CIRJE-F-1144

A Mean Field Game Approach to Equilibrium Pricing with Market Clearing Condition

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March 2020; Revised in October 2020

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First version: 6 March, 2020
This version: 3 October, 2020

Abstract

In this work, we study an equilibrium-based continuous asset pricing problem which seeks to form a price process endogenously by requiring it to balance the flow of sales-and-purchase orders in the exchange market, where a large number of agents $1 \leq i \leq N$ are interacting through the market price. Adopting a mean field game (MFG) approach, we find a special form of forward-backward stochastic differential equations of McKean-Vlasov type with common noise whose solution provides a good approximate of the market price. We show the convergence of the net order flow to zero in the large N-limit and get the order of convergence in N under some conditions. We also extend the model to a setup with multiple populations where the agents within each population share the same cost and coefficient functions but they can be different population by population.

Keywords : FBSDE of McKean-Vlasov type, common noise, general equilibrium

1 Introduction

One of the most important problems in the financial economics is to understand how the asset price processes are formed through the interaction among a large number of rational competitive agents. In this paper, using a stylized model of security exchange, we try to explicitly form an approximate market price process which balances the flow of sales-and-purchase orders from a large number of rational financial institutions. If we directly force the price process to balance the net order flow, the strategies of the agents become strongly coupled and the problem is hardly solvable. In fact, it is even unclear how to make the cost functions of the agents well-defined, since the market price results in a very complicated recursive functional of strategies of all the agents that makes it difficult to guarantee the convexity of the cost functions. In order to circumvent this problem, we make use of the recent developments of mean field games.

Since its inception brought by the pioneering works of Lasry & Lions [20, 21, 22] and Huang, Malhame & Caines [18], mean field game has rapidly developed into one of the most actively studied topics in the field of probability theory, applied mathematics, engineering, finance and economics. The greatest strength of the mean field game approach is to render notoriously

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difficult problems of stochastic differential games among many agents tractable by transforming it to a simpler form of stochastic control problem. There exist two approaches to the mean field games, one is analytic approach using partial differential equations (PDEs), and the others is probabilistic approach based on forward-backward stochastic differential equations (FBSDEs). For details of analytic approach and its applications, the interested readers may consult the works of Bensousssan, Frehse & Yam [3], Gomes, Nurbekyan & Pimentel [14], Achdou et.al. [1], Gomes, Pimentel & Voskanyan [15] and also Kolokoltsov & Malafeyev [19]. On the other hand, the probabilistic approach was developed by the series of works of Carmona & Delarue [4, 5, 6] and the recent two volumes of monograph [7, 8] provide its full mathematical details and many references for a wide array of applications of mean field games.

Interestingly, from the perspective of equilibrium asset pricing, the number of applications of mean field games is quite limited. In most of the existing literature, the authors have given a response function of the price process exogenously and searched an approximate Nash equilibrium among agents. See, for example, applications to optimal trading as well as liquidation of portfolio, exploitation of exhaustible resources and related issues among many agents [10, 11, 12, 23], or an application to electricity pricing with smart grids [2, 9]. In the work [17], the authors treat explicitly the balance of demand and supply in the oil market, but the demand is exogenously given as a function of the oil price. One notable exception is the work of Gomes & Saude [16], in which the authors explicitly force demand and supply to balance and endogenously construct the market clearing electricity price. They use the analytic approach and the resultant equilibrium price process becomes deterministic due to the absence of common noise.

In the current paper, we extend the work [16] by adopting the probabilistic approach. In order to understand the price processes, in particular those of financial assets, including systemic signals which impacts all the agents is crucially important. We find an interesting form of FBSDEs of McKean-Vlasov type with common noise as a limit problem. Although it involves dependence in conditional law, it only appears as a conditional expectation. This allows us to adopt the well-known Peng-Wu's continuation method [24] to prove the existence of a unique strong solution. The resultant candidate of the market price process is derived completely endogenously by the optimal trading strategies of the agents facing systemic information (including securities' coupon stream) as well as idiosyncratic noise. Another benefit of probabilistic approach is that it allows us to quantify the relation between the actual game with finite number of agents and its large population limit. In a similar manner to the standard mean field games in proving ε-Nash equilibrium, we show that the solution of the mean-field limit problem actually provides asymptotic market clearing in the large-N limit. Under additional integrability conditions, Glivenko-Cantelli convergence theorem in the Wasserstein distance even provides a specific order of convergence in terms of the number of agents N. We also discuss the extension of the model to the situation with multiple populations where the agents share the same cost and coefficient functions within each population but they can be different population by population. This will provide an important tool to study the price formation in the presence of different type of agents such as Buy-side and Sell-side institutions, for example.

The organization of the paper is as follows: After explaining the notation in Section 2, we give an intuitive derivation of the limit problem from the game of finite number of agents in Section 3, which motivates the readers to study the special type of FBSDEs of MKV-type. The solvability of the FBSDE is studied in Section 4. Using the derived regularity of the solution, we prove the asymptotic market clearing in Section 5. In Section 6, we discuss the extension of the model to the setup with multiple populations. Finally, in Section 7, we give concluding remarks. We discuss further extensions of the model and future directions of research.

2 Notation

We introduce (N+1) complete probability spaces:

$$(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0) \quad \text{and} \quad (\overline{\Omega}^i, \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^i)_{i=1}^N ,$$

endowed with filtrations $\overline{\mathbb{F}}^i := (\overline{\mathcal{F}}_t^i)_{t\geq 0}$, $i \in \{0, \cdots, N\}$. Here, $\overline{\mathbb{F}}^0$ is the completion of the filtration generated by d^0-dimensional Brownian motion $\boldsymbol{W}^0$ (hence right-continuous) and, for each $i \in \{1, \cdots, N\}$, $\overline{\mathbb{F}}^i$ is the complete and right-continuous augmentation of the filtration generated by d-dimensional Brownian motions $\boldsymbol{W}^i$ as well as a $\boldsymbol{W}^i$-independent n-dimensional square-integrable random variables (ξ^i). $(\xi^i)_{i=1}^N$ are supposed to have the same law. We also introduce the product probability spaces

$$\Omega^i = \overline{\Omega}^0 \times \overline{\Omega}^i, \quad \mathcal{F}^i, \quad \mathbb{F}^i = (\mathcal{F}_t^i)_{t\geq 0}, \quad \mathbb{P}^i \ , i \in \{1, \cdots, N\}$$

where $(\mathcal{F}^i, \mathbb{P}^i)$ is the completion of $(\overline{\mathcal{F}}^0 \otimes \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^0 \otimes \overline{\mathbb{P}}^i)$ and $\mathbb{F}^i$ is the complete and right-continuous augmentation of $(\overline{\mathcal{F}}_t^0 \otimes \overline{\mathcal{F}}_t^i)_{t\geq 0}$. In the same way, we define the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions as a product of $(\overline{\Omega}^i, \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^i; \overline{\mathbb{F}}^i)_{i=0}^N$.

Throughout the work, the symbol L and L_ϖ denote given positive constants, the symbol C a general positive constant which may change line by line. When we want to emphasize that C depends only on some specific variables, say a and b, we use the symbol $C(a, b)$. For a given constant $T > 0$, we use the following notation for frequently encountered spaces:
• $\mathbb{L}^2(\mathcal{G}; \mathbb{R}^d)$ denotes the set of $\mathbb{R}^d$-valued $\mathcal{G}$-measurable square integrable random variables.
• $\mathbb{S}^2(\mathbb{G}; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathbb{G}$-adapted continuous processes $\boldsymbol{X}$ satisfying

$$||X||_{\mathbb{S}^2} := \mathbb{E}\big[\sup_{t\in[0,T]} |X_t|^2\big]^{\frac{1}{2}} < \infty .$$

• $\mathbb{H}^2(\mathbb{G}; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathbb{G}$-progressively measurable processes $\boldsymbol{Z}$ satisfying

$$||Z||_{\mathbb{H}^2} := \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)\Big]^{\frac{1}{2}} < \infty .$$

• $\mathcal{L}(X)$ denotes the law of a random variable X.
• $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
• $\mathcal{P}_p(\mathbb{R}^d)$ with $p \geq 1$ is the subset of $\mathcal{P}(\mathbb{R}^d)$ with finite p-th moment; i.e., the set of $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfying

$$M_p(\mu) := \Big(\int_{\mathbb{R}^d} |x|^p \mu(dx)\Big)^{\frac{1}{p}} < \infty .$$

We always assign $\mathcal{P}_p(\mathbb{R}^d)$ with $(p \geq 1)$ the p-Wasserstein distance W_p, which makes $\mathcal{P}_p(\mathbb{R}^d)$ a complete separable metric space. As an important property, for any $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$, we have

$$W_p(\mu, \nu) = \inf\Big\{\mathbb{E}[|X - Y|^p]^{\frac{1}{p}}; \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu\Big\} , \tag{2.1}$$

where "inf" is taken over all random variables with laws equal to μ and ν, respectively. For more details, see Chapter 5 in [7]. We frequently omit the arguments such as $(\mathbb{G},\mathbb{R}^d)$ in the above definitions when there is no confusion from the context.

3 Intuitive Derivation of the Mean Field Problem

In this section, in order to introduce the special form of forward-backward stochastic differential equations of McKean-Vlasov type to be studied in this paper, we give a heuristic derivation of the mean-field limit problem from the corresponding equilibrium problem with finite number of agents. As a motivating example, we consider the equilibrium-based pricing problem of n types of securities, which are continuously traded in the security exchange in the presence of a large number of participating agents indexed by $i\in\{1,\cdots,N\}$. Every agent is supposed to have many small clients who can only trade directly to the agent via over-the-counter markets (OTC) and have no access to the security exchange.

We suppose that each agent $i\in\{1,\cdots,N\}$ tries to solve the problem

$$\inf_{\boldsymbol{\alpha}^i\in\mathbb{A}^i} J^i(\boldsymbol{\alpha}^i) \tag{3.1}$$

with some cost functional

$$J^i(\boldsymbol{\alpha}^i):=\mathbb{E}\Big[\int_0^T f(t,X_t^i,\alpha_t^i,\varpi_t,c_t^0,c_t^i)dt+g(X_T^i,\varpi_T,c_T^0,c_T^i)\Big]\ ,$$

subject to the dynamic constraint:

$$dX_t^i=\Big(\alpha_t^i+l(t,\varpi_t,c_t^0,c_t^i)\Big)dt+\sigma_0(t,\varpi_t,c_t^0,c_t^i)dW_t^0+\sigma(t,\varpi_t,c_t^0,c_t^i)dW_t^i$$

with $X_0^i=\xi^i\in\mathbb{L}^2(\overline{\mathcal{F}}_0^i;\mathbb{R}^n)$. Here, $(X_t^i)_{t\ge0}$ is an $\mathbb{R}^n$-valued process denoting the time-t position of the n securities for the agent i with the initial position ξ^i. $(c_t^0)_{t\ge0}\in\mathbb{H}^2(\overline{\mathbb{F}}^0;\mathbb{R}^n)$ denotes the coupon payments from the securities or the market news commonly available to all the agents, while $(c_t^i)_{t\ge0}\in\mathbb{H}^2(\overline{\mathbb{F}}^i;\mathbb{R}^n)$ denotes some independent factors affecting only on the agent i. Moreover, $(c_t^i)_{t\ge0}$ are also assumed to have the common law for all $1\le i\le N$. We further suppose c_T^0 and c_T^i are square integrable to handle the terminal cost g. Each agent controls $(\alpha_t^i)_{t\ge0}$ denoting the trading speed though the security exchange. The remaining terms (l,σ_0,σ) denote the order flow to the agent from his/her clients through over-the-counter (OTC) markets. $(\varpi_t)_{t\ge0}$ is the market price of the n securities. The space of admissible strategies $\mathbb{A}^i$ of the agent i is the set of processes $(\alpha_t^i)_{t\ge0}$ adapted to the complete right-continuous augmentation of the filtration $\big(\sigma\{\varpi_s:s\le t\}\vee\mathcal{F}_t^i\big)_{t\ge0}$ satisfying

$$\mathbb{E}\int_0^T|\alpha_t^i|^2dt<\infty\ .$$

In contrast to the standard optimization problems with a given market price process, we want to understand the fundamental mechanism of financial market which determines the market price by the equilibrium condition. The equilibrium price $(\varpi_t)_{t\ge0}$ adapted to the filtration $\mathbb{F}$ is determined *endogenously* so that the optimal strategies of the agents $(\widehat{\alpha}_t^i)_{i=1}^N$ satisfy the market clearing condition for every $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\sum_{i=1}^{N} \widehat{\alpha}_t^i = 0 \ , \tag{3.2}$$

which denotes the balance point of demand and supply at the security exchange.

Although we have already made simplistic assumptions such that the cost functions as well as the coefficient functions of all the agents are common, the problem is still hardly solvable. Due to the clearing condition (3.2), we cannot adopt an open-loop equilibrium approach. In particular, $(\varpi_t)_{t\geq 0}$ becomes a complicated functional of the agents' trading strategies and hence the problem for each agent is highly recursive with respect to $(\alpha_t^i)_{t\geq 0, 1\leq i\leq N}$. It is even unclear how to guarantee the cost function well-defined by making it convex with respect to the controls.

In order to obtain some insight, let us consider a much simpler situation. It is natural to suppose that the impact to the market price $(\varpi_t)_{t\in[0,T]}$ from the individual agent becomes negligibly small when N is sufficiently large. Moreover, $(\varpi_t)_{t\in[0,T]}$ is likely to be given by $\overline{\mathbb{F}}^0$-progressively measurable process since the effects from the idiosyncratic parts from many agents are expected to be canceled out. If this is the case, the problem for each agent reduces to the standard stochastic optimal control problem in a given random environment $(\varpi_t, c_t^0, c_t^i)_{t\in[0,T]}$ with $\mathbb{F}^i$-adapted trading strategy $(\alpha_t^i)_{t\in[0,T]}$. Let us first investigate this simple problem in details. We introduce the cost functions: $f : [0,T]\times(\mathbb{R}^n)^5 \to \mathbb{R}$, $g : (\mathbb{R}^n)^4 \to \mathbb{R}$, $\overline{f} : [0,T]\times(\mathbb{R}^n)^4 \to \mathbb{R}$ and $\overline{g} : (\mathbb{R}^n)^3 \to \mathbb{R}$, which are measurable functions such that

$$\begin{aligned}
&f(t,x,\alpha,\varpi,c^0,c) := \langle \varpi, \alpha\rangle + \frac{1}{2}\langle \alpha, \Lambda\alpha\rangle + \overline{f}(t,x,\varpi,c^0,c),\\
&g(x,\varpi,c^0,c) := -\delta\langle\varpi, x\rangle + \overline{g}(x,c^0,c) \ .
\end{aligned}$$

Assumption 3.1. (MFG-a)
(i) Λ *is a positive definite* $n\times n$ *symmetric matrix with* $\underline{\lambda} I_{n\times n} \leq \Lambda \leq \overline{\lambda} I_{n\times n}$ *in the sense of 2nd-order form where* $\underline{\lambda}$ *and* $\overline{\lambda}$ *are some constants satisfying* $0<\underline{\lambda}\leq\overline{\lambda}$.
(ii) For any (t,x,ϖ,c^0,c),

$$|\overline{f}(t,x,\varpi,c^0,c)| + |\overline{g}(x,c^0,c)| \leq L(1+|x|^2+|\varpi|^2+|c^0|^2+|c|^2) \ .$$

(iii) $\overline{f}$ *and* $\overline{g}$ *are continuously differentiable in* x *and satisfy, for any* (t,x,x',ϖ,c^0,c),

$$|\partial_x\overline{f}(t,x',\varpi,c^0,c) - \partial_x\overline{f}(t,x,\varpi,c^0,c)| + |\partial_x\overline{g}(x',c^0,c) - \partial_x\overline{g}(x,c^0,c)| \leq L|x'-x| \ ,$$

and $|\partial_x\overline{f}(t,x,\varpi,c^0,c)| + |\partial_x\overline{g}(x,c^0,c)| \leq L(1+|x|+|\varpi|+|c^0|+|c|)$.
(iv)The functions $\overline{f}$ *and* $\overline{g}$ *are convex in* x *in the sense that for any* (t,x,x',ϖ,c^0,c),

$$\begin{aligned}
&\overline{f}(t,x',\varpi,c^0,c) - \overline{f}(t,x,\varpi,c^0,c) - \langle x'-x, \partial_x\overline{f}(t,x,\varpi,c^0,c)\rangle \geq \frac{\gamma^f}{2}|x'-x|^2 \ ,\\
&\overline{g}(x',c^0,c) - \overline{g}(x,c^0,c) - \langle x'-x, \partial_x\overline{g}(x,c^0,c)\rangle \geq \frac{\gamma^g}{2}|x'-x|^2 \ ,
\end{aligned}$$

with some constants $\gamma^f, \gamma^g \geq 0$.
(v) l, σ_0, σ *are the measurable functions defined on* $[0,T]\times(\mathbb{R}^n)^3$ *and are* $\mathbb{R}^n, \mathbb{R}^{n\times d^0}$ *and* $\mathbb{R}^{n\times d}$-

valued, respectively. Moreover they satisfy the linear growth condition:

$$|(l,\sigma_0,\sigma)(t,\varpi,c^0,c)| \le L(1+|\varpi|+|c^0|+|c|)$$

for any (t,ϖ,c^0,c).
(vi) $\delta\in[0,1)$ *is a given constant.*

The first term $\langle \varpi,\alpha\rangle$ of f denotes the direct cost incurred by the sales and purchase of the securities and the second term $\frac{1}{2}\langle\alpha,\Lambda\alpha\rangle$ is some fee to be paid to the exchange depending on the trading speed, or may be interpreted as some internal cost. The first term of g denotes the mark-to-market value at the closing time with some discount factor $\delta<1$.[1] $\overline{f}$ and $\overline{g}$ denote the running as well as the terminal cost which are affected by the market price, coupon streams, or the news.

Remark 3.1. *If we think* c^0 *as a coupon stream of the securities, one may consider for example,*

$$\overline{f}(t,x,\varpi,c^0,c) = -\langle c^0,x\rangle + \overline{f}'(t,x,\varpi,c)$$

as a running cost with an appropriate measurable function $\overline{f}'$*. For securities with a given maturity* T *with exogenously specified payoff* c^0*, such as bonds and futures, it is natural to consider*

$$g(x,c^0)=\overline{g}(x,c^0)=-\langle c^0,x\rangle$$

as the terminal cost.

For this problem, the (reduced) Hamiltonian is given by

$$H(t,x,y,\alpha,\varpi,c^0,c)=\langle y,\alpha+l(t,\varpi,c^0,c)\rangle + f(t,x,\alpha,\varpi,c^0,c)\ .$$

Since $\partial_\alpha H(t,x,y,\alpha,\varpi,c^0,c)=y+\varpi+\Lambda\alpha$, the minimizer of the Hamiltonian is

$$\widehat{\alpha}(y,\varpi) := -\overline{\Lambda}(y+\varpi) \tag{3.3}$$

where $\overline{\Lambda}:=\Lambda^{-1}$. The adjoint FBSDE associated with the stochastic maximal principle for each agent $1\le i\le N$ is thus given by,

$$\begin{aligned} dX_t^i &= \Big(\widehat{\alpha}(Y_t^i,\varpi_t)+l(t,\varpi_t,c_t^0,c_t^i)\Big)dt+\sigma_0(t,\varpi_t,c_t^0,c_t^i)dW_t^0+\sigma(t,\varpi_t,c_t^0,c_t^i)dW_t^i\ ,\\ dY_t^i &= -\partial_x\overline{f}(t,X_t^i,\varpi_t,c_t^0,c_t^i)dt+Z_t^{i,0}dW_t^0+Z_t^idW_t^i\ , \end{aligned} \tag{3.4}$$

with $X_0^i=\xi^i$ and $Y_T^i=\partial_x g(X_T^i,\varpi_T,c_T^0,c_T^i)$.

Theorem 3.1. *Under Assumption* (MFG-a) *and a given* $(\varpi_t)_{t\in[0,T]}\in\mathbb{H}^2(\overline{\mathbb{F}}^0;\mathbb{R}^n)$*, the problem* (3.1) *for each agent is uniquely characterized by the FBSDE* (3.4) *which is strongly solvable with a unique solution* $(X^i,Y^i,Z^{i,0},Z^i)\in\mathbb{S}^2(\mathbb{F}^i;\mathbb{R}^n)\times\mathbb{S}^2(\mathbb{F}^i;\mathbb{R}^n)\times\mathbb{H}^2(\mathbb{F}^i;\mathbb{R}^{n\times d^0})\times\mathbb{H}^2(\mathbb{F}^i;\mathbb{R}^{n\times d})$.

Proof. Since the cost functions are jointly convex with (x,α) and strictly convex in α, the problem is the special situation investigated in Section 1.4.4 in [8]. Note that, in our case, the diffusion terms σ_0,σ are independent of (X^i,α^i). The proof is the direct result of Theorem 1.60 in the same reference. □

[1] We shall see that the condition $\delta<1$ is necessary to obtain well-defined terminal condition for the limit problem.

Using the above solution, the optimal strategy of each agent is given by

$$\widehat{\alpha}_t^i = -\overline{\Lambda}(Y_t^i + \varpi_t), \quad t \in [0,T] \ .$$

Let us check the market clearing condition. In the current situation, (3.2) is equivalent to

$$\varpi_t = -\frac{1}{N}\sum_{i=1}^N Y_t^i$$

which is of course inconsistent with the our simplifying assumption that requires $(\varpi_t)_{t\geq 0}$ to be an $\overline{\mathbb{F}}^0$-adapted process. However, in the current setup, for any $t \in [0,T]$, $(Y_t^i)_{i=1}^N$ are exchangeable random variables due to the construction of the probability space, common coefficient functions, and the fact that $(\xi^i)_{i=1}^N$ as well as $(c_t^i, t \in [0,T])_{i=1}^N$ are assumed to be i.i.d. Thus De Finetti's theory of exchangeable sequence of random variables tells,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N Y_t^i = \mathbb{E}\Bigl[Y_t^1 | \bigcap_{k\geq 1} \sigma\{Y_t^j, j \geq k\}\Bigr] \quad \text{a.s.}$$

See for example Theorem 2.1 in [8]. It also seems natural to expect that the tail σ-field is reduced to $\overline{\mathcal{F}}_t^0$. Therefore we can expect that, in the large-N limit, the market price of the securities may be given by $\varpi_t = -\mathbb{E}[Y_t^1|\overline{\mathcal{F}}_t^0]$.

The above observation motivates us to consider the following FBSDE:

$$\begin{aligned}
dX_t &= \Bigl(\widehat{\alpha}\bigl(Y_t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]\bigr) + l\bigl(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0], c_t^0, c_t\bigr)\Bigr)dt \\
&\quad + \sigma_0\bigl(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0], c_t^0, c_t\bigr)dW_t^0 + \sigma\bigl(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0], c_t^0, c_t\bigr)dW_t^1 \ , \\
dY_t &= -\partial_x \overline{f}\bigl(t, X_t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0], c_t^0, c_t\bigr)dt + Z_t^0 dW_t^0 + Z_t dW_t^1 \ ,
\end{aligned}$$

with $X_0 = \xi$ with $Y_T = \dfrac{\delta}{1-\delta}\mathbb{E}\bigl[\partial_x \overline{g}(X_T, c_T^0, c_T)|\overline{\mathcal{F}}_T^0\bigr] + \partial_x \overline{g}(X_T, c_T^0, c_T)$. To simplify the notation, we have omitted the superscript 1 from Y^1, X^1, ξ^1 and c^1. Let us remark on the terminal condition. $Y_T = \partial_x g(X_T, -\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0], c_T^0, c_T)$ is not yet fully specified. Taking the conditional expectation in the both sides gives

$$\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0] = \delta\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0] + \mathbb{E}\bigl[\partial_x \overline{g}(X_T, c_T^0, c_T)|\overline{\mathcal{F}}_T^0\bigr] \ ,$$

which implies $\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0] = \frac{1}{1-\delta}\mathbb{E}\bigl[\partial_x \overline{g}(X_T, c_T^0, c_T)|\overline{\mathcal{F}}_T^0\bigr]$. Substituting this expression for $\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0]$ in $\partial_x g$, we get the above specification of the terminal condition.

This is the FBSDE we are going to study in the following. It is of McKean-Vlasov type with common noise, and similar to the FBSDEs relevant for the extended mean field games. In the following, we are going to prove the existence of a unique solution to the above FBSDE under appropriate conditions and then show that $-\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]$ is actually a good approximate of the market price by investigating how accurately it achieves the market clearing condition (3.2) when N increases.

4 Solvability of the mean-field FBSDE

We now investigate the solvability of the FBSDE derived in the last section

$$\begin{aligned}
dX_t &= \Big(\widehat{\alpha}\big(Y_t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t]\big) + l\big(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t], c^0_t, c_t\big)\Big)dt \\
&\quad + \sigma_0\big(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t], c^0_t, c_t\big)dW^0_t + \sigma\big(t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t], c^0_t, c_t\big)dW^1_t \ , \\
dY_t &= -\partial_x \overline{f}\big(t, X_t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t], c^0_t, c_t\big)dt + Z^0_t dW^0_t + Z_t dW^1_t \ ,
\end{aligned} \tag{4.1}$$

with $X_0 = \xi$ with $Y_T = \dfrac{\delta}{1-\delta}\mathbb{E}\big[\partial_x \overline{g}(X_T, c^0_T, c_T)|\overline{\mathcal{F}}^0_T\big] + \partial_x \overline{g}(X_T, c^0_T, c_T)$. $\widehat{\alpha}$ is defined as in (3.3). $(c^0_t)_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^0; \mathbb{R}^n)$ and $(c_t)_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^1; \mathbb{R}^n)$ with square integrable c^0_T, c_T are given as inputs. Let us remind the notation to write $\xi = \xi^1$ and $c = c^1$.

4.1 Unique existence for small T

Assumption 4.1. (MFG-b)
For any $(t, x, c^0, c) \in [0, T] \times (\mathbb{R}^n)^3$ and any $\varpi, \varpi' \in \mathbb{R}^n$, the coefficient functions l, σ_0, σ and $\overline{f}$ satisfy

$$|(l, \sigma_0, \sigma)(t, \varpi, c^0, c) - (l, \sigma_0, \sigma)(t, \varpi', c^0, c)| + |\partial_x \overline{f}(t, x, \varpi, c^0, c) - \partial_x \overline{f}(t, x, \varpi', c^0, c)| \leq L_\varpi |\varpi - \varpi'| \ .$$

Due to the Lipschitz continuity and the absence of (Z^0, Z) in the diffusion coefficients of the forward SDE, we have the following short-term existence result.

Theorem 4.1. *Under Assumptions (MFG-a,b), there exists some constant $\tau > 0$ which depends only on $(L, L_\varpi, \underline{\lambda}, \delta)$ such that for any $T \leq \tau$, there exists a unique strong solution $(X, Y, Z^0, Z) \in \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d^0}) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d})$ to the FBSDE* (4.1)*.*

Proof. Although there exist terms involving $\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t]$, one can adopt the standard technique for the Lipschitz FBSDE. See, for example, the proof of Theorem 1.45 [8]. □

4.2 Unique existence for general T

In order to obtain existence result for general T, we are going to apply the technique developed by Peng & Wu [24]. In the case of the standard optimization problem, the joint convexity in the state and control variables combined with strict convexity in the control variable are enough to obtain the unique existence. Interestingly however, we need a strict convexity also in the state variable X in our problem. As we shall see, this is because the term $-\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t]$ which appears due to the clearing condition weakens the convexity.

Assumption 4.2. *(MFG-c1)*
(i) The functions σ_0 and σ are independent of the argument ϖ.
(ii) For any $t \in [0, T]$, any random variables $x, x', c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ and any sub-σ-field $\mathcal{G} \subset \mathcal{F}$, the function l satisfies the monotone condition with some positive constant $\gamma^l > 0$:

$$\mathbb{E}\Big[\langle l(t, \mathbb{E}[x|\mathcal{G}], c^0, c) - l(t, \mathbb{E}[x'|\mathcal{G}], c^0, c), x - x'\rangle\Big] \geq \gamma^l \mathbb{E}\big[\mathbb{E}[x - x'|\mathcal{G}]^2\big] \ .$$

(iii) There exists a strictly positive constant γ satisfying $0 < \gamma \leq \Big(\gamma^f - \frac{L_\varpi^2}{4\gamma^l}\Big) \wedge \gamma^g$. Moreover, for any $x, x', c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ and any sub-σ-field $\mathcal{G} \subset \mathcal{F}$, the function $\overline{g}$ satisfies

$$\gamma^g \mathbb{E}[|x - x'|^2] + \frac{\delta}{1-\delta} \mathbb{E}\Big[\langle \mathbb{E}\big[\partial_x \overline{g}(x, c^0, c) - \partial_x \overline{g}(x', c^0, c) | \mathcal{G}\big], x - x' \rangle\Big] \geq \gamma \mathbb{E}[|x - x'|^2] \ .$$

Remark 4.1. *If l and $\partial_x \overline{g}$ have separable forms such as $h(x) + h^c(c^0, c)$ with some functions h and h^c, then the conditions (ii) and (iii) are satisfied when the function h is monotone. Economically speaking, the condition (ii) implies that the demand from the individual OTC clients of each agent toward the security decreases when its market price rises.*

The next theorem is the first main existence result.

Theorem 4.2. *Under Assumptions* (MFG-a,b,c1)*, there exists a unique strong solution $(X, Y, Z^0, Z) \in \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n \times d^0}) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n \times d})$ to the FBSDE* (4.1).

Proof. In order to simplify the notation, let us define the functionals B, F and G for any $y, x, c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ by

$$\begin{aligned}
B(t, y, c^0, c) &:= \Big(-\overline{\Lambda}(y - \mathbb{E}[y|\overline{\mathcal{F}}^0_t]) + l(t, -\mathbb{E}[y|\overline{\mathcal{F}}^0_t], c^0, c)\Big), \\
F(t, x, y, c^0, c) &:= -\partial_x \overline{f}\big(t, x, -\mathbb{E}[y|\overline{\mathcal{F}}^0_t], c^0, c\big), \\
G(x, c^0, c) &:= \frac{\delta}{1-\delta} \mathbb{E}\big[\partial_x \overline{g}(x, c^0, c) | \overline{\mathcal{F}}^0_T\big] + \partial_x \overline{g}(x, c^0, c) \ .
\end{aligned} \tag{4.2}$$

With the convention $\Delta y := y - y'$, $\Delta x := x - x'$, one can easily confirms

$$\begin{aligned}
&\mathbb{E}\big[\langle B(t, y, c^0, c) - B(t, y', c^0, c), \Delta y \rangle\big] \leq -\gamma^l \mathbb{E}\big[\mathbb{E}[\Delta y|\overline{\mathcal{F}}^0_t]^2\big] \ , \\
&\mathbb{E}\big[\langle F(t, x, y, c^0, c) - F(t, x', y', c^0, c), \Delta x \rangle\big] \leq -\Big(\gamma^f - \frac{L_\varpi^2}{4\gamma^l}\Big) \mathbb{E}[|\Delta x|^2] + \gamma^l \mathbb{E}\big[\mathbb{E}[\Delta y|\overline{\mathcal{F}}^0_t]^2\big] \ , \\
&\mathbb{E}\big[\langle G(x, c^0, c) - G(x', c^0, c), \Delta x \rangle\big] \geq \gamma \mathbb{E}[|\Delta x|^2],
\end{aligned} \tag{4.3}$$

where the first estimate follows from (MFG-c1)(ii) and Jensen's inequality, the second from (MFG-a)(iv), (MFG-b) and Cauchy-Schwarz inequality. The third one is the direct consequence of (MFG-c)(iii).

We first make the following hypothesis: there exists some constant $\varrho \in [0, 1)$ such that, for any $(I^b_t)_{t \geq 0}$, $(I^f_t)_{t \geq 0}$ in $\mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^n)$ and any $\eta \in \mathbb{L}^2(\mathcal{F}^1_T; \mathbb{R}^n)$, there exists a unique solution $(x^\varrho, y^\varrho, z^{0,\varrho}, z^\varrho) \in \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n \times d^0}) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n \times d})$ to the FBSDE:

$$\begin{aligned}
dx^\varrho_t &= \big(\varrho B(t, y^\varrho_t, c^0_t, c_t) + I^b_t\big) dt + \sigma_0(t, c^0_t, c_t) dW^0_t + \sigma(t, c^0_t, c_t) dW^1_t \ , \\
dy^\varrho_t &= -\big((1-\varrho)\gamma x^\varrho_t - \varrho F(t, x^\varrho_t, y^\varrho_t, c^0_t, c_t) + I^f_t\big) dt + z^{0,\varrho}_t dW^0_t + z^\varrho_t dW^1_t \ ,
\end{aligned} \tag{4.4}$$

with $x^\varrho_0 = \xi$ and $y^\varrho_T = \varrho G(x^\varrho_T, c^0_T, c_T) + (1-\varrho) x^\varrho_T + \eta$. Note that when $\varrho = 0$ we have a decoupled set of SDE and BSDE and hence the hypothesis trivially holds. Our goal is to extend the ϱ up to 1 by following Peng-Wu's continuation method [24]. Now, for an arbitrary set of inputs

$(x, y, z^0, z) \in \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n)^2 \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d^0}) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d})$ and constant $\zeta \in (0,1)$, consider

$$\begin{aligned} dX_t &= \big[\varrho B(t, Y_t, c_t^0, c_t) + \zeta B(t, y_t, c_t^0, c_t) + I_t^b\big]dt + \sigma_0(t, c_t^0, c_t)dW_t^0 + \sigma(t, c_t^0, c_t)dW_t^1 \ , \\ dY_t &= -\big[(1-\varrho)\gamma X_t - \varrho F(t, X_t, Y_t, c_t^0, c_t) + \zeta(-\gamma x_t - F(t, x_t, y_t, c_t^0, c_t)) + I_t^f\big]dt \\ &\quad + Z_t^0 dW_t^0 + Z_t dW_t^1 \ , \end{aligned} \tag{4.5}$$

with $X_0 = \xi$ and $Y_T = \varrho G(X_T, c_T^0, c_T) + (1-\varrho)X_T + \zeta(G(x_T, c_T^0, c_T) - x_T) + \eta$. The existence of the solution $(X, Y, Z^0, Z) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ is guaranteed by the previous hypothesis. We are going to prove the map $(x, y, z^0, z) \mapsto (X, Y, Z^0, Z)$ defined above becomes strict contraction when $\zeta > 0$ is chosen small enough.

For two set of inputs (x, y, z^0, z) and $(x', y', z^{0\prime}, z')$, let us denote the corresponding solutions to (4.5) by (X, Y, Z^0, Z) and $(X', Y', Z^{0\prime}, Z')$, respectively. We put $\Delta X_t := X_t - X_t'$, $\Delta Y_t := Y_t - Y_t'$ and similarly for the others. Applying Itô's formula to $\langle \Delta X_t, \Delta Y_t\rangle$ and using the estimates (4.3), we obtain

$$\begin{aligned} \mathbb{E}\big[\langle \Delta X_T, \Delta Y_T\rangle\big] &\le -\gamma \mathbb{E}\int_0^T |\Delta X_t|^2 dt \\ &\qquad + \zeta C \mathbb{E}\int_0^T \Big[|\Delta Y_t|(|\Delta y_t| + \mathbb{E}[\Delta y_t|\overline{\mathcal{F}}_t^0]) + |\Delta X_t|(|\Delta x_t| + \mathbb{E}[|\Delta y_t|\overline{\mathcal{F}}_t^0])\Big]dt \ , \\ \mathbb{E}\big[\langle \Delta X_T, \Delta Y_T\rangle\big] &\ge (\varrho\gamma + (1-\varrho))\mathbb{E}[|\Delta X_T|^2] - \zeta C \mathbb{E}\big[|\Delta X_T|(|\Delta x_T| + \mathbb{E}[|\Delta x_T||\overline{\mathcal{F}}_T^0])\big] \ , \end{aligned}$$

with some ϱ-independent constant C. Let us set $\gamma_c := \min(1, \gamma) > 0$. Then one easily confirms $0 < \gamma_c \le \varrho\gamma + (1-\varrho)$ for any $\varrho \in [0,1)$. Then the above estimates yields

$$\begin{aligned} \gamma_c \mathbb{E}\Big[|\Delta X_T|^2 + \int_0^T |\Delta X_t|^2 dt\Big] \quad &\le \quad \zeta C \mathbb{E}\big[|\Delta X_T|(|\Delta x_T| + \mathbb{E}[|\Delta x_T||\overline{\mathcal{F}}_T^0])\big] \\ &\qquad + \zeta C \mathbb{E}\int_0^T \Big[|\Delta Y_t|(|\Delta y_t| + \mathbb{E}[\Delta y_t|\overline{\mathcal{F}}_t^0]) + |\Delta X_t|(|\Delta x_t| + \mathbb{E}[|\Delta y_t|\overline{\mathcal{F}}_t^0])\Big]dt \ . \end{aligned}$$

Using Young's inequality and and a new constant C, we get

$$\mathbb{E}[|\Delta X_T|^2] + \mathbb{E}\int_0^T |\Delta X_t|^2 dtd \le \zeta C \mathbb{E}\int_0^T \big(|\Delta Y_t|^2 + (|\Delta x_t|^2 + |\Delta y_t|^2)\big)dt + \zeta C \mathbb{E}[|\Delta x_T|^2] \ . \tag{4.6}$$

Treating X, X' as inputs, the standard estimates for the Lipschitz BSDEs (see, for example, Theorem 4.2.3 in [25]) gives

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t\in[0,T]} |\Delta Y_t|^2 + \int_0^T (|\Delta Z_t^0|^2 + |\Delta Z_t|^2)dt\Big] \\ &\qquad \le C\mathbb{E}\Big[|\Delta X_T|^2 + \int_0^T |\Delta X_t|^2 dt\Big] + \zeta C \mathbb{E}\Big[|\Delta x_T|^2 + \int_0^T (|\Delta x_t|^2 + |\Delta y_t|^2)dt\Big] \ . \end{aligned}$$

Combining with (4.6) and choosing $\zeta > 0$ small, we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\Delta Y_t|^2 + \int_0^T (|\Delta Z_t^0|^2 + |\Delta Z_t|^2)dt\Big] \le \zeta C \mathbb{E}\Big[|\Delta x_T|^2 + \int_0^T (|\Delta x_t|^2 + |\Delta y_t|^2)dt\Big] \ . \tag{4.7}$$

By the similar procedures, we also have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t|^2\Big] \le \zeta C\mathbb{E}\Big[|\Delta x_T|^2 + \int_0^T (|\Delta x_t|^2 + |\Delta y_t|^2)dt\Big] \ . \tag{4.8}$$

From (4.7) and (4.8), we obtain

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t|^2 + \sup_{t\in[0,T]}|\Delta Y_t|^2 + \int_0^T (|\Delta Z_t^0|^2 + |\Delta Z_t|^2)dt\Big]\\ &\le \zeta C\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta x_t|^2 + \sup_{t\in[0,T]}|\Delta y_t|^2 + \int_0^T (|\Delta z_t^0|^2 + |\Delta z_t|^2)dt\Big].\end{aligned}$$

Thus there exists $\zeta > 0$, being independent of the size of ϱ, that makes the map $(x, y, z^0, z) \mapsto (X, Y, Z^0, Z)$ strict contraction. Therefore the initial hypothesis holds true for $(\varrho + \zeta)$, which establishes the existence. The uniqueness follows from the next proposition. $\square$

Proposition 4.1. *Given two set of inputs* $(\xi, c^0, c), (\xi', c^{0\prime}, c')$*, coefficients* $(\delta, \Lambda), (\delta', \Lambda')$ *and the coefficient functions* $(l, \sigma_0, \sigma, \overline{f}, \overline{g}), (l', \sigma_0', \sigma', \overline{f}', \overline{g}')$ *satisfying Assumptions* (MFG-a,b,c1)*, let us denote the corresponding solutions to* (4.1) *by* (X, Y, Z^0, Z) *and* $(X', Y', Z^{0\prime}, Z')$*, respectively. We also define the functionals* (B, F, G)*and* (B', F', G') *by* (4.2) *with corresponding coefficients, respectively. Then, we have the following stability result:*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t|^2 + \sup_{t\in[0,T]}|\Delta Y_t|^2 + \int_0^T (|\Delta Z_t^0|^2 + |\Delta Z_t|^2)dt\Big]\\ &\le C\mathbb{E}\Big[|\Delta\xi|^2 + |\overline{G}|^2 + \int_0^T \Big(|\overline{F}(t)|^2 + |\overline{B}(t)|^2 + |\overline{\sigma}_0(t)|^2 + |\overline{\sigma}(t)|^2\Big)dt\Big] \ ,\end{aligned}$$

where C is a constant depending only on T as well as the Lipschitz constants of the system, and

$$\begin{aligned}&\overline{B}(t) := B(t, Y_t', c_t^0, c_t) - B'(t, Y_t', c_t^{0\prime}, c_t'),\\ &\overline{F}(t) := F(t, X_t', Y_t', c_t^0, c_t) - F'(t, X_t', Y_t', c_t^{0\prime}, c_t'),\\ &(\overline{\sigma}_0, \overline{\sigma})(t) = (\sigma_0(t, c_t^0, c_t) - \sigma_0'(t, c_t^{0\prime}, c_t'), \sigma(t, c_t^0, c_t) - \sigma'(t, c_t^{0\prime}, c_t')),\\ &\overline{G} := G(X_T', c_T^0, c_T) - G'(X_T', c_T^{0\prime}, c_T') \ ,\end{aligned}$$

and $\Delta\xi := \xi - \xi'$, $\Delta X_t := X_t - X_t'$ *and similarly for the other variables.*

Proof. Let us put $\Delta B(t) := B(t, Y_t, c_t^0, c_t) - B(t, Y_t', c_t^0, c_t)$, $\Delta F(t) := F(t, X_t, Y_t, c_t^0, c_t) - F(t, X_t', Y_t', c_t^0, c_t)$ and $\Delta G := G(X_T, c_T^0, c_T) - G(X_T', c_T^0, c_T)$. We get by Itô's formula that

$$\begin{aligned}\mathbb{E}\big[\langle \Delta X_T, \Delta G + \overline{G}\rangle\big] &= \mathbb{E}\Big[\langle \Delta\xi, \Delta Y_0\rangle + \int_0^T \Big(\langle \overline{F}(t), \Delta X_t\rangle + \langle \overline{B}(t), \Delta Y_t\rangle\\ &+\langle \overline{\sigma}_0(t), \Delta Z_t^0\rangle + \langle \overline{\sigma}(t), \Delta Z_t\rangle + \big(\langle \Delta F(t), \Delta X_t\rangle + \langle \Delta B(t), \Delta Y_t\rangle\big)\Big)dt\Big] \ .\end{aligned}$$

Using (4.3), we obtain

$$\gamma\mathbb{E}\Big[|\Delta X_T|^2+\int_0^T|\Delta X_t|^2dt\Big]\le\mathbb{E}\Big[\langle\Delta\xi,\Delta Y_0\rangle-\langle\Delta X_T,\overline{G}\rangle$$
$$+\int_0^T\Big(\langle\overline{F}(t),\Delta X_t\rangle+\langle\overline{B}(t),\Delta Y_t\rangle+\langle\overline{\sigma}_0(t),\Delta Z_t^0\rangle+\langle\overline{\sigma}(t),\Delta Z_t\rangle\Big)dt\Big]~. \tag{4.9}$$

On the other hand, the standard estimates for Lipschitz SDEs and BSDEs give

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta Y_t|^2+\int_0^T(|\Delta Z_t^0|^2+|\Delta Z_t|^2)dt\Big]\\&\quad\le C\mathbb{E}\Big[|\overline{G}|^2+\int_0^T|\overline{F}(t)|^2dt\Big]+C\mathbb{E}\Big[|\Delta X_T|^2+\int_0^T|\Delta X_t|^2dt\Big]~,\\&\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t|^2\Big]\le C\mathbb{E}\Big[|\Delta\xi|^2+\int_0^T\big[|\overline{B}(t)|^2+|\overline{\sigma}_0(t)|^2+|\overline{\sigma}(t)|^2\big]dt\Big]+C\mathbb{E}\int_0^T|\Delta Y_t|^2dt~.\end{aligned}\tag{4.10}$$

Combining the above inequalities (4.9) and (4.10) gives

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t|^2+\sup_{t\in[0,T]}|\Delta Y_t|^2+\int_0^T(|\Delta Z_t^0|^2+|\Delta Z_t|^2)dt\Big]\\&\le C\mathbb{E}\Big[|\Delta\xi|^2+|\overline{G}|^2+\int_0^T\big[|\overline{F}(t)|^2+|\overline{B}(t)|^2+|\overline{\sigma}_0(t)|^2+|\overline{\sigma}(t)|^2\big]dt\Big]\\&+C\mathbb{E}\Big[\langle\Delta\xi,\Delta Y_0\rangle-\langle\Delta X_T,\overline{G}\rangle+\int_0^T\big[\langle\overline{F}(t),\Delta X_t\rangle+\langle\overline{B}(t),\Delta Y_t\rangle+\langle\overline{\sigma}_0(t),\Delta Z_t^0\rangle+\langle\overline{\sigma}(t),\Delta Z_t\rangle\big]dt\Big]~.\end{aligned}$$

Now simple application of Young's inequality establishes the claim. □

Corollary 4.1. *Under Assumptions* (MFG-a,b,c1)*, the solution* (X,Y,Z^0,Z) *to the FBSDE* (4.1) *satisfies the following estimate:*

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^2+\sup_{t\in[0,T]}|Y_t|^2+\int_0^T(|Z_t^0|^2+|Z_t|^2)dt\Big]&\le C\mathbb{E}\Big[|\xi|^2+|\partial_x\overline{g}(0,c_T^0,c_T)|^2\\&+\int_0^T\Big(|\partial_x\overline{f}(t,0,0,c_t^0,c_t)|^2+|l(t,0,c_t^0,c_t)|^2+|(\sigma_0,\sigma)(t,c_t^0,c_t)|^2\Big)dt\Big]~,\end{aligned}$$

where C is a constant depending only on T,δ and Lipschitz constants of the system.

Proof. By quick inspection of the proof for Proposition 4.1, one can confirm that as long as there exists a solution $(X',Y',Z^{0\prime},Z')\in\mathbb{S}^2\times\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{H}^2$, their coefficients need not satisfy Assumption (MFG-a,b,c1). In particular, by putting ξ' and $(l',\sigma_0',\sigma',\overline{f}',\overline{g}')$ all zero, we have a trivial solution $(X',Y',Z^{0'},Z')=(0,0,0,0)$. The desired estimate now follows from Proposition 4.1. □

Securities of maturity T with exogenously specified payoff

If we consider the exchange markets of bonds and futures, or other financial derivatives with maturity T, those securities cease to exist at T after paying exogenously specified amount of cash c^0_T. In this case, it is natural to consider with $\delta = 0$ and

$$g(x, c^0) = \overline{g}(x, c^0) := -\langle c^0, x\rangle \ , \tag{4.11}$$

since there is no reason to put penalty on the outstanding volume at T. In this case, the terminal function g in (4.11) does not have the strict convexity. Fortunately, even in this case, we can prove the unique existence as well as the stability result of the same form.

Assumption 4.3. *(MFG-c2)*
(i) The functions σ_0 and σ are independent of the argument ϖ.
(ii) For any $t \in [0, T]$, any random variables $x, x', c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ and any sub-σ-field $\mathcal{G} \subset \mathcal{F}$, the function l satisfies the monotone condition with some positive constant $\gamma^l > 0$:

$$\mathbb{E}\Big[\langle l(t, \mathbb{E}[x|\mathcal{G}], c^0, c) - l(t, \mathbb{E}[x'|\mathcal{G}], c^0, c), x - x'\rangle\Big] \geq \gamma^l \mathbb{E}\big[\mathbb{E}[x - x'|\mathcal{G}]^2\big] \ .$$

(iii) $\gamma := \gamma^f - \frac{L_\varpi^2}{4\gamma^l}$ is strictly positive and the terminal function g is given by (4.11) *with $\delta = 0$.*

Theorem 4.3. *Under Assumptions* (MFG-a,b,c2)*, there exists a unique strong solution $(X, Y, Z^0, Z) \in \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}^1; \mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d^0}) \times \mathbb{H}^2(\mathbb{F}^1; \mathbb{R}^{n\times d})$ to the FBSDE* (4.1)*. Moreover, the same form of stability and $\mathbb{L}^2$ estimates given in Proposition 4.1 and Corollary 4.1 hold.*

Proof. Note that, in this case, the terminal condition for the BSDE is independent of X_T. Thus, as in Theorem 2.3 [24], we put $y^\varrho_T = Y_T = -c^0_T$ in (4.4) and (4.5), respectively. Using the fact that $\langle \Delta X_T, \Delta Y_T\rangle = 0$, one can follow the same arguments to get the desired result. The proof of the stability result can also be done in almost exactly the same way. $\square$

5 Asymptotic Market Clearing

We are now ready to investigate if our FBSDE (4.1) actually provides a good approximate of the market price and if so, how accurate it is. By Theorem 3.1, if we use $(-\mathbb{E}\big[Y_t|\overline{\mathcal{F}}^0_t\big])_{t\in[0,T]}$ as the input $(\varpi_t)_{t\in[0,T]}$, where $(Y_t)_{t\in[0,T]}$ is the unique solution to the FBSDE (4.1) with the convention $\xi = \xi^1$ and $c = c^1$, the optimal strategy of the individual agent is given by

$$\widehat{\alpha}^i_{\rm mf}(t) := \widehat{\alpha}(Y^i_t, -\mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t]) = -\overline{\Lambda}(Y^i_t - \mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t]) \tag{5.1}$$

where $(Y^i_t)_{t\in[0,T]}$ is the solution to (3.4) with $(\varpi_t = -\mathbb{E}\big[Y_t|\overline{\mathcal{F}}^0_t\big])_{t\in[0,T]}$.

Theorem 5.1. *If the conditions for Theorem 4.1, Theorem 4.2 or Theorem 4.3 are satisfied then we have*

$$\lim_{N\to\infty} \mathbb{E}\int_0^T \Big|\frac{1}{N}\sum_{i=1}^N \widehat{\alpha}^i_{\rm mf}(t)\Big|^2 dt = 0 \ .$$

Moreover if there exists some constant Γ *such that* $\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t|^q\big]^{\frac{1}{q}} \leq \Gamma < \infty$ *for some* $q > 4$, *then there exists some constant C independent of N such that*

$$\mathbb{E}\int_0^T \Big|\frac{1}{N}\sum_{i=1}^N \widehat{\alpha}^i_{\rm mf}(t)\Big|^2 dt \leq C\Gamma^2\epsilon_N, \tag{5.2}$$

where $\epsilon_N := N^{-2/\max(n,4)}\big(1+\log(N)\mathbf{1}_{\{n=4\}}\big)$.

Proof. Let us consider the following set of FBSDEs with $1 \le i \le N$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ constructed in Section 2.

$$\begin{aligned} d\underline{X}^i_t &= \Big(-\overline{\Lambda}(\underline{Y}^i_t - \mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t]) + l(t, -\mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t], c^0_t, c^i_t)\Big)dt \\ &\qquad +\sigma_0(t, -\mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t], c^0_t, c^i_t)dW^0_t + \sigma(t, -\mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t], c^0_t, c^i_t)dW^i_t, \\ d\underline{Y}^i_t &= -\partial_x\overline{f}(t, \underline{X}^i_t, -\mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t], c^0_t, c^i_t)dt + \underline{Z}^{i,0}_t dW^0_t + \underline{Z}^i_t dW^i_t, \end{aligned}$$

with $\underline{X}^i_0 = \xi^i$ and $\underline{Y}^i_T = \delta/(1-\delta)\mathbb{E}[\partial_x\overline{g}(\underline{X}^i_T, c^0_T, c^i_T)|\overline{\mathcal{F}}^0_T] + \partial_x\overline{g}(\underline{X}^i_T, c^0_T, c^i_T)$. Thanks to the existence of unique strong solution, Yamada-Watanabe Theorem for FBSDEs (see, Theorem 1.33 [8]), there exists some measurable function Φ such that for every $1 \le i \le N$,

$$(\underline{X}^i_t, \underline{Y}^i_t)_{t\in[0,T]} = \Phi\Big((c^0_t)_{t\in[0,T]}, (W^0_t)_{t\in[0,T]}, \xi^i, (c^i_t)_{t\in[0,T]}, (W^i_t)_{t\in[0,T]}\Big)\ .$$

Hence, conditionally on $\overline{\mathcal{F}}^0$, the set of proceses $(\underline{X}^i_t, \underline{Y}^i_t)_{t\in[0,T]}$ with $1 \le i \le N$ are independently and identically distributed. In particular, we have $\mathbb{P}$-a.s.

$$\begin{aligned} &\mathbb{E}[\underline{Y}^i_t|\overline{\mathcal{F}}^0_t] = \mathbb{E}[Y_t|\overline{\mathcal{F}}^0_t], \quad \forall t \in [0,T], \\ &\mathbb{E}[\partial_x\overline{g}(\underline{X}^i_T, c^0_T, c^i_T)|\overline{\mathcal{F}}^0_T] = \mathbb{E}[\partial_x\overline{g}(X_T, c^0_T, c_T)|\overline{\mathcal{F}}^0_T]\ . \end{aligned} \tag{5.3}$$

Note that, under the convention $\xi^1 = \xi$ and $c^1 = c$, we actually have $(\underline{X}^1, \underline{Y}^1) = (X, Y)$. From (5.3), we conclude that $(X^i_t, Y^i_t, Z^{i,0}_t, Z^i_t)_{t\in[0,T]} = (\underline{X}^i_t, \underline{Y}^i_t, \underline{Z}^{i,0}_t, \underline{Z}^i_t)_{t\in[0,T]}$ in $\mathbb{S}^2(\mathbb{F}^i) \times \mathbb{S}^2(\mathbb{F}^i) \times \mathbb{H}^2(\mathbb{F}^i) \times \mathbb{H}^2(\mathbb{F}^i)$. Therefore,

$$\frac{1}{N}\sum_{i=1}^N \widehat{\alpha}^i_{\rm mf}(t) = -\overline{\Lambda}\Big(\frac{1}{N}\sum_{i=1}^N \underline{Y}^i_t - \mathbb{E}[\underline{Y}^1_t|\overline{\mathcal{F}}^0_t]\Big)\ . \tag{5.4}$$

We can easily check that

$$\mathbb{E}\Big[W_2\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\underline{Y}^i_t}, \mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)\Big)^2\Big|\overline{\mathcal{F}}^0_t\Big] \le \frac{2}{N}\sum_{i=1}^N \mathbb{E}\big[|\underline{Y}^i_t|^2|\overline{\mathcal{F}}^0_t\big] + 2\mathbb{E}\big[|\underline{Y}^1_t|^2|\overline{\mathcal{F}}^0_t\big] = 4\mathbb{E}\big[|\underline{Y}^1_t|^2|\overline{\mathcal{F}}^0_t\big]\ .$$

Since $(\underline{Y}^i_t)_{1\le i\le N}$ are $\overline{\mathcal{F}}^0_t$-conditionally independently and identically distributed and also $\underline{Y}^1 \in \mathbb{S}^2$, the same arguments leading to (2.14) in [8] imply that the pointwise convergence holds:

$$\lim_{N\to\infty} \mathbb{E}\Big[W_2\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\underline{Y}^i_t}, \mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)\Big)^2\Big] = 0\ . \tag{5.5}$$

We are now going to show that the set of functions, $(f_N)_{N\in\mathbb{N}}$ defined by

$$[0,T] \ni t \mapsto f_N(t) := \mathbb{E}\big[W_2\big(\overline{\mu}_t, \mu_t\big)^2\big] \in \mathbb{R}$$

with $\overline{\mu}_t := \frac{1}{N}\sum_{i=1}^N \delta_{\underline{Y}^i_t}$ and $\mu_t := \mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)$ are precompact in the set $\mathcal{C}([0,T];\mathbb{R})$ endowed with the topology of uniform convergence. In fact, uniformly in N,

$$\sup_{t\in[0,T]}|f_N(t)|\le 4\sup_{t\in[0,T]}\mathbb{E}\big[|\underline{Y}^1_t|^2\big]\le C<\infty \tag{5.6}$$

where C is given by the estimate in Corollary 4.1. Moreover, for any $0\le t,s\le T$, Cauchy-Schwarz, (5.6) and the triangular inequalities give

$$\begin{aligned}|f_N(t)-f_N(s)|&\le\mathbb{E}\Big[\Big(W_2(\overline{\mu}_t,\mu_t)+W_2(\overline{\mu}_s,\mu_2)\Big)^2\Big]^{\frac12}\mathbb{E}\Big[\Big(W_2(\overline{\mu}_t,\mu_t)-W_2(\overline{\mu}_s,\mu_2)\Big)^2\Big]^{\frac12}\\ &\le C\mathbb{E}\Big[\Big(W_2(\overline{\mu}_t,\mu_t)-W_2(\overline{\mu}_s,\mu_s)\Big)^2\Big]^{\frac12}\le C\mathbb{E}\Big[W_2(\overline{\mu}_t,\overline{\mu}_s)^2+W_2(\mu_t,\mu_s)^2\Big]^{\frac12}\ .\\ &\le C\mathbb{E}\Big[\frac1N\sum_{i=1}^N|\underline{Y}^i_t-\underline{Y}^i_s|^2+|\underline{Y}^1_t-\underline{Y}^1_s|^2\Big]^{\frac12}\\ &\le C\mathbb{E}\big[|\underline{Y}^1_t-\underline{Y}^1_s|^2\big]^{\frac12}\ ,\end{aligned}$$

uniformly in N, where we have used the fact that $(\underline{Y}^i)_{i\ge1}$ are conditionally i.i.d at the last inequality. Since $(\underline{Y}^1_t)_{t\in[0,T]}$ is a continuous process, the above estimate tells that $(f_N)_{N\in\mathbb{N}}$ is equicontinuous, which is also uniformly equicontinuous since we are working on the finite interval. Now, Arzela-Ascoli theorem implies the desired precompactness.

Combining with the pointwise convergence (5.5), we thus conclude

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\Big[W_2\Big(\frac1N\sum_{i=1}^N\delta_{\underline{Y}^i_t},\mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)\Big)^2\Big]=0\ . \tag{5.7}$$

From the definition of Wasserstein distance (2.1), we have

$$\Big|\frac1N\sum_{i=1}^N\underline{Y}^i_t-\mathbb{E}[\underline{Y}^1_t|\overline{\mathcal{F}}^0_t]\Big|\le W_1\Big(\frac1N\sum_{i=1}^N\delta_{\underline{Y}^i_t},\mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)\Big)\ ,$$

and hence, from (5.4),

$$\mathbb{E}\int_0^T\Big|\frac1N\sum_{i=1}^N\widehat{\alpha}^i_{\rm mf}(t)\Big|^2dt\le C\sup_{t\in[0,T]}\mathbb{E}\Big[W_2\Big(\frac1N\sum_{i=1}^N\delta_{\underline{Y}^i_t},\mathcal{L}(\underline{Y}^1_t|\overline{\mathcal{F}}^0_t)\Big)^2\Big]\ . \tag{5.8}$$

The first conclusion now follows from (5.7). The latter claims directly follows from the expression (5.8) and the (Fourth Step) in the proof of Theorem 2.12 in [8]. □

Theorem 5.1 justifies our intuitive understanding and a special type of FBSDEs (4.1) derived in Section 3 as a reasonable model to approximate the market clearing price. When there exists higher integrability, Glivenko-Cantelli convergence theorem in the Wasserstein distance even provides a specific order ϵ_N of convergence in terms of the number of agents N (5.2). See Theorem 5.8 and Remark 5.9 in [7] for more details.

Remark 5.1. *Consider the situation treated in Theorem 4.3, for example, a market model of a Futures contract. If the contract pays unit amount of the underlying asset per contract whose value is exogenously given by c^0_T, our mean-field limit model* (4.1) *gives $Y_T=-c^0_T$. This means* *that the modeled Futures price satisfies $\varpi_T = -\mathbb{E}[Y_T|\overline{\mathcal{F}}^0_T] = c^0_T$, which guarantees the convergence of the modeled price to the value of the underlying asset at the maturity T. This is a crucially important feature that any market model of this type of securities must satisfy.*

6 Extension to Multiple Populations

The main limitation of the last model is that there exists only one type of agents who share the common cost functions as well as the coefficient functions for their state dynamics. Interestingly, it is rather straightforward to extend the model to the situation with multiple populations, where the agents in each population share the same cost and coefficient functions but they can be different population by population. From the perspective of the practical applications, this is a big advantage since we can analyze, for example, the interactions between the Sell-side and Buy-side institutions for financial applications, or consumers and producers for economic applications. For general issues of mean field games as well as mean field type control problems in the presence of multiple populations without common noise, see Fujii [13]. Although there exists a common noise in the current model, the conditional law only enters as a form of expectation. Therefore, as long as the system of FBSDEs is Lipschitz continuous, there exists a unique strong solution at least for small T. For general T, although it is rather difficult to find appropriate set of assumptions, it is still possible for some simple cases. In this section, our main task is to find an appropriate limit model that extends (4.1) for multiple populations and the sufficient conditions that make appropriate monotone conditions hold, which guarantees the existence of unique solution.

In the following, we shall treat m populations indexed by $p \in \{1, \cdots, m\}$. For each p, $N_p \geq 1$ agents are assumed to belong to the population. We denote by (p, i) the ith agent in the population p. First, let us enlarge the probability space constructed in Section 2. In addition to $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0; \overline{\mathbb{F}}^0)$, we introduce $(\overline{\Omega}^{p,i}, \overline{\mathcal{F}}^{p,i}, \overline{\mathbb{P}}^{p,i}; \overline{\mathbb{F}}^{p,i})$ with $1 \leq i \leq N_p$ and $1 \leq p \leq m$, each of which is generated by $(\xi^{p,i}, \boldsymbol{W}^{p,i})$ with d-dimensional Brownian motion $\boldsymbol{W}^{p,i}$ and a $\boldsymbol{W}^{p,i}$-independent $\mathbb{R}^n$-valued square integrable random variable $\xi^{p,i}$. For each p, $(\xi^{p,i})_{i=1}^{N_p}$ are assumed to have the common law. We define $(\Omega^{p,i}, \mathcal{F}^{p,i}, \mathbb{P}^{p,i}; \mathbb{F}^{p,i})$ as the product of $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0; \overline{\mathbb{F}}^0)$ and $(\overline{\Omega}^{p,i}, \overline{\mathcal{F}}^{p,i}, \overline{\mathbb{P}}^{p,i}; \overline{\mathbb{F}}^{p,i})$. Finally $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ is defined as a product of all the spaces $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0; \overline{\mathbb{F}}^0)$ and $(\overline{\Omega}^{p,i}, \overline{\mathcal{F}}^{p,i}, \overline{\mathbb{P}}^{p,i}; \overline{\mathbb{F}}^{p,i})$, $1 \leq i \leq N_p, 1 \leq p \leq m$, and $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i; \mathbb{F}^i)$ as a product of $(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0; \overline{\mathbb{F}}^0)$ and $(\overline{\Omega}^{p,i}, \overline{\mathcal{F}}^{p,i}, \overline{\mathbb{P}}^{p,i}; \overline{\mathbb{F}}^{p,i})$ with $1 \leq p \leq m$. Every probability space is assumed to be complete and every filtration is assumed to be complete and right-continuously augmented to satisfy the usual conditions.

As we have done in Section 3, we first assume that the market price of n securities is given exogenously by $\varpi_t \in \mathbb{H}^2(\overline{\mathbb{F}}^0; \mathbb{R}^n)$. Under this setup, we consider the control problem for each (p, i) agent defined by

$$\inf_{\boldsymbol{\alpha}^{p,i} \in \mathbb{A}^{p,i}} J^{p,i}(\boldsymbol{\alpha}^{p,i}) , \tag{6.1}$$

with

$$J^{p,i}(\boldsymbol{\alpha}^{p,i}) := \mathbb{E}\Big[\int_0^T f_p(t, X^{p,i}_t, \alpha^{p,i}_t, \varpi_t, c^0_t, c^{p,i}_t) dt + g_p(X^{p,i}_T, \varpi_T, c^0_T, c^{p,i}_T)\Big] ,$$

subject to the dynamic constraint:

$$dX_t^{p,i} = \Big(\alpha_t^{p,i} + l_p(t, \varpi_t, c_t^0, c_t^{p,i})\Big)dt + \sigma_{p,0}(t, \varpi_t, c_t^0, c_t^{p,i})dW_t^0 + \sigma_p(t, \varpi_t, c_t^0, c_t^{p,i})dW_t^{p,i}$$

with $X_0^{p,i} = \xi^{p,i}$. As before we assume $(c_t^0)_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^0; \mathbb{R}^n)$ and $(c_t^{p,i})_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^{p,i}; \mathbb{R}^n)$. In addition, within each population p, the random sources $(c_t^{p,i})_{t\geq 0}$ are assumed to have a common law $1 \leq i \leq N_p$. Admissible strategies $\mathbb{A}^{p,i}$ is the space $\mathbb{H}^2(\mathbb{F}^{p,i}; \mathbb{R}^n)$. The measurable functions $f_p : [0,T] \times (\mathbb{R}^n)^5 \to \mathbb{R}$, $g_p : (\mathbb{R}^n)^4 \to \mathbb{R}$, $\overline{f}_p : [0,T] \times (\mathbb{R}^n)^4 \to \mathbb{R}$ and $\overline{g}_p : (\mathbb{R}^n)^3 \to \mathbb{R}$ are given by

$$\begin{aligned} f_p(t, x, \alpha, \varpi, c^0, c) &:= \langle \varpi, \alpha \rangle + \frac{1}{2}\langle \alpha, \Lambda_p \alpha \rangle + \overline{f}_p(t, x, \varpi, c^0, c) \ , \\ g_p(x, \varpi, c^0, c) &:= -\delta \langle \varpi, x \rangle + \overline{g}_p(x, c^0, c) \ . \end{aligned}$$

Assumption 6.1. (MFG-A) *We assume the following conditions uniformly in* $p \in \{1, \cdots, m\}$.
(i) Λ_p *is a positive definite* $n \times n$ *symmetric matrix with* $\underline{\lambda} I_{n\times n} \leq \Lambda_p \leq \overline{\lambda} I_{n\times n}$ *in the sense of 2nd-order form where* $\underline{\lambda}$ *and* $\overline{\lambda}$ *are some constants satisfying* $0 < \underline{\lambda} \leq \overline{\lambda}$.
(ii) For any (t, x, ϖ, c^0, c),

$$|\overline{f}_p(t, x, \varpi, c^0, c)| + |\overline{g}_p(x, c^0, c)| \leq L(1 + |x|^2 + |\varpi|^2 + |c^0|^2 + |c|^2) \ .$$

(iii) $\overline{f}_p$ *and* $\overline{g}_p$ *are continuously differentiable in* x *and satisfy, for any* $(t, x, x', \varpi, c^0, c)$,

$$|\partial_x \overline{f}_p(t, x', \varpi, c^0, c) - \partial_x \overline{f}_p(t, x, \varpi, c^0, c)| + |\partial_x \overline{g}_p(x', c^0, c) - \partial_x \overline{g}_p(x, c^0, c)| \leq L|x' - x| \ ,$$

and $|\partial_x \overline{f}_p(t, x, \varpi, c^0, c)| + |\partial_x \overline{g}_p(x, c^0, c)| \leq L(1 + |x| + |\varpi| + |c^0| + |c|)$.
(iv)The functions $\overline{f}_p$ *and* $\overline{g}_p$ *are convex in* x *in the sense that for any* $(t, x, x', \varpi, c^0, c)$,

$$\begin{aligned} &\overline{f}_p(t, x', \varpi, c^0, c) - \overline{f}_p(t, x, \varpi, c^0, c) - \langle x' - x, \partial_x \overline{f}_p(t, x, \varpi, c^0, c) \rangle \geq \frac{\gamma^f}{2}|x' - x|^2 \ , \\ &\overline{g}_p(x', c^0, c) - \overline{g}_p(x, c^0, c) - \langle x' - x, \partial_x \overline{g}_p(x, c^0, c) \rangle \geq \frac{\gamma^g}{2}|x' - x|^2 \ , \end{aligned}$$

with some constants $\gamma^f, \gamma^g \geq 0$.
(v) $l_p, \sigma_{p,0}, \sigma_p$ *are the measurable functions defined on* $[0,T] \times (\mathbb{R}^n)^3$ *and are* $\mathbb{R}^n, \mathbb{R}^{n\times d^0}$ *and* $\mathbb{R}^{n\times d}$*-valued, respectively. Moreover they satisfy the linear growth condition:*

$$|(l_p, \sigma_{p,0}, \sigma_p)(t, \varpi, c^0, c)| \leq L(1 + |\varpi| + |c^0| + |c|)$$

for any (t, ϖ, c^0, c).
(vi) $\delta \in [0, 1)$ *is a given constant.*

Under Assumption (MFG-A), Theorem 3.1 guarantees that the control problem (6.1) for each agent (p, i) is uniquely characterized by

$$\begin{aligned} dX_t^{p,i} &= \Big(\widehat{\alpha}_p(Y_t^{p,i}, \varpi_t) + l_p(t, \varpi_t, c_t^0, c_t^{p,i})\Big)dt + \sigma_{p,0}(t, \varpi_t, c_t^0, c_t^{p,i})dW_t^0 + \sigma_p(t, \varpi_t, c_t^0, c_t^{p,i})dW_t^{p,i}, \\ dY_t^{p,i} &= -\partial_x \overline{f}_p(t, X_t^{p,i}, \varpi_t, c_t^0, c_t^{p,i})dt + Z_t^{p,i,0}dW_t^0 + Z_t^{p,i}dW_t^{p,i}, \end{aligned} \tag{6.2}$$

with $X_0^{p,i} = \xi^{p,i}$ and $Y_T^{p,i} = -\delta\varpi_T + \partial_x \overline{g}_p(X_T^{p,i}, c_T^0, c_T^{p,i})$. We have defined $\widehat{\alpha}_p(y, \varpi) := -\overline{\Lambda}_p(y + \varpi)$

and $\overline{\Lambda}_p := (\Lambda_p)^{-1}$ as before. There exists a unique strong solution $(X_t^{p,i}, Y_t^{p,i}, Z_t^{p,i,0}, Z_t^{p,i})_{t\in[0,T]} \in \mathbb{S}^2(\mathbb{F}^{p,i};\mathbb{R}^n)\times\mathbb{S}^2(\mathbb{F}^{p,i};\mathbb{R}^n)\times\mathbb{H}^2(\mathbb{F}^{p,i};\mathbb{R}^{n\times d^0})\times\mathbb{H}^2(\mathbb{F}^{p,i};\mathbb{R}^{n\times d})$, and the optimal trading strategy for the agent (p,i) is given by

$$\widehat{\alpha}_t^{p,i} = \widehat{\alpha}_p(Y_t^{p,i}, \varpi_t) \ , \forall t \in [0,T].$$

Let us check the market clearing condition under this setup. In order to balance the demand and supply of securities at the exchange, we need to have $\sum_{p=1}^m \sum_{i=1}^{N_p} \widehat{\alpha}(Y_t^{p,i}, \varpi_t) = 0$. This requires the market price to satisfy

$$\varpi_t = -\Big(\sum_{p=1}^m n_p \overline{\Lambda}_p\Big)^{-1} \sum_{p=1}^m n_p \overline{\Lambda}_p \Big(\frac{1}{N_p}\sum_{i=1}^{N_p} Y_t^{p,i}\Big) \ ,$$

where $N = \sum_{p=1}^m N_p$ and $n_p := N_p/N$. At the moment, this is inconsistent to the initial assumption that requires $(\varpi_t)_{t\geq 0}$ to be $\overline{\mathbb{F}}^0$-adapted. However, since for each $1 \leq p \leq m$, $(Y_t^{p,i})_{i=1}^{N_p}$ are $\overline{\mathcal{F}}^0$-conditionally independently and identically distributed, we may follow the same arguments used in Section 3. If we take $N \to \infty$ while keeping the relative size of populations n_p constant, we can expect to obtain

$$\varpi_t = -\hat{\Xi}\sum_{p=1}^m \hat{\Lambda}_p \mathbb{E}[Y_t^{p,1}|\overline{\mathcal{F}}_t^0] \tag{6.3}$$

in the large population limit where

$$\hat{\Lambda}_p := n_p \overline{\Lambda}_p, \qquad \hat{\Xi} := \Big(\sum_{p=1}^m \hat{\Lambda}_p\Big)^{-1} \ .$$

Remark 6.1. *When $\Lambda_p = \Lambda$ for every population p, one can easily check that* (6.3) *becomes*

$$\varpi_t = -\sum_{p=1}^m n_p \mathbb{E}[Y_t^{p,1}|\overline{\mathcal{F}}_t^0] \ .$$

Since Y of the adjoint equation represents the marginal cost i.e., the first order derivative of the value function with respect to the state variable x, the above expression of ϖ implies that the market price may be given by the population-weighted average of the marginal benefit (-cost) across the entire populations.

6.1 Limit problem with multiple populations

By the observation we have just made, we are motivated to study the following limit problem with $1 \leq p \leq m$:

$$\begin{aligned} dX_t^p &= \Big(\widehat{\alpha}_p\big(Y_t^p, \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0])\big) + l_p\big(t, \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]), c_t^0, c_t^p\big)\Big)dt \\ &\quad + \sigma_{p,0}\big(t, \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]), c_t^0, c_t^p\big)dW_t^0 + \sigma_p\big(t, \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]), c_t^0, c_t^p\big)dW_t^{p,1}, \\ dY_t^p &= -\partial_x \overline{f}_p\big(t, X_t^p, \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]), c_t^0, c_t^p\big)dt + Z_t^{p,0}dW_t^0 + Z_t^p dW_t^{p,1} \ , \end{aligned} \tag{6.4}$$

with $X_0^p = \xi^p$ and

$$Y_T^p = \frac{\delta}{1-\delta}\hat{\Xi}\sum_{p=1}^m \hat{\Lambda}_p \mathbb{E}\big[\partial_x \overline{g}_p(X_T^p, c_T^0, c_T^p)|\overline{\mathcal{F}}_T^0\big] + \partial_x \overline{g}_p(X_T^p, c_T^0, c_T^p)\ .$$

We put as before $\xi^p := \xi^{p,1}$ and $c^p := c^{p,1}$ to lighten the notation. Here,

$$\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]) := -\hat{\Xi}\sum_{p=1}^m \hat{\Lambda}_p \mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0], \quad \widehat{\alpha}_p(y,\varpi) := -\overline{\Lambda}_p(y+\varpi)$$

and hence (6.4) is actually an m-coupled system of FBSDEs of McKean-Vlasov type. One can derive the terminal condition from

$$Y_T^p = -\delta\varpi(\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0]) + \partial_x \overline{g}_p(X_T^p, c_T^0, c_T^p)\ , \tag{6.5}$$

by summing over $1 \leq p \leq m$ after taking conditional expectation given $\overline{\mathcal{F}}_T^0$. In the following, we use the notation

$$(X_t, Y_t, Z_t^0, Z_t)_{t\in[0,T]} = \Big((X_t^p)_{p=1}^m, (Y_t^p)_{p=1}^m, (Z_t^{p,0})_{p=1}^m, (Z_t^p)_{p=1}^m\Big)_{t\in[0,T]}\ . \tag{6.6}$$

6.2 Solvability for small T

For small T, Lipschitz continuity suffices to guarantee the existence of a unique solution.

Assumption 6.2. (MFG-B)
Uniformly in $p \in \{1,\cdots,m\}$, for any $(t,x,c^0,c) \in [0,T]\times(\mathbb{R}^n)^3$ and any $\varpi,\varpi' \in \mathbb{R}^n$, the coefficient functions $l_p, \sigma_{p,0}, \sigma_p$ and $\overline{f}_p$ satisfy

$$\begin{aligned}&|(l_p,\sigma_{p,0},\sigma_p)(t,\varpi,c^0,c) - (l_p,\sigma_{p,0},\sigma_p)(t,\varpi',c^0,c)| \\ &\qquad +|\partial_x\overline{f}_p(t,x,\varpi,c^0,c) - \partial_x\overline{f}_p(t,x,\varpi',c^0,c)| \leq L_\varpi|\varpi-\varpi'|\ .\end{aligned}$$

The following theorem follows exactly in the same way as Theorem 4.1.

Theorem 6.1. *Under Assumptions (MFG-A,B), there exists some constant $\tau > 0$ which depends only on $(L, L_\varpi, \delta, n_p, \Lambda_p)$ such that for any $T \leq \tau$, there exists a unique strong solution $(X,Y,Z^0,Z) \in \mathbb{S}^2\big(\mathbb{F}^1;(\mathbb{R}^n)^m\big)\times\mathbb{S}^2\big(\mathbb{F}^1;(\mathbb{R}^n)^m\big)\times\mathbb{H}^2\big(\mathbb{F}^1;(\mathbb{R}^{n\times d^0})^m\big)\times\mathbb{H}^2\big(\mathbb{F}^1;(\mathbb{R}^{n\times d})^m\big)$ to the FBSDE* (6.4)*.*

Remark 6.2. *Note that the above system of FBSDEs becomes a linear-quadratic form by choosing $(l_p,\sigma_{p,0},\sigma_p,\overline{f}_p,\overline{g}_p)$ appropriately. In this case, the problem reduces to solving ordinary differential equations of Riccati type. Therefore, the existence of a solution for a given T can be tested, at lest numerically, by checking the absence of a "blow up" in its solution.*

6.3 Solvability for general T

We now move on to the existence result of a unique solution for general T. It is very difficult to find general existence criteria for fully-coupled multi-dimensional FBSDEs. A the moment, in order to apply well-known Peng-Wu's method, let us put the following simplifying assumptions.

Assumption 6.3. (MFG-C1)
(i) For every $1 \le p \le m$, the functions $\sigma_{p,0}$ and σ_p are independent of the argument ϖ.
(ii) Λ_p=Λ and $n_p = 1/m$ for every p.
(iii) For any $t \in [0,T]$, any random variables $x^p, x^{p\prime}, c^0, c^p \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^n)$ and any sub-σ-field $\mathcal{G} \subset \mathcal{F}$, the functions $(l_p)_{p=1}^m$ satisfy with some positive constant $\gamma^l > 0$,

$$\sum_{p=1}^m \mathbb{E}\Big[\big\langle l_p\big(t,\mathbb{E}[\overline{x}|\mathcal{G}],c^0,c^0\big) - l_p\big(t,\mathbb{E}[\overline{x}'|\mathcal{G}],c^0,c^p\big), x^p - x^{p\prime}\big\rangle\Big] \ge m\gamma^l \mathbb{E}\big[\mathbb{E}[\overline{x}-\overline{x}'|\mathcal{G}]^2\big],$$

where $\overline{x} := \frac{1}{m}\sum_{p=1}^m x^p$ and similarly for $\overline{x}'$.
(iv) There exists a strictly positive constant γ satisfying $0 < \gamma \le \Big(\gamma^f - \frac{L_\varpi^2}{4\gamma^l}\Big) \wedge \gamma^g$. Moreover, the functions $(\overline{g}_p)_{p=1}^m$ satisfy for any $x^p, x^{p\prime}, c^0, c^p \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^n)$ and any sub-σ-field $\mathcal{G} \subset \mathcal{F}$,

$$\begin{aligned}&\frac{\delta}{1-\delta} m^{-1}\mathbb{E}\Big[\big\langle \sum_{p=1}^m \mathbb{E}[\partial_x \overline{g}_p(x^p,c^0,c^p) - \partial_x \overline{g}_p(x^{p\prime},c^0,c^p)|\mathcal{G}], \sum_{p=1}^m (x^p - x^{p\prime})\big\rangle\Big] \\ &\qquad + \gamma^g \sum_{p=1}^m \mathbb{E}[|x^p - x^{p\prime}|^2] \ge \gamma \sum_{p=1}^m \mathbb{E}[|x^p - x^{p\prime}|^2] \ .\end{aligned}$$

Remark 6.3. *The conditions (iii) and (iv) in the above assumption are rather restrictive. The condition (iii) is satisfied, for example, if l_p has a separable form $l_p = h(x) + h_p(c_t^0, c_t^p)$ with some function h, which is common to every population and strictly monotone. (iv) is also satisfied by requiring similar structure. Or, since $\partial_x \overline{g}_p$ is Lipschitz continuous in x, the absolute value of the first term is bounded by $\frac{\delta}{1-\delta}\max((L_p)_{p=1}^m)\sum_{p=1}^m \mathbb{E}|x^p - x^{p\prime}|^2$, where the L_p is the Lipschitz constant for $\partial_x \overline{g}_p$. Thus the condition (iv) is satisfied if $\delta \max((L_p)_{p=1}^m)$ is sufficiently small.*

The next result is the counterpart of Theorem 4.2.

Theorem 6.2. *Under Assumptions* (MFG-A,B,C1)*, there exists a unique strong solution $(X,Y,Z^0,Z) \in \mathbb{S}^2\big(\mathbb{F}^1;(\mathbb{R}^n)^m\big) \times \mathbb{S}^2\big(\mathbb{F}^1;(\mathbb{R}^n)^m\big) \times \mathbb{H}^2\big(\mathbb{F}^1;(\mathbb{R}^{n\times d^0})^m\big) \times \mathbb{H}^2\big(\mathbb{F}^1;(\mathbb{R}^{n\times d})^m\big)$ to the FBSDE* (6.4)*. Moreover, the same form of stability and $\mathbb{L}^2$ estimates given in Proposition 4.1 and Corollary 4.1 hold.*

Proof. Under Assumption (MFG-C1), (6.4) can be written as

$$\begin{aligned} dX_t^p &= \Big\{-\overline{\Lambda}\Big(Y_t^p - \frac{1}{m}\sum_{p=1}^m \mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0]\Big) + l_p\Big(t, -\frac{1}{m}\sum_{p=1}^m \mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0], c_t^0, c_t^p\Big)\Big\}dt \\ &\quad + \sigma_{p,0}(t,c_t^0,c_t^p)dW_t^0 + \sigma_p(t,c_t^0,c_t^p)dW_t^{p,1}, \\ dY_t^p &= -\partial_x \overline{f}_p\Big(t, X_t^p, -\frac{1}{m}\sum_{p=1}^m \mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0], c_t^0, c_t^p\Big)dt + Z_t^{p,0}dW_t^0 + Z_t^p dW_t^{p,1}, \end{aligned}$$

with $X_0^p = \xi^p$ and

$$Y_T^p = \frac{\delta}{1-\delta}\frac{1}{m}\sum_{p=1}^m \mathbb{E}\big[\partial_x \overline{g}_p(X_T^p,c_T^0,c_T^p)|\overline{\mathcal{F}}_T^0\big] + \partial_x \overline{g}_p(X_T^p,c_T^0,c_T^p) \ .$$

For each p, let us define the functionals B_p, F_p and G_p for any $y^p, x^p, c^0, c^p \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^n)$ with $y := (y^p)_{p=1}^m$, $x := (x^p)_{p=1}^m$ and $c := (c^p)_{p=1}^m$ by

$$
\begin{aligned}
&B_p(t,y,c^0,c^p) := -\overline{\Lambda}\Big(y^p - \frac{1}{m}\sum_{p=1}^m \mathbb{E}[y^p|\overline{\mathcal{F}}^0_t]\Big) + l_p\Big(t, -\frac{1}{m}\sum_{p=1}^m \mathbb{E}[y^p|\overline{\mathcal{F}}^0_t], c^0, c^p\Big)\\
&F_p(t,x^p,y,c^0,c^p) := -\partial_x \overline{f}\Big(t, x^p, -\frac{1}{m}\sum_{p=1}^m \mathbb{E}[y^p|\overline{\mathcal{F}}^0_t], c^0, c^p\Big),\\
&G_p(x,c^0,c) := \frac{\delta}{1-\delta}\frac{1}{m}\sum_{p=1}^m \mathbb{E}[\partial_x \overline{g}_p(x^p,c^0,c^p)|\overline{\mathcal{F}}^0_T] + \partial_x \overline{g}_p(x^p,c^0,c^p)\ ,
\end{aligned}
$$

and set $B(t,y,c^0,c) := (B_p(t,y,c^0,c^p))_{p=1}^m$, $F(t,x,y,c^0,c) := (F_p(t,x^p,y,c^0,c^p))_{p=1}^m$ and $G(x,c^0,c) := (G_p(x,c^0,c))_{p=1}^m$. With $\Delta y := y - y'$ and $\Delta x := x - x'$, we have from (MFG-C1)(iii),

$$
\begin{aligned}
&\mathbb{E}\Big[\langle B(t,y,c^0,c) - B(t,y',c^0,c), \Delta y\rangle\Big] := \sum_{p=1}^m \mathbb{E}\Big[\langle B_p(t,y,c^0,c) - B_p(t,y',c^0,c), \Delta y^p\rangle\Big]\\
&\le -\sum_{p=1}^m \mathbb{E}[\langle \Delta y^p, \overline{\Lambda}\Delta y^p\rangle] + \frac{1}{m}\mathbb{E}\Big[\langle \sum_{p=1}^m \mathbb{E}[\Delta y^p|\overline{\mathcal{F}}^0_t], \overline{\Lambda}\sum_{p=1}^m \Delta y^p\rangle\Big] - m\gamma^l \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{p=1}^m \mathbb{E}[\Delta y^p|\overline{\mathcal{F}}^0_t]\Big)^2\Big]\\
&\le -m\gamma^l \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{p=1}^m \mathbb{E}[\Delta y^p|\overline{\mathcal{F}}^0_t]\Big)^2\Big].
\end{aligned}
\tag{6.7}
$$

There exists a orthogonal matrix P such that $P^\top \overline{\Lambda} P$ becomes diagonal. Then working on the new basis $\hat{y}^p = P^\top \Delta y^p$, $1 \le p \le m$, the last inequality of (6.7) can be checked component by component $1 \le i \le n$ by the fact $(\sum_{p=1}^m \hat{y}^p_i)^2 \le m\sum_{p=1}^m |\hat{y}^p_i|^2$. Second, from (MFG-A)(iv), (MFG-B) and Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\mathbb{E}\Big[\langle F(t,x,y,c^0,c) - F(t,x',y',c^0,c), \Delta x\rangle\Big]\\
&\qquad \le -\Big(\gamma^f - \frac{L_\varpi^2}{4\gamma^l}\Big)\mathbb{E}[|\Delta x|^2] + m\gamma^l \mathbb{E}\Big[\Big(\frac{1}{m}\sum_{p=1}^m \mathbb{E}[\Delta y^p_t|\overline{\mathcal{F}}^0_t]\Big)^2\Big].
\end{aligned}
\tag{6.8}
$$

Finally, from (MFG-A, C1)(iv), we immediately get

$$
\mathbb{E}\Big[\langle G(x,c^0,c) - G(x',c^0,c), \Delta x\rangle\Big] \ge \gamma \mathbb{E}[|\Delta x|^2]\ .
$$

Now we have established the monotone conditions corresponding to (4.3) for the current model. We can now repeat the same procedures in the proof of Theorem 4.2 and Proposition 4.1. □

Let us give the results for the securities of maturity T with exogenously specified payoff.

Assumption 6.4. (MFG-C2)
(i) For every $1 \le p \le m$, the functions $\sigma_{p,0}$ and σ_p are independent of the argument ϖ.
(ii) Λ_p=Λ and $n_p = 1/m$ for every p.
(iii) For any $t \in [0,T]$, any random variables $x^p, x^{p\prime}, c^0, c^p \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^n)$ and any sub-σ-field

$\mathcal{G} \subset \mathcal{F}$, the functions $(l_p)_{p=1}^m$ satisfy with some positive constant $\gamma^l > 0$,

$$\sum_{p=1}^m \mathbb{E}\Big[\big\langle l_p\big(t, \mathbb{E}[\overline{x}|\mathcal{G}], c^0, c^0\big) - l_p\big(t, \mathbb{E}[\overline{x}'|\mathcal{G}], c^0, c^p\big), x^p - x^{p\prime}\big\rangle\Big] \geq m\gamma^l \mathbb{E}\big[\mathbb{E}[\overline{x} - \overline{x}'|\mathcal{G}]^2\big],$$

where $\overline{x} := \frac{1}{m}\sum_{p=1}^m x^p$ and similarly for $\overline{x}'$.
(iv) $\gamma := \gamma^f - \frac{L_\varpi^2}{4\gamma^l}$ is strictly positive. Moreover, $\delta = 0$ and the terminal function g_p is given by

$$g_p(x, c^0) = \overline{g}_p(x, c^0) := -\langle c^0, x\rangle \tag{6.9}$$

for every $1 \leq p \leq m$.

Theorem 6.3. *Under Assumptions* (MFG-A,B,C2)*, there exists a unique strong solution $(X, Y, Z^0, Z) \in \mathbb{S}^2\big(\mathbb{F}^1; (\mathbb{R}^n)^m\big) \times \mathbb{S}^2\big(\mathbb{F}^1; (\mathbb{R}^n)^m\big) \times \mathbb{H}^2\big(\mathbb{F}^1; (\mathbb{R}^{n\times d^0})^m\big) \times \mathbb{H}^2\big(\mathbb{F}^1; (\mathbb{R}^{n\times d})^m\big)$ to the FBSDE* (6.4)*. Moreover, the same form of the stability and $\mathbb{L}^2$ estimates given in Proposition 4.1 and Corollary 4.1 holds.*

Proof. Using the inequalities (6.7) and (6.8) with $\sum_{p=1}^m \langle \Delta X_T^p, \Delta Y_T^p\rangle = 0$, we can follow the same arguments in the proof of Theorem 4.3. □

6.4 Asymptotic market clearing for multi-population model

At the last part of this section, we investigate the asymptotic market clearing in the presence of multiple populations. As in Section 5, we define $(\varpi_t)_{t\in[0,T]}$ using the solution to the system of the mean-field FBSDEs:

$$\varpi_t = \varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]) := -\hat{\Xi}\sum_{p=1}^m \hat{\Lambda}_p \mathbb{E}\big[Y_t^p|\overline{\mathcal{F}}_t^0\big]$$

where $(Y_t^p)_{p=1}^m$ is the solution of (6.4). In order to test the accuracy of the above $(\varpi_t)_{t\in[0,T]}$ as a market clearing price, we solve the individual agent problem (6.1) with this ϖ as an input. The corresponding individual problem (6.1) for the agent (p, i) is given by the unique strong solution $(X^{p,i}, Y^{p,i}, Z^{p,i,0}, Z^{p,i})$ of (6.2). The optimal strategy for the agent (p, i) is then given by

$$\widehat{\alpha}_{\rm mf}^{p,i}(t) := -\overline{\Lambda}_p\Big(Y_t^{p,i} - \hat{\Xi}\sum_{q=1}^m \hat{\Lambda}_q \mathbb{E}\big[Y_t^q|\overline{\mathcal{F}}_t^0\big]\Big)\ , \forall t \in [0, T]\ .$$

Theorem 6.4. *If the conditions for Theorem 6.1, Theorem 6.2 or Theorem 6.3 are satisfied then we have*

$$\lim_{N\to\infty} \mathbb{E}\int_0^T \Big|\frac{1}{N}\sum_{p=1}^m \sum_{i=1}^{N_p} \widehat{\alpha}_{\rm mf}^{p,i}(t)\Big|^2 dt = 0\ ,$$

where $N := \sum_{p=1}^m N_p$ and the limit is taken while keeping $(n_p := N_p/N)_{1\leq p\leq m}$ constant. Moreover if there exists some constant Γ such that $\sup_{t\in[0,T]} \mathbb{E}\big[|Y_t|^q\big]^{\frac{1}{q}} \leq \Gamma < \infty$ for some $q > 4$, then *there exists some constant C independent of N such that*

$$\mathbb{E}\int_0^T \Big|\frac{1}{N}\sum_{p=1}^m\sum_{i=1}^{N_p}\widehat{\alpha}_{\rm mf}^{p,i}(t)\Big|^2 dt \le C\Gamma^2\epsilon_N,$$

where $\epsilon_N := N^{-2/\max(n,4)}\big(1+\log(N)\mathbf{1}_{\{n=4\}}\big)$.

Proof. By the definition of $\widehat{\alpha}_{\rm mf}^{p,i}$, we have

$$\begin{aligned}\frac{1}{N}\sum_{p=1}^m\sum_{i=1}^{N_p}\widehat{\alpha}_{\rm mf}^{p,i}(t) &= -\frac{1}{N}\sum_{p=1}^m\sum_{i=1}^{N_p}\overline{\Lambda}_p\Big(Y_t^{p,i}-\hat{\Xi}\sum_{q=1}^m\hat{\Lambda}_q\mathbb{E}[Y_t^q|\overline{\mathcal{F}}_t^0]\Big)\\ &= -\sum_{p=1}^m\hat{\Lambda}_p\Big(\frac{1}{N_p}\sum_{i=1}^{N_p}Y_t^{p,i}-\mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0]\Big)\ . \end{aligned}\tag{6.10}$$

On the other hand, we have for each $1\le p\le m$, $1\le i\le N_p$,

$$\begin{aligned} dX_t^{p,i} &= \Big(\widehat{\alpha}_p\big(Y_t^{p,i},\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0])\big)+l_p\big(t,\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]),c_t^0,c_t^{p,i}\big)\Big)dt\\ &\quad +\sigma_{p,0}\big(t,\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]),c_t^0,c_t^{p,i}\big)dW_t^0+\sigma_p\big(t,\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]),c_t^0,c_t^{p,i}\big)dW_t^{p,i},\\ dY_t^{p,i} &= -\partial_x\overline{f}_p\big(t,X_t^{p,i},\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0]),c_t^0,c_t^{p,i}\big)dt+Z_t^{p,i,0}dW_t^0+Z_t^{p,i}dW_t^{p,i}\ , \end{aligned}$$

with $X_0^{p,i}=\xi^{p,i}$,

$$Y_T^{p,i}=-\delta\varpi(\mathbb{E}[Y_T|\overline{\mathcal{F}}_T^0])+\partial_x\overline{g}_p(X_T^{p,i},c_T^0,c_T^{p,i})\ .$$

By the unique strong solvability, Yamada-Watanabe theorem implies that there exists some function Φ_p for each $1\le p\le m$ such that for every $1\le i\le N_p$,

$$(Y_t^{p,i})_{t\in[0,T]}=\Phi_p\Big(c^0,(W_t^0)_{t\in[0,T]},(\mathbb{E}[Y_t^q|\overline{\mathcal{F}}_t^0]_{t\in[0,T]})_{1\le q\le m},\xi^{p,i},(c_t^{p,i})_{t\in[0,T]},(W_t^{p,i})_{t\in[0,T]}\Big).$$

Hence $(Y_t^{p,i})_{t\in[0,T],1\le i\le N_p}$ are independently and identically distributed conditionally on $\overline{\mathcal{F}}^0$. In particular, we have $\mathbb{E}[Y_t^{p,i}|\overline{\mathcal{F}}_t^0]=\mathbb{E}[Y_t^{p,1}|\overline{\mathcal{F}}_t^0]$.

We now compare $(X_t^{p,1},Y_t^{p,1},Z_t^{p,1,0},Z_t^{p,1})_{t\in[0,T]}$ with $(X_t^p,Y_t^p,Z_t^{p,0},Z_t^p)_{t\in[0,T]}$ by treating $\varpi(\mathbb{E}[Y_t|\overline{\mathcal{F}}_t^0])$ as external inputs. Note that the terminal condition of the latter satisfies the relation (6.5). Then the standard stability result of the Lipschitz FBSDEs implies $(Y_t^{p,1})_{t\in[0,T]}=(Y_t^p)_{t\in[0,T]}$ in $\mathbb{S}^2(\mathbb{F}^{p,1};\mathbb{R}^n)$. As a result we have obtained $\mathbb{E}[Y_t^p|\overline{\mathcal{F}}_t^0]=\mathbb{E}[Y_t^{p,1}|\overline{\mathcal{F}}_t^0]$. Using the expression (6.10), we obtain

$$\frac{1}{N}\sum_{p=1}^m\sum_{i=1}^{N_p}\widehat{\alpha}_{\rm mf}^{p,i}(t) = -\sum_{p=1}^m\hat{\Lambda}_p\Big(\frac{1}{N_p}\sum_{i=1}^{N_p}Y_t^{p,i}-\mathbb{E}[Y_t^{p,1}|\overline{\mathcal{F}}_t^0]\Big)\ .$$

We can now repeat the last part of the proof for Theorem 5.1. □

7 Concluding Remarks and Further Extensions

In this work, we have studied endogenous formation of market clearing price using a stylized model of a security exchange. We have derived a special type of FBSDE of McKean-Vlasov type with common noise whose solution provides a good approximate of the equilibrium price. In addition to the existence of strong unique solution to the FBSDE, we have proved that the modeled price asymptotically clear the market in the large N-limit. We also gave the order of convergence ϵ_N when the solution of the FBSDE possesses higher order of integrability. In the following, let us list up of a further extension of our technique and some interesting topics for future projects:

• **Dependence on the conditional law of the state**: For applications to energy and commodity markets, or economic models with producers and consumers, one may want to study the cost functions $(\overline{f}, \overline{g})$ depending on the empirical distribution of the sate X of the agents such as $\overline{f}\Big(t, X_t^i, \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}, \varpi_t, c_t^0, c_t^i\Big)$. Under the setup with conditional independence, the cost function for the limit problem is naturally given by $\overline{f}\Big(t, X_t, \mathcal{L}(X_t|\overline{\mathcal{F}}_t^0), \varpi_t, c_t^0, c_t\Big)$. Even in this case, the resultant FBSDE (4.1) is solvable, at least for small T, if $(\partial_x \overline{f}, \partial_x \overline{g})$ are Lipschitz continuous in the measure argument with respect to W_2-distance. Under the stronger assumption guaranteeing the monotone conditions (4.3), one can even achieve the existence of unique solution in general T. As long as the source of common noise is solely from the filtration $\overline{\mathbb{F}}^0$ generated by $\boldsymbol{W}^0$, we can avoid subtleties regarding the admissibility (so-called H-hypothesis). See Remark 2.10 in [8] as a useful summary for this issue.

• **Explicit solution**: If we chose $\overline{f}, \overline{g}$ as quadratic functions and l, σ_0, σ as affine functions, we obtain a linear-quadratic mean field game with common noise. In this case, an explicit solution may be available where the coefficients functions are given as the solutions to differential equations of Riccati type.

• **Property of market price process**: It seems interesting to study the properties of the market clearing price theoretically and numerically. For example, if $n = d^0$ the equivalent martingale measure (EMM) can be uniquely determined. Based on the payoff distribution c^0 and the cost functions of the agents $(\overline{f}, \overline{g})$, one may study how the market price process under the EMM behaves, for example, the relation between the skew of its implied volatility and the risk-averseness of the agents.

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