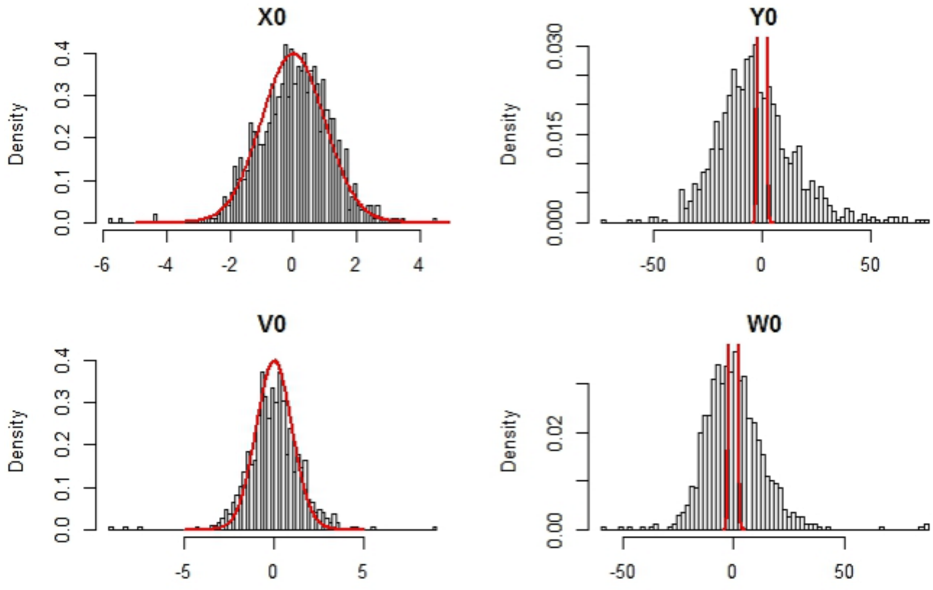

Figure 3 : Effects of Noise for CLT (1st Jump-Test)

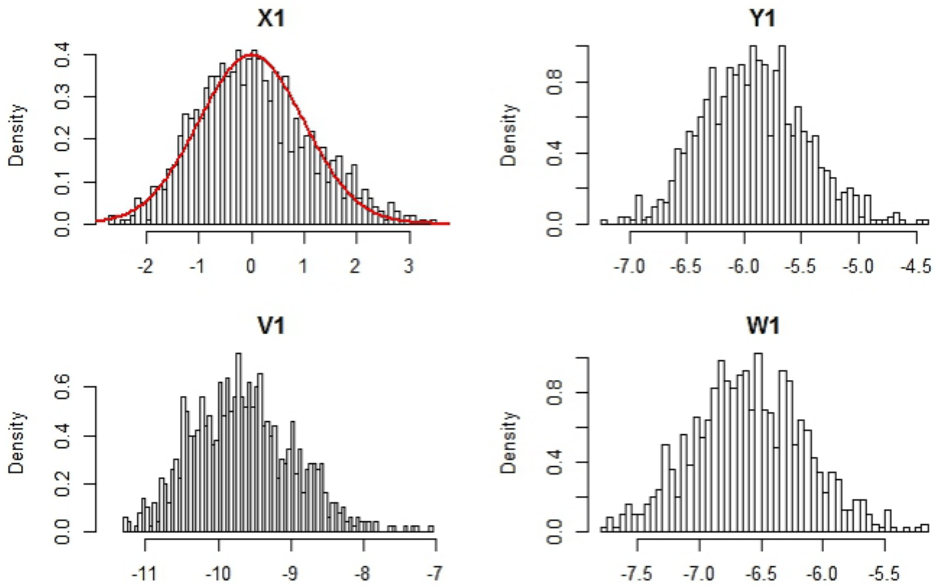

Figure 4 : Effects of Noise for CLT (2nd Jump-Test))

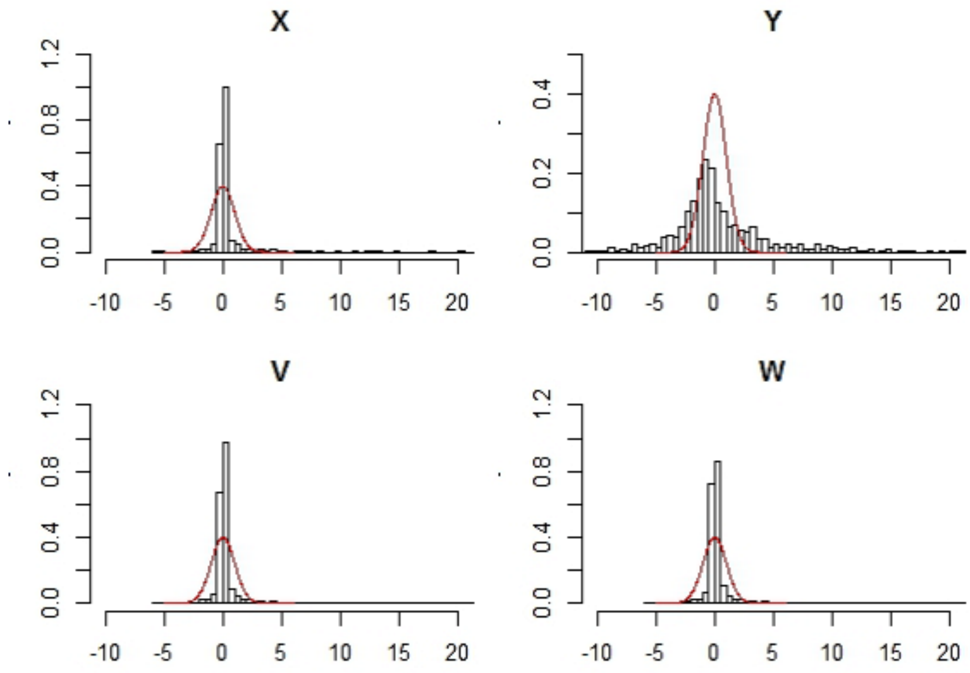

Figure 5 : Empirical distribution of the test statistics proposed in Ait-Sahalia, Jacod and Li(2012) when $1/\Delta_n = 10000$, $c = 10$, $k_n = 120$ and $v_i \sim \mathrm{N}(0, 1)$.

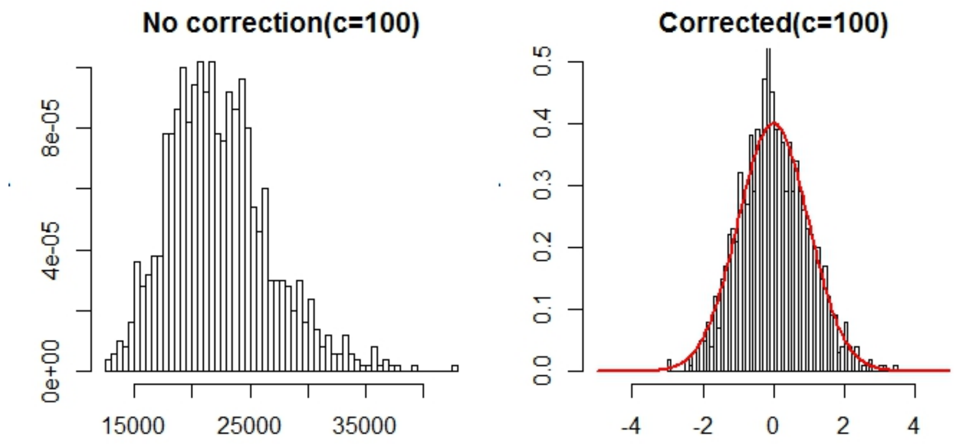

Figure 6 : Effects of Noise (c=100, Theorem 1)