CIRJE-F-1082

Attribute-Based Inferences in Subjective State Spaces: A Dissatisficing-Averse Utility Representation

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April 2018

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First Draft: January 2015; This Draft: April 10, 2018

Abstract

This paper studies preferences over menus, and presents a theory of attribute-based inferences, i.e., a decision-making based on attributes of alternatives. Attribute-based inferences can often lead to systematic violations of rationality, namely, WARP (*Weak Axiom of Revealed Preference*). The *Compromise Effect* (Simonson (1989)) is a typical example for the deviation from WARP. In this paper, we introduce plausible new axioms for attribute-based inferences: *Dominance*, *Dissatisfaction*, and *Contemplation*. The three key axioms characterize a dissatisficing-averse utility representation, in which the decision maker determines the optimal weight on the objective attribute space to minimize the deviation from each attribute-best option. We find out that the *aversion* to the increase in the trade-off across attributes, stated as *Dissatisfaction* can lead to the Compromise effect. Moreover, by imposing on *Dominance*, we argue that the *Attraction Effect* (Huber et al. (1982)), another typical behavioral regularity, may stem from a different cognitive mechanism. As an extended analysis, this paper studies a pair of preferences over menus and choice correspondences to provide behavioral foundations for the ex-post choices of the dissatisficing-averse utility representation, and to consider a relationship between menu-preferences and choices explicitly.

Keywords: Attribute-Based Inferences; the Compromise effect; Reasoned Choices; Preferences over Menus.

JEL Classification Numbers: D01, D11, D90.

*I am indebted to my adviser Akihiko Matsui for his unique guidance, constant supports, and encouragement. I would like to thank In-Koo Cho, Youichiro Higashi, Kazuya Hyogo, Atsushi Kajii, Michihiro Kandori, Nobuo Koida, Hitoshi Matsushima, Daisuke Nakajima, Daisuke Oyama, and Norio Takeoka for their invaluable suggestions and discussions. I am also grateful for the seminar participants at the University of Tokyo (3rd Summer School of Econometric Society), Kyoto (the BBL workshop) and Waseda (2016 Autumn Meeting of Japanese Economic Association). I gratefully acknowledge financial support from the Japan Society for the Promotion of Science (Grant-in-Aid for JSPS Fellows (02605576)). I would like to thank Editage (www.editage.jp) for English language editing. All remaining errors are mine.

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1 Introduction

In real life, people often find it difficult to make choices, due to the fact that there are various criteria for ranking alternatives such as attributes of alternatives. In attribute-based inferences, attributes are interpreted as criterion or dimensions for decision-making. Classically, Rosen (1974) introduces and applies attribute-based inferences into economic analysis.

There are mainly two steps for decision-making under attribute-based inferences. First, the decision maker chooses an attribute space. Even though the decision maker may be better to take all attributes into account, it appears to be cognitively demanding. This issue is related to the notion of *bounded rationality*. Second, the decision maker needs to resolve the trade-off between attributes. For example, consider the decision-making of booking a hotel. If the decision maker tries to choose a hotel with a good location for sightseeing, booking such a hotel might be beyond the budget of the decision maker.

It is widely recognized that attribute-based inferences can lead to systematic violations of rationality. The *Compromise Effect* is a typical example in well-known behavioral regularities such as violations of WARP (*Weak Axiom of Revealed Preference*). The Compromise effect was introduced by Simonson (1989). The Compromise effect states that given a choice set, the choice probability of an "intermediate" alternative increases, when an "extreme" alternative is added. Recently, various models have been developed, which are consistent with the compromise effect (de Clippel and Eliaz (2012); Tserenjigmid (2017), etc.).[1] Most models are axiomatically characterized by relaxing WARP, which makes it possible to provide testable implications.

The robust tendency to exhibit the Compromise effect stems from attribute-based inferences. In particular, the task of considering the trade-off across attributes is closely related to the resulting behaviors. Since it is the difficult task, the decision maker may be averse to the trade-off, and then choose a *moderate* alternative across attributes. Thus, we can interpret the Compromise effect as the result of reasoning in attribute-based inference. However, it is not easy to capture such an aspect in terms of choice functions/correspondences as primitives, so we explicitly consider a relationship between preferences and the Compromise effect as a choice behavior.

The motivation of this paper is to explore a cognitive foundation behind the Compromise effect. In this paper, we study preferences over menus, and explore plausible axioms for attribute-based inferences under the trade-off across attributes.[2] We provide new axioms for attribute-based inferences, and study how the decision maker chooses a menu under attribute-based inferences. In the framework of preferences over menus, various models have already been developed, ranging from self-control preferences (Gul and Pesendorfer (2001)), regret

[1] Both de Clippel and Eliaz (2012) and Tserenjigmid (2017) are also consistent with the Attraction effect (Huber et al. (1982)), which is a behavioral regularity as a preference reversal.

[2] This framework was introduced in Kreps (1979), and generalized by Dekel et al. (2001).

(Sarver (2008)), to social image (Dillenberger and Sadowski (2012)).

Traditionally, in attribute-based inferences, Krantz et al. (1971) is regarded as the seminal literature on attribute-based inferences. They provide an axiomatic foundation for additively separable utility representations in attribute-based inferences. Keeney and Raiffa (1976) refer to the importance of the procedure of determining the optimal weight on attribute spaces (Section 3.2.4; page 74 in the second edition). Indeed, the optimal weight is changeable, depending on choice opportunities, i.e., *menus*. Consequently, the Compromise effect can occur if the weight on the attribute space is reference-dependent/ menu-dependent. To caputure such a procedural aspect, we need a richer structure such as preferences over menus. The framework of preferences over menus has been explored to study subjective uncertainty such as taste uncertainty (Dekel et al. (2001)). The process of determining the optimal weight on attribute spaces can be regarded as a type of taste uncertainty.

The contribution of this paper is to provide an axiomatic foundation for attribute-based inferences, by showing that exploring the *best* option on the *efficient* frontier in each menu on the attribute-based utility space is equivalent to exploring the *optimal* weight on the attribute space. Behaviorally, this model is consistent with the Compromise effect.[3] We show that the *aversion* to the increase in the trade-off across attributes is related to a class of *preferences for commitment* (preferring smaller menus). To capture the procedural aspect in attribute-based inferences, we consider the following new axioms explicitly.

First, we introduce the axiom of *Dominance*. To understand this axiom, let us note that an attribute space is exogenously given. We say that an option *dominates* another option if it is desirable in terms of all attribute-based evaluations. This axiom says that, if an option $\mathbf{q}$ is dominated by some option $\mathbf{p}$ in a menu A, then adding it into the menu does not change the ranking, i.e., $A \sim A \cup \{\mathbf{q}\}$. In this case, any trade-off between attributes does not occur. In the example of booking a hotel, even if a higher priced hotel, located farther away is added, the trade-off is not produced. Moreover, this axiom rules out the *Attraction Effect* (Huber et al. (1982)). This axiom implies that a different cognitive mechanism can lead to the Attraction effect.

Second, we introduce the key axiom of *Dissatisfaction*. Intuitively, the axiom states that the decision maker dislikes increasing the trade-off between attributes, when a new option is added. Formally, the axiom states that the decision maker dislikes adding an option $\mathbf{q}$ into a menu A, i.e., $A \succeq A \cup \{\mathbf{q}\}$, if the added option has the following property.[4] First, there are attributes $i, j \in \mathbb{A}$ such that, in an attribute i, there is an option $\mathbf{p}$ in a menu A that is strictly desirable in terms of the attribute i, but, in another attribute j, $\mathbf{q}$ is strictly desirable

[3]Notice that this paper is not consistent with the Attraction effect. The Attraction effect appears to be related to limited attention. See, for example, Ok et al. (2015). To study limited attention on preferences over menus, we need a richer structure.

[4]This axiom is included into a class of commitment preferences: *Set-Betweenness* (Gul and Pesendorfer (2001)), *Dominance* (Sarver (2008)), *Exclusion* (Stovall (2010)), etc.

than $\mathbf{p}$ in terms of the attribute j. Second, $\mathbf{p}$ is preferred to $\mathbf{q}$ for some $\mathbf{p}$ in the menu A.

Intuitively, adding the option $\mathbf{q}$ may increase the trade-off across attributes due to the first condition in *Dissatisfaction*. As a result, when making a choice, the "dissatisfaction" due to reasoning on attributes may increase. For example, again, consider a decision-making relating to booking a hotel. Suppose that the decision maker considers a menu A, and the decision maker thinks that a hotel $\mathbf{p}$ is the best in the menu A. Then, the decision maker finds a new hotel on the web where the location is great for sightseeing, but the hotel $\mathbf{p}$ in the menu A is still better (the second condition in *Dissatisfaction*). If the new hotel is added, then the criterion of attribute-based inferences can change, and the relative ranking may also change. The decision maker dislikes producing new trade-off across attributes.

Third, we introduce the axiom of *Contemplation* for weights on the objective attribute space. This axiom is a *weaker* version of the axiom of *Independence* (Dekel et al. (2001)). This axiom has the following two conditions. The first condition states that in singleton menus, there is no need for contemplation. This condition is standard, since in singleton menus, the decision maker just chooses the option. The second condition refers to a property, when the decision maker needs a contemplation. Consider a menu A. if both a menu A and another menu B do not have a "desirable" option which dominates other options in terms of attribute-based inferences, then the λ-mixture of A and B is preferred to A, i.e., for all $\lambda \in [0, 1]$, $\lambda A + (1 - \lambda)B \succeq A$. That is, the contemplation is required.[5] This axiom captures a natural procedure of attribute-based inferences. Intuitively, in the example of booking a hotel, if there is no hotel that dominates all hotels, the decision maker contemplates which hotel is better.

The key axioms, *Dominance*, *Dissatisfaction*, and *Contemplation*, along with other basic axioms, characterize a dissatisficing-averse utility representation in attribute-based inferences. The utility representation depicts the decision maker who determines the optimal weight on the objective attribute space to minimize the deviation from each attribute-best option. Since we relax the axiom of *Independence*, and impose on the key axiom of *Dissatisfaction*, the resulting choice behaviors that reflect on menu preferences can deviate from the postulate of *rationality*, i.e., WARP.

The new axioms for attribute-based inferences are closely related to the Compromise effect in the cognitive mechanism. In psychology, it is observed that both the Attraction effect and the Compromise effect can stem from the same heuristic. Simonson (1989) states that the same group of subjects exhibits both effects in roughly the same magnitude. For this evidence, both de Clippel and Eliaz (2012) and Tserenjigmid (2017) propose the single choice

[5]In the framework of preferences over menus, contemplations in attribute-based inferences are captured by *contingent planning*. Consider two menus A, B. Then, consider an objective lottery of menus $\lambda A + (1 - \lambda)B$. The *contingent planning* means that the decision maker considers choosing an option $\mathbf{p}$ from the menu A if the menu A is realized, and choosing an option $\mathbf{q}$ from the menu B if the menu B is realized.

model to explain both effects. In this paper, on the other hand, to allow for the Attraction effect, we need to rule out the axiom of *Dominance*. In this sense, the Attraction effect can occur due to a cognitive mechanism different from attribute-based inferences including the trade-off between attributes.

Some may think that the compromise effect is reminiscent of *convex preferences*. In Mas-Colell, et al. (1995), the convexity of preference relations is interpreted as the formal expression of a basic inclination of decision makers for "diversification". In fact, this paper relaxes the axiom of *Independence*. Mathematically, *Independence* leads to the linear structure of utility representations. However, the axiom of *Contemplation* itself cannot explain the Compromise effect. Hence, the aversion to the increase in the trade-off between attributes, captured by the axiom of *Dissatisfaction*, can lead to the Compromise effect.

This paper takes the framework of preferences over menus, and discusses the relationship between raw preferences and reasoned choices. Gilboa (2009) refers to this issue in terms of "feeling" and "reasoning."[6] In this paper, if an option *dominates* another option, then there is no need for reasoning (see the axiom of *Dominance*). This can correspond to "raw preference" because there is no inference for decision-making. On the other hand, this paper requires that if such a *dominance* option does not exist, then contemplation is necessary. By taking preferences over menus as primitives, the difference between raw preferences and reasoned choices is axiomatically captured. The difference can be revealed by the resulting choice behaviors.

As an extended analysis, this paper studies a pair of preferences over menus and choice correspondence, i.e., $\langle \succeq, C \rangle$, to study ex-post choices of the dissatisficing-averse utility representation, i.e., the resulting behaviors in the model. We characterize the ex-post choice, by relaxing the postulate of rationality, namely, WARP, along with other basic axioms. We study a relationship between menu-preferences and choices.

The rest of this paper is organized as follows. In Section 2, we introduce the axioms of this chapter. In Section 3, we state the representation theorem (Theorem 1), the uniqueness result (Proposition 1), the proof outline of Theorem 1, comparative statics (Proposition 2), and the characterization of ex-post choices (Proposition 3). In Section 4, we discuss literature review and present concluding comments. All proofs are in the Appendix.

2 Axioms

Let $X := \Pi_{i=1}^{n} X_i$ be a finite set of all alternatives, where X_i is a domain of attribute i of alternatives. This domain corresponds to an attribute i's evaluation for alternatives. We

[6] See Gilboa et al. (2012). Zajonc (1980) uses the phrase "preference requires no inference."

assume that an attribute space $\mathbb{A}$ is finite, i.e., $\mathbb{A} = \{1, \cdots, n\}$ and $|\mathbb{A}| = n$.[7] The elements of X_i are denoted by $x_i, y_i, z_i \in X_i$. For each i, let $\Delta(X_i)$ be a set of probability distributions over X_i, endowed with the Euclidean metric d. Since each X_i is finite, the topology generated by the Euclidean metric d is equivalent to the weak* topology on $\Delta(X_i)$. The elements of $\Delta(X_i)$ are denoted by $p_i, q_i, r_i \in \Delta(X_i)$.[8]

An *option* is denoted by $\mathbf{p} := (p_1, \cdots, p_n)$. Let $\mathcal{X} := \prod_{i=1}^{n} \Delta(X_i)$ be a set of all options.[9] Let $\mathcal{A}$ be the set of all non-empty closed and compact subsets of $\mathcal{X}$, endowed with the Hausdorff metric. The Hausdorff metric is defined by

$$d_h(A, B) := \max \left\{ \max_{\mathbf{p} \in A} \min_{\mathbf{q} \in B} d(\mathbf{p}, \mathbf{q}), \max_{\mathbf{p} \in B} \min_{\mathbf{q} \in A} d(\mathbf{p}, \mathbf{q}) \right\}.$$

Menus are denoted by $A, B, C \in \mathcal{A}$. Define the convex combinations in the standard manner: For any $A, B \in \mathcal{A}$ and for any $\lambda \in [0, 1]$,

$$\lambda A + (1 - \lambda) B := \{\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \mid \mathbf{p} \in A, \mathbf{p} \in B\}.$$

The primitive of the model is a binary relation $\succeq$ over $\mathcal{A}$. The binary relation $\succeq$ describes the decision maker's menu-preferences under attribute-based inferences. The asymmetric and symmetric parts of $\succeq$ are denoted by $\succ$ and $\sim$ respectively.

First, we provide standard requirements in decision theory. The first condition says that all menus are comparable. Plus, the ranking of menu-preferences is consistent. The second condition states that there is no jump for menu preferences. The third condition requires that there exists a singleton menu which is strictly preferred.

Axiom (Standard Preferences): $\succeq$ is (i) a *weak order*, (ii) *continuous*, and (iii) *non-degenerate*:

(i) (Weak Order): $\succeq$ is *complete* and *transitive.*

(ii) (Continuity): The sets $\{A \in \mathcal{A} \mid A \succeq B\}$ and $\{A \in \mathcal{A} \mid B \succeq A\}$ are closed (in the Hausdorff metric d_h) .

(iii) (Strict Non-Degeneracy): There exist $\mathbf{p}, \mathbf{q} \in \mathcal{X}$ such that $\{\mathbf{p}\} \succ \{\mathbf{q}\}$.

By assumption, an *objective* attribute space $\mathbb{A} = \{1, \cdots, n\}$ is exogenously given. For notational convenience, we write $\mathcal{X}_i := \Delta(X_i)$ and $\mathcal{X}_{-i} := \prod_{j \neq i} \Delta(X_j)$, for each $i \in \mathbb{A}$. The

[7] We assume that $|\mathbb{A}| > 1$. In the case of $|\mathbb{A}| = 1$, the result in this paper holds. I thank Daisuke Oyama for his comment.

[8] A lottery p_i is interpreted as follows. For example, suppose that a university has application letters from Ph.D. applicants with TOEFL iBT scores. However, the university is still uncertain how well the Ph.D. applicants can speak English. The lottery p_i captures a risky prospect of the university for a candidate's attribute-based evaluation.

[9] In this setting, each element in options is independent.

following axiom is a separable condition between attribute-based rankings.

Axiom (Separability): For any $p_i, q_i \in \mathcal{X}_i$ and $r_{-i}, r'_{-i} \in \mathcal{X}_{-i}$,

$$\{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\} \Rightarrow \{(p_i, r'_{-i})\} \succeq \{(q_i, r'_{-i})\}.$$

This axiom requires that the decision maker can consider each attribute-based evaluation *separately*. Consider two options (p_i, r_{-i}) and (q_i, r_{-i}) that are only different from an attribute i. If the decision maker has a taste $\{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\}$, this taste should stem from the difference between p_i and q_i.

The definition of $\succsim_i$ on $\mathcal{X}_i$ is introduced in the following way, which is induced by the primitive of the model, i.e., $\succeq$ on $\mathcal{A}$.

Definition 1. For any $p_i, q_i \in \mathcal{X}_i$,

$$p_i \succsim_i q_i \Leftrightarrow \{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\},$$

for any $r_{-i} \in \mathcal{X}_{-i}$.

Notice that each ranking for an attribute i, $\succsim_i$, is *well-defined* (see Appendix A). By using the definition of $(\succsim_i)_{i \in \mathbb{A}}$, we provide the following monotonic condition.

Axiom (Dominance): If for any $\mathbf{q} \in \mathcal{X}$, there exists $\mathbf{p} \in A$ such that for any $i \in \mathbb{A}$, $p_i \succsim_i q_i$, then $A \sim A \cup \{\mathbf{q}\}$.

This axiom says that if an option $\mathbf{q}$ is dominated by some alternative $\mathbf{p}$, in terms of attribute-based inferences, i.e., for all $i \in \mathbb{A}$, $p_i \succsim_i q_i$, then the option $\mathbf{q}$ is not chosen, even if adding $\mathbf{q}$ into A.[10] Intuitively, since two menus A and $A \sim A \cup \{\mathbf{q}\}$ are *indifferent*, the existence of the option $\mathbf{q}$ does not produce any "dissatisfaction." Consider the example of booking a hotel. Suppose that a decision maker considers two attributes of hotels; (i) accommodation fees and (ii) locations. Even if an *inferior* hotel is added into a menu which the decision maker faces, the choice of booking hotels does not change.

One remark is that this axiom rules out the *Attraction* effect introduced by Huber et al. (1982). The Attraction effect says that, given a choice set (*menu*), when an alternative that is asymmetry dominated or relatively inferior is added into the choice set, the probability of choosing the *dominating* alternative increases. However, the axiom requires that adding such an option into menus does not change the ranking.

We introduce a new axiom, called *Dissatisfaction*.

Axiom (Dissatisfaction): For any $\mathbf{q} \in \mathcal{X}$, if

[10] In the case of $|\mathbb{A}| = 1$, the axiom corresponds to a weaker version of the axiom of *Strategic Rationality* in Kreps (1979): $A \succeq B \Rightarrow A \sim A \cup B$. Our axiom only considers the case of menu $|B| = 1$.

(i) there exist $i, j \in \mathbb{A}$ such that

(a) for some $\mathbf{p} \in A$, $p_i \succ_i q_i$;

(b) for any $\mathbf{p} \in A$, $q_j \succ_j p_j$; and

(ii) $\{\mathbf{p}\} \succeq \{\mathbf{q}\}$ for some $\mathbf{p} \in A$,

then

$$A \succeq A \cup \{\mathbf{q}\}.$$

This axiom says that, for any options $\mathbf{q}$, there exist attributes $i, j \in \mathbb{A}$ such that $p_i \succ_i q_i$ and $p_j \prec_j q_j$ for some $\mathbf{p}$ in a menu A, and $\{\mathbf{p}\} \succeq \{\mathbf{q}\}$. Then, the decision maker dislikes adding $\mathbf{q}$ into the menu A.[11]

The notion of "dissatisfaction" is interpreted as follows. By the axiom of *Dominance*, if for any alternatives $\mathbf{q}$ in a menu A, there exists $\mathbf{p}$ in the menu A such that for all attributes $i \in \mathbb{A}$, $p_i \succ_i q_i$, then $\mathbf{p}$ is chosen from the menu A. However, such an "ideal" option rarely exists in reality. Under the trade-off among attributes, the decision maker tries to have an agreement or a compromise among attributes, to make a choice. This contemplation is related to decrease the dissatisfaction from the ideal criterion.

The axiom states that, intuitively, if for some $j \in \mathbb{A}$, $\mathbf{q}$ is $\succsim_j$-best in the menu $A \cup \{\mathbf{q}\}$, then the dissatisfaction can increase. To avoid this, the decision maker dislikes adding $\mathbf{q}$ into the menu A. This is the attitude toward the aversion to the increase in the trade-off between attributes.

Consider the example of booking a hotel. Suppose, again, that a decision maker considers two attributes of hotels; accommodation fees and locations. A decision maker considers whether a hotel is added into her consideration or not. In terms of attributes of hotels, the hotel is not dominated by any hotels in her consideration. That is, there is a trade-off between attributes (the conditions (a) and (b)). If for any hotels in her consideration, the new hotel is desirable, the hotel is added into her consideration. However, if there is already a hotel in her consideration that is better than the new hotel, the new hotel is not probably chosen (the condition (ii)). Hence, the decision maker does not have to add the new hotel into her consideration.

Finally, we provide the axiom of *Contemplation*, a weaker version of *Independence*. The axiom of *Independence* is stated as follows: For any $A, B \in \mathcal{A}$ and $\lambda \in [0, 1]$,

$$A \succeq B \Rightarrow \lambda A + (1 - \lambda)C \succeq \lambda B + (1 - \lambda)C.$$

[11] In the case of $|\mathbb{A}| = 1$, the condition (i) is redundant. The axiom is equivalent to the axiom of *Dominance*in Sarver (2008). I would like to thank Daisuke Oyama for his comment.

The axiom of *Contemplation* has two conditions.[12] The first condition says that the axiom of *Independence* holds under any mixtures of singleton menus. Singleton menus have no requirement for reasoning. The second condition says that if both a menu A and another menu B do not have a "desirable" option that dominates other options in terms of attribute-based inferences, then the λ-mixture of A and B is preferred to A. Formally, the axiom is stated as follows.

Axiom (Contemplation): $\succeq$ satisfies the following two conditions:

(i) (No Need for Contemplation): For any $A, B \in \mathcal{A}$, $\mathbf{p} \in \mathcal{X}$, and $\lambda \in [0, 1]$,

$$A \succeq B \Rightarrow \lambda A + (1-\lambda)\{\mathbf{p}\} \succeq \lambda B + (1-\lambda)\{\mathbf{p}\}.$$

(ii) (Contemplation Seeking)[13]: For any $A, B \in \mathcal{A}$,

(a) if there exist $i, j \in \mathbb{A}$ such that

- for any $\mathbf{p} \in A$, there exists $\mathbf{q} \in B$ such that $q_i \succ_i p_i$;
- for any $\mathbf{q} \in B$, there exists $\mathbf{p} \in A$ such that $p_j \succ_j q_j$; and

(b) $A \sim B$,

then, for any $\lambda \in [0, 1]$,

$$\lambda A + (1-\lambda)B \succeq A.$$

Notice that there is no cost of reading options in a menu, so, the size of menus does not matter for the contemplation in attribute-based inferences. First, in singleton menus, there is no need for contemplation in decision-making. The axiom of *Independence* holds under the mixtures of singletons.

Second, for any two menus, A and B, if both A and B do not have a "desirable" option that dominates other options in terms of attribute-based inferences, then the contemplation

[12]This "contemplation" is different from the costly contemplation in Ergin and Sarver (2010a). They study the decision maker who explores a *subjective state space* with costly contemplation. On the other hand, this paper studies the decision maker who explores the optimal weight on the objective attribute space via contemplation.

[13]In the case $|\mathbb{A}| = 1$, the condition (a) is redundant. The condition (b) says that if the two menus are indifferent, then the λ-mixture as a contingent planning is preferred. This leads to the consideration in the minimum requirement in attribute-based inferences. In the example of booking a hotel, suppose the decision maker considers three attributes of hotels: accommodation fees, locations, and services. If the condition (a) is ruled out, the axiom of *Contemplation Seeking* leads to the consideration in the pessimistic evaluations under attribute-based inferences. That is, the decision maker only considers one attribute such as locations. This contemplation is interpreted as the minimum requirement (*reasoning*) in decision-making. I would like to thank Hitoshi Matsushima for his comment.

in contingent planning, i.e., choosing $\mathbf{p}$ in A with probability λ and choosing $\mathbf{q}$ in A with probability $1-\lambda$, is needed. Remember that, by letting $\lambda \in [0,1]$,

$$\lambda A + (1-\lambda)B = \{\lambda \mathbf{p} + (1-\lambda)\mathbf{q} \mid \mathbf{p} \in A, \mathbf{p} \in B\}.$$

Then, contemplation-seeking can lead to the complementary across attributes.

3 Result

3.1 Representation Theorem

We state the main result. Let $\Lambda(\mathbb{A})$ be the set of all non-empty compact subsets of non-negative measures on $\mathbb{A}$. Given a menu $A \in \mathcal{A}$, let $\boldsymbol{u}^*(A) := (\max_{\mathbf{p}\in A} u_i(p_i))_{i\in\mathbb{A}}$ be the *ideal* option of the menu A.

Theorem 1. *The following statements are equivalent:*
(a) $\succeq$ *on* $\mathcal{A}$ *satisfies Standard Preferences, Separability, Dominance, Dissatisfaction, and Contemplation.*
(b) There exists a pair $\langle\ \mathcal{U}, \mathcal{M}\ \rangle$ *where* $\mathcal{U} = (u_1, \cdots, u_i, \cdots, u_n)$ *is a set of non-constant utility functions where* $u_i : \mathcal{X}_i \to \mathbb{R}$*, and* $\mathcal{M}$ *is a set of non-negative measures on* $\mathbb{A}$ *defined by* $\mathcal{M} : \mathbb{R}^n \to \Lambda(\mathbb{A})$*, such that* $\succeq$ *is represented by* $V : \mathcal{A} \to \mathbb{R}$ *defined by*

$$V(A) = \max_{\mathbf{p}\in A}\Big[\sum_{i\in\mathbb{A}} u_i(p_i) - \min_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{u}^*(A))} \sum_{i\in\mathbb{A}} \mu_i\big(\max_{\mathbf{q}\in A}(u_i(q_i) - u_i(p_i))\big)\Big],$$

and the following conditions hold:

(i) $\mathcal{M}$ *is consistent: for each* $\mu, \mu' \in \mathcal{M}$ *and* $\mathbf{p} \in \mathcal{X}$, $\sum_{i\in\mathbb{A}} \mu_i u_i(p_i) = \sum_{i\in\mathbb{A}} \mu'_i u_i(p_i)$*;*

(ii) $\mathcal{M}$ *is minimal: for any compact subset* $\mathcal{M}'$ *of* $\mathcal{M}$*, the function* V' *obtained by replacing* $\mathcal{M}$ *with* $\mathcal{M}'$ *no longer represents* $\succeq$*.*

We explain about the interpretation of the main result of this chapter. We call the utility representation the *dissatisficing-averse utility* representation if $\succeq$ satisfies the axioms in Theorem 1. This utility representation has the following two terms. In the similar way with Tversky and Simonson (1993), we have a linear combination of two terms: the reference-independent value of options and the impact of relative evaluations of options in each menu.

The first term is the aggregation of each attribute-based value $(u_i)_{i\in\mathbb{A}}$ without *reference-dependence*. In singleton menus, for each $\{\mathbf{p}\} \in \mathcal{A}$,

$$V(\{\mathbf{p}\}) = U(\mathbf{p}) = \sum_{i\in\mathbb{A}} u_i(p_i).$$

The second term is the key part in the dissatisficing-averse utility representation. For each attribute $i \in \mathbb{A}$, $\max_{\mathbf{q}\in A}(u_i(q_i) - u_i(p_i))$ captures the dissatisfaction in an attribute i. The decision maker has a set of weights on the attribute space that depends on the *ideal* option in the menu A. Different menus have different *ideal* options. The decision maker chooses the optimal weight to minimize the dissatisfaction from the *ideal* option in attribute-based inferences.

3.2 Uniqueness Result

We state the uniqueness result.

Proposition 1. *Suppose that two dissatisficing-averse model* $\langle \mathcal{U}, \mathcal{M} \rangle$ *and* $\langle \mathcal{U}', \mathcal{M}' \rangle$ *represent the same* $\succeq$*. Then, the following statements hold: There exists* $\alpha > 0$ *and* $\beta_i \in \mathbb{R}$ *for each* $i \in \mathbb{A}$ *such that*

(i) for any $i \in \mathbb{A}$, $u_i = \alpha u_i' + \beta_i$.

(ii) $\mathcal{M} = \alpha \mathcal{M}'$.

3.3 Proof Overview

We provide a proof outline of the sufficiency part in Theorem 1. We have mainly three steps.

In Step 1, first, we can show that, by the axiom of *Standard Preferences*, $\succeq$ on $\mathcal{A}$ is represented by $V : \mathcal{A} \to \mathbb{R}$, i.e., for any $A, B \in \mathcal{A}$,

$$A \succeq B \Leftrightarrow V(A) \geq V(B).$$[14]

Next, we show that each $\succsim_i$ on $\mathcal{X}_i$ is represented by $u_i : \mathcal{X}_i \to \mathbb{R}$, by mainly using the axioms of *Standard Preferences* and *Separability*. To do so, we show that each $\succsim_i$ ($i \in \mathbb{A}$) is *well-defined*, and that each $\succsim_i$ ($i \in \mathbb{A}$) satisfies the axiom of *Independence* (in the vNM Expected Utility Theorem), by using the axiom of *No Need for Contemplation* in the key axiom of *Contemplation*. Then, we obtain the following. For each $i \in \mathbb{A}$, for any $p_i, q_i \in \mathcal{X}_i$,

$$p_i \succsim_i q_i \Leftrightarrow u_i(p_i) \geq u_i(q_i).$$

Finally, we can show that a binary relation $\succsim$ on $\mathcal{X}$ satisfies the axioms of additively separable utility representations in Krantz et al. (1971). This corresponds to the case of singleton-menu comparisons. That is, for any $\mathbf{p}, \mathbf{q} \in \mathcal{X}$,

$$\{\mathbf{p}\} \succeq \{\mathbf{q}\} \Leftrightarrow \sum_{i \in \mathbb{A}} u_i(p_i) \geq \sum_{i \in \mathbb{A}} u_i(q_i).$$

In Step 2, by mainly using the axiom of *Contemplation*, we show that a utility of a menu has certain desired properties (See Lemma 4). First, we consider a set of attribute-based

[14]The mathematical backgroud is in Debreu (1959).

utilities of options on an attribute-based utility space. To do so, by Step 1, we can use the property of positive affine transformations for each u_i. Without loss of generality, consider $u_i : \mathcal{X}_i \to \mathbb{R}_+$ for each $i \in \mathbb{A}$. For any $A \in \mathcal{A}$, define

$$\boldsymbol{u}(A) := \Big\{ \Big(\frac{u_1(p_1)}{\sum_{i\in\mathbb{A}} u_i(p_i)}, \cdots, \frac{u_n(p_n)}{\sum_{i\in\mathbb{A}} u_i(p_i)}\Big) \in \mathbb{R}^n \mid \mathbf{p} = (p_1, \cdots, p_n) \in A \Big\}.$$

Let $\{\boldsymbol{u}(A) \mid A \in \mathcal{A}\ \}$. Notice that each $A \in \mathcal{A}$ is compact. $\boldsymbol{u}(A)$ is also compact by the continuity of u_i ($i \in \mathbb{A}$). Define $\succeq^*$ on $\mathcal{A}^*$ in the following way:

$$A^* \succeq^* B \text{ if } A \succeq B,$$

where $A^* = \boldsymbol{u}(A)$ and $B^* = \boldsymbol{u}(B)$. The asymmetric and symmetric parts of $\succeq^*$ are denoted by $\succ^*$ and $\sim^*$, respectively. Then, we show that $\succeq^*$ is *well-defined*. Next, we introduce the axiom of *Translation Invariance*. We define the set of *translations* in the following way.

$$\Theta := \Big\{\theta \in \mathbb{R}^n \mid \sum_{i=1}^{n} \theta_i = 0\ \Big\}.$$

For any $A^* \in \mathcal{A}^*$ and $\theta \in \Theta$, define $A^* + \theta := \{\boldsymbol{u} + \theta \mid \boldsymbol{u} \in A^*\ \}$.

Axiom* (Translation Invariance*): For any $A^*, B^* \in \mathcal{A}^*$ and $\theta \in \Theta$ such that $A^*+\theta, B^*+\theta \in \mathcal{A}^*$,

$$A^* \succeq^* B^* \Rightarrow A^* + \theta \succeq^* B^* + \theta.$$

Finally, by using the axiom, we show that a utility of a menu has certain properties. We construct a value function $V^* : \mathcal{A}^* \to \mathbb{R}$ that represents $\succeq^*$ on $\mathcal{A}^*$. We say that V^* is *translation linear* if for all $A^* \in \mathcal{A}$ and $\theta \in \Theta$, there exists $v \in \mathbb{R}^n$ such that

$$V^*(A^* + \theta) = V^*(A^*) + v \cdot \theta.$$

We verify that V^* has certain properties. If $\succeq^*$ is a continuous weak order that satisfies Dominance*, Dissatisfaction*, and Contemplation*, then there exists $V^* : \mathcal{A}^* \to \mathbb{R}$ with the following properties:

(i) For any $A^*, B^* \in \mathcal{A}^*$, $A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*)$.

(ii) V^* is continuous, concave, and translation linear.

(iii) For any $A^*, B^* \in \mathcal{A}^*$, if $V^*(A^*) \geq V^*(B^*) \Leftrightarrow V'^*(A^*) \geq V'^*(B^*)$, then there exists $a > 0$ and $b \in \mathbb{R}$ such that $V'^* = aV^* + b$.

In Step 3, we complete the dissatisficing-averse utility representation. To do so, we identify a subjective state space of the representation. We show that the state space is finite, by applying the result of Kopylov (2009). We have the state space such that the number of the state space is $n+1$. The first n states are related to the *objective* attribute space. Each state is related to each attribute $i \in \mathbb{A}$. The rest is related to the procedure of contemplation in attribute-based inferences. For this state, we apply the duality result in convex analysis.

Let $U(\mathbf{p}) = \sum_{i \in \mathbb{A}} u_i(p_i)$ for each $\mathbf{p} \in \mathcal{X}$. Let $\boldsymbol{u}(\mathbf{p}) = (u_1(p_1), \cdots, u_n(p_n))$ for each $\mathbf{p} \in \mathcal{X}$. Define a functional $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$J(\boldsymbol{u}^*(A), \boldsymbol{u}^*(A) - \boldsymbol{u}(\mathbf{p})) := \underbrace{U(\mathbf{p}) - V^*(A^*)}_{\text{dissatisfaction}},$$

where $\boldsymbol{u}^*(A) = (\max_{\mathbf{p} \in A} u_i(p_i))_{i \in \mathbb{A}}$ and for some $\mathbf{p} \in A$.

First, we show that J is *well-defined*. Second, we show that J has certain properties. We show that J is *monotonic*, i.e., given $a \in \mathbb{R}^n$, if $b \geq b'$, then $J(a,b) \geq J(a,b')$. Moreover, by definition, J is continuous in the second arguments, i.e., for any $a \in \mathbb{R}^n$, $J(a,\cdot)$ is continuous. We also show that J is homogeneous of degree one with respect to the second arguments, i.e., for any pair (a,b) and $\lambda \in (0,1)$, $J(a,\lambda b) = \lambda J(a,b)$. Thus, we can define a functional $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by, for all $A \in \mathcal{A}$ with $\mathbf{p} \in A$,

$$J(\boldsymbol{u}^*(A), \boldsymbol{u}^*(A) - \boldsymbol{u}(\mathbf{p})) := \widehat{J}(\boldsymbol{u}^*(A))(\boldsymbol{u}^*(A) - \boldsymbol{u}(\mathbf{p})),$$

for some $\widehat{J} : \mathbb{R}^n \to \mathbb{R}^n$.

The case of $|\mathbb{A}| = 2$, given a menu $A \in \mathcal{A}$ is described in the following way. For simplicity, let $u_1 = u_1(p_1)$, $u_2 = u_2(p_2)$,$u_1^* = \max u_1$, and $u_2^* = \max u_2$. Assume $|A| > 1$.

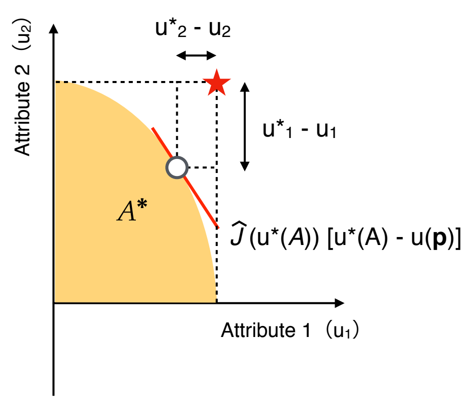

Figure 1: the functional $\widehat{J}$: the case $|\mathbb{A}| = 2$

We apply the duality result into the desired utility representation. That is, exploring the *best* option on the Pareto frontier in each menu on the attribute-based utility space is equivalent to exploring the *optimal* weight on the attribute space. To apply the Riesz's representation theorem with the duality result, let us introduce some notation.

Let $\mathcal{U}$ be the set of profiles of continuous real-valued functions on the attribute space $\mathbb{A}$, denoted by $\mathcal{U} = \{(u_i)_{i\in\mathbb{A}} | u_i : \mathcal{X}_i \to \mathbb{R}, \text{ for all } i \in \mathbb{A}\}$. Let $\sigma_A : \mathcal{U} \to \mathbb{R}$ defined by

$$\sigma_A(\boldsymbol{u}) = \max_{\mathbf{p}\in A} \boldsymbol{u} \cdot \mathbf{p} = \max_{\mathbf{p}\in A} u_1(p_1) + \cdots + u_n(p_n).$$

Let $C(\mathcal{U})$ be the set of continuous real-valued functions on $\mathbb{A}$. Let $\Sigma = \{\sigma_A \in C(\mathcal{U}) | A \in \mathcal{A}\}$. Let $\langle \sigma, \boldsymbol{\mu} \rangle = \sum_{i\in\mathbb{A}} u_i \mu_i$. Let $C^*(\mathcal{U})$ be the set of all finite Borel *non-negative* measures on $\mathbb{A}$. The non-negativity of measures follows from the monotonicity of J.

Moreover, let us introduce the following notation. Let $\mathcal{M} \subset C^*(\mathcal{U})$ be the set of *weights* (non-negative measures) on the attribute space. We have the following claim: $\mathcal{M}(\boldsymbol{u}^*(A)) \subset C^*(\mathcal{U})$ is weak* compact (we apply the result of Ergin and Sarver (2010a)). Moreover, for any weak* compact $\mathcal{M} \subset C^*(\mathcal{U})$,

$$\widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \min_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{u}^*(A))} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big], \ \forall \sigma \in \Sigma.$$

By the definition of J, for all $A^* \in \mathcal{A}^*$,

$$V^*(A^*) = \max_{\boldsymbol{u}\in A^*} \sum_i u_i - \min_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{u}^*(A))} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big].$$

Now, we apply the result in Ergin and Sarver (2010b) (Corollary 2). For any $A^* \in \mathcal{A}^*$,

$$V^*(A^*) \text{ subject to } c(\boldsymbol{\mu}) \leq k,$$

for some $c : C^*(\mathcal{U}) \to \mathbb{R}$ and $k \in \mathbb{R}$. Then, let $\overline{\mathcal{M}}(\boldsymbol{u}^*(A)) = \{ \ \boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A)) \mid c(\boldsymbol{\mu}) \leq k \ \}$. Since c is lower semicontinuous (Ergin and Sarver (2010b)), $\overline{\mathcal{M}}(\boldsymbol{u}^*(A))$ is compact. Hence, the set of non-negative measures on $\mathcal{U}$ is obtained.

By the preceding steps, we obtain the following functional form. For any $A^* \in \mathcal{A}^*$,

$$V^*(A^*) = \max_{\boldsymbol{u}\in A^*} \sum_{i\in\mathbb{A}} u_i + \min_{\boldsymbol{\mu}\in\overline{\mathcal{M}}(\boldsymbol{u}^*(A))} \max_{\boldsymbol{u}\in A^*} \Big(\sum_{i\in\mathbb{A}} u_i \mu_i \Big).$$

For any $A \in \mathcal{A}$, define $V(A) = V^*(A^*)$. Then, we have $A \succeq B \Leftrightarrow A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*) \Leftrightarrow V(A) \geq V(B)$. By arranging the terms, we have the desired representation.

3.4 Characterization of Weights on Attribute Spaces

We study a comparative statics on the set of weights $\mathcal{M}$ on an objective attribute space $\mathbb{A}$. Remember that the set of weights $\mathcal{M}$ is *reference-dependent*, especially, *menu-dependent*. To study a comparative attitude toward weighting on attributes, fix an arbitrary menu $A \in \mathcal{A}$. Consider two decision makers: Mr.X and Mr.Y. Let $\succeq^j$ be the binary relation over $\mathcal{A}$ of Mr.j ($j \in \{X, Y\}$). Assume that the two decision makers have the same attribute functions, i.e., for all $i \in \mathbb{A}$, $u_i^X = u_i^Y$.

Definition 2. $\succeq^X$ exhibits a stronger contemplation-seeking than $\succeq^Y$ if, for all $A \in \mathcal{A}$ and $p \in A$,

$$A \succeq^Y \{p\} \Rightarrow A \succeq^X \{p\}.$$

This definition states that Mr.X prefers a menu A to an alternative p in the menu A, whenever Mr.Y does. Intuitively, each decision maker prefers making a choice with contemplation from the menu A to choosing any option in the menu A as a singleton.

Definition 3. $\mathcal{M}^X$ exhibits more extreme-aversion than $\mathcal{M}^Y$ if for any $\boldsymbol{\mu}^Y \in \mathcal{M}^Y$, there exists $\boldsymbol{\mu}^X \in \mathcal{M}^X$ such that for each $\mathbf{p} \in A$, $\sum_{i \in \mathbb{A}} \mu_i^X u_i(p_i) \geq \sum_{i \in \mathbb{A}} \mu_i^Y u_i(p_i)$.

Proposition 2. *Suppose that $\succeq^j$ ($j \in \{X, Y\}$) is represented by a pair $\langle \mathcal{U}, \mathcal{M}^j \rangle$. Then, the following statements are equivalent:*

(i) $\succeq^X$ exhibits a stronger contemplation-seeking than $\succeq^Y$.

(ii) $\mathcal{M}^X$ exhibits more extreme-aversion than $\mathcal{M}^Y$.

This result states that $\succeq^X$ exhibits a stronger contemplation-seeking than $\succeq^Y$ if and only if Mr.X has a weight on the attribute space which leads to extreme-averse more than that of Mr.Y.

3.5 Ex-Post Choices

To study ex-post choices, we investigate the pair $\langle \succeq, C \rangle$ where C is a choice correspondence. We say that a correspondence $C : \mathcal{A} \rightrightarrows \mathcal{X}$ is a choice correspondence if for all $A \in \mathcal{A}$, $C(A) \subseteq A$ and $C(A) \neq \varnothing$. We say that $\mathbf{p}$ is *ideal* in A if for all $\mathbf{q} \in A \setminus \{\mathbf{p}\}$, $p_i \succsim_i q_i$ for all $i \in \mathbb{A}$. Let

$$\mathcal{A}_{\mathbf{p}} := \{A \in \mathcal{A} \mid \mathbf{p} \text{ is } \textit{ideal} \text{ in } A\}.$$

The first axiom is a weaker version of WARP (*Weak Axiom of Revealed Preference*). This axiom states that if two menus A and B share the *ideal* option in terms of attribute-based inferences, then WARP holds. Under such two menus, the decision maker leads to the same level of contemplation in attribute-based inferences.

Axiom (WARP with Attributes): For any $A, B \in \mathcal{A}_{\mathbf{p}}$ and $\mathbf{p}, \mathbf{q} \in A \cap B$, if $\mathbf{p} \in C(A)$, and $\mathbf{q} \in C(B)$, then $\mathbf{p} \in C(B)$.

The second axiom is a standard continuity condition. Since $\mathcal{X}$ is compact, and $\mathcal{A}$ is endowed with the Hausdorff metric, the property of Closed Graph is equivalent to the *upper hemicontinuity* of C (Aliprantis and Border (2006)). It is postulated that C is continuous.

Axiom (Closed Graph): The set $\{(\mathbf{p}, A) \mid \mathbf{p} \in C(A) \text{ and } A \in \mathcal{A}\}$ is closed in $\mathcal{X} \times \mathcal{A}$.

The third condition is a consistency condition between the ex-ante menu preference $\succeq$ and the ex-post choice C. This axiom states that if $\mathbf{p}$ is strictly preferred at the ex-ante stage under $A \cup \{p\}$, then $\mathbf{p}$ should be chosen at the ex-post stage.

Axiom (Consistency): If for any $\mathbf{p} \in \mathcal{X}$ and $A \in \mathcal{A}$,

$$A \cup \{\mathbf{p}\} \succ A \Rightarrow C(A \cup \{\mathbf{p}\}) = \{\mathbf{p}\}.$$

Proposition 3. *Let $\succeq$ be represented by a pair $\langle \mathcal{U}, \mathcal{M} \rangle$. Then, a choice correspondence C satisfies WARP with Attributes, Closed Graph, and Consistency if and only if, for any $A \in \mathcal{A}$,*

$$C(A) = \arg\max_{\mathbf{p} \in A} \Big[\sum_{i \in \mathbb{A}} u_i(p_i) - \max_{\boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A))} \sum_{i \in \mathbb{A}} \mu_i \big(\max_{\mathbf{q} \in A} (u_i(q_i) - u_i(p_i)) \big) \Big].$$

4 Discussion, Literature Review, and Concluding Remarks

4.1 Subjective State Spaces

Kreps (1979) develops the framework of preferences over menus, and Dekel et al. (2001) generalize the framework. Kreps (1979) introduces the axiom of *Monotonicity*: $B \subseteq A \Rightarrow A \succeq B$. This axiom requires that the decision maker prefers larger menus, which reflects on *preferences for flexibility*. On the other hand, in this paper, to capture the aversion to trade-off across attribute-based evaluations, the axiom of *Dissatisfaction* is introduced.

To study a class of preferences for commitment, Gul and Pesendorfer (2001) introduce the axiom of *Set-Betweenness*: $A \succeq B \Rightarrow A \succeq A \cup B \succeq B$ to investigate a relationship between temptation and self-control. Stovall (2010) extends the theory of self-control preferences into multiple temptations by providing a weaker version *Set-Betweenness*. Sarver (2008) studies a theory of anticipated regret. Sarver (2008) introduces the axiom of *Dominance*: If $\{p\} \succeq \{q\}$ for some $p \in A$, then $A \succeq A \cup \{q\}$. In this paper, given an attribute space *exogenously*, a type of commitment preferences is introduced to study dissatisficing-averse preferences.

By imposing on the axiom of *Monotonicity*, we can develop a theory of attribute-based inferences, in which a state-dependent weight on an objective attribute space is elicited. Since the axiom of *Dissatisfaction* is not consistent with the axiom of *Monotonicity*, we need to rule out the axiom of *Dissatisfaction*, and consider a new axiom related to *preferences for flexibility*.[15] This is a future task.

[15] For example, we can consider "subjective" uncertainty under attribute-based inferences. If there is a trade-off between attributes, the decision maker may prefer larger menus to resolve the trade-off, by considering a lot of options.

4.2 Reasoned Choices

Decision-making with "contemplation" is related to reasoned choices. Gilboa (2009) refers to a relationship between raw preferences and reasoned choices. Raw preferences are tastes with no need for inferences (Zajonc (1980)). In attribute-based inferences, raw preferences are interpreted as follows: for all $\mathbf{p}, \mathbf{q} \in\in \mathcal{X}$,

$$(\forall i \in \mathbb{A})\ p_i \succsim_i q_i \Rightarrow \{\mathbf{p}\} \succeq \{\mathbf{q}\}.$$

This statement is obtained from the axiom of *Dominance*. If an option dominates another option, then the decision maker does not have to contemplate which one is desirable.

In general, there is often a trade-off across attribute-based evaluations. Consumers often face a problem of choosing a product under two options: one is an option $\mathbf{p}$ with high-prices but high-qualities and another is an option $\mathbf{q}$ with low-prices but low-qualities. Under this menu $\{\mathbf{p}, \mathbf{q}\}$, the decision maker needs a contemplation for decision-making. In this paper, the difference between raw preferences and reasoned choices may be captured by ex-post choices (observed resulting choice behaviors) such as preference reversals. Due to contemplation in decision-making process, weights on an objective attribute space are *reference-dependent*.

In the study of subjective state spaces, Ergin and Sarver (2010a) study a theory of contemplation in subjective state spaces. In their model, state spaces are endogenous, and the decision maker explores her own subjective state space with contemplation. Dillenberger et al. (2014) study a theory of subjective learning. In Dillenberger et al. (2014), an information signal on an objective state space is obtained for the decision maker, but it is not observable for the decision analyst. Since this paper considers an objective attribute space, a contemplation for attribute spaces, or a subjective learning for attribute spaces is a future task.

4.3 Preference Reversal: the Compromise Effect

Simonson (1989) reports that decision makers have a tendency to exhibit extremeness-aversion in attribute-based inferences. Such a behavioral regularity is called the *Compromise Effect*. We verify that the ex-post choice of the dissatisficing-averse utility representation can allow for the Compromise effect. To study the Compromise effect, consider the pair $\langle \succeq, C \rangle$ in the subsection 3.5. For simplicity, assume that an attribute space $\mathbb{A}$ has two attributes, i.e., $\mathbb{A} = \{1, 2\}$. Define the Compromise effect in the following.

Definition 4. Suppose that $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathcal{X}$ such that $q_1 \succ_1 p_1 \succ_1 r_1$ and $r_2 \succ_2 p_2 \succ_2 q_2$, and that $C(\{\mathbf{p}, \mathbf{r}\}) = \mathbf{p}$. Then, the pair $\langle \succeq, C \rangle$ exhibits the compromise effect if

$$\mathbf{q} \in C(\{\mathbf{p}, \mathbf{q}\}) \text{ and } \mathbf{p} \in C(\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}).$$

As mentioned above, since weights on an objective attribute space are *reference-dependent*, the contemplation might produce the Compromise effect. To study a relationship between the Compromise effect and the dissatisficing-averse utility representation, Notice that the set of weights where $\mathcal{M}$ depends on the *ideal point* of each menu defined by $\boldsymbol{u}^*(A) = (\max_{\mathbf{p}\in A} u_i(p_i))_{i\in\mathbb{A}}$.

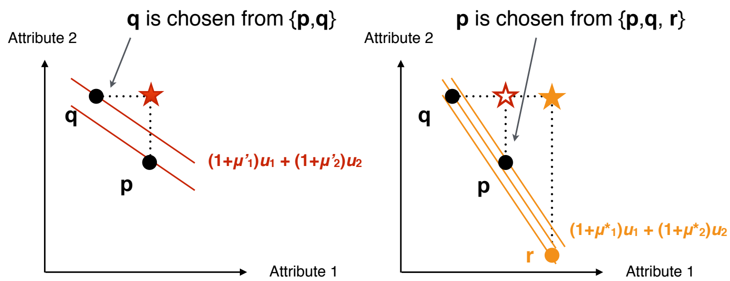

Figure 2: the Compromise Effect

Consider $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathcal{X}$ such that $q_1 \succ_1 p_1 \succ_1 r_1$ and $r_2 \succ_2 p_2 \succ_2 q_2$. Suppose that $C(\{\mathbf{p}, \mathbf{r}\}) = \{\mathbf{p}\}$. Then, $\{\mathbf{p}\} \succ \{\mathbf{r}\}$. Given a menu $\{\mathbf{p}, \mathbf{q}\}$, consider adding an option $\mathbf{r}$ into $\{\mathbf{p}, \mathbf{q}\}$. By the axiom of *Dissatisfaction*, $\{\mathbf{p}, \mathbf{q}\} \succeq \{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$. From the ex-ante preference $\succeq$, the different options can be chosen at the ex-post stage. That is, $C(\{\mathbf{p}, \mathbf{q}\}) = \{\mathbf{q}\}$, and $C(\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}) = \{\mathbf{p}\}$.

4.4 Attribute Spaces

In this paper, an attribute space is exogenously given. This makes it possible for us to consider plausible axioms on attribute-based inferences. However, in general, decision analysts cannot observe attribute spaces directly. Ok et al. (2015) elicits an endogenous attribute space by using the framework of revealed preference theory. Ok et al. (2015) also has an endogenous reference point that is one of feasible alternatives in a given menu. Their reference-dependent choice model allows for the Attraction effect. Notice that we provide the axiom of *Dominance*, which rules out the attraction effect. Since Ok et al. (2015) is not consistent with the compromise effect, there can be a different cognitive mechanism behind the Attraction and the Compromise effects. This is a future task to get a deeper understanding for the two effects.

4.5 Literature Review

In relation to the literature, this paper has the following advantages. First, we study preferences over menus, and present a theory of attribute-based inferences. The axiom of *Dissat-*

isfaction is closely related to the Compromise effect, by requiring that the decision maker is averse to the increase in the trade-off between attributes. Moreover, the axiom of *Dominance* rules out the Attraction effect. This implies that there is a different cognitive mechanism behind the Attraction effect. Second, this framework makes it possible to extend the further tasks such as intertemporal choices, stochastic choices, and so on. To access the contributions of this paper, we discuss literature review below.

Regret

The representation in this paper is a disatisficing-averse utility representation. We mention that a dissatisfaction stems from each attribute-best option in choice sets. In the theory of non-Expected Utility, the notion of *regret* is similar to that of *dissatisficing*. One remark is that regret stem from each state-contingent best act in choice sets. State spaces are different from attribute spaces, so the interpretations are also different. Such a regret occurs after a state is realized. On the other hand, a dissatisficing is driven by attribute-based inferences, which is different from (objective) state spaces.

In the main literature on regret aversion, Hayashi (2008) proposes an axiomatic model of decision-making under uncertainty in which the decision maker is driven by anticipated ex post regrets. Hayashi (2008) takes a choice function as a primitive. The axiomatic characterization is also different from the characterization in this paper. For other studies on regret, see the related literature of Hayashi (2008) therein.

Choice Theory

Recently, various axiomatic models have been developed to explain the Attraction effect or the Compromise effect, or both, by relaxing WARP.

Ok et al. (2015) propose a reference-dependent model that allows for the Attraction effect, but not the Compromise effect. In their model, given a choice set A, the decision maker maximizes a utility function by forming a consideration set intentionally or not. In their model, a feasible alternative in choice sets can be a reference point, and the decision maker chooses the best alternative in terms of the utility function u from the consideration set that is better than the reference point. This cognitive mechanism is different from this paper. First, in this paper, the reference point is interpreted as an *ideal* option in each choice set. Second, in this paper, the utility function u in Ok et al. (2015) corresponds to the aggregation of attribute-based functions $(u_i)_{i\in\mathbb{A}}$ with the weight $\boldsymbol{\mu}$. In this paper, the decision maker explores the optimal weight on the attribute space. Compared with Ok et al. (2015), the different procedure is captured. Ok et al. (2015) pays much attention to attention.

de Clippel and Eliaz (2012) and Tserenjigmid (2017) are consistent with both the Attraction and the Compromise effects. de Clippel and Eliaz (2012) consider the difficulty in the trade-off between attributes. They propose the reason-based choice model to view the resolution of the trade-off as a cooperative solution to an *intra*personal bargaining problem among attributes. Since this paper takes a different framework, and imposes on the axiom of *Dominance*, this paper rules out the Attraction effect. To explain the Compromise effect, we provide the axiom of *Dissatisfaction*, which requires that the decision maker dislikes the increase in the trade-off between attributes, when adding an option. Compared with de Clippel and Eliaz (2012), this paper has a richer structure, and the decision maker evaluates options involving a notion of *intensity*. The intensity stems from the aversion to the increase in the trade-off between attributes, which leads to dissatisficing-aversion.

Tserenjigmid (2017) proposes a reference-dependent model, in which the reference point is determined as a minimum of each attribute of choice sets. The decision maker takes such a reference point, and exhibits a non-linear reaction from the reference point. Tserenjigmid (2017) show an "equivalence" between the Attraction/Compromise effects and the diminishing sensitivity. Compared with Tserenjigmid (2017), the position of reference points is different. In this paper, the *ideal* option of choice sets is perceived as a reference point. Also, this paper requires that the decision maker satisfies *Dominance*, so the Attraction effect is ruled out. Under the framework of preferences over menus, it is not clear how reference points are formed, including a minimum and a maximum of each attribute of choice sets. We need to consider a plausible axiom to capture a formation of such reference points.

4.6 Concluding Remarks

This paper studies attribute-based inferences in the framework of preferences over menus. To capture the trade-off across attributes, we have introduced new axioms of *Dominance*, *Dissatisfaction*, and *Contemplation* (Section 2). Attribute-based inferences capture a relationship between raw preferences and reasoned choices (Subsection 4.2). The ex-post choices of the dissatisficing-averse utility representation are characterized by a weaker version of WARP (Subsection 3.5). The ex-post choices of the dissatisficing-averse utility representation allow for the Compromise effect.

There are some further tasks. First, we consider the axiom of *Dominance*, which rules out the Attraction effect. This implies that there is a different cognitive mechanism behind the Attraction effect. Interestingly, Ok et al. (2015) develop a reference-dependent model to be consistent with the Attraction effect. The reference-dependence stems from limited attention. However, Simonson (1989) argues that both the Attraction effect and the Compromise effect may stem from the same heuristic related to reason-based choices. To get a deeper understanding for these behavioral regularities, we need much evidence about them. The experimental study to explore the possibility to distinguish the cognitive mechanism behind the Compromise effect from that of the Attraction effect is a future task.

Second, as mentioned in Introduction, there are mainly two steps for decision-making under attribute-based inferences. In this paper, we have focused on the resolution of the trade-off between attributes in the latter process. We have assumed that an attribute space is exogenously given. The next issue is to study the dynamics of attribute spaces; that is, we take into account that the decision maker's focus on attributes may change dynamically. Since taking all attributes into accoing is cognitively demanding, the focus on attributes can change with history-dependence. This process should be studied in intertemporal choices. This is also a future task.

A Proof of Theorem 1

A.1 Sufficiency Part

We show the sufficiency part. Suppose that $\succeq$ satisfies the axioms in the main theorem (Theorem 1).

Step 1

In Step 1, first, we show that each induced binary relation $\succsim_i$ on $\Delta(X_i)$, for each $i \in \mathbb{A}$, is *well-defined*. We use the notation $\mathcal{X}_i := \Delta(X_i)$ for each $i \in \mathbb{A}$. Next, we show that $\succsim_i$ on $\Delta(X_i)$ ($i \in \mathbb{A}$) satisfies the axiom of *Independence* in the vNM-type expected utility (vNM-type EU) theorem. Finally, we represent the first term of the dissatisficing-averse utility representation.

Remember that for each $i \in \mathbb{A}$, we define $\succsim_i$ on $\mathcal{X}_i$ as follows. Fix $i \in \mathbb{A}$. The asymmetric and symmetric part of $\succsim_i$ are described by $\succ_i$ and $\sim_i$, respectively. For all $p_i, q_i \in \mathcal{X}_i$, we say that $p_i \succsim_i q_i$ if

$$\{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\},$$

for some $r_{-i} \in \mathcal{X}_{-i}$. We show that each $\succsim_i$ is *well-defined.* We show that $p_i \succsim_i q_i$ if $\{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\}$ for *any* $r_{-i} \in \mathcal{X}_{-i}$. Take $p_i, q_i \in \mathcal{A}_i$, and consider some $r_{-i} \in \mathcal{X}_{-i}$ such that $\{(p_i, r_{-i})\} \succeq \{(q_i, r_{-i})\}$. Suppose that there exists $r'_{-i} \in \mathcal{X}_{-i}$ such that $\{(p_i, r'_{-i})\} \prec \{(q_i, r'_{-i})\}$. Take $\lambda \in (0, 1)$, and consider $\lambda r_{-i} + (1-\lambda) r'_{-i}$. If $\{(p_i, \lambda r_{-i} + (1-\lambda) r'_{-i})\} \prec \{(q_i, \lambda r_{-i} + (1-\lambda) r'_{-i})\}$, then $p_i \prec_i q_i$. This is a contradiction. Hence, $\succsim_i$ is *well-defined*, for each $i \in \mathbb{A}$.

We show that for each $i \in \mathbb{A}$, $\succsim_i$ satisfies the axiom of *Independence* in the expected utility theorem (EUT). Consider $\succsim_i$. By the axiom of *Standard Preferences*, $\succeq$ satisfies the axioms of *Completeness*, *Transitivity*, and *Continuity*, so it is easily verified that $\succsim_i$ satisfies *Completeness*, *Transitivity*, and *Mixture Continuity.* We show that $\succsim_i$ satisfies the axiom of *Independence*: For any p_i, q_i, r_i and $\lambda \in [0, 1]$, $p_i \succsim_i q_i$ if and only if $\lambda p_i + (1-\lambda) r_i \succsim_i \lambda q_i + (1-\lambda) r_i$.

Fix $p_i, q_i, r_i \in \Delta(X_i)$ and $\lambda \in [0, 1]$. Then, for any $p_{-i}, q_{-i} \in \mathcal{X}_{-i}$,

$$\begin{aligned}
p_i \succsim_i q_i \quad &\Leftrightarrow \quad \{(p_i, p_{-i})\} \succeq \{(q_i, p_{-i})\} \\
&\Leftrightarrow \quad \lambda\{(p_i, p_{-i})\} + (1-\lambda)\{(r_i, q_{-i})\} \succeq \{\lambda(q_i, p_{-i}) + (1-\lambda)(r_i, q_{-i})\} \\
&\Leftrightarrow \quad \{(\lambda p_i + (1-\lambda) l_i, \lambda p_{-i} + (1-\lambda) q_{-i})\} \succeq \{(\lambda q_i + (1-\lambda) l_i, \lambda p_{-i} + (1-\lambda) q_{-i})\} \\
&\Leftrightarrow \quad \lambda p_i + (1-\lambda) r_i \succsim_i \lambda q_i + (1-\lambda) r_i
\end{aligned}$$

It is shown that $\succsim_i$ satisfies the axiom of *Independence.* In the same way, we can show that each $\succsim_j$, $j \in \mathbb{A}$, satisfies the axiom of *independence.*

By the von Neumann-Morgenstern's Expected Utility Theorem (EUT), for each $i \in \mathbb{A}$, there exists a continuous and mixture linear utility function $u_i : \Delta(X_i) \to \mathbb{R}$ which represents $\succsim_i$. We can apply the result of Theorem 13 (Chapter 6) in Krantz et al. (1971), i.e., the n-component additive conjoint structure.

Consider the compact set of singleton menus $\mathcal{A}_s \subseteq \mathcal{A}$ where s means singletons. It is equivalent to consider a binary relation $\succsim$ on $\mathcal{X}$. Then, it is easily shown that the primitive of this paper $\succeq$ satisfies Definition 13 (p.301) in Krantz et al. (1971). By the axiom of *Standard Preferences*, i.e., (i) Weak Order, $\succeq$ on $\mathcal{A}_s$ satisfies the first condition (Weak Ordering) in Definition 13. By the axiom of *Separability*, $\succeq$ on $\mathcal{A}_s$ satisfies the second condition (Independence) in Definition 13. By the axiom of *Standard Preferences*, i.e., (ii) Continuity and (iii) Strict Non-Degeneracy, since $\mathcal{X}_i$ is compact, $\succeq$ on $\mathcal{A}_s$ satisfies the third condition in Definition 13. By the axiom of *Standard Preferences*, i.e., (ii) Continuity, $\succeq$ on $\mathcal{A}_s$ satisfies the forth condition in Definition 13. By the axiom of *Standard Preferences*, i.e., (ii) Continuity and (iii) Strict Non-Degeneracy, since $\mathcal{X}_i$ is compact, $\succeq$ on $\mathcal{A}_s$ satisfies the fifth condition in Definition 13. Hence, we obtain the first term of the desired representation, i.e., for all $\mathbf{p} \in \mathcal{X}$, $\succeq$ on $\mathcal{A}_s$ is represented by $U(\mathbf{p}) = \sum_{i \in \mathbb{A}} u_i(p_i)$.

Notice that $\mathcal{A}$ is *connected* and *separable*. Since $\succeq$ is a continuous weak order, by Debreu (1959), there exists a continuous function $V : \mathcal{A} \to \mathbb{R}$ such that, for any $A, B \in \mathcal{A}$, $A \succeq B \Leftrightarrow V(A) \geq V(B)$. Notice that, for all $\mathbf{p} \in \mathcal{X}$, $V(\{\mathbf{p}\}) = U(\mathbf{p}) = \sum_{i \in \mathbb{A}} u_i(p_i)$.

Step 2

In Step 2, by mainly using the axiom of *Contemplation*, we show that a utility of a menu has certain desired properties (Lemma 4). First, we introduce a set of utilities of option on an attribute-based utility space. Next, we introduce the axiom of *Translation Invariance*. Finally, by using this axiom, we show that a utility of a menu has certain properties.

We consider a set of utilities of options on a utility space in each menu A. To do so, by Step 1, we can use the property of positive affine transformations for each u_i. Without loss of generality, consider $u_i : \mathcal{X}_i \to \mathbb{R}_+$ for each $i \in \mathbb{A}$. For any $A \in \mathcal{A}$, define

$$\boldsymbol{u}(A) := \Big\{ \Big(\frac{u_1(p_1)}{\sum_{i \in \mathbb{A}} u_i(p_i)}, \cdots, \frac{u_n(p_n)}{\sum_{i \in \mathbb{A}} u_i(p_i)} \Big) \in \mathbb{R}^n \mid \mathbf{p} = (p_1, \cdots, p_n) \in A \Big\}.$$

Let $\{\boldsymbol{u}(A) \mid A \in \mathcal{A}\}$. Notice that each $A \in \mathcal{A}$ is compact. Then, $\boldsymbol{u}(A)$ is also compact by the continuity of u_i ($i \in \mathbb{A}$). $\mathcal{A}^*$ is a set of compact subsets of $\mathbb{R}^n$, endowed with the Hausdorff metric d_h. Define $\succeq^*$ on $\mathcal{A}^*$ in the following way:

$$A^* \succeq^* B \text{ if } A \succeq B,$$

where $A^* = \boldsymbol{u}(A)$ and $B^* = \boldsymbol{u}(B)$. The asymmetric and symmetric parts of $\succeq^*$ are denoted by $\succ^*$ and $\sim^*$, respectively.

First, we show that $\succeq^*$ is *well-defined.*

Lemma 1. $\succeq^*$ *is well-defined.*

Proof. Suppose $A^* = B^*$, i.e., $\boldsymbol{u}(A) = \boldsymbol{u}(B)$. We show that $A \sim B$. Then, for any $\mathbf{p} \in A$ there exists $\mathbf{q} \in B$ such that for all $i \in \mathbb{A}$, $p_i \sim_i q_i$. By the axiom of *Dominance*, we have $A \sim A \cup B$. In the same way, we have $B \sim A \cup B$. By the axiom of *Transitivity*, we obtain $A \sim B$. □

Consider the axioms in Theorem 1 in the above attribute-based utility space. We show that $\succeq^*$ satisfies the following axioms. The following axioms are introduced.

Axiom* (Dominance*) : For any $\boldsymbol{v} \in B^*$ there exists $\boldsymbol{u} \in A^*$ such that $u_1 > v_1$ and $u_S > v_S$, then $A^* \sim^* A^* \cup B^*$.

Axiom* (Dissatisfaction*) : For any $\boldsymbol{v} \in \mathbb{R}^n$, if there exists $i, j \in \mathbb{A}$ such that $u_i > v_i$ and $u_j < v_j$, and $\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\}$ for some $\boldsymbol{u} \in A^*$, then

$$A^* \succeq^* A^* \cup \{\boldsymbol{v}\}.$$

Axiom* (Contemplation*) : $\succeq^*$ satisfies the following two conditions:

(i) (No Need for Contemplation*): For any $A^*, B^* \in \mathcal{A}^*$, $\boldsymbol{u} \in \mathbb{R}^n$, and $\lambda \in [0, 1]$,

$$A^* \succeq^* B^* \Rightarrow \lambda A^* + (1 - \lambda)\{\boldsymbol{u}\} \succeq^* \lambda B^* + (1 - \lambda)\{\boldsymbol{u}\}.$$

(ii) (Contemplation Seeking*): If there does not exist $\boldsymbol{u} \in A^*$ such that for any $\boldsymbol{v} \in B^*$, $u_i > v_i$ for all $i \in \mathbb{A}$, then, for any $\lambda \in [0, 1]$,

$$A^* \succeq^* B^* \Rightarrow \lambda A^* + (1 - \lambda)B^* \succeq A^*.$$

We have the following lemma.

Lemma 2. $\succeq^*$ *is a continuous weak order that satisfies Dominance*, Dissatisfaction*, and Contemplation*.*

We omit the proof of Lemma 2. Since we suppose that $\succeq$ satisfies the axioms in Theorem 1, by the definition of $\succeq^*$, it is easily verified that $\succeq^*$ satisfies the axioms in the attribute-based utility space.

We define the set of *translations* in the following way.

$$\Theta := \left\{ \theta \in \mathbb{R}^n \mid \sum_{i=1}^{n} \theta_i = 0 \right\}.$$

For any $A^* \in \mathcal{A}^*$ and $\theta \in \Theta$, define $A^* + \theta := \{\boldsymbol{u} + \theta \mid \boldsymbol{u} \in A^* \}$. We introduce the axiom of *Translation Invariance*.

Axiom* (Translation Invariance*): For any $A^*, B^* \in \mathcal{A}^*$ and $\theta \in \Theta$ such that $A^*+\theta, B^*+\theta \in \mathcal{A}^*$,

$$A^* \succeq^* B^* \Rightarrow A^* + \theta \succeq^* B^* + \theta.$$

Lemma 3. *$\succeq^*$ satisfies No Need for Contemplation* if and only if it satisfies Translation Invariance*.*

The proof of Lemma 3 is in the Proposition 1 of Ergin and Sarver (2010b). We omit it.

We construct a value function $V^* : \mathcal{A}^* \to \mathbb{R}$ that represents $\succeq^*$ on $\mathcal{A}^*$. We say that V^* is *translation linear* if for all $A^* \in \mathcal{A}$ and $\theta \in \Theta$, there exists $v \in \mathbb{R}^n$ such that

$$V^*(A^* + \theta) = V^*(A^*) + v \cdot \theta.$$

We verify that V^* has certain properties.

Lemma 4. *If $\succeq^*$ is a continuous weak order that satisfies Dominance*, Dissatisfaction*, and Contemplation*, then there exists $V^* : \mathcal{A}^* \to \mathbb{R}$ with the following propeties:*

(i) For any $A^*, B^* \in \mathcal{A}^*$, $A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*)$.

(ii) V^* is continuous, concave, and translation linear.

(iii) For any $A^*, B^* \in \mathcal{A}^*$, if $V^*(A^*) \geq V^*(B^*) \Leftrightarrow V'^*(A^*) \geq V'^*(B^*)$, then there exists $a > 0$ and $b \in \mathbb{R}$ such that $V'^* = aV^* + b$.

To show Lemma 4, we show the following claims (Claim 1-9). Before proceeding, define the following subset of $\mathcal{A}^*$.

$$\mathcal{A}^\circ := \{\ A^* \in \mathcal{A}^* \mid \forall\theta \in \Theta\ \exists\alpha > 0 \text{ such that } A^* + \alpha\theta \in \mathcal{A}^*\ \}.$$

Claim 1. *Suppose that $\succeq^*$ is a continuous weak order that satisfies Translation Invariance*. If $A^* \in \mathcal{A}^*$, $\theta \in \Theta$, and $\alpha \in (0, 1)$, then*

$$A^* \succeq^* A^* + \theta \Leftrightarrow A^* \succeq^* A^* + \alpha\theta \Leftrightarrow A^* + \alpha\theta \succeq^* A^* + \theta.$$

Proof. There exists $m, n \in \mathbb{N}$ with $m < n$ such that $A^* + \frac{m-1}{n}\theta \succeq^* A^* + \frac{m}{n}\theta$. Taking $\frac{1}{n}\theta$, by the axiom of *Translation Invariance*, we have $A^* + \frac{m}{n}\theta \succeq^* A^* + \frac{m+1}{n}\theta$. Suppose that $A^* \succeq^* A^* + \frac{1}{n}\theta$. By the axiom of *Transitivity* in *Standard Preference*,

$$A^* \succeq^* A^* + \frac{1}{n}\theta \succeq^* \cdots \succeq^* A^* + (1 - \frac{1}{n})\theta \succeq^* A^* + \theta.$$

Conversely, if $A^* \prec^* A^* + \frac{1}{n}\theta$, then

$$A^* \prec^* A^* + \frac{1}{n}\theta \prec^* \cdots \prec^* A^* + (1 - \frac{1}{n})\theta \prec^* A^* + \theta.$$

Hence, for any $m, n \in \mathbb{N}$ with $1 \leq m < n$,

$$\begin{aligned} A^* \succeq^* A^* + \frac{1}{n}\theta \quad &\Leftrightarrow \quad A^* \succeq^* A^* + \theta \\ &\Leftrightarrow \quad A^* \succeq^* A^* \frac{m}{n}\theta \\ &\Leftrightarrow \quad A^* + \frac{m}{n}\theta \succeq^* A^* + \theta. \end{aligned}$$

This holds for $\alpha \in (0,1) \cap \mathbb{Q}$. By the axiom of *Continuity* in *Standard Preferences*, we can show that this holds for all $\alpha \in (0,1)$. □

Consider the following axiom, a weaker version of *No Need for Contemplation**.

Axiom* (Strong Singleton Independence*): For any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\tau} \in \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\} \Leftrightarrow \lambda\{\boldsymbol{u}\} + (1-\lambda)\{\boldsymbol{\tau}\} \succeq^* \lambda\{\boldsymbol{v}\} + (1-\lambda)\{\boldsymbol{\tau}\}.$$

Claim 2. *Suppose that $\succeq^*$ satisfies Completeness*, Transitivity*, Continuity*, and Translation Invariance*, then it also satisfies Strong Singleton Independence*.*

Proof. Take $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ with $\boldsymbol{u} - \boldsymbol{v} = \theta \in \Theta$, and take $\boldsymbol{\tau} \in \mathbb{R}^n$ with $\theta' = (1-\lambda)(\boldsymbol{\tau} - \boldsymbol{u})$. Suppose the following.

$$\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\} \Leftrightarrow \{\boldsymbol{u}\} \succeq^* \{\boldsymbol{u}\} + \theta.$$

By Claim 1, $\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{u}\} + \lambda\theta \Leftrightarrow \{\boldsymbol{u}\} \succeq^* (1-\lambda)\{\boldsymbol{u}\} + \{\boldsymbol{v}\}$. By the axiom of *Translation Invariance** with θ', $\{\boldsymbol{u}\} + \theta' = \lambda\{\boldsymbol{u}\} + (1-\lambda)\{\boldsymbol{\tau}\}$. And, $(1-\lambda)\{\boldsymbol{u}\} + \{\boldsymbol{v}\} = \lambda\{\boldsymbol{v}\} + (1-\lambda)\{\boldsymbol{\tau}\}$. Hence, we have

$$\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\} \Leftrightarrow \lambda\{\boldsymbol{u}\} + (1-\lambda)\{\boldsymbol{\tau}\} \succeq^* \lambda\{\boldsymbol{v}\} + (1-\lambda)\{\boldsymbol{\tau}\}.$$

By using the axiom of *Transitivity* in *Standard Preferences*, we can show that for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\tau} \in \mathbb{R}^n$,

$$\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\} \Leftrightarrow \lambda\{\boldsymbol{u}\} + (1-\lambda)\{\boldsymbol{\tau}\} \succeq^* \lambda\{\boldsymbol{v}\} + (1-\lambda)\{\boldsymbol{\tau}\}.$$

□

We show the following claim for translations.

Claim 3. *Suppose that $\succeq^*$ satisfies Completeness*, Transitivity*, Continuity*, and Translation Invariance*. Let $A^*, B^* \in \mathcal{A}^\circ$ with $A^* \sim^* B^*$ and $\alpha, \beta \in \mathbb{R}$ with $A^* + \alpha\theta, B^* + \beta\theta \in \mathcal{A}^*$. Then,*

$$A^* + \alpha\theta \succeq^* B^* + \beta\theta \Leftrightarrow \alpha \geq \beta.$$

Proof. Take $\boldsymbol{u}^*, \boldsymbol{u}_* \in [0,1]^n$ with $\boldsymbol{u}^* - \boldsymbol{u}_* \in \Theta$. Fix it. Let $\theta^* = \boldsymbol{u}^* - \boldsymbol{u}_*$. First, we show that for any $A^* \in \mathcal{A}^*$ there exists $\alpha > 0$ such that $A^* + \alpha\theta^* \succ^* A^*$. Fix $A^* \in \mathcal{A}^*$. We can find out that there exists $A^{*'} \in \mathcal{A}^\circ$ and $\alpha > 0$ such that $A^* = (1-\alpha)A^{*'} + \alpha\{\boldsymbol{u}_*\}$. By the axiom of *Continuity** and θ^*,

$$A^* + \alpha\theta^* = (1-\alpha)A^{*'} + \alpha\{\boldsymbol{u}^*\} \succ^* (1-\alpha)A^{*'} + \alpha\{\boldsymbol{u}_*\} = A^*.$$

This holds for any $\alpha > 0$ with $A^* + \alpha\theta^* \in \mathcal{A}^*$. We obtain $A^* + \alpha\theta^* \succ^* A^*$. In the same way, we can show that if $A^* \in \mathcal{A}^\circ$ and $\alpha < 0$ with $A^* + \alpha\theta \in \mathcal{A}^*$, then we have $A^* \succ^* A^* + \alpha\theta^*$.

Let us move on to the case $A^* \sim^* B^*$ and $A^* + \alpha\theta^*, B^* + \beta\theta^* \in \mathcal{A}^*$. We show that the statement in Claim 3 holds. First, consider the case of $\alpha = \beta$. Then, we have $\alpha\theta^* = \beta\theta^*$. By the axiom of *Translation Invariance**, $A^* + \alpha\theta^* \sim^* B^* + \beta\theta^*$.

Next, consider the case of $\alpha > \beta$. First, suppose $\alpha > \beta \geq 0$. Then, $0 < \alpha - \beta \leq \alpha$. We have

$$A^* + \alpha\theta^* = [A^* + (\alpha - \beta)\theta^*] + \beta\theta^* \succ^* B^* + \beta\theta^*.$$

Next, suppose $0 \geq \alpha > \beta$. Then, in the same way, we have $\beta \leq \beta - \alpha < 0$. Then, $B^* + (\beta - \alpha)\theta^* \in \mathcal{A}^*$. Hence,

$$A^* \sim^* B^* \succ^* B^* + (\beta - \alpha)\theta^*.$$

By the axiom of *Translation Invariance**, $A^* + \alpha\theta^* \succ^* [B^* + (\beta - \alpha)\theta^*] + \alpha\theta^* = B^* + \beta\theta^*$. Third, suppose $\alpha > 0 > \beta$. Then, we obtain

$$A^* + \alpha\theta^* \succ^* A^* \sim^* B^* \succ^* B^* + \beta\theta^*.$$

By the three cases, the case of $\alpha > \beta$ is shown.

Finally, consider the case of $\beta > \alpha$. This case is also shown in the same way. We omit the proof of this case. □

Let us introduce some notation for the preceding claims. Take $\boldsymbol{u}^*, \boldsymbol{u}_* \in \mathbb{R}^n$ with $\boldsymbol{u}^* - \boldsymbol{u}_* \in \Theta$. Fix them. Let $\theta^* = \boldsymbol{u}^* - \boldsymbol{u}_*$. Consider the set of singleton menus $\mathcal{A}^*_s \subset \mathcal{A}^*$. Let

$$\mathcal{A}^*_s := \{\ \{\boldsymbol{u}\} \mid \boldsymbol{u} \in \mathbb{R}^n\ \}.$$

Then, let $\mathcal{A}^*_0 := \mathcal{A}^\circ \cap \mathcal{A}^*_s$. Define, for all $i \geq 0$,

$$\overline{\mathcal{A}}^*_i := \{\ A^* \in \mathcal{A}^\circ \mid A^* \sim^* B^* \text{ for some } B^* \in \mathcal{A}^*_i\ \},$$

and,

$$\mathcal{A}^*_i := \{\ A^* \in \mathcal{A}^\circ \mid A^* = B^* + \alpha\theta^* \text{ for some } \alpha \in \mathbb{R} \text{ and } B^* \in \overline{\mathcal{A}}^*_i\ \}.$$

Note that $\mathcal{A}^*_0 \subset \overline{\mathcal{A}}^*_0 \subset \mathcal{A}^*_1 \subset \overline{\mathcal{A}}^*_1 \subset \cdots$. Define, for all $i \geq 0$, $V^*_i : \mathcal{A}^*_i \to \mathbb{R}$, and $\overline{V}^*_i : \overline{\mathcal{A}}^*_i \to \mathbb{R}$ by

(i) let $V_0^* := v|_{\mathcal{A}_0^*}$;

(ii) for $i \geq 0$, if $A^* \in \overline{\mathcal{A}}_i^*$, then $A^* \sim^* B^*$ for some $B^* \in \mathcal{A}^*$. Define $\overline{V}_i^*(A^*) := V_i^*(B^*)$; and

(iii) for $i \geq 1$, if $A^* \in \mathcal{A}_i^*$, then $A^* = B^* + \alpha\theta^*$ for some $\alpha \in \mathbb{R}$ and $B^* \in \overline{\mathcal{A}}_{i-1}^*$. Define V_i^* by $V_i^*(A^*) := \overline{V}_{i-1}^*(B^*) + \alpha(v \cdot \theta^*)$.

Claim 4. *The following statements hold:*

(i) If $A^* \in \mathcal{A}_i^*$ and $\theta \in \Theta$, then there exists $\overline{\alpha} > 0$ such that $A^* + \alpha\theta \in \mathcal{A}_i^*$ for all $\alpha \in [0, \overline{\alpha}]$.

(ii) For any $A^*, B^* \in \mathcal{A}_i^*$, $C^* \in \mathcal{A}^\circ$, if $A^* \succeq^* C^* \succeq^* B^*$, then $C^* \in \overline{\mathcal{A}}_i^*$.

Proof. We show (i). By the definition of $\mathcal{A}^\circ$, we obtain the following: for any $A^* \in \mathcal{A}_0^*$ and $\theta \in \Theta$, there exists $\overline{\alpha} > 0$ such that $A^* + \alpha\theta \in \mathcal{A}_0^*$ for all $\alpha \in [0, \overline{\alpha}]$. We show (i) by induction. Remember that $\mathcal{A}_0^* = \mathcal{A}^\circ \cap \mathcal{A}_s^*$. Take $A^* \in \mathcal{A}_0^*$. Recall that, by definition, $A^* = \{\boldsymbol{u}\}$. Take $\overline{\alpha} > 0$ such that the statement in (i) holds. Then, for all $\alpha \in [0, \overline{\alpha}]$, $\boldsymbol{u} + \alpha\theta \in \mathbb{R}^n$. This implies $A^* + \alpha\theta \in \mathcal{A}_0^*$. Suppose that this property holds for $\mathcal{A}_i^*$. Then, we show that it also holds for $\mathcal{A}_{i+1}^*$. Take $A^* \in \mathcal{A}_{i+1}^*$ and $\theta \in \Theta$. Then, there exists $B^* \in \overline{\mathcal{A}}_i^*$ and $\beta \in \mathbb{R}$ such that $A^* = B^* + \beta\theta^*$. Hence, $B^* \sim^* C^*$ for some $C^* \in \mathcal{A}_i^*$. Choose $\overline{\alpha} > 0$ that satisfies the statement in (ii) for A^* and B^*, and that satisfies the statement in (i) for C^*. Fix $\alpha \in [0, \overline{\alpha}]$. Then, $C^* + \alpha\theta \in \mathcal{A}_i^*$. Thus, we have $B^* + \alpha\theta + \beta\theta^* = A^* + \alpha\theta \in \mathcal{A}_{i+1}^*$.

We show (ii). We prove it by induction. Begin with $\overline{\mathcal{A}}_0^*$. Suppose $A^*, B^* \in \overline{\mathcal{A}}_0^*$ and $A^* \succeq^* C^* \succeq^* B^*$ for some $C^* \in \mathcal{A}^\circ$. First, we show $C^* \in \overline{\mathcal{A}}_0^*$. Since $A^*, B^* \in \overline{\mathcal{A}}_0^*$, there exists $\{\boldsymbol{u}\}, \{\boldsymbol{v}\} \in \overline{\mathcal{A}}_0^*$ such that $\{\boldsymbol{u}\} \sim^* A^* \succeq^* C^* \succeq^* B^* \sim^* \{\boldsymbol{v}\}$. By the axiom of *Continuity**, $\{\lambda\boldsymbol{u} + (1-\lambda)\boldsymbol{v}\} \sim^* C^*$. By the convexity of $\mathcal{A}_0^* = \mathcal{A}^\circ \cap \mathcal{A}_s^*$, and the definition of $\overline{\mathcal{A}}_0^*$, we have $C^* \in \overline{\mathcal{A}}_0^*$. Next, we show the following: suppose that $\overline{\mathcal{A}}_i^*$ satisfies the statement in (ii). Then, $\overline{\mathcal{A}}_{i+1}^*$ does. Suppose $A^*, B^* \in \overline{\mathcal{A}}_{i+1}^*$ and $A^* \succeq^* C^* \succeq^* B$ for some $C^* \in \mathcal{A}^\circ$. We show $C^* \in \overline{\mathcal{A}}_{i+1}^*$. If there exist $A^*, B^* \in \overline{\mathcal{A}}_i^*$ such that $A^* \succeq^* C^* \succeq^* B$, then $C^* \in \overline{\mathcal{A}}_i^* \subset \overline{\mathcal{A}}_{i+1}^*$. Without loss of generality, suppose $C^* \succ^* A^{*'}$ for all $A^{*'} \in \overline{\mathcal{A}}_i^*$. Since $A^* \in \overline{\mathcal{A}}_{i+1}^*$, there exists $A^{*'} \in \mathcal{A}_{i+1}^*$ such that $A^{*'} \sim^* A^*$. Since $A^{*'} \in \overline{\mathcal{A}}_{i+1}^*$, there exists $\alpha \in \mathbb{R}$ and $A^{*''} \in \overline{\mathcal{A}}_i^*$ such that $A^{*''} + \alpha\theta^* \succeq^* C^* \succ^* A^{*''}$. By Claim 3, $\alpha > 0$. By the axiom of *Continuity**, there exists $\alpha' \in [0, \overline{\alpha}]$ such that $A^{*''} + \alpha'\theta^* \sim^* C^*$. However, $A^{*''} + \alpha'\theta^* \in \mathcal{A}_{i+1}^*$. Then, it must be $C^* \in \overline{\mathcal{A}}_{i+1}^*$. □

In the following two claims, we show that for each $i \geq 0$, V_i^* and $\overline{V}_i^*$ have desired properties.

Claim 5. *For all $i \geq 0$, if V_i^* is (i) well-defined, (ii) translation linear, and (iii) $\succeq^*$ on $\mathcal{A}_i^*$ is represented by V_i^*, then $\overline{V}_i^*$ is (i) well-defined, (ii) translation linear, and (iii) $\succeq^*$ on $\overline{\mathcal{A}}_i^*$ is represented by $\overline{V}_i^*$.*

Proof. We show (i). Suppose that $A^* \in \overline{\mathcal{A}}_i^*$, and $B^*, B^{*'} \in \mathcal{A}^*$ such that $A^* \sim^* B^*$ and $A^* \sim^* B^{*'}$. Since V_i^* represents $\succeq^*$ on $\mathcal{A}_i^*$ and $\succeq^*$ is *transitive*, we have $V_i^*(B^*) = V_i^*(B^{*'})$. Hence, $\overline{V}_i^*(A^*)$ is uniquely identified.

We show (iii). If $A^*, A^{*'} \in \overline{\mathcal{A}}_i^*$, then there exist $B^*, B^{*'} \in \mathcal{A}_i^*$ such that $A^* \sim^* B^*$ and $A^{*'} \sim^* B^{*'}$. Therefore,

$$\begin{aligned} \overline{V}_i^*(A^*) = V_i^*(B^*) \quad &\geq \quad V^*(B^{*'}) = \overline{V}_i^*(A^{*'}) \\ \Leftrightarrow B^* \quad &\succeq^* \quad B^{*'} \\ \Leftrightarrow A^* \quad &\succeq^* \quad A^{*'}. \end{aligned}$$

Thus, $\overline{V}_i^*$ represents $\succeq^*$ on $\overline{\mathcal{A}}_i^*$.

We show (ii). By (ii) of Claim 4, we have the following fact.

Fact. If $\theta \in \Theta$ and $A^*, A^* + \theta \in \overline{\mathcal{A}}_i^*$, then $A^* + \alpha\theta \in \overline{\mathcal{A}}_i^*$ for all $\alpha \in [0, 1]$.

Then, we consider a weaker version of *translation linearity*: For any $A^* \in \overline{\mathcal{A}}_i^*$, $\theta \in \Theta$ with $A^* + \theta \in \overline{\mathcal{A}}_i^*$, there exists $\overline{\alpha} > 0$ such that for all $\alpha \in [0, \overline{\alpha}]$,

$$\overline{V}_i^*(A^* + \alpha\theta) = V_i^*(A^*) + \alpha(v \cdot \theta).$$

Suppose $\theta \in \Theta$, and $A^*, A^* + \theta \in \overline{\mathcal{A}}_i^*$. By the definition of $\overline{\mathcal{A}}_i^*$, there exists $B^* \in \mathcal{A}_i^*$ such that $A^* \sim^* B^*$. By (i) of Claim 4, there exists $\overline{\alpha} \in (0, 1]$ such that $B^* + \alpha\theta \in \mathcal{A}_i^*$ for all $\alpha \in [0, \overline{\alpha}]$. Fix $\alpha \in [0, \overline{\alpha}]$. By the axiom of *Translation Invariance**, $A^* \sim^* B^*$ implies $A^* + \alpha\theta \sim^* B^* + \alpha\theta$. Hence, by the translation linearity of V_i^* on $\mathcal{A}_i^*$, we have $V_i^*(A^* + \alpha\theta) = V_i^*(B^* + \alpha\theta) = V_i^*(B^*) + \alpha(v \cdot \theta) = V_i^*(A^*) + \alpha(v \cdot \theta)$.

We show that if the weaker version of translation linearity holds, then the translation linearity holds. Fix $A^* \in \overline{\mathcal{A}}_i^*$ and $\theta \in \Theta$ with $A^* + \theta \in \overline{\mathcal{A}}_i^*$. Let $\alpha^* := \sup\{\overline{\alpha} | \overline{V}_i^*(A^* + \alpha\theta) = \overline{V}_i^*(A^*) + \alpha(v \cdot \theta),\ \forall \alpha \in [0, \overline{\alpha}]\ \}$. Note that $\overline{V}_i^*(A^* + \alpha\theta) = \overline{V}_i^*(A^*) + \alpha^*(v \cdot \theta)$. Consider the two cases: (i) $\alpha = 0$ and (ii) $\alpha > 0$. Consider the first case of (i) $\alpha = 0$. This is obvious. Consider the second case of (ii) $\alpha > 0$. Let $A^{*'} = A^* + \alpha^*\theta \in \overline{\mathcal{A}}_i^*$ and $\theta' = -\alpha^*\theta$. Then, there exists $\overline{\alpha} > 0$ such that $\overline{V}_i^*(A^* + \alpha^*\theta - \alpha\theta) = \overline{V}_i^*(A^* + \alpha^*\theta) - \overline{\alpha}(v \cdot \theta)$. Hence, $\overline{V}_i^*(A^* + \alpha^*\theta 0 + \overline{V}_i^*(A^* + (\alpha^* - \overline{\alpha})\theta) + \overline{\alpha}(v \cdot \theta) = \overline{V}_i^*(A^*) + (\alpha^* - \overline{\alpha})(v \cdot \theta) + \overline{\alpha}(v \cdot \theta) = \overline{V}_i^*(A^*) + \alpha^*(v \cdot \theta)$.

Finally, we show $\alpha = 1$. We prove it by the way of contradiction. Suppose not. Then, $A^{*'} + \alpha^*\theta \in \overline{\mathcal{A}}_i^*$ and $\theta' = (1 - \alpha^*)\theta$. This implies that there exists $\overline{\alpha} > 0$ such that for all $\alpha \in [0, \overline{\alpha}]$, $\overline{V}_i^*(A^* + \alpha^*\theta + \alpha\theta) = \overline{V}_i^*(A^* + \alpha^*\theta) + \alpha(v \cdot \theta) = \overline{V}_i^*(A^*) + (\alpha^* + \alpha)(v \cdot \theta)$. This implies $\alpha^* \geq \alpha^* + \overline{\alpha}$. This is a contradiction. □

In the similar way, we show another claim.

Claim 6. *For all $i \geq 0$, if $\overline{V}_i^*$ is (i) well-defined, (ii) translation linear, and (iii) $\succeq^*$ on $\overline{\mathcal{A}}_{i-1}^*$ is represented by $\overline{V}_i^*$, then V_i^* is (i) well-defined, (ii) translation linear, and (iii) $\succeq^*$ $\mathcal{A}^*$ is represented by V_i^*.*

Proof. We show (i). Suppose $A^* \in \mathcal{A}_i^*$ and $A^* = B^* + \alpha\theta^* = B^{*'} + \alpha'\theta^*$ for $B^*, B^{*'} \in \overline{\mathcal{A}}_{i-1}^*$ and $\alpha, \alpha' \in \mathbb{R}$. Then, $B^* = B^{*'} + (\alpha' - \alpha)\theta$. By the translation linearity of $\overline{V}_{i-1}^*$, $\overline{V}_{i-1}^* = \overline{V}_{i-1}^*(B^{*'}) + (\alpha' - \alpha)(v \cdot \theta^*)$. Hence, $\overline{V}_{i-1}^*(B^*) + \alpha(v \cdot \theta) = \overline{V}_{i-1}^*(B^*) + \alpha'(v \cdot \theta^*)$. Thus, $V_i^*(A^*)$ is uniquely identified.

We show (ii). Suppose $\theta \in \Theta$ and $A^*, A^* + \theta \in \mathcal{A}_i^*$. Then, there exist $B^*, B^{*'} \in \overline{\mathcal{A}}_{i-1}^*$ and $\alpha, \alpha' \in \mathbb{R}$ such that $A^* = B^* + \alpha\theta^*$ and $A^* + \theta = B^{*'} + \alpha'\theta^*$. Then, we have $B^{*'} = B^* + (\alpha - \alpha')\theta^* + \theta$. The translation linearity of $\overline{V}_{i-1}^*$ implies that $\overline{V}_{i-1}^*(B^{*'}) = \overline{V}_{i-1}^*(B^*) + v \cdot [(\alpha - \alpha')\theta^* + \theta]$. By the definition of V_i^*, $V_i^*(A^* + \theta) = \overline{V}_{i-1}^*(B^{*'}) + \alpha'(v \cdot \theta^*) = \overline{V}_{i-1}^*(B^*) + \alpha(v \cdot \theta^*) = V_i^*(A^*) + v \cdot \theta$.

We show (iii). Suppose that $A^*, A^{*'} \in \mathcal{A}_i^*$ with $A^* = B^* + \alpha\theta^*$ and $A^{*'} = B^{*'} + \alpha'\theta^*$ for some $B^*, B^{*'} \in \overline{\mathcal{A}}_{i-1}^*$ and $\alpha, \alpha' \in \mathbb{R}$. Consider $A^*, A^{*'} \succeq^* B^{*'} \succeq^* B^*$. By Claim 4, $\alpha \geq 0$. By the axiom of *Continuity**, there exists $\alpha'' \in\in [0, \alpha]$ such tha $B^* + \alpha''\theta^* \sim^* B^{*'}$. Then, we have $B^* + \alpha''\theta^* \in \overline{\mathcal{A}}_{i-1}^*$. By Remark 4 and the definition of V_i^*, $A^* \succeq^* A^{*'} \Leftrightarrow \alpha - \alpha'' \geq \alpha' \Leftrightarrow V_i^*(A^*) = \overline{V}_{i-1}^*(B^* + \alpha''\theta^*) + (\alpha - \alpha'')(v \cdot \theta^*) = \overline{V}_i^*(B^{*'}) + (\alpha - \alpha'')(v \cdot \theta^*) \geq \overline{V}_{i-1}^*(B^*) + \alpha'(v \cdot \theta^*) = V_i^*(A^{*'})$. □

Now, define $\widehat{V}_i^* : \cup_i \mathcal{A}^* \to \mathbb{R}$ by $\widehat{V}_i^*(A^*) := V_i^*(A^*)$ if $A^* \in \mathcal{A}_i^*$ such that $\widehat{V}^*$ is (i) well-defined, (ii) transltation linear, and (iii) $\succeq^*$ on $\cup_i \mathcal{A}_i^*$ is represented by $\widehat{V}^*$.

Claim 7. *$\mathcal{A}^\circ = \cup_i \mathcal{A}_i^*$.*

Proof. We need to show both (i) $\cup_i \mathcal{A}_i^* \subset \mathcal{A}^\circ$ and (ii) $\mathcal{A}^\circ \subset \cup_i \mathcal{A}_i^*$. By the definition of $\mathcal{A}_i^*$, it is immediately shown that $\cup_i \mathcal{A}_i^* \subset \mathcal{A}^\circ$. We show $\mathcal{A}^\circ \subset \cup_i \mathcal{A}_i^*$.

Consider $\boldsymbol{u} \in A^*$. By the definition of $\mathcal{A}^\circ$, there exists $\alpha > 0$ such that $A^* + \alpha\theta^* \in \mathcal{A}^\circ$ and $A^* - \alpha\theta^* \in \mathcal{A}^\circ$. Fix $\boldsymbol{u} \in A^*$. We have $\{\boldsymbol{u}\} + \alpha\theta^* \in \mathcal{A}_0^* \subset \mathcal{A}^\circ$ and $\{\boldsymbol{u}\} - \alpha\theta^* \in \mathcal{A}_0^* \subset \mathcal{A}^\circ$. For every $\lambda \in [0, 1]$, define $A^*(\lambda) \equiv \lambda A^* + (1 - \lambda)\{\boldsymbol{u}\}$. Note that $A^*(\lambda) + \alpha\theta^* \in \mathcal{A}^\circ$ and $A^*(\lambda) - \alpha\theta^* \in \mathcal{A}^\circ$. By the convexity of $\mathcal{A}^\circ$, $A^*(\lambda) - \alpha\theta^* = \lambda A^* + (1 - \lambda)\{\boldsymbol{u}\} + \alpha\theta^* = \lambda(A^* + \alpha\theta^*) + (1 - \lambda)(\{\boldsymbol{u}\} + \alpha\theta^*)$. By Claim 3, for any $\lambda \in [0, 1]$, $A^*(\lambda) + \alpha\theta^* \succ^* A^* \succ^* A^*(\lambda) - \alpha\theta^*$.

By the axiom of *Continuity**, for each $\lambda \in [0, 1]$, there exists an open interval $e(\lambda)$ such that $\lambda \in e(\lambda)$ and for all $\lambda' \in e(\lambda)$, $A^*(\lambda) + \alpha\theta^* \succ^* A^*(\lambda') \succ^* A^*(\lambda) - \alpha\theta^*$. Thus, $\{e(\lambda) | \lambda \in [0, 1]\}$ is an open cover of $[0, 1]$. By the compactness of $[0, 1]$, there exists a finite subcover of $\{e(\lambda_1), \cdots, e(\lambda_n)\}$. Assume that $e(\lambda_i) \cap e(\lambda_{i+1}) \neq \varnothing$, $0 \in e(\lambda_1)$, and $1 \in e(\lambda_n)$. We can prove that $A^*(\lambda_1) \in \mathcal{A}_1^*$. $A^*(\lambda_1) \succ^* A^*(0) = \{\boldsymbol{u}\} \succ^* A^*(\lambda_1) - \alpha\theta^*$. Then, by the axiom of *Continuity**, there exists $\alpha' \in (-\alpha, \alpha)$ such that $A^*(\lambda_1) + \alpha'\theta^* \sim^* \{\boldsymbol{u}\}$. This implies $A^*(\lambda_1) + \alpha'\theta^* \in \overline{\mathcal{A}}_0^*$. Hence, $A^*(\lambda_1) \in \mathcal{A}_1^*$.

Now, we show that if $A^*(\lambda_i) \in \mathcal{A}_i^*$, then $A^*(\lambda_{i+1}) \in \mathcal{A}_{i+1}^*$. If $A^*(\lambda_i) \in \mathcal{A}_i^*$, then, for all $\alpha' \in (-\alpha, \alpha)$, $A^*(\lambda_i) + \alpha'\theta^* \in \mathcal{A}_i^*$. Since $e(\lambda_i) \cap e(\lambda_{i+1}) \neq \varnothing$, choose $\lambda \in e(\lambda_i) \cap e(\lambda_{i+1})$.

Then, $A^*(\lambda_i) \succ^* A^*(\lambda) \succ^* A^*(\lambda_i) - \alpha\theta^*$ and $A^*(\lambda_{i+1}) \succ^* A^*(\lambda) \succ^* A^*(\lambda_{i+1}) - \alpha\theta^*$. By the axiom of *Continuity**, there exist $\alpha', \alpha'' \in (-\alpha, \alpha)$ such that $A^*(\lambda_i) + \alpha'\theta^* \sim^* A^*(\lambda) \sim^* A^*(\lambda_{i+1}) - \alpha''\theta^*$. Then, we obtain $A^*(\lambda_{i+1}) - \alpha''\theta^* \in \overline{\mathcal{A}}_i^*$. Hence, $A^*(\lambda_{i+1}) \in \mathcal{A}_{i+1}^*$. By the way of induction, $A^*(\lambda_i) \in \mathcal{A}_i^*$ $(i = 1, \cdots, n)$, so $A^* \in \overline{\mathcal{A}}_n^* \subset \mathcal{A}_{i+1}^* \subset \cup_i \mathcal{A}_i^*$. □

By extending $\mathcal{A}^\circ$ to $\mathcal{A}^*$, we show that $V^* : \mathcal{A}^* \to \mathbb{R}$ is translation linear. Let $A^*, A^* + \theta$ for some $\theta \in \Theta$. Fix $\boldsymbol{u} \in \mathbb{R}$ such that $u_i > 0$ for all $i \in \mathbb{A}$. For all $n \in \mathbb{N}$, define $A_n^* \equiv (1 - \frac{1}{n})A^* + \frac{1}{n}\{\boldsymbol{u}\}$ and $\theta_n := (1 - \frac{1}{n})\theta$.

Fact. For all $n \in \mathbb{N}$, $A_n^* \in \mathcal{A}^\circ$ and $A_n^* + \theta_n = (1 - \frac{1}{n})(A^* + \theta) + \frac{1}{n}\{\boldsymbol{u}\} \in \mathcal{A}^\circ$.

Moreover, as $n \to \infty$, we have $A_n^* \to A^*$ and $A_n^* + \theta_n \to A^* + \theta$. Hence,

$$\begin{aligned} V^*(A^* + \theta) - V^*(A^*) &= \lim_{n\to\infty} [\widehat{V}^*(A_n^* + \theta_n) - \widehat{V}^*(A_n^*)] \\ &= \lim_{n\to\infty} v \cdot \theta_n \\ &= v \cdot \theta. \end{aligned}$$

We show that V^* is concave.

Claim 8. $\widehat{V}^*$ *is concave.*

Proof. Let $A_0^* \in \mathcal{A}^\circ$. Take $B_\varepsilon(A_0^*) := \{A^* | d_h(A^*, A_0^*) < \varepsilon\}$ for some $\varepsilon > 0$. For all $\theta \in \Theta$, $A^* \in \mathcal{A}^\circ$ with $A^* + \theta$ and $d_h(A^*, A^* + \theta) = \|\theta\|$. Then, there exists $\theta \in \Theta$ such that $\|\theta\| < \varepsilon$ and $v \cdot \theta > 0$. Then $A_0^* + \theta \in B_\varepsilon(A_0^*)$ and $A_0^* + \theta \succ^* A_0^*$.

By the axiom of *Continuity**, there exists $\rho \in (0, \frac{1}{2})$ such that for all $A^* \in B_{\rho\varepsilon}(A_0^*)$, $|\widehat{V}^*(A^*) - \widehat{V}^*(A_0^*)| < \frac{1}{2}(v \cdot \theta)$. Hence, if $A^*, B^* \in B_{\rho\varepsilon}(A_0^*)$, then $|\widehat{V}^*(A^*) - \widehat{V}^*(A_0^*)| \leq |\widehat{V}^*(A^*) - \widehat{V}^*(A_0^*)| + |\widehat{V}^*(A_0^*) - \widehat{V}^*(B^*)| < \frac{1}{2}(v \cdot \theta)$. Let

$$\alpha := \frac{|\widehat{V}^*(A^*) - \widehat{V}^*(A_0^*)|}{v \cdot \theta}.$$

Then, $|\alpha| < \frac{1}{2}$. We have

$$\begin{aligned} d_h(A_0^*, B^* + \alpha\theta) &\leq d_h(A_0^*, B^*) + d_h(B^*, B^* + \alpha\theta) \\ &< \rho\varepsilon + \|\alpha\theta\| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Hence, $B^* + \alpha\theta \in B_\varepsilon(A_0^*) \subset \mathcal{A}^\circ$. Then, $\widehat{V}$ is defined at $B^* + \alpha\theta$. Note that $\alpha(v \cdot \theta) = \widehat{V}(A^*) - \widehat{V}^*(B^*)$, so that $\widehat{V}^*(B^* + \alpha\theta) = \widehat{V}^*(B^*) + \alpha(v \cdot \theta) = \widehat{V}^*(A^*)$.

By the axiom of *Contemplation-Seeking*, for all $\lambda \in [0, 1]$, $\widehat{V}^*(A^*) \leq \widehat{V}^*(\lambda A^* + (1-\lambda)(B^* + \alpha\theta)) = \widehat{V}^*(\lambda A^* + (1-\lambda)B^*) + (1-\lambda)\alpha(v \cdot \theta) = \widehat{V}^*(\lambda A^* + (1-\lambda)B^*) + (1-\lambda)(\widehat{V}^*(A^*) - \widehat{V}^*(B^*))$. Therefore, we obtain $\lambda\widehat{V}^*(A^*) + (1-\lambda)\widehat{V}^*(B^*) \leq \widehat{V}^*(\lambda A^* + (1-\lambda)B^*)$. □

It is straightforward to extend $\widehat{V}^*$ to V^*. We omit the proof.

Claim 9. *If for any $A^*, B^* \in \mathcal{A}^*$, $V^*(A^*) \geq V^*(B^*) \Leftrightarrow V^{*'}(A^*) \geq V^{*'}(B^*)$, then there exists $a > 0$ and $b \in \mathbb{R}$ such that $V^{*'} = aV^* + b$.*

Proof. By the axiom of *Continuity**, V^* is continuous. Since $\succeq^*$ is a continuous weak order, we have, for any $A^*, B^* \in \mathcal{A}^*$, $A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*)$. Since V^* is *translation linear*, V^* is affine on singleton sets. Then, we have the following: if $\succeq^*$ is represented by V^* and $V^{*'}$, then $V^{*'}|_s = aV^*|_s + b$ for some $a > 0$ and $b \in \mathbb{R}$. By the way of induction, consider $\mathcal{A}_0^*$. By Remark, $\mathcal{A}^\circ = \cup_i \mathcal{A}_i^*$. Then, since $A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*)$, $V^{*'}|_{\mathcal{A}_i^*} = aV^*|_{\mathcal{A}_i^*} + b$ for all $i \geq 0$. Since $\mathcal{A}^\circ = \cup_i \mathcal{A}_i^*$, in the similar way, $V^{*'}|_{\mathcal{A}^\circ} = aV^*|_{\mathcal{A}^\circ} + b$. By the continuity of V^*, we have $V^{*'}|_{\mathcal{A}^*} = aV^*|_{\mathcal{A}^*} + b$. □

Step 3

In Step 3, we complete the desired representation, i.e., the dissatisficing-averse utility representation. To do so, we apply the duality result in convex analysis (Rockafellar (1970)). Especially, we apply the duality result into the contemplation part (second term) in the utility representation. That is, exploring the *best* option on the *Pareto* frontier in each menu on the attribute-based utility space is equivalent to exploring the *optimal* weight on the attribute space.

To represent the contemplation part in the utility representation, we introduce a functional $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. First, we show that the functional J has certain desirable properties (Claim 10 and 11). Next, by applying both the duality result and the result in Ergin and Sarver (2010b), we show that we represent the dissatisficing utility with the contemplation part (Claim 12). Then, we obtain the set of *weights* on the attribute space with menu-dependence ($\mathcal{M}$). Moreover, we show that $\mathcal{M}$ has certain properties (Claim 13, 14, and 15). Finally, we obtain the desired result, by arranging the terms.

Applying the Duality Result

Let $U(\mathbf{p}) = \sum_{i \in \mathbb{A}} u_i(p_i)$. Let $\boldsymbol{u}(\mathbf{p}) = (u_1(p_1), \cdots, u_n(p_n))$. Define a functional $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$J(\boldsymbol{u}^*(A), \boldsymbol{u}^*(A) - \boldsymbol{u}(\mathbf{p})) := U(\mathbf{p}) - V^*(A^*),$$

where $\boldsymbol{u}^*(A) = (\max_{\mathbf{p} \in A} u_i(p_i))_{i \in \mathbb{A}}$ and for some $\mathbf{p} \in A$.

We show that the functional J is *well-defined*. Take $A^*, A^{*'} \in \mathcal{A}^*$ with $A^* \sim^* A^{*'}$ and $\boldsymbol{u}^*(A) = \boldsymbol{u}^*(A')$. By definition, $A \sim A'$. Then, $V^*(A^*) = V^*(A^{*'})$. Suppose that $\mathbf{p}$ is chosen from A, and that $\mathbf{p}'$ is chosen from A'. By the definition of J, we obtain $U(\mathbf{p}) - V^*(A^*) = U(\mathbf{p}') - V^*(A^{*'})$. Moreover, for any $\mathbf{r} \in \mathcal{X}$ and $\lambda \in [0, 1]$, $\lambda A + (1 - \lambda)\{\mathbf{r}\} \sim \lambda A' + (1 - \lambda)\{\mathbf{r}\}$. Notice that the λ-mixture menus have the ideal options in the following way: $\boldsymbol{u}^*(\lambda A + (1-\lambda)\{\mathbf{r}\}) = \boldsymbol{u}^*(\lambda A' + (1-\lambda)\{\mathbf{r}\})$. By *No Need for Contemplation**, J is well-defined.

In the following, we show that J has certain properties. First, we show that J is *monotonic*, i.e., given $a \in \mathbb{R}^n$, if $b \geq b'$, then $J(a,b) \geq J(a,b')$.

Claim 10. *J is monotonic.*

Proof. By the axiom of *Dominance**, if $u_i \geq v_i$ for all $i \in \mathbb{A}$, then $\{\boldsymbol{u}\} \succeq^* \{\boldsymbol{v}\}$. Hence, $V^*(\{\boldsymbol{u}\}) \geq V^*(\{\boldsymbol{v}\}) \Leftrightarrow \boldsymbol{u} \geq \boldsymbol{v}$. Consider a menu $A^* \in \mathcal{A}^*$ with $\boldsymbol{u}, \boldsymbol{v} \in A^*$. Let $a = \boldsymbol{u}^*(A)$ and $b = \boldsymbol{u}^*(A) - \boldsymbol{u}$, and $b' = \boldsymbol{u}^*(A) - \boldsymbol{v}$. By the axiom of *Dominance** and the definition of J, since $b \geq b'$, we obtain $J(a,b) \geq J(a,b')$. □

Moreover, by definition, J is continuous in the second arguments, i.e., for any $a \in \mathbb{R}^n$, $J(a,\cdot)$ is continuous.

We show that J is homogeneous of degree one with respect to the second arguments, i.e., for any pair (a,b) and $\lambda \in (0,1)$, $J(a,\lambda b) = \lambda J(a,b)$.

Claim 11. *J is homogeneous of degree one with respect to the second arguments.*

Proof. Consider a menu $A^* \in \mathcal{A}^*$. Let $a = \boldsymbol{u}^*(A)$ and $b = \boldsymbol{u}^*(A) - \boldsymbol{u}(\mathbf{p})$. By the definition of J, $J(a,b) = V^*(A^*) - U(\mathbf{p})$. By taking $\lambda \in (0,1)$, $J(a,\lambda b) = V^*(\lambda A^* + (1-\lambda)\{\boldsymbol{u}\}) - U(\mathbf{p}) = \lambda V^*(A^*) + (1-\lambda)V^*(\{\boldsymbol{u}\}) - U(\mathbf{p}) = \lambda(V^*(A^*) - \mathbf{u}(\mathbf{p})) = \lambda J(a,b)$. The second equality holds by the property of *translation linearity.* □

Let $\lambda \in (0,1)$. Then, by the property of *homogeneity of degree one*, we have $J(a,\lambda 0) = \lambda J(a,0) = 0$. Hence, $J(a,0) = 0$. Consider a pair (a,b). Take $b < \bar{b}$. Then, $J(a,b) = J(a, \frac{b}{\bar{b}}\bar{b})$. Notice that $\frac{b}{\bar{b}} \in (0,1)$. By the property of homogeneity of degree one, we have $\frac{b}{\bar{b}} J(a,\bar{b})$. Then, $J(a,b) = \frac{b}{\bar{b}} J(a,\bar{b}) = \frac{1}{\bar{b}} J(a,\bar{b}) \cdot b$. Define $\widehat{J}: \mathbb{R}^n \to \mathbb{R}^n_+$ by for any pair (a,b),

$$J(a,b) = \widehat{J}(a) \cdot b.$$

By Claim 10 and Claim 11, we can define a functional $J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$J(\boldsymbol{u}^*(A), \cdot) := \widehat{J}(\boldsymbol{u}^*(A))(\cdot),$$

for some $\widehat{J} : \mathbb{R}^n \to \mathbb{R}^n$.

To apply the duality result, let us introduce some notation. Let $\mathcal{U}$ be the set of profiles of continuous real-valued functions on the attribute space $\mathbb{A}$, denoted by $\mathcal{U} = (\mathcal{U}_i)_{i \in \mathbb{A}}$ where for each $i \in \mathbb{A}$, $\mathcal{U}_i = \{u_i | u_i : \mathcal{X}_i \to \mathbb{R}\}$. Define σ_A defined by, given $A \in \mathcal{A}$,

$$\sigma_A(\boldsymbol{u}) = \max_{\mathbf{p} \in A} \boldsymbol{u} \cdot \mathbf{p} = \max_{\mathbf{p} \in A} u_1(p_1) + \cdots + u_n(p_n).$$

Let $C(\mathcal{U})$ be the set of continuous real-valued functions on $\mathbb{A}$. Let $\Sigma = \{\sigma_A \in C(\mathcal{U}) | A \in \mathcal{A}\}$. Let $\langle \sigma, \boldsymbol{\mu} \rangle = \sum_{i \in \mathbb{A}} u_i \mu_i$. Let $C^*(\mathcal{U})$ be the set of all finite Borel *non-negative* measures on $\mathbb{A}$. The non-negative follows from the monotonicity of J.

Fix $A \in \mathcal{A}$. For $\sigma \in \Sigma$, the *superdifferential* of $\widehat{J}(\boldsymbol{u}^*(A))$ at σ is defined to be

$$\partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma) := \{ \ \boldsymbol{\mu} \in C^*(\mathcal{U}) \mid \langle \sigma' - \sigma, \boldsymbol{\mu}' - \boldsymbol{\mu} \rangle \leq \widehat{J}(\boldsymbol{u}^*(A))(\sigma') - \widehat{J}(\boldsymbol{u}^*(A))(\sigma) \ \forall \sigma \in \mathcal{C}(\mathcal{U}) \ \}.$$

The *conjugate* of $\widehat{J}(\boldsymbol{u}^*(A))$ is the function $\widehat{J}^*(\boldsymbol{u}^*(A)) : C^*(\mathcal{U}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) := \inf_{\sigma \in \Sigma} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}(\boldsymbol{u}^*(A))(\sigma) \Big].$$

The superdifferential of $\widehat{J}(\boldsymbol{u}^*(A))$ at σ captures a set of weights on the attribute space $\mathbb{A}$ at σ. The conjugate of $\widehat{J}(\boldsymbol{u}^*(A))$ is the cost function of choosing a weight on $\mathbb{A}$. We use the following fact.

Fact. (Ergin and Sarver (2010b)): The following statements hold:

(i) $\widehat{J}^*(\boldsymbol{u}^*(A))$ is lower semicontinuous in the weak* topology.[16]

(ii) $\widehat{J}(\boldsymbol{u}^*(A))(\sigma) \leq \langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu})$ for all σ and $\boldsymbol{\mu}$.

(iii) $\widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Leftrightarrow \boldsymbol{\mu} \in \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$.

Let us introduce the following:

- $\Sigma_{\widehat{J}(\boldsymbol{u}^*(A))} = \{\sigma \in \Sigma | \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$ is a singleton$\}$.
- $\mathcal{C}_{\widehat{J}(\boldsymbol{u}^*(A))} = \{\boldsymbol{\mu} \in C^*(\mathcal{U}) | \boldsymbol{\mu} \in \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$ for some $\sigma \in \Sigma_{\widehat{J}(\boldsymbol{u}^*(A))}\}$.
- $\mathcal{M}_{\widehat{J}(\boldsymbol{u}^*(A))} = \overline{\mathcal{C}}_{\widehat{J}(\boldsymbol{u}^*(A))}$ is the closure taken with respect to the weak* topology.

The first notation $\Sigma_{\widehat{J}(\boldsymbol{u}^*(A))}$ is the set of support functions σ such that the superdifferential of $\widehat{J}(\boldsymbol{u}^*(A))$ at σ is a singleton. That is, the weight at σ is uniquely determined. The second notation is, given the ideal option of A, a set of weights. Third notation is the closure of the set. We have the following claim.

Claim 12. *$\mathcal{M}_{\widehat{J}(\boldsymbol{u}^*(A))}$ is weak* compact. Moreover, for any weak* compact $\mathcal{M} \subset C^*(\mathcal{U})$,*

$$\mathcal{M}_{\widehat{J}(\boldsymbol{u}^*(A))} \subset \mathcal{M} \Leftrightarrow \widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \min_{\boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A))} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big], \ \forall \sigma \in \Sigma.$$

[16]See Royden and Fitzpatrick (1968).

The proof of Claim 12 is in Ergin and Sarver (2010b). By Claim 12, for all $\sigma \in \Sigma$,

$$\widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \min_{\boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A))} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big].$$

Hence, by the definition of J, for all $A^* \in \mathcal{A}^*$,

$$V^*(A^*) = \max_{\boldsymbol{u} \in A^*} \sum_i u_i + \min_{\boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A))} \Big[\max_{\boldsymbol{u} \in A^*} (\sum_i u_i \mu_i) - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big].$$

We apply the result in Ergin and Sarver (2010b) (Corollary 2). For any $A^* \in \mathcal{A}^*$,

$$V^*(A^*) \text{ subject to } c(\boldsymbol{\mu}) \leq k,$$

for some $c : C^*(\mathcal{U}) \to \mathbb{R}$ and $k \in \mathbb{R}$. Then, let $\overline{\mathcal{M}}(\boldsymbol{u}^*(A)) = \{ \boldsymbol{\mu} \in \mathcal{M}_{\boldsymbol{u}^*(A)} \mid c(\boldsymbol{\mu}) \leq k \}$. Since c is lower semicontinuous, $\overline{\mathcal{M}}(\boldsymbol{u}^*(A))$ is compact. Hence, the set of non-negative measures on $\mathcal{U}$ is obtained.

We show that the properties of *consistency* and *minimality* of $\mathcal{M}$ are satisfied. First, we show the *consistency* of $\overline{\mathcal{M}}(\boldsymbol{u}^*(A))$ by the following two claims. Let $\overline{\mathcal{M}}(\boldsymbol{u}^*(A)) = \overline{J}(\boldsymbol{u}^*(A))$.

Claim 13. *If $A \in \mathcal{A}$ and $\boldsymbol{\mu} \in \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma_A)$, then for all $\boldsymbol{u} \in \mathbb{R}^n$ there exists $\boldsymbol{v} \in \mathbb{R}^n$ with $\theta = \boldsymbol{u} - \boldsymbol{v} \in \Theta$ such that $\langle \sigma, \boldsymbol{\mu} \rangle = v \cdot \theta$.*

Proof. Fix $A \in \mathcal{A}$ and $\boldsymbol{\mu} \in \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma_A)$. Take $\theta = \boldsymbol{u} - \boldsymbol{v} \in \Theta$. Then, $\sigma_{\{\theta\}}(u) = u \cdot \theta$. It is easily verified that, for any $A \in \mathcal{A}$ and $\theta \in \Theta$, $\sigma_{A+\theta} = \sigma_A + \sigma_{\{\theta\}}$.

We show that for all $\theta \in \Theta$, $\langle \sigma_{\{\theta\}}, \beta \rangle = v \cdot \theta$. Fix $\theta \in \Theta$. There exists $\alpha > 0$ such that $A^* + \alpha\theta, A^* - \alpha\theta \in \mathcal{A}^*$. By the translation linearity of V^*, $\alpha(v \cdot \theta) = V^*(A^* + \alpha\theta) - V^*(A^*) = \overline{J}(\overline{\boldsymbol{u}})(\sigma_{A+\alpha\theta}) - \overline{J}(\overline{\boldsymbol{u}})(\sigma_A)$.

By claims, $\alpha(v \cdot \theta) \geq \langle \sigma_{A+\alpha\theta}, \beta \rangle - \langle \sigma_A, \mu \rangle = \langle \sigma_{\alpha\theta}, \mu \rangle = \alpha \langle \sigma_\theta, \mu \rangle$. In the same way, we have $-\alpha(v \cdot \theta) = \overline{J}(\boldsymbol{u}^*(A))(\sigma_{A+\alpha\theta}) - \overline{J}(\boldsymbol{u}^*(A))(\sigma_A) \geq -\alpha \langle \sigma_\theta, \mu \rangle$. Hence, $\alpha(v \cdot \theta) = \langle \sigma_{\{\theta\}}, \mu \rangle$
□

Moreover, we show another claim.

Claim 14. *If $\boldsymbol{\mu} \in \mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))}$, then for all $\boldsymbol{u} \in \mathbb{R}^n$ there exists $\boldsymbol{v} \in \mathbb{R}^n$ with $\theta = \boldsymbol{u} - \boldsymbol{v} \in \Theta$ such that $\langle \sigma, \boldsymbol{\mu} \rangle = v \cdot \theta$.*

Proof. Define $\mathcal{M} \subset \mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))}$ by

$$\mathcal{M} := \{ \boldsymbol{\mu} \in \mathcal{M}(\overline{J}_{\boldsymbol{u}^*(A)}) | \langle \sigma_{\{\mathbf{p}\}}, \boldsymbol{\mu} \rangle = v \cdot (\mathbf{p} - \mathbf{q}) \ \forall \mathbf{p} \in \mathcal{X} \}.$$

$\mathcal{M}$ is a closed subset of $\mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))}$, and $\mathcal{M}$ is compact. We show $\mathcal{M} = \mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))}$. That it, we show $\mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))} \subset \mathcal{M}$. By Claim, we need to show

$$\widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \max_{\boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{u}^*(A))} \Big[\langle \sigma, \boldsymbol{\mu} \rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu}) \Big],$$

for all $\sigma \in \Sigma$ (we can normalize it).

Take $\sigma \in \Sigma$. For any $\lambda \in (0,1)$, $\lambda A_\sigma + (1-\lambda)\{\mathbf{q}\} \in \mathcal{A}$. Note that $\sigma_{\lambda A_\sigma + (1-\lambda)\{\mathbf{q}\}} = \lambda\sigma(A_\sigma) + (1-\lambda)\sigma_{\{\mathbf{q}\}} = \lambda\sigma$. Hence, for all $\lambda \in (0,1)$, $\mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))} \cap \partial\widehat{J}(\boldsymbol{u}^*(A))(\lambda\sigma) \subset \mathcal{M}$. Then, there exists $\boldsymbol{\mu} \in \mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))}$ such that $\widehat{J}(\boldsymbol{u}^*(A))(\sigma) = \langle\lambda\sigma, \boldsymbol{\mu}\rangle - \widehat{J}^*(\boldsymbol{u}^*(A))(\boldsymbol{\mu})$. Then, $\boldsymbol{\mu} \in \partial\widehat{J}(\boldsymbol{u}^*(A))(\lambda\sigma)$. Therefore, $\mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))} \cap \partial\widehat{J}(\boldsymbol{u}^*(A))(\lambda\sigma) \neq \varnothing$.

Take a net $\{\lambda_t\}_{t\in T}$ such that $\lambda_t \to 1$. Let $\sigma_t := \lambda_t\sigma$. Then, $\sigma_t \to \sigma$. For all $t \in T$, there exists $\boldsymbol{\mu} \in \mathcal{M}_{\overline{J}(\boldsymbol{u}^*(A))} \cap \partial\widehat{J}(\boldsymbol{u}^*(A))(\lambda\sigma) \subset \mathcal{M}$. $\mathcal{M}$ is weak* compact. Then, every net has convergent subnet.

Without loss of generality, suppose $\boldsymbol{\mu}_t \xrightarrow{w^*} \boldsymbol{\mu}$ for some $\boldsymbol{\mu} \in \mathcal{M}$. For all $\sigma' \in \Sigma$, $\langle\sigma' - \sigma, \boldsymbol{\mu}' - \boldsymbol{\mu}\rangle = \lim_t\langle\sigma' - \sigma_t, \boldsymbol{\mu}' - \boldsymbol{\mu}_t\rangle \leq [\widehat{J}(\boldsymbol{u}^*(A))(\sigma') - \widehat{J}(\boldsymbol{u}^*(A))(\sigma_t)] = \widehat{J}(\boldsymbol{u}^*(A))(\sigma') - \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$. Hence, we have $\boldsymbol{\mu} \in \partial\widehat{J}(\boldsymbol{u}^*(A))(\sigma)$. Thus, the desired statement is shown. □

Finally, we show that $\overline{\mathcal{M}}(\boldsymbol{u}^*(A))$ is *minimal*.

Claim 15. *$\overline{\mathcal{M}}(\boldsymbol{u}^*(A))$ is minimal.*

Proof. Suppose $\mathcal{M}' \subset \overline{\mathcal{M}}(\boldsymbol{u}^*(A))$. And, suppose that $(\mathcal{M}', \overline{J}(\boldsymbol{u}^*(A))|_{\mathcal{M}'})$ represents $\succeq^*$. We show $\mathcal{M}' = \overline{\mathcal{M}}(\boldsymbol{u}^*(A))$

$\overline{J}_{\boldsymbol{u}^*(A)} = \max_{\boldsymbol{\mu}\in\mathcal{M}'}\langle\sigma, \boldsymbol{\mu}\rangle$. There exists $\mathbf{p}^*, \mathbf{p}_* \in \mathcal{X}$ such that $\{\mathbf{p}^*\} \succ \{\mathbf{p}_*\}$. Then, $\langle\sigma_{\{\mathbf{p}^*\}} - \sigma_{\{\mathbf{p}_*\}}\rangle = \langle\sigma_{\{\mathbf{p}^*\}}, \boldsymbol{\mu}\rangle - \langle\sigma_{\{\mathbf{p}_*\}}, \boldsymbol{\mu}\rangle = v \cdot (\mathbf{p}^* - \mathbf{p}_*) > 0$. By applying Proposition S.1 in Ergin and Sarver (2010b) with $\overline{x} = \sigma_{\{\mathbf{p}^*\}} - \sigma_{\{\mathbf{p}_*\}}$, we conclude $\mathcal{M}' = \mathcal{M}(\boldsymbol{u}^*(A))$. □

Identifying a Subjective State Space

Now, we identify the subjective state space of the dissatisficing-averse utility representation. We show that $\succeq$ satisfies the axiom of *Finiteness* in Kopylov (2009).

Axiom (Finiteness): For any sequence $\{A_n\}$ of $\mathcal{A}$, there exists a positive integer N such that $\cup_{n=1}^{N} A_n \sim \cup_{n=1}^{N+1} A_n$.

To show the axiom of *Finiteness*, we verify that the following holds: for any $\mathbf{q} \in B$, there exists $\mathbf{p} \in A$ such that $p_1 \succsim_i q_i$ for all $i \in \mathbb{A}$, then $A \sim A \cup B$. For any $\mathbf{q} \in A \cup B$, there exists $\mathbf{p} \in A$ such that $p_i \sim_i q_i$ for all $i \in \mathbb{A}$. Then, by the axiom of *Dominance*, we have $A \sim A \cup B$.

Take $A_1, \cdots, A_n, A_{n+1} \in \mathcal{A}$. Let

$$\mathbf{p}_i \in \arg\max_{\mathbf{p}\in A_1\cup\cdots\cup A_n} u_i(p_i).$$

Without loss of generality, assume that $\mathbf{p}_i \in A_i$ for each $i \in \mathbb{A}$. Suppose $A_1 \cup \cdots \cup A_n \succeq A_1 \cup \cdots \cup A_{n-1} \cup A_{n+1}$. Then, by the axiom of *Dominance*, we have $A_1 \cup \cdots \cup A_n \sim A_1 \cup \cdots \cup A_n \cup A_{n+1}$. Hence, the axiom of *Finiteness* is satisfied with $N = n$.

By the preceding steps, we obtain the following functional form. For any $A^* \in \mathcal{A}^*$,

$$V^*(A^*) = \max_{\boldsymbol{u} \in A^*} u_1 + \cdots + \max_{\boldsymbol{u} \in A^*} u_n - \min_{\boldsymbol{\mu} \in \overline{\mathcal{M}}(\boldsymbol{u}^*)} \max_{\boldsymbol{u} \in A^*} \Big(\sum_{i \in \mathbb{A}} u_i \mu_i \Big).$$

For any $A \in \mathcal{A}$, define $V(A) = V^*(A^*)$. Then, we have $A \succeq B \Leftrightarrow A^* \succeq^* B^* \Leftrightarrow V^*(A^*) \geq V^*(B^*) \Leftrightarrow V(A) \geq V(B)$. by arranging the terms, we have the desired representation. □

A.2 Necessity Part

We show the necessity part. We show that the dissatisficing-averse utility representation V satisfies the axioms of *Dominance* and *Dissatisfaction*. Other axioms are easily verified. We omit it.

First, we show the necessity of *Dominance*. Take an arbitrary menu $A \in \mathcal{A}$. Take $\mathbf{p} \in A$. Suppose that there exists $\mathbf{q} \in \mathcal{X}$ such that for all $i \in \mathbb{A}$, $p_i \succ_i q_i$. Then, $A \succeq \{\mathbf{q}\}$. We have $V(A) - V(\{\mathbf{q}\}) \geq 0$. Notice that, for all $i \in \mathbb{A}$, $\max_{\mathbf{p}' \in A} u_i(p'_i) = \max_{\mathbf{p}' \in A \cup \{\mathbf{q}\}} u_i(p'_i)$. Hence, $V(A) - V(A \cup \{\mathbf{q}\}) = 0$. Thus, V satisfies the axiom of *Dominance*.

Next, we show the necessity of *Dissatisfaction*. Take an arbitrary menu $A \in \mathcal{A}$. Take $\mathbf{p} \in A$. Suppose that there exists $\mathbf{q} \in \mathcal{X}$ such that there exist $i, j \in \mathbb{A}$ with $p_i \succ_i q_i$ and $p_j \prec_j q_j$. Then, $\max_{\mathbf{p}' \in A} u_j(p'_j) < \max_{\mathbf{p}' \in A \cup \{\mathbf{q}\}} u_j(p'_j)$. Then, we obtain $V(A) > V(A \cup \{\mathbf{q}\})$. Thus, V satisfies the axiom of *Dissatisfaction*. □

B Proof of Propositions

B.1 Proof of Proposition 1

The first condition is standard for additively separable utility representations. See Krantz et al. (1971) (Theorem 2, 13 in Chapter 6). To show the second condition, suppose that two dissatisficing-averse utility representations $\langle \mathcal{U}, \mathcal{M} \rangle$ and $\langle \mathcal{U}', \mathcal{M}' \rangle$ represent the same binary relation $\succeq$. We can show the uniqueness result by using Lemma 4. Both $\langle \mathcal{U}, \mathcal{M} \rangle$ and $\langle \mathcal{U}', \mathcal{M}' \rangle$ satisfy the uniqueness part of Lemma 4. Then, there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $V' = \alpha V + \beta$. With the first condition, this implies that, by definition, $\widehat{J}' = \alpha \widehat{J} + \beta$. For all $\boldsymbol{\mu} \in C(\mathcal{U})$ and $\sigma, \sigma' \in \Sigma$, $\langle \sigma' - \sigma, \mu \rangle \leq \widehat{J}(\boldsymbol{u}^*(A))(\sigma') - \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$. Then, $\langle \sigma' - \sigma, \boldsymbol{\mu} \rangle \leq \widehat{J}'(\boldsymbol{u}^*(A))(\sigma') - \widehat{J}'(\boldsymbol{u}^*(A))(\sigma)$. Fix $A \in \mathcal{A}$. Hence, we obtain $\partial \widehat{J}'(\boldsymbol{u}^*(A))(\sigma) = \alpha \partial \widehat{J}(\boldsymbol{u}^*(A))(\sigma)$. We obtain $\mathcal{M}_{\widehat{J}'(\boldsymbol{u}^*(A))} = \alpha \mathcal{M}_{\widehat{J}(\boldsymbol{u}^*(A))}$. In Theorem 1, we have $\overline{\mathcal{M}}' = \mathcal{M}_{\widehat{J}'(\boldsymbol{u}^*(A))}$ and $\overline{\mathcal{M}} = \mathcal{M}_{\widehat{J}(\boldsymbol{u}^*(A))}$. Therefore, we obtain $\mathcal{M}' = \alpha \mathcal{M}$. □

B.2 Proof of Proposition 2

First, we show the sufficiency part. For each $h \in \{X, Y\}$, $\succeq^h$ is represented by a pair $\langle \mathcal{U}, \mathcal{M}^h \rangle$. Take an arbitrary menu $A \in \mathcal{A}$. Suppose that $\succeq^X$ exhibits a stronger contemplation-seeking than $\succeq^Y$. Then, $V^Y(A) \geq U(\mathbf{p}) \Rightarrow V^X(A) \geq U(\mathbf{p})$. We have $V^Y(A) - U(\mathbf{p}) \geq 0 \Rightarrow V^X(A) - U(\mathbf{p}) \geq 0$. Hence, $V^Y(A) \leq V^X(A)$. If $V^Y(B) = V^X(A)$, then $\mathcal{M}^Y = \mathcal{M}^X$. If $V^Y(A) < V^X(A)$, for any $\boldsymbol{\mu}^Y \in \mathcal{M}^Y$, there exists $\boldsymbol{\mu}^X \in \mathcal{M}^X$ such that for each $\mathbf{p} \in A$, $\sum_{i \in \mathbb{A}} \mu_i^X u_i(p_i) \geq \sum_{i \in \mathbb{A}} \mu_i^Y u_i(p_i)$. Thus, $\mathcal{M}^X$ exhibits more extreme-aversion than $\mathcal{M}^Y$.

Next, we show the necessity part. Take an arbitrary menu $A \in \mathcal{A}$. Consider two binary relations on $\mathcal{A}$: $\succeq^X, \succeq^Y$. Suppose that $V^X(A) \geq V^Y(A)$. And, suppose that for all $\mathbf{p} \in A$, $A \succeq^Y \{\mathbf{p}\}$. Then, $V^Y(A) \geq V^Y(\{\mathbf{p}\}) = U(\mathbf{p})$. Hence, we have $V^X(A) - U(\mathbf{p}) \geq V^Y(A) - U(\mathbf{p})$. Thus, $A \succeq^X \{\mathbf{p}\}$. It is shown that $\succeq^X$ exhibits a stronger contemplation-seeking than $\succeq^Y$. □

B.3 Proof of Proposition 3

Let $\succeq$ be represented by a pair $\langle \mathcal{U}, \mathcal{M} \rangle$. We show the sufficiency part. Take $\mathbf{p}_0, \mathbf{p}_1 \in \mathcal{X}$. Let $U_0(\mathbf{p}) := \sum_{i \in \mathbb{A}} u_i(p_i)$, and $U_1(\mathbf{p}) := \sum_{i \in \mathbb{A}} \mu_i^* u_i(p_i)$, for each $\mathbf{p} \in A \in \mathcal{A}_{\mathbf{p}_*}$, where $\boldsymbol{\mu}^* = (\mu_i)_{i \in \mathbb{A}}$ is the maximizer of $\mathcal{M}$. Take an arbitrary menu $A \in \mathcal{A}$ with $\mathbf{p}_* \in A$ such that $U_1(\mathbf{p}_*) \geq U_1(\mathbf{q})$ for all $\mathbf{q} \in A$.

Suppose that a choice correspondence C satisfies the axioms of *WARP with Contemplation*, *Closed Graph*, and *Consistency*. Fix $\lambda \in (0, 1)$. Let $\mathbf{r} = \lambda \mathbf{p}_1 + (1 - \lambda)\mathbf{p}_*$. Define a menu $B \in \mathcal{A}$ by $B := \{\mathbf{r}\} \cup (\lambda \mathbf{p}_0 + (1 - \lambda)A)$. Then, we have $U_1(\mathbf{r}) > U_1(\mathbf{q})$ for all $\mathbf{q} \in B \setminus \{\mathbf{r}\}$. Moreover, $U_0(\mathbf{r}) > U_0(\lambda \mathbf{p}_0 + (1 - \lambda)\mathbf{p}_*)$. Then, $B \notin \mathcal{A}_{\mathbf{p}_*}$. By the axiom of *Consistency*, $B \succ B \setminus \{\mathbf{r}\}$. As $\lambda \to 0$, by the axiom of *Closed Graph*, we obtain $\mathbf{p}_* \in C(A)$.

Consider $\mathbf{q} \in A$ such that $U_1(\mathbf{q}) < U_1(\mathbf{p}_*)$. Let $\mathcal{C} = \{\mathbf{p} \in \mathcal{X} \mid U_1(\mathbf{q}) < U_1(\mathbf{p}) \leq U_1(\mathbf{p}_*)\}$. Let $B' = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{q}\}$. Then, $B' \succ B' \setminus \{\mathbf{p}_1\}$. Hence, $C(B') = \{\mathbf{p}_1\}$ by the axiom of *Consistency*.

Let $A' = A \cup \{\mathbf{p}_1\}$. Then, $A, A' \in \mathcal{A}_{\mathbf{p}_*}$. Notice that $\mathbf{p}_* \in \arg\max_{\mathbf{p} \in A'} U_1(\mathbf{p})$. We have $\mathbf{p}_* \in C(A')$. Hence, we obtain $\mathbf{p} \in C(A)$ if and only if $U_1(\mathbf{p}) \geq U_1(\mathbf{q})$ for all $\mathbf{q} \in A$.

We show the necessity part. For any $A \in \mathcal{A}$, let

$$C(A) = \arg\max_{\mathbf{p} \in A} \Big[\sum_{i \in \mathbb{A}} u_i(p_i) - \max_{\boldsymbol{\mu} \in \mathcal{M}(A)} \sum_{i \in \mathbb{A}} \mu_i \big(\max_{\mathbf{q} \in A} (u_i(q_i) - u_i(p_i)) \big) \Big].$$

Take two menus $A, B \in \mathcal{A}_{\mathbf{p}}$. By definition, $\mathbf{p} \in C(A)$. If $\mathbf{q} \in C(B)$, then $\mathbf{p} \in C(B)$. C satisfies the axiom of *WARP with Contemplation*. Consider a sequence of pairs $\{\mathbf{p}_n, A_n\}$ such that $\mathbf{p}_n \to \mathbf{p}$ and $A_n \to A$. Suppose that, for each n, $\mathbf{p}_n \in C(A_n)$. For any $\mathbf{q} \in A$, there exists $\mathbf{q}_n \in A_n$ such that $\mathbf{q}_n \to \mathbf{q}$. Since each u_i is continuous, we have, as $n \to \infty$, $U_1(\mathbf{p}) \geq U_1(\mathbf{q})$. Hence, $\mathbf{p} \in C(A)$, and then C satisfies the axiom of *Closed Graph*.

Take $A \in \mathcal{A}$ and $\mathbf{p} \in \mathcal{X}$ such that $A \cup \{\mathbf{p}\} \succ A$. Then, $U_1(\mathbf{p}) > U_1(\mathbf{q})$ for all $\mathbf{q} \in A$. Hence, $C(A \cup \{\mathbf{p}\}) = \{\mathbf{p}\}$. Thus, C satisfies the axiom of *Consistency*. □

References

Aliprantis, C. D., and Border, K. (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer Science and Business Media.

Debreu, G. (1959): *Theory of Value: An Axiomatic Analysis of General Equilibrium*, Yale University Press, New Haven, CT.

de Clippel, G., and Eliaz, K (2012): "Reason-based Choice: A Bargaining Rationale for the Attraction and Compromise Effects," *Theoretical Economics*: 7(1), 125-162.

Dekel, E., Lipman, B. L., and Rustichini, A. (2001): "Representing Preferences with a Unique Subjective State Space," *Econometrica*: 69(4), 891-934.

Dillenberger, D., and Sadowski, P. (2012): "Ashamed to be Selfish," *Theoretical Economics*: 7(1), 99-124.

Dillenberger, D., Lleras, J. S., Sadowski, P., and Takeoka, N. (2014): "A Theory of Subjective Learning," *Journal of Economic Theory*: 153, 287-312.

Ergin, H., and Sarver, T. (2010a): "A Unique Costly Contemplation Representation," *Econometrica*: 78(4), 1285-1339.

Ergin, H., and Sarver, T. (2010b): "Supplement to 'A Unique Costly Contemplation Representation'," *Econometrica*.

Gilboa, I. (2009): *Theory of Decision under Uncertainty* (Vol. 1), Cambridge: Cambridge University Press.

Gilboa, I., Postlewaite, A., and Schmeidler, D. (2012): "Rationality of Belief or: Why Savage's Axioms Are Neither Necessary Nor Sufficient For Rationality," *Synthese*: 187(1), 11-31.

Gul, F., and Pesendorfer, W. (2001): "Temptation and Self-Control," *Econometrica*: 69(6), 1403-1435.

Hayashi, T. (2008): "Regret Aversion and Opportunity Dependence," *Journal of Economic Theory*: 139(1), 242-268.

Huber, J., Payne, J. W., and Puto, C. (1982): "Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis," *Journal of Consumer Research*: 9(1), 90-98.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995): *Microeconomic Theory*, New York: Oxford University Press.

NOOR, J., AND TAKEOKA, N. (2015): "Menu-Dependent Self-Control," *Journal of Mathematical Economics*: 61, 1-20.

KEENEY, R. L., AND RAIFFA, H. (1976): *Decisions with Multiple Objectives: Preferences and Value Trade-Offs*, Wiley (2nd edition: 1993, Cambridge University Press).

KOPYLOV, I. (2009): "Finite Additive Utility Representations for Preferences over Menus," *Journal of Economic Theory*: 144(1), 354-374.

KRANTZ, D., LUCE, D., SUPPES, P., AND TVERSKY, A. (1971): *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*, Academic Press (2nd edition: 2007, Dover publications).

KREPS, D. M. (1979): "A Representation Theorem for "Preference for Flexibility"," *Econometrica: Journal of the Econometric Society*: 565-577.

OK, E. A., ORTOLEVA, P., AND RIELLA, G. (2015): "Revealed (P) Reference Theory," *American Economic Review*: 105(1), 299-321.

ROCKAFELLAR, R. T. (1970): *Convex Analysis*, NJ: Princeton University Press.

ROSEN, S. (1974): "Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition," *Journal of Political Economy*: 82(1), 34-55.

ROYDEN, H. L., AND FITZPATRICK, P. (1968): *Real Analysis*, New York: Macmillan.

SARVER, T. (2008): "Anticipating Regret: Why Fewer Options May Be Better," *Econometrica*: 76(2), 263-305.

SIMONSON, I. (1989): "Choice Based on Reasons: The Case of Attraction and Compromise effects," *Journal of Consumer Research*: 158-174.

STOVALL, J. E. (2010): "Multiple Temptations," *Econometrica*: 78(1), 349-376.

TSERENJIGMID, G. (2017): "Choosing with the Worst in Mind: A Reference-Dependent Model," Working Paper.

TVERSKY, A., AND SIMONSON, I. (1993): "Context-Dependent Preferences," *Management Scienc*: 39(10), 1179-1189.

ZAJONC, R. B. (1980): "Feeling and Thinking: Preferences Need No Inferences," *American Psychologist*: 35(2), 151-175.