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The Evolution of Money
-- A Search-Theoretic Foundation of Monetary Economics --

by

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Abstract

The paper develops a model of decentralized economy with many heterogeneous individuals seeking to exchange their endowments with needful goods, and analyses both barter and monetary exchange processes as two different forms of exchange equilibrium. It argues that the reason why money is able to surmount the difficulty of barter lies in its being the product of a bootstraps mechanism or in the fact that money is used as money merely because it is used as money by everybody else in the economy. Indeed, it shows that while barter equilibrium does not always exist, such bootstraps mechanism allows the economy to have as many monetary equilibria as the number of durable goods in it. The paper also demonstrates, however, that there is a fundamental difficulty in the natural evolution of money and that in order for the economy to achieve one of its potential monetary equilibria it is necessary to have a large initial disturbance which breaks the intrinsic symmetry of barter exchange.

Part I: On the Bootstraps Mechanism of Money

1. What is money?

Money is a medium which is accepted in exchange, not to be used directly for consumption or production, but to be exchanged for some other good with some other person at some other time. Money also serves as a measure of and a store of value, but these functions, though inseparable from money, can be and actually are shared by many other goods in the economy. Money is, however, not a mere medium of exchange; it is the "universal medium of exchange" which is accepted by anybody at any place at any time. It is thus able to overcome the essential difficulties of barter exchange, which requires a "double coincidence of wants" -- a situation in which one individual has the goods that the other individual needs and needs the goods that the other individual has.[1] This is of course a matter of common knowledge, and we have been told and retold since cradle a story which begins: "Once upon a time there was a barter economy which was very inconvenient ....."[2]

It is not the intention of the present paper to add anything new to this already well-understood nature of money and monetary economy, except for its explicit modeling of the processes of barter and monetary exchanges within a formal search theory.[3] Its major objective is rather to apply the method of search theory to demonstrate that the reason why money is able to surmount the requirement of double coincidence of wants lies at the deeper level in the fact that it is the product of a "bootstraps mechanism", or in the fact that:

(1)Money is used as the universal medium of exchange merely because it is used as the universal medium of exchange by everybody else.

The difficulties of barter arises from its being constrained by the "real" conditions of the economy. The double coincidence of wants is precisely the condition for the way "technology and preferences" are distributed among different individuals in the economy. Money, on the other hand, is a "social contrivance" which is able to sustain itself by its own bootstraps.[4] And it is precisely because money is such a social contrivance that it is able to fill the lack of double coincidence of wants and make the otherwise impossible exchanges possible. Ancient Greek knew this when they called money "nomisma." Money, in other words, is able to overcome the "real" constraints of the economy by no other reason than that it requires no "real" foundations to support itself.

There is of course no place for a free lunch in economics, Indeed, once the economy has severed itself from the real constraints of barter by having one of its goods exclusively used as money, people's exchange activities have to face a new form of constraint, which requires them to have money in hand whenever they want to acquire (that is, to "buy") any other good in the economy.[5] It is only at the expense of introducing this monetary constraint that the real constraints of barter have been eliminated from the economy.

Now, the classical literature on money is abound with the colorful descriptions of almost infinite variety of goods which have served as money and pseudo-money in the past and with the ingenious and sometimes outrageous explanations of why certain commodities, especially some precious metals, have almost universally come to be chosen as the ultimate monetary goods. But, if money does not require any real foundations to support itself, then any good (as long as it is durable) can become money and any economy (as long as it is well-defined) can have money. The second objective of the present paper is to demonstrate within a formal search theory the proposition that:

(2) An economy can have as many monetary equilibria (whose precise meaning

will be given later) as the number of (durable) goods in it.

Money is thus seen to be potentially an ubiquitous entity. But, as any student of the speculative philosophy knows, there is a wide gap between the potentiality and the actuality, and, however tempting it is, we cannot immediately jump from this proposition about the potential ubiquity of money to the assertion that "therefore, money evolves naturally in any economy." Indeed, it is the third objective of the present paper to argue that:

(3) In order for an economy to actually attain one of its potential monetary equilibria, it is necessary to have a large initial disturbance which breaks the intrinsic symmetry of barter exchange.

While barter treats all goods symmetrically, monetary exchange creates an artificial asymmetry in the economy by assigning a social role to an arbitrarily chosen good. When the economy is trapped in a barter system, it can be shown to have a locally stable exchange structure. Unless, therefore, some outside enforcement or some historical accident or some other form of large disturbance were to displace the economy far from the symmetry of barter and set the bootstraps mechanism in motion, an asymmetric monetary economy would never come into existence. And, once the economy has been pushed into one of its potential monetary equilibria by such disturbance, it then becomes extremely difficult to stop and reverse the on-going bootstraps mechanism sustaining it. There is a certain irreversibility in the evolutionary process of money.

In his well-known work on money Carl Menger insisted on the "natural" origin of money. He said that "as each economizing individual becomes increasingly more aware of his economic interest, he is led by this interest, without any agreement, without legislative compulsion, and even without regard to the public interest, to give his commodities in exchange for the other, more saleable, commodities, even if he does not need them for any immediate consumption purpose," and then argued that "with economic progress, we can everywhere observe the phenomena of a certain number of goods ... becoming acceptable to everyone in trade," that is, becoming money.[6] Menger was quite to the point in his theoretical description of the (bootstraps) mechanism which endows a certain number of goods with the characteristics of money, but he was wrong when he applied something like the Invisible Hand mechanism to the explanation of the evolution of money. Money never evolves "naturally" from a barter system even if people become aware of their economic interests, and there must have been a large symmetry-breaking disturbance to create it "in the beginning".

But we seem to have anticipated too much of what follows. We better start presenting our search-theoretic model of barter and monetary exchange processes at once.

2. The model of search.

Consider an economy with many types of individuals. Each individual is born into the economy with an endowment of one unit of one good but with a need to consume another good. (This is an extreme example of the division of labor.) The goods are assumed to be durable and will be carried from period to period until they reach the hands of those who have real need of them.[7] It is also assumed that one unit of any good is equivalent in value to one unit of any other good and that there is no room for bargaining about the terms of exchange among individuals.

Individuals in this economy therefore have to search for trading partners in order to exchange the goods they have for the goods they need. There are almost unlimited number of models which are capable of describing the way people meet each other and trade with each other. The simplest (at least at the level of assumption) is to assume that people meet each other randomly. This is perhaps the most decentralized form of exchange economy, and will be studied in detail in a forthcoming paper.[8] The present paper, however, will take up the case of a slightly more organized form of the economy. In fact, by introducing some pre-existing structure into the trading process, we shall be able to obtain much sharper (and perhaps the sharpest) results than the case of completely random search for trading partners.[9]

Imagine then that the economy has N(N-1)/2 different trading zones (or islands) to exchange N different goods. Goods to be traded are indexed by integers: 1, 2, 3, ..., N. Each trading zone carries a sign of two goods to be exchanged in it. In a trading zone with a sign of (i,j) one unit of good i and one unit of good j are exchanged, and in a trading zone with a sign of (k,h) one unit of good k and one unit of good h are exchanged. (We regard (i,j) trading zone and (j,i) trading zone as identical.)

In order to find a trading partner, therefore, each individual has to look around in the economy and search for an appropriate trading zone to visit. For instance, an individual who is endowed with good i but needs good j for consumption may make a direct trip to the (i,j) trading zone and search for a complementary trading partner who is willing to exchange good j for good i. This is the strategy of barter exchange which requires the condition of the "double coincidence of wants" for its successful conclusion. The same purpose may, however, be accomplished in a roundabout way. Indeed, an i-endowed, j-consumer may first choose to make a trip to the (i,k) trading zone and search for a trading partner who is willing to exchange good k for good i. After having found such a partner and successfully exchanged good i for good k, this individual will then make a second trip to the (k,j) trading zone and search for a trading partner who expresses the desire to exchange good j for good k. This is the strategy of indirect exchange which uses good k as a "medium of exchange". The same individual may even seek a longer sequence of indirect exchanges which uses two or more goods as media of exchange.

We still need a description of the way people meet each other in each trading zone. Accordingly, let us denote by q_{ij} the frequency of individuals (relative to the total population) who want to supply good i in exchange of good j in the (i,j) trading zone and call it the "supply-demand frequency." (By construction $q_{ii} = 0$.) We then assume that the probability of meeting one of such i-supplying, j-demanders in the (i,j) trading zone within a period of time is equal to q_{ij}, the frequency of that individual type. The logic behind this assumption is as follows.

In the (i,j) trading zone only those individuals who want to exchange good i for good j and good j for good i are searching, so that there are altogether $q_{ij}+q_{ji}$ frequency of individuals in it. Among these individuals the relative proportion of i-supplying, j-demanders can be calculated as $q_{ij}/(q_{ij}+q_{ji})$ and the relative proportion of j-supplying, i-demanders as $q_{ji}/(q_{ij}+q_{ji})$. If the trading zone is sparsely populated by searchers (and this is the reason for our having called it a trading "zone" rather than a trading "post"), it is reasonable to suppose that the probability of meeting another individual in each trading zone is proportional to the total frequency of individuals searching in that zone. It is also reasonable to suppose that the conditional probability that the person one meets in that zone turns out to be of a given type is equal to the relative proportion of that type among those searching in the zone. Then, the probability that one meets an i-supplying, j-demander in the (i,j) trading zone becomes proportional to

$(q_{ij}+q_{ji})\{q_{ij}/(q_{ij}+q_{ji})\} = q_{ij}$, the frequency of that type of individuals. If we choose the unit of time appropriately, the meeting probability of an i-supplying, j-demander can be made equal to q_{ij} within a unit period of time.[10] (We shall adopt the method of period analysis in what follows.)

The notion of the supply-demand frequency q_{ij} we have just introduced captures at least a part of what Carl Menger called the "marketability" or "saleability" of commodities in his theory of the money.[11] A commodity has, according to Menger, a greater marketability if its "possession would considerably facilitate the individual's search for persons who have just the goods he needs." In the context of our exchange model, the possession of good j with a higher value of q_{ij} would considerably facilitate the individual's search process by increasing the chance of meeting a trading partner who wants to accept good j in exchange of good i.

We shall suppose until section 13 that all the individuals in the economy are in their expectational equilibria, so that their subjective estimates of these supply-demand frequencies are identical with their objective values.

We are now able to analyze the individual search strategy in this somewhat loosely organized exchange economy.

3. The individual search strategy.

Consider an individual who currently holds good i but has a need to consume good j. This i-supplying, j-consumer may not be the one who was born with an endowment of good i (because good i may have come to one's possession by a previous exchange), and may not be the one who demands good j in the current period (because some other good may be demanded as a medium of exchange). In order to analyze the search strategy of this individual, we need to introduce more notations.

Accordingly, let us denote by u the utility of consuming the good one needs, by b the cost (in terms of utility) associated with an act of exchange, and by c the cost (in terms of utility) associated with spending one period in search. In the present paper we shall not search into the concrete contents of these utility and costs and simply assume their constancy over time and uniformity across both individuals and goods.[12] No good is predestined to become money.

Then, let us denote by V_{ij} the maximum life-time expected utility of our i-supplying, j-consumer. As a convention we can put $V_{jj} = u$, for the j-consumer will retire from the economy in order to enjoy the utility u of consumption as soon as good j is acquired. Let us explain how our fellow individual computes this maximum expected utility V_{ij}.

Suppose that our i-supplying, j-consumer has decided to seek the first exchange in the (i,k) trading zone. Note that when $k = j$ this is the case of a direct barter and when $k \neq j$ this is the case of an indirect exchange which uses good k as a medium of exchange. Our fellow individual will start searching for a trading partner who is willing to exchange good k for good i in the (i,k) trading zone. Since the probability of meeting such a trading partner in each period is given by q_{ki} (do not confuse this with q_{ik} !), the expected search period can be calculated as $1+(1-q_{ki})+(1-q_{ki})^2+\ldots = 1/q_{ki}$ and the expected search cost as c/q_{ki}, as long as q_{ki} remains constant over time. As soon as a trading partner is found, our fellow individual exchanges one unit of good i with one unit of good k at the expense of b and becomes a holder of good k. When $k = j$ this marks the end of search and our fellow individual will happily retire from the economy in order to enjoy the utility u of consuming the needful good j. If we ignore the time discount factor, this individual's life-time expected utility in this case can then be calculated simply as $u-b-c/q_{ji}$. On the other hand, when $k \neq j$, our fellow individual has to start a search activity afresh as a k-supplying, j-consumer. In fact, if all the demand-supply frequencies are assumed to be invariant over time, this newly transformed k-supplying, j-consumer is able to expect the utility level of V_{kj} from the search activity to start at the beginning of the next period. Hence, the life-time expected utility of an i-supplying, j-consumer who has decided to visit the (i,k) trading zone first can be calculated (at least implicitly) as $V_{kj}-b-c/q_{ki}$.

Let us then recall the convention: $u=V_{jj}$ we introduced a short moment ago. This of course allows us to write the life-time expected utility of the i-supplying, k-consumer summarily as $V_{kj}-b-c/q_{ki}$ for any k including j (that is, both for barter and indirect exchanges). Since the task of our fellow individual is to choose the trading zone to visit first which would maximize the life-time expected utility, we have in fact obtained the following proposition concerning the individual search strategy in our decentralized exchange economy.

<u>Proposition 1</u>: The maximum expected utility of an i-supplying, j-consumer is given by the following functional equation, as long as the demand-supply frequencies $\{q_{ij}\}$ remain constant over time[13]:

$$(1) \quad V_{ij} = \underset{k}{\text{Max}}[\ V_{kj} - b - c/q_{ki}\ ] .$$

The above functional equation (*à la* Bellman) will play the key role in our subsequent analysis of individual search behavior. There is, however, a slightly more intuitive way of calculating V_{ij}, which will also prove useful in what follows.

Accordingly, let us denote by V^0_{ij} the expected utility of an i-supplying, j-consumer who has committed to a barter exchange; by V^1_{ij} the maximum expected utility of an i-supplying, j-consumer who has committed to an indirect exchange which uses one medium of exchange; by V^2_{ij} the maximum expected utility of an i-supplying, j-consumer who has committed to a sequence of indirect exchanges which uses two media of exchange; and so on. Then, the same sort of argument we advanced above leads us to the following functional equations for these strategy-specific expected utilities.

(2a) $V^0_{ij} = u - b - c/q_{ji}$,

(2b) $V^1_{ij} = \underset{k}{\text{Max}}[V^0_{kj} - b - c/q_{ki}] = u - b - \underset{k}{\text{Min}}[b + c/q_{jk} + c/q_{ki}]$,

(2c) $V^2_{ij} = \underset{k}{\text{Max}}[V^1_{kj} - b - c/q_{ki}] = u - b - \underset{k,h}{\text{Min}}[2b + c/q_{jh} + c/q_{hk} + c/q_{ki}]$,

and so on for longer sequences of indirect exchanges.

Hence, the expected utility of a barter exchange is equal to the net utility of direct consumption, u-b, minus the expected search cost for such occasion; the maximum expected utility of one medium indirect exchange is equal to the net utility of consumption minus the minimum sum of the costs of searching in two trading zones and the cost of conducting one indirect exchange; the maximum expected utility of two media indirect exchange is equal to the net utility of consumption minus the minimum sum of the costs of searching in three trading zones and the cost of conducting two indirect exchanges; and so on. Since the maximum expected utility V_{ij} must be the maximum of maxima, we can also express it as

(3) $V_{ij} = \underset{n}{\text{Max}}[V^n_{ij}]$.

In order to complete the description of individual search behavior, we need to specify a tie-breaking rule to decide which exchange strategy is actually chosen when more than one strategies are expected to give the same life-time utility. In what follows, we shall adopt a (half-lexicographic and half-randomizing) rule which says that when the tied strategies have different lengths of exchange sequence the one with the least indirect sequence is actually chosen and when the tied strategies have the same length of exchange sequence a coin-tossing or some randomization device picks up one of them.

4. The conditions for the universal barter exchange.

The individual exchange process can in general be quite complex. But there are at least two special cases which have simple exchange patterns. They are of course barter exchange and monetary exchange. We now take up the barter exchange first.

It is tautological to say that an individual chooses to barter if and only if it would guarantee the higher expected utility than any sequence of indirect exchanges. This tautological statement can be formalized compactly as $V_{ij} = u-b-c/q_{ji}$, or more intuitively as $V^0_{ij} \geqq V^n_{ij}$ for any n = 1, 2, 3.... . Indeed, by comparing (2a) and (2b), we can rewrite the condition for preferring a direct barter to any of the one-medium indirect exchange (that is, $V^0_{ij} \geqq V^1_{ij}$) simply as

$$\text{(4a)} \quad c/q_{ji} \leqq b + c/q_{jk} + c/q_{ki}$$

for any k. And by comparing (2a) and (2c), we can also rewrite the condition for preferring to barter directly rather than to use two media of exchanges (that is, $V^0{}_{ij} \geqq V^2{}_{ij}$) as

(4b) $c/q_{ji} \leqq 2b + c/q_{jh} + c/q_{hk} + c/q_{ki}$

for any k and h, and so on.

It can be shown, however, that if the first inequality (4a) holds for any i and $j(\neq i)$, all the other inequalities involving longer sequences of indirect exchanges become redundant as sufficient conditions for barter exchange. For by repeatedly applying (4a) we obtain the following series of inequalities:

(4c) $c/q_{ji} \leqq b + c/q_{jk} + c/q_{ki} \leqq 2b + c/q_{jk} + c/q_{kh} + c/q_{hi} \leqq \cdots\cdots$

Since the necessity of (4a) is obvious, we have in fact obtained the following proposition concerning the universal barter exchange.

Proposition 2: Every possible individual type chooses to barter if and only if the following set of inequalities hold for any i, $j(\neq i)$ and $k(\neq i$ and $j)$:

(5) $c/q_{ji} \leqq b + c/q_{jk} + c/q_{ki}$.

5. The conditions for the universal monetary exchange.

At the other extreme lies the exchange process where one particular good is used as the medium of exchange by everybody, except of course by those who have real need of it and by those who already have it in hand. That particular good is functioning as the exclusive and universal medium of exchange -- money. Without loss of generality, we can let this particular good be the m-th good of the economy and state the necessary and sufficient conditions for such universal monetary exchange in the following manner.

Proposition 3: A particular good m is used as the exclusive medium of exchange by every possible individual type in the economy, except by those who already have it in hand and by those who have real needs of it, if and only if the following set of inequalities are satisfied;

(6a) $c/q_{ji} > b + c/q_{jm} + c/q_{mi}$,

for any $i(\neq m)$ and $j(\neq m$ and $i)$;

(6b) $b + c/q_{jm} + c/q_{mi} < b + c/q_{jk} + c/q_{ki}$,

for any $i(\neq m)$, $j(\neq m$ and $i)$ and $k(\neq m$, i and $j)$;

(6c) $c/q_{jm} \leqq b + c/q_{jk} + c/q_{km}$,

for any $j(\neq m)$ and $k(\neq m$ and $j)$; and

(6d) $c/q_{mi} \leqq b + c/q_{mk} + c/q_{ki}$,

for any $i(\neq m$ and $j)$ and $k(\neq m$ and $i)$.

The first inequality has the complementary form to the inequality condition

(5) for barter exchange and says simply that it is less costly to use good m as a medium of exchange than to barter directly. The second inequality says that among all the possible candidates for the medium of an indirect exchange good m is the least costly to use and no other good can rival it. The third inequality then says that once good m has come into one's possession (either by an exchange or by an endowment) it is less costly to exchange it directly for the good one needs than to seek another indirect exchange by using some other good as a medium. And the fourth inequality says that for an individual who has real need of consuming good m it is less costly to seek a direct barter than to use some other good as a medium for obtaining it. These inequalities appear almost self-explanatory. But in order to prove their necessity and sufficiency for the universality of monetary exchanges rigorously, we unfortunately need a rather long argument. The impatient reader could skip the rest of the present section without losing the main thread of the argument.

Let us first prove the sufficiency of these inequalities. Indeed, since the expected utility of using good m as the sole medium of exchange can be calculated simply as $u-2b-c/q_{jm}-c/q_{mi}$, it is only necessary to show that this value is the unique maximum of V_{ij} for any $i(\neq m)$ and $j(\neq m$ and $i)$. (This is of course a formal way of saying that a particular good m is used as the exclusive medium of exchange by every possible individual type in the economy, except by those who already have it in hand and by those who have real needs of it.) But we need some preparatory steps.

<u>Lemma 1</u>: The individual who has good m will never seek any indirect exchange if (6c) holds for any $j(\neq m)$ and $k(\neq m$ and $j)$.

(Proof): By repeatedly applying (6c) to its own right-hand-side, we obtain a series of inequalities: $c/q_{jm} \leqq b+c/q_{jk}+c/q_{km} \leqq 2b+c/q_{jk}+c/q_{kh}+c/q_{hm} \leqq \cdots$, for any k ($\neq$ m) and h ($\neq$ m) and so on. They of course imply that V^0_{mj} = u-b-c/q_{jm} is the unique maximizer of V_{mj}, or that the individual who currently holds good m will never seek any indirect exchange. (QED)

Lemma 2: The individual who has the real need of good m will never seek any indirect exchange if (6d) holds for any i($\neq$ m and j) and k($\neq$ m and i).

(Proof): By repeatedly applying (6d) to its own right-hand-side, we obtain: $c/q_{mi} \leqq b+c/q_{mk}+c/q_{ki} \leqq 2b+c/q_{mh}+c/q_{hk}+c/q_{ki} \leqq \cdots$, for any k ($\neq$ m) and h ($\neq$ m) and so on. These series of inequalities imply that V^0_{im} = u-b-c/q_{mi} is the unique maximum of V_{im}, or that the individual who has real need of good m will never seek any indirect exchange. (QED)

We are now ready to prove the sufficiency part of Proposition 3.

(Proof of the sufficiency part of Proposition 3): Now, for any i($\neq$ m) and j($\neq$ m and i), the first inequality (6a) says that the expected utility of using good m as the sole medium of exchange, u-2b-c/q_{jm}-c/q_{mi}, is strictly greater than V^0_{ij} = u-b-c/q_{ji}, and the second inequality (6b) says that it is the unique maximum of V^1_{ij}. What remains to be proved is only that it is also strictly greater than V^n_{ij} for any $n \geqq 2$. Suppose not. Then, there is a sequence of n ($\geqq$ 2) indirect exchanges which use goods k,..., h as media, such that u-2b-c/q_{jm}-c/q_{mi} < u-(n+1)b-c/q_{jk}-...-c/q_{hi}. Assume for the time being that one of the supposed media, k,..., h, happens to be the good m. Then, the above inequalities can be rewritten as: u-2b-c/q_{jm}-c/q_{mi} < (u-2b)-(n_1b+c/q_{jk}+...+c/q_{rm})-(n_2b+c/q_{ms}+...+c/q_{hi}) where n_1+1 is the length of the sequence: j, k, ...,r and m and n_2+1 is the length of the sequence: m, s, ...h and i. But by applying one of the inequality in the proof of Lemma 1 to the second bracket and one of the inequality in the proof of Lemma 2 to the third bracket in the right-hand-side of this inequality, we obtain another inequality: $u-2b-c/q_{jm}-c/q_{hi} < u-2b-c/q_{jm}-c/q_{mi}$, which is an outrigt contradiction. Assume next that the supposed media, k, ..., h, do not include m. Then, by applying the inequality (6a) to the last term in the right-hand-side of the supposed inequality: $u-2b-c/q_{jm}-c/q_{mi} < u-(n+1)b-c/q_{jk}-\ldots-c/q_{hi}$, we obtain a new inequality: $u-2b-c/q_{jm}-c/q_{mi} < u-(n+2)b-c/q_{jk}-\ldots-c/q_{hm}-c/q_{mi}$. We are thus back to the previous case where one of the supposed media happens to be the good m, which of course leads to a contradiction. (QED)

We now turn to the proof of the necessity part of Proposition 3, although it is of little use in the main text of the present paper. For this purpose we now have to deduce all the inequalities, (6a) -- (6d), from the fact that the expected utility of using good m as the sole medium of exchange, $u-2b-c/q_{jm}-c/q_{mi}$, is the unique maximum of V_{ij}. We again need some Lemmata to do this. Indeed, the fact that the good m is used as the exclusive medium of exchange by every possible individual type, except by those who have come to possess it and by those who have real needs of it, turns out to have obvious but important implications for the exchange activities of those individuals just excluded.

<u>Lemma</u> 3: In order for good m to be used as the exclusive and universal medium of exchange, no holders of good m must seek any indirect exchange.

(Proof): Suppose not. Then, there is a sequence of n (≥ 1) indirect exchanges such that $u-(n+1)b-c/q_{jk}-c/q_{kl}-\ldots-c/q_{gh}-c/q_{hm} > u-b-c/q_{jm}$ for some $h \neq m$. By adding $-b-c/c/q_{mi}$ to both sides of the inequality, we have $u-(n+2)b-c/q_{jk}-c/q_{kl}-\ldots-c/q_{gh}-c/q_{hm}-c/q_{mi} > u-2b-c/q_{jm}-c/q_{mi}$. But the right-hand-side of this inequality is nothing but V_{ij}, which implies that a sequence of n+1 indirect exchanges gives a higher expected utility than the supposed maximum of V_{ij}. A contradiction. (QED)

Lemma 4: In order for good m to be used as the exclusive and universal medium of exchange, no consumers of good m must seek any indirect exchange.

(Proof): Suppose not. Then, there is again a sequence of n (≥ 1) indirect exchanges such that $V_{im} = u-(n+1)b-c/q_{ml}-c/q_{lk}-\ldots-c/q_{gh}-c/q_{hi} > u-b-c/q_{mi}$ for some $k \neq m$. By adding $-b-c/c/q_{jm}$ to both sides of the inequality, we have $u-(n+2)b-c/q_{jm}-c/q_{ml}-c/q_{lk}-\ldots-c/q_{gh}-c/q_{hi} > u-2b-c/q_{jm}-c/q_{mi} = V_{ij}$. We have thus obtained the same contradiction as Lemma3. (QED)

We are now able to deduce the necessity part of Proposition 3.

(Proof of the necessity part of Proposition 3): In the first place, the fact that $u-2b-c/q_{jm}-c/q_{mi}$ is the unique maximum of V_{ij} immediately implies (6a) and (6b). Next, by Lemma 3 we have $V_{mj} = u-b-c/q_{jm} \geq u-b-\mathrm{Min}_k[b+c/q_{jk}+c/q_{km}]$. This leads to (6c). Finally, by Lemma 4 we have $V_{im} = u-b-c/q_{mi} \geq u-b-\mathrm{Min}_k[b+c/q_{mk}+c/q_{ki}]$. This leads to (6d). (QED)

Needless to say, the case of universal barter exchange and the case of universal monetary exchange we have examined above are two of the most extreme exchange patterns. Indeed, our search model allows an embarrassingly rich variety of "impure" exchange patterns between (and perhaps besides) these two cases. (Examples are bi-monetary system, hierachical monetary system, multiple monetary areas, local monies, etc.) But these impure exchange patterns are not easily amenable to analysis at least at present, and we have to suppress our taxonomic instinct in the present paper.

6. The endowment-need distribution and supply-need distribution of the economy.

So far we have devoted ourselves to the analysis of the individual search behavior when the set of supply-demand frequencies, $\{q_{ij}\}$, are given exogenously. But, the actual values of these frequencies are in turn determined by the way these individuals search for appropriate exchange partners in the economy. Our task then is to characterize possible equilibrium patterns of exchange which render the individual search behaviors all consistent with each other in the economy.

For this purpose, we have to introduce, in addition to the supply-demand frequencies $\{q_{ij}\}$, two more representations of the economy's population structure. To begin with, let us denote by e_{ij} the relative frequency of individuals in the economy who were born with an endowment of good i and with a need of good j and call it the "endowment-need frequency". Note that $e_{ii} = 0$ by assumption and $\Sigma_i \Sigma_{j \neq i} e_{ij} = 1$ by construction.

The set of endowment-need frequencies $\{e_{ij}\}$ represents the distribution of individual types in the economy, classified in accordance with their original constitutions. It is the "real" data of our exchange economy, in the sense that it summarizes the basic structure of "technology and tastes", or what may be called the "fundamentals" of the underlying economy. It should be, however, noted here that these frequencies are in general subject to changes over time. For new individuals are constantly entering the economy freshly endowed with goods and needs, and some old individuals are constantly retiring from the economy in order to consume the goods they have longed for. But in order to simplify the exposition, we shall suppose in what follows that, as soon as an i-endowed, j-consumer retires from the economy a new i-endowed, j-consumer enters into the economy.[14] Such a parent-child succession of each individual type will keep the frequency e_{ij} of i-endowed, j-consumers constant over time, irrespective of the way people exchange with each other.

Let us then introduce one important definition concerning the way endowments and needs are distributed among individuals.

Definition: An economy is said to be connected if for any i and j ($\neq$ i) we have a sequence of positive endowment-need frequencies such that $e_{jk} > 0$, $e_{kl} > 0$, ..., $e_{gh} > 0$ and $e_{hi} > 0$.

If an economy is connected, every individual born in it is in principle able to reach the good in need by a suitable exchange sequence.[15] The connectedness is therefore the minimum requirement for an economy to be meaningfully called "an" economy. If there is an economy which lacks the connectedness, it should rather be regarded as a collection of several economies, each having some sort of self-sufficiency.

There are two special population structures which deserve special attention here. First, if the economy is symmetric with respect of both endowments and needs, we have

(7) $e_{ij} = e_{ji} = 1/N(N-1)$ for any i and $j \neq i$.

Needless to say, this economy is abundantly connected.

At the other extreme is the population structure whose endowments and needs do not allow any double coincidence of wants but still are minimally connected,or

(8) $e_{12} = e_{23} = .. = e_{N-1,N} = e_{N1} = 1/N$; all the other e_{ij}'s = 0.

This is the example Cass and Yaari used as a static counterpart of Samuelson's consumption-loan model of intertemporal exchanges.[16]

It should be noted, however, that the frequency distribution of endowments and needs $\{e_{ij}\}$ -- the "real" data of our exchange economy -- is not the relevant data for the actual exchange process going on. Except for the case of barter exchange, some individuals are bound to exchange their initial endowments for some other goods in order to use them as media of exchange, and it is the good one holds at the moment and not the good one was originally endowed with that is relevant for the individual decision.

Accordingly, let us denote by f_{ij} the relative frequency of individuals who currently hold good i and need to consume good j and call it the "supply-need frequency". Note that $f_{ii} = 0$ by assumption and that $\Sigma_i \Sigma_{j \neq i} f_{ij} = 1$ by construction.

There are therefore three different sets of frequencies in our model -- $\{q_{ij}\}$, $\{f_{ij}\}$ and $\{e_{ij}\}$ -- each representing a different layer of the economy. At the surface level is the set of supply-demand frequencies $\{q_{ij}\}$, which is what we can observe in the daily processes of exchanges. Indeed, it is on the basis of their observations that the individuals in this economy decide their daily search activities. The middle layer is occupied by the set of supply-need frequencies $\{f_{ij}\}$ we have just introduced. It is the description of the partly visible and partly invisible behavioral characteristics of the individuals -- the goods they actually hold and the goods they need to consume -- which determine the forms of their search strategies in the economy. And at the deepest level lies the set of endowment-need frequencies $\{e_{ij}\}$. It is the summary of the "technology and tastes" of the economy and represents its most invariant and most invisible structure.

To sort out the hierarchical relationship among these three frequency distributions is in general a very complex task, except in the case of barter equilibrium to be discussed in section 8. Indeed, the analysis of the relationship between the supply-need frequencies $\{f_{ij}\}$ and the endowment-need frequencies $\{e_{ij}\}$ would require a much fuller understanding of the dynamic evolution of the processes of exchange than that we have at the moment. For the time being we therefore have no other choice but to concentrate on the analysis of the relationship between the supply-need frequencies $\{f_{ij}\}$ and the supply-demand frequencies $\{q_{ij}\}$ and proceed until section 10 as if the former frequencies are given data of our economy.

7. The definition of exchange equilibrium.

In section 2 we supposed that the probability of meeting an individual who supplies good i and demands good j in exchange in the (i,j) trading zone is equal to the frequency q_{ij} of the individuals of that type within the whole economy. In the previous section, on the other hand, we introduced the population parameter f_{ij} which represents the frequency of individuals who currently hold good i but need to consume good j. It goes without saying that they are not equal in general. If the i-supplying, k-consumers have found it profitable to use good j as a medium of exchange, they join the (i,j) trading zone, adding their frequency f_{ik} to the frequency q_{ij} of i-supplying, j-demanders; and if the i-supplying, j-consumers have found it more profitable to use some non-j good as a medium of exchange rather than to seek a direct barter, they vacate the (i,j) trading zone, subtracting their frequency f_{ij} from the frequency q_{ij} of i-supplying, j-demanding individuals. Now the condition that the i-supplying, k-consumers demand good j as a medium of exchange can be written simply as $V_{jk}-b-c/q_{ji} = V_{ik}$, and the condition that i-supplying, j-consumers demand good j in order to barter with good i can be written simply as $u-b-c/q_{ji} = V_{ij}$. If we take care not to count the individuals with zero frequency, we can determine the supply-demand frequency q_{ij} in the following manner:

$$(9) \quad q_{ij} = \sum_{\{k\neq j:f_{ik}>0\ \&\ V_{jk}-b-c/q_{ji}=V_{ik}\}} f_{ik} + \begin{cases} f_{ij} & \text{when } f_{ij}>0\ \&\ u-b-c/q_{ji}=V_{ij}, \\ 0 & \text{otherwise,} \end{cases}$$

for any i and $j\neq i$.[17]

We have now closed the circle at least in the formal sense. Given a set of supply-demand frequencies $\{q_{ij}\}$, equation (1) of section 2 in principle allows us to calculate the expected utility V_{ij} of any individual whose search activity in the economy is dictated by the objective of maximizing it. Then, the counting-up equation (9) we have just written down in turn determines these supply-demand frequencies $\{q_{ij}\}$ as the aggregate outcomes of these individual search behaviors. We can thus state that:

Definition: An economy is said to be in a state of exchange equilibrium if its supply-demand frequencies $\{q_{ij}\}$ satisfy both

(1) $V_{ij} = \max_{k}[V_{kj} - b - c/q_{ki}]$.

(9) $q_{ij} = \sum_{\{k \neq j: f_{ik}>0 \ \& \ V_{jk}-b-c/q_{ji}=V_{ik}\}} f_{ik} + \begin{cases} f_{ij} & \text{when } f_{ik}>0 \ \& \ u-b-c/q_{ji}=V_{ij}, \\ 0 & \text{otherwise} \end{cases}$,

for any i and j ($\neq$ i).[18]

It is one thing to define an equilibrium of the economy, but it is another to analyze it. There are indeed a large number of possible equilibrium patterns of exchange in our economy, and every one of them deserves a special attention. But we cannot be exhaustive in the present paper and shall concentrate mostly on the analysis of the relationship between two of the most important equilibrium patterns, to be defined below.[19]

Definition: An exchange equilibrium is said to be a barter equilibrium if every individual in the economy seeks a barter in it.

Definition: An exchange equilibrium is said to be a monetary equilibrium if a particular good is used as the exclusive medium of exchange by every individual in the economy, except by those who have real need of it and by those who already have it.

As it turns out, even if we have confined our analysis to these two forms of exchange equilibrium, we still have to face the situation of embarras de richesses.

8. The difficulties of barter equilibrium.

The first question we would like to pose is: what are the conditions for an economy to have a barter equilibrium?.

As was pointed out in the last section, the meeting probability q_{ij} in general deviates from supply-need frequency f_{ij} either because some i-supplying individuals may demand good j for the purpose of using it as a medium of exchange or because i-supplying, j-consumers may demand some other good for the purpose of using it as a medium of exchange. But in the case of barter exchange, to demand good j is to consume it, and we have $q_{ij} = f_{ij}$ for any i and j with $f_{ij} > 0$. Since q_{ij} is equal to 0 for the remaining i and j with $f_{ij} = 0$, we indeed have

(10) $q_{ij} = f_{ij}$,

for any i and j.

Next, as we argued in section 6, the supply-need frequency f_{ij} in general deviates from the endowment-need frequency e_{ij} because some i-endowed, j-consumers may exchange their initially endowed good i for some other goods in order to use them as media of exchange. But in the case of barter exchange, no individuals change their initial endowments until they barter them for the goods in need, and we have

(11) $f_{ij} = e_{ij}$,

for any i and j.

Having thus reduced all the supply-demand frequencies in a barter equilibrium to the endowment-need frequencies -- the "real" data of the economy --, we are now in a position of stating one necessary condition for the existence of a barter equilibrium, which should look familiar to every student of monetary economics.

Proposition 4: In order for a connected economy to have a barter equilibrium, the endowment-need frequency e_{ji} must be strictly positive whenever its complementary frequency e_{ij} is strictly positive.

(Proof): Suppose $e_{ji} = 0$ for some j and i ($\neq$ j) such that $e_{ij} > 0$. Then, by the connectedness of the economy, there is a sequence of n endowment-need frequencies such that $e_{jk}>0$, $e_{kl}>0$, ..., $e_{gh}>0$ and $e_{hi}>0$, and we have $(n-1)b+c/e_{jk}+c/e_{kl}+....+c/e_{gh}+c/e_{hi} < \infty = c/0 = c/e_{ji}$. This implies that the i-endowed, j-consumers (who are active searchers) will find it less costly to seek a sequence of n indirect exchanges than to barter directly. The economy cannot stay in a barter equilibrium. (QED)

What is stipulated in this Proposition is nothing but the requirement of "double coincidence of wants". It says that the exchange economy cannot have a barter equilibrium unless every active participant in it is endowed with a good that some other individual needs and at the same time needs the good that the same individual is endowed with. It is indeed easy to see that the economy with a minimally connected endowment-need distribution, given by (8) in section 6, lacks the condition for double coincidence of wants and has no barter equilibrium in it.

There is one Corollary to the above Proposition, which guarantees us that if there is a barter equilibrium in our exchange economy, it must be a nontrivial one, or

Cororally: In a connected economy, the situation of autarky cannot be sustained as a barter equilibrium.

(Proof): $V_{ij} = u-b-c/e_{ji} > -\infty$ in a barter equilibrium, because $e_{ji} > 0$ for any i and j such that $e_{ij} > 0$. (QED)

The double coincidence of wants, however, is only a necessary condition for the existence of a barter equilibrium. In order to understand the nature of barter equilibrium more fully, we now have to look for the sufficiency conditions for the existence of a barter equilibrium.

For this purpose, let us recall the set of inequalities (5) in Proposition 2, which has given us the range of the supply-demand frequencies $\{q_{ij}\}$ which would be both necessary and sufficient for every possible individual type to seek a barter trade. Of course, as long as some of the individual types are absent from the economy in the sense that their endowment-need frequency e_{ij} is 0, the necessity part of the proposition becomes useless. But the sufficiency part is still applicable even to such situation. Accordingly, if we apply the equality between q_{ij} and f_{ij} given by (10) and then the equality between f_{ij} and e_{ij} given by (11) to these inequalities, we are able to obtain the sufficient conditions for the existence of a barter equilibrium in the form of:

Proposition 5: An economy has a barter equilibrium if its endowment-need frequency distribution $\{e_{ij}\}$ satisfies the following set of inequalities for any triplet of i, j ($\neq$ i) and k ($\neq$ i and $\neq$ j):

(12) $c/e_{ji} \leqq b + c/e_{ki} + c/e_{jk}$.

If, moreover, all the endowment-need frequencies $\{e_{ij}\}$ happen to be strictly positive, the above set of inequalities become necessary conditions as well.

From this proposition, we can readily make the following observations concerning the existence of a barter equilibrium.

Starting from the least interesting case, we can observe that if $e_{ij} \geqq c/b$ for all i and $j(\neq i)$ a barter equilibrium always exists. This is the case where a complementary trading partner is easily found in the barter trading zone and nobody bothers to seek indirect exchange. However, since the frequencies of individual types must satisfy the obvious constraint: $\Sigma_i \Sigma_{j \neq i} e_{ij} = 1$, this case disappears if $N(N-1) > b/c$, or if the number of goods N is sufficiently large and/or the ratio between exchange and search costs b/c is sufficiently small.

More interesting is the case of totally symmetric population structure. If, as in (7), $e_{ij} = e_{ji} = 1/\{N(N-1)\}$ for any i and $j(\neq i)$, the above set of inequalities are all trivially satisfied. Thus, economies with totally symmetric population structure have always a barter equilibrium. But, as the frequency distribution of individual types becomes more and more dispersed, the chance that some of the inequalities in (12) are violated becomes higher and higher (except for the case of $e_{ij} > c/b$ for any i and $j(\neq i)$ mentioned above). And once they are actually violated, some individuals may attempt to use goods with relatively higher demands as media of exchange, and the existing barter equilibrium may break down. Indeed, we already know from Proposition 4 that the economy with a minimally connected population structure, given in (8), is not able to have any barter equilibrium.

Except for the case of $e_{ij} > c/b$ for any i and j ($\neq$ i), economies without barter equilibrium are not the exceptions but the rules.

Fig. 1 illustrates the above discussion for the case of $b/c < N(N-1)$. In this figure three curved surfaces are drawn in a diagram with an arbitrarily chosen triplet of e_{ji}, e_{ki} and e_{jk} as three axes. The first surface corresponds to the condition $c/e_{ji} \leqq b+c/e_{jk}+c/e_{ki}$, the second to $c/e_{ki} \leqq b+c/e_{kj}+c/e_{ji}$ and the third to $c/e_{jk} \leqq b+c/e_{ji}+c/e_{ik}$. The fourth plane which slices up these three surfaces represents an obvious constraint $\Sigma_i \Sigma_{j\neq i}\, e_{ij} = 1$, which implies $e_{jk}+e_{ki}+e_{ij} \leqq 1$. The shaded area enclosed by these surfaces represents the set of endowment-need frequencies which allow the economy to have a barter equilibrium.[20]

9. The bootstraps mechanism of monetary equilibrium.

Let us now turn to the analysis of the "monetary equilibrium" -- a form of exchange equilibrium which uses one particular good as the exclusive and universal medium of exchange.

Let this particular good be good m, as before. Then, if the economy is in a monetary equilibrium, its supply-demand frequencies $\{q_{ij}\}$ are determined as the aggregative outcomes of individuals' search behaviors in the following manner. First, since every holder of good i demands good m either for direct consumption or as a medium of exchange, the frequency q_{im} of i-supplying, m-demanders consists of the frequencies f_{ik} (>0) of all the individual types who are currently supplying good i. Since the value of q_{im} is unaffected by the addition of zero frequencies, we can simply write it as

(13a) $q_{im} = \sum_k f_{ik}$ for any $i (\neq m)$.

Second, since every holder of good m does not seek any indirect exchange, we would have $q_{mj} = f_{mj}$ for j such that $f_{mj} > 0$. Since q_{mj} is zero for j with $f_{mj} = 0$, we indeed have

(13b) $q_{mj} = f_{mj}$ for any $j (\neq m)$.

Finally, since every holder of non-monetary good does not demand the good in need immediately, and since the inactive individual type does not demand any good any way, we have

(13c) $q_{ij} = 0$ for $i \neq m$ and $j \neq m$.

These three relations are no more than the explicit representations of the counting-up equation (9) in the case of monetary equilibrium.

Having thus calclulated the supply-demand frequencies $\{q_{ij}\}$ in a monetary equilibrium explicitly, we are in a position of asking the following question: are they consistent with each other and actually constituting a monetary equilibrium? In fact, we are now able to state the first fundamental proposition concerning the existence of a monetary equilibrium:

Proposition 6: Suppose $f_{mi} > 0$ for any $i \neq m$ and $\sum_k f_{jk} > 0$ for any $j \neq m$. Then, the economy has a monetary equilibrium, with good m used as the exclusive and universal medium of exchange.

(Proof): All we have to do is to demonstrate that the set of supply-demand frequencies $\{q_{ij}\}$, given in (13a) -- (13c), satisifies the set of inequalities (6a) -- (6d) stipulated in Proposition 3. This is because these inequalities are nothing but the (sufficient) conditions which induce (or, literally, force) everybody in the economy to use good m as the exclusive and universal medium of exchange (except, of course, those who already have good m and who have needs to consume good m). In the first place, all the inequalities in (6a) are trivially satisfied, for if $f_{mi} > 0$ and $\Sigma_k f_{jk} > 0$ we have

$$b+c/q_{jm}+c/q_{mi} = b+c/\sum_k f_{jk}+c/f_{mi} < c/q_{ji} = c/0 = \infty,$$

for any $i(\neq m)$ and $j(\neq m$ and $i)$. Secondly, all the inequalities in (6b) are again trivially satisfied, for

$$b+c/q_{jm}+c/q_{mi} = b+c/\sum_k f_{jk}+c/f_{mi} < b+c/q_{jk}+c/q_{ki} = b+c/0+c/0 = \infty,$$

for any $i(\neq m)$, $j(\neq m$ and $i)$ and $k(\neq m$, i and $j)$. Thirdly, all the inequalities in (6c) are again trivially satisfied, for we have

$$c/q_{jm} = c/\sum_k f_{jk} \leqq b+c/q_{jk}+c/q_{km} = b+c/0+/f_{km} = \infty.$$

for any $j(\neq m)$ and $k(\neq m$ and $j)$. And fourth, all the inequalities in (6d) are once again trivially satisfied, for if $f_{mi} > 0$ we have

$$c/q_{mi} = c/f_{mi} \leqq b+c/q_{mk}+c/q_{ki} = b+c/f_{mk}+c/0 = \infty.$$

for any $i(\neq m)$ and $k(\neq m$ and $i)$. (QED)

What we have seen in the above proof is the working of the "bootstraps mechanism" which supports itself by its own inner forces! Everybody in the economy (except those who either own good m or need good m) switches their immediate demands from the goods in need to good m in order to use it as the exclusive medium of exchange, merely because everybody else has switched their immediate demands from the goods in need to good m in order to use it as the exclusive medium of exchange. A particular good m becomes the exclusive and universal medium of exchange -- money -- by no other reason than that it is used by everybody in the economy as the exclusive and universal medium of exchange -- money.

Indeed, there is no intrinsic reason why a particular good should be demanded as money in the first place. To see this, let us rewrite the counting-up equation : $q_{im} = \Sigma_k f_{ik}$ of (13a) as $f_{im} + \Sigma_{k \neq m} f_{ik}$. In this expression which determines the i-holders' demand for good m in a monetary equilibrium, f_{im} represents the "real" demand for good m as an object of consumption and $\Sigma_{k \neq m} f_{ik}$ represents the sum of all the "social" demands for good m as the medium of exchange. Indeed, as long as the "social" demands $\Sigma_{k \neq m} f_{ik}$ remains non-zero, a monetary equilibrium is able to sustain itself without any "real" demand f_{im} for money. This, for instance, implies the possibility of having a monetary equilibrium with a useless piece of paper used as a "fiat money" or a large round stone sunken deep in the sea used as a "symbolic money".

Money is money simply because it is used as money. It is a "social contrivance" which supports itself by its own "bootstraps".

We must, however, conclude the present section by pointing to the fact that our Proposition 6 just falls short of establishing the purely social nature of money. The existence of a monetary equilibrium still requires one condition: $f_{mi} > 0$ for all $i \neq m$, which amounts to saying that for any non-monetary good in the economy there must always be some individual who wants to "supply" money in exchange for it. (Another condition: $\Sigma_k f_{jk} > 0$ is trivially satisfied in the case of connected economy.) In fact, if the distribution of supply-need frequencies among individuals happens to have a minimally connected structure in the sense that $f_{12} = f_{23} = \ldots = f_{N-1,N} = f_{N1} = 1/N$, the above condition for monetary equilibrium is clearly violated. Since this is a reasonable-looking distribution (because it has formally the same structure as that of the endowment-need distribution, given by (8)), it seems to imply that in the final analysis the institution of money still requires some minimal support of the "real" conditions of the economy. Does this mean that money is a social product, but not purely a social product?

The answer is, however, "no". But in order to explain the purely social nature of money, we need to analyze the interaction between the dynamic process of monetary exchange and the evolutionary process of the supply-need frequencies in some detail -- the interaction we have so far ignored. To this problem, we now have to turn.

10. The steady-state distribution of supply-need frequencies.

In the preceding section we have characterized the monetary equilibrium in terms of the distribution of supply-need frequencies $\{f_{ij}\}$ among individuals. But, as we have repeatedly pointed out, the real data of our exchange economy consist in the distribution $\{e_{ij}\}$ of endowments and needs. Of course, in the case of barter exchange, no individual will ever change the initial holdings of the endowed goods until the very day of their retirement, and we have $f_{ij} = e_{ij}$. But, in general and in the case of monetary equilibrium in particular, most individuals will in the course of time exchange their initial endowments for some other goods in order to use them as media of exchange, and f_{ij} will in general deviate from e_{ij}. In order to understand the nature of monetary equilibria more fully, we therefore need to understand how the supply-need frequencies $\{f_{ij}\}$ evolve over time, as the very outcomes of the on-going processes of monetary exchanges.

Although it is possible to write down a set of difference equations which describes the entire evolutionary history of $\{f_{ij}\}$, it is difficult to solve it explicitly.[21] In the present paper we have to be content with the examination of long-run "steady-state" behavior. Indeed, as far as the steady-state values of $\{f_{ij}\}$ are concerned, there is a way to compute them without having recourse to the analysis of intricate difference equations.

To see this, let us consider an individual who has just been thrown into a monetary equilibrium with an endowment of good i and a need to consume good j. If neither good i nor good j are monetary good m, this individual first seeks to exchange good i for good m and then seeks to exchange the latter for good j. Since the probability of meeting an m-supplying, i-consumer is q_{mi} and the probability of meeting a j-supplying, m-demander is q_{jm}, we can calculate this individual's expected waiting period for the first exchange as $1/q_{mi}$ and for the final exchange after the acquisition of money as $1/q_{jm}$. Hence, the total expected waiting period is equal to $1/q_{mi}+1/q_{jm}$. It then follows that within this life time the i-endowed, j-consumer is expected to spend a fraction $(1/q_{mi})/(1/q_{mi}+1/q_{jm}) = q_{jm}/(q_{jm}+q_{mi})$ of time period as an i-supplying, j-consumer and a fraction $(1/q_{jm})/(1/q_{mi}+1/q_{jm}) = q_{mi}/(q_{jm}+q_{mi})$ of time period as an m-supplying, j-consumer.

Suppose now that the economy has settled down to a steady state such that each frequency f_{ij} of ownership and need converges to its steady-state value, to be denoted by f^*_{ij}. Then, the Ergodic theorem in probability theory enables us to identify (with probability 1) the time-series fraction $q_{jm}/(q_{jm} + q_{mi})$ with the cross-sectional fraction of i-endowed, j-consumers who are currently holding i-good.[22] Hence, for $i(\neq m)$ and $j(\neq m$ and $i)$, we have

$$(14a)\quad f^*_{ij} = \frac{q_{jm}}{q_{jm} + q_{mi}} e_{ij} .$$

By the same token, we can also identify (with probability 1) the time-series fraction $q_{mi}/(q_{jm} + q_{mi})$ with the cross-sectional fraction of i-endowed, j-consumers who are currently holding money. Since the m-supplying, j-consumers consist of m-endowed, j-consumers as well as all the other j-consumers who are currently holding good m as the medium of exchange, we have for $j(\neq m)$

$$(14b)\quad f^*_{mj} = e_{mj} + \sum_{k\neq m} \frac{q_{mk}}{q_{jm} + q_{mi}} e_{kj} .$$

Finally, since the consumers of good m remain as such until their retirement, we have for $i(\neq m)$

$$(14c)\quad f^*_{im} = e_{im} .$$

The above set of equations relate the steady-state values of the supply-need frequencies $\{f^*_{ij}\}$ to both the supply-demand frequencies $\{q_{ij}\}$ and the endowment-need frequencies $\{e_{ij}\}$. But we know from equations (13a), (13b) and (13c) of section 9 that in a monetary equilibrium the supply-demand frequencies $\{q_{ij}\}$ are in turn determined by the distribution of supply-need frequencies $\{f_{ij}\}$. Keeping these two sets of relationships in mind, we can determine the steady-state value of each of the supply-demand frequenciyes, to be denote by q^*_{ij}, in the following implicit manner.

$$(15a)\quad q^*_{im} = \sum_k f^*_{ik} = e_{im} + \sum_{k\neq m} \frac{q^*_{km}}{q^*_{km} + q^*_{mi}} e_{ik} \qquad \text{for } i\ (\neq m) .$$

$$(15b)\quad q^*_{mj} = f^*_{mj} = e_{mj} + \sum_{k \neq m} \frac{q^*_{mk}}{q^*_{jm} + q^*_{mk}} e_{kj} \qquad \text{for } j\ (\neq m)\ .$$

$$(15c)\quad q^*_{ij} = 0 \qquad \text{for } i\ (\neq m) \text{ and } j\ (\neq m \text{ and } i).$$

If we substitute back these equations into (14a), (14b) and (14c), we can also determine the steady-state supply-need frequencies $\{f^*_{ij}\}$ at least implicitly.

We have finally reduced the surface layer of the economy, represented by the steady-state supply-demand frequencies $\{q^*_{ij},\}$ and the intermediary layer of the economy, represented by the steady-state supply-need frequencies $\{f^*_{ij}\}$, to the endowment-need frequencies $\{e_{ij}\}$ -- the "real" conditions of the economy lying at its deepest layer. Having equipped with these results, we are now going to establish the second fundamental proposition concerning the existence of monetary equilibria.

11. The complete ubiquity of monetary equilibria in the long-run.

As was remarked at the end of section 9, Proposition 6 just fell short of establishing the purely social nature of money and monetary equilibrium, because it still requires the condition that for any non-monetary good in the economy there must always be some individual who wants to supply money in exchange for it, in the sense that $f_{mi} > 0$ for all i ($\neq$ m). But, let us look at their steady-state values f^*_{mj} we just obtained in (14b). They now consist not only of the original m-endowed, j-consumers, e_{mj}, but also of all the other j-consumers who are temporarily holding good m in order to use it as the medium of further exchanges. And, of course, this is no more than the very consequence of using good m as the exclusive and universal medium of exchange -- as money! Even if the "real" data of the economy $\{e_{ij}\}$ lacks the conditions for the existence of a monetary equilibrium, the very processes of monetary exchanges will be able to create these conditions in the long-run. This is the second "bootstraps mechanism" working in a monetary equilibrium.

We first need to state a Lemma in order to formaize this second bootstraps mechanism of monetary equilibrium.

Lemma 5: If an economy is connected, there is a closed loop of strictly positive endowment-need frequencies which "connects" all the indices of the goods in the economy (possibly more than once).

(Proof): The connectedness of the economy implies that we have a sequence of strictly positive endowment-need frequencies connecting good 1 and good 2, good 2 and good 3, and so on until we finally have a sequence connecting good N and good 1. We thus have a loop of strictly positive endowment-need frequencies, which starts from good 1, connects all the other goods in the middle (possibly more than once) and returns to good 1. (QED)

We are now able to state our thesis of the complete ubiquity of monetary equilibrium in the form of:

Proposition 7: In any connected economy there are as many monetary equilibria as the number of goods in the long-run.

(Proof): Without loss of generality, let m be the index of the monetary good. By Lemma 5 above, we have a closed loop of strictly positive endowment-need frequencies $e_{mn} > 0$, $e_{no} > 0$, $e_{op} > 0$, ... $e_{kl} > 0$ and $e_{lm} > 0$, such that the closed loop of the connected indices, m, n, o, ..., k, l and m, contains all the goods in the economy at least once. Let us first substitute $j = n$ to (15b), and we have $q^*_{mn} = f^*_{mn} \geqq e_{mn} > 0$. Let us next substitute $j = o$ again to (15b), and we have $q^*_{mo} = f^*_{mo} \geqq \{q^*_{mn}/(q^*_{om}+q^*_{mn})\}e_{no}$. Since $e_{no} > 0$ by the connectedness of the economy and $q^*_{mn} > 0$ as we have just stated, we have $q^*_{mo} = f^*_{mo} > 0$. By repeating the same argument through $j = p, \ldots,$ k to l, we prove $q^*_{mj} = f^*_{mj} > 0$ for all $j \neq m$. (If the same index is included more than once in the closed loop, the positivity of the corresponding steady-state frequency is in fact demonstrated more than once.) This is nothing but the first condition stated in Proposition 6. The second condition: $q^*_{jm} = \Sigma_k f^*_{jk} > 0$ for all $j \neq m$ can also be demonstrated by applying the same method of argument to (15a) backwardly from $j = l, k, \ldots p, o$ to n. It then follows from Proposition 6 that the economy has a monetary equilibrium with good m used as money. Furthermore, since our choice of good m as money has been completely arbitrary, we have indeed established the existence of N different monetary equilibria, each of which uses one of N goods as money. (QED)

A monetary equilibrium is thus able to sustain itself without any "real" foundations. Even if there is no "real" demand for it and even if there is no "real" supply of it, the bootstraps mechanism is able to endow any (durable) good in the economy with all the characteristics of the exclusive and universal medium of exchange -- money. The very process of monetary exchanges not only creates the universal demand for money in exchange of every other good, f_{jm}, but also creates the universal supply of money in exchange for every other good, f^*_{mi}, at least in the long-run.[23]

12. The Essential Properties of Money and Markets.

Money is a "social contrivance" whose existence owes nothing to the "technology and preferences" of the economy. But it is precisely for this lack of "real" foundations that money is able to surmount the requirement of double coincidence of wants in barter exchange and make the otherwise impossible exchanges possible among searching individuals.

Now, the resulting exchange system created by such bootstraps mechanism is a full-fledged market economy in which everybody first seeks to sell the good in hand in exchange for money and then seeks to buy the good in need in exchange of money. What we call "markets" where goods are "sold" and "bought" in our daily sense of the words have thus emerged endogenously in our model of exchange economy. Markets as economic institutions are therefore no more than the joint product of the bootstraps mechanism which has produced money as a social contrivance.

Indeed, once money and markets have come into existence simultaneously in this manner, all the barter trading zones become defunct, except, of course, for those of money! For it is the barter trading zones for money that now function as the "markets" for all the other goods, and money continues to be exchanged with all the other goods on the barter basis. It is the common habit of economists to talk about the "demand and supply of money" as if the conventional demand and supply analysis can be applied to money with little modification. Of course, apple has its own market, automobile has its own market, foreign money has its own market, and even future money has its own market. But, money itself cannot have its own "market" by the very fact that it is used as the medium of exchange in all the "other markets" in the economy. This is precisely what is meant by the break-down of Say's law in the classical monetary economics.[24]

Moreover, such virtual disappearance of all the barter trading zones except for those of money has a fundamental implication for the way people organize their exchange activities in a monetary economy. For in a monetary equilibrium "money buys goods and goods buy money; but goods do not buy goods". All the individuals' exchange activities have now become subject to the so-called "Clower constraint", which requires them to have money in order to obtain (that is, to buy) any other goods in the economy.[25] It should, however, be hastely added here that the Clower constraint we have obtained is not the superimposed condition which presupposes the monetary structure of the economy from the beginning (just as Clower did) but is the very outcome of the interacting exchange processes among searching individuals.[26] Or

> One man is king only because other men stand in the relation of subjects to him. They, on the contrary, imagine that they are subjects because he is king.[27]

Part II: On the Difficulty of the Evolution of Money.

13. The evolution of monetary equilibrium.

In Part I of this paper we established the complete ubiquity of money and monetary equilibrium in our model of exchange economy. We are thus tempted to jump to the conclusion that: "therefore, it is only a matter of time for an economy to attain one of its potential monetary equilibria." But, life (and history) is not so simple, and there is no simple "therefore" in our economy.

Indeed, we are now going to argue that there is a fundamental difficulty in the natural evolution of money and monetary equilibrium.

So far we have implicitly assumed that every individual correctly estimates the supply-demand frequencies $\{q_{ij}\}$ and acts on their basis. But, in order to embark upon the detailed analysis of the dynamic relationship between barter and monetary exchanges, it is now time to drop this implicit assumption.

Accordingly, let us distinguish the individuals' subjective estimates of the supply-demand frequencies from their objective values q_{ij} and denote the former by $\hat{q}_{ij}$. (These subjective estimates are all assumed to be uniform across individuals. Of course their uniformity does not necessarily guarantee their correctness.)

Since the individual search behavior has no other choice but to be guided by the subjective estimates of frequencies $\hat{q}_{ij}$, we now have to rewrite the functional equation (1) explicitly in their terms.

$$(1)' \quad V_{ij} = \underset{k}{\mathrm{Max}}[V_{kj} - b - c/\hat{q}_{ki}],$$

for any i and $j \neq i$. By the same token, we also have to rewrite equation (9) which determines the objective values of supply-demand frequencies in the following manner.

$$(9)' \quad q_{ij} = \sum_{\{k \neq j: f_{ik}>0 \ \& \ V_{jk}-b-c/\hat{q}_{ji}=V_{ik}\}} f_{ik} + \begin{cases} f_{ij} & \text{when } f_{ij}>0 \ \& \ u-b-c/\hat{q}_{ji}= V_{ij}, \\ 0 & \text{otherwise}, \end{cases}$$

for any i and $j(\neq i)$.[28]

If the possibility of the divergence between subjective and objective frequencies is taken into account in this manner, we need the third set of conditions to define an exchange equilibrium of the economy. These are the conditions that require the equality between subjective and objective distribution of supply-demand frequencies, or

(16) $\hat{q}_{ij} = q_{ij}$

for any i and j ($\neq$ i). Needless to say, the above new set of equilibrium conditions, (1)', (9)' and (16), are formally equivalent to the old set of equilibrium conditions, (1) and (9), in section 7.

But it is for the sake of analyzing the dynamic evolution of the exchange system that we have introduced the conceptual distinction between subjective and objective supply-demand frequencies. And it is for this reason that we have to introduce a certain expectation formation process explicitly. Now, the intrinsic multiplicity of exchange equilibria we have endeavored to show in the first part of this paper prevents us from jumping to the hypothesis of rational expectations, whose logical consistency requires the uniqueness of equilibrium. We thus have no other choice but to introduce some ad hocness into our modeling of expectation formation process. The simplest and hence the most ad hoc is the case of static expectations, which suppose that the current state of affairs will continue to hold in the next period, or if we let t = 1, 2, 3, ... denote time periods,

(17) $\hat{q}_{ij}(t+1) = q_{ij}(t)$.

There are of course many other possible formulations of expectation-formation process, but this simplest hypothesis is sufficient for our purpose at least until section 16.

Suppose now that an economy has a barter equilibrium and that it indeed finds itself in it. The question is: will the economy be able to evolve from this barter equilibrium to one of the monetary equilibria? Will a change, say, in the people's subjective estimates of supply-demand frequencies uproot the economy from a barter equilibrium and implant it into a monetary equilibrium?

In order to prepare ourselves to answer such question, we first state the following two propositions concerning the local stability of both barter and monetary equilibria.

Proposition 8: Each of the possible monetary equilibria is locally stable.

(Proof): A small change in q_{ij} or $\hat{q}_{ij}$ will not violate the inequalities in (6a), (6b), (6c) and (6d), thus immediately sending the economy back to the original monetary equilibrium. (QED)

Proposition 9: Suppose that a barter equilibrium exists in the economy and that it strictly satisfies all the inequalities in (5), then it is locally stable.

(Proof): A small change in q_{ij} or $\hat{q}_{ij}$ will not violate the inequalities in (5), thus immediately sending the economy back to the barter equilibrium. (QED)

Proposition 8 is a good news, for it says that once the economy finds itself in one of possible monetary equilibria, it can not easily be displaced from it. Proposition 9, on the other hand, is a bad news, for it says that as long as the disturbance is small, the economy can never escape the tyranny of its own "real" conditions which dictate the barter equilibrium. Contrary to what Menger said, there is no "natural" evolution from barter to monetary equilibrium.[29] We need some "unnatural" disturbance if such an evolution will ever be possible, as we shall now see.

14. An informal discussion on the bootstraps evolutionary process of monetary equilibrium.

In the present section we shall study rather informally the possible evolutionary process from a barter to a monetary equilibrium in an economy with strictly positive endowment-need frequencies $\{e_{ij}\}$. (When some of the endowment-need frequencies are zero, the evolution of monetary equilibrium involves far more complications, which force us to postpone its discussion until section 16.) Since this section is rather tedious, the impatient reader may directly proceed to the next section.

Imagine that an economy has been in a barter equilibrium with $\hat{q}_{ij} = q_{ij} = f_{ij}$ $(= e_{ij})$ for a long period of time. Suppose first that in period t people suddenly become optimistic about j-holders' demand for good m, $\hat{q}_{jm}$, and at the same time become pessimistic about their demands for some other goods, $\hat{q}_{ji}$. Now, if this is merely a local disturbance in the sense that none of the following inequalities are violated,

$$(18) \quad c/\hat{q}_{ji}(t) \leqq c/\hat{q}_{jm}(t) + c/\hat{q}_{mk}(t) + b \ ,$$

for any i ($\neq$ j and m), then under the hypothesis of static expectations (17) the economy will return to the original barter equilibrium in the next period.

The barter equilibrium is in general quite robust to local disturbances.

If, however, the disturbance of $\hat{q}_{jm}$ is large enough to reverse all of the above inequalities, so that we have

$$(19) \quad c/\hat{q}_{jm}(t) + c/\hat{q}_{mi}(t) + b < c/\hat{q}_{ji}(t) .$$

for all i ($\neq$ j and m), then the original barter equilibrium will disintegrate itself locally. All the consumers of good j now find it less costly to use good m as a medium of exchange rather than to seek a direct barter for good j. Their demands for good j now switch to good m, so that $q_{im}(t)$, which used to be equal to f_{im}, now becomes equal to $f_{im}+f_{ij}$, and all the $q_{ij}(t)$'s, which used to be equal to f_{ij}, now becomes equal to 0 for i ($\neq$ m). If people form their expectations in accordance with the rule of static expectations (17), their subjective frequencies in period t+1 will be adjusted to these new frequencies, and as long as $f_{mj} > 0$ we then have:

$$(20) \quad c/\hat{q}_{im}(t+1) + c/\hat{q}_{mj}(t+1) + b = c/(f_{im}+f_{ij}) + c/f_{mj} + b$$
$$< c/\hat{q}_{ij}(t+1) = c/0 = \infty$$

for all i ($\neq$ j and m). Hence, all the j-supplying individuals begin to switch their demands to good m in order to use it as a medium of exchange, so that $q_{jm}(t+1)$ becomes equal to $\Sigma_k f_{jk}$ and $q_{ji}(t+1)$ becomes equal to 0 for all i ($\neq$ m and j). People's subjective frequencies in period t+2 are then adjusted to these values, and we finally have

$$(21) \quad c/\hat{q}_{jm}(t+2) + c/\hat{q}_{mi}(t+2) + b = c/\Sigma_k f_{jk} + c/f_{mi} + b$$
$$< c/\hat{q}_{ji}(t+2) = c/0 = \infty ,$$

for all i ($\neq$ m and j). This miraculously justifies the initially made unfounded optimism about $\hat{q}_{jm}(t)$ and unfounded pessimism about $\hat{q}_{ji}(t)$!

A large enough disturbance of the people's subjective estimates of supply-demand frequencies has thus seen to have a self-fulfilling property.

This is, unfortunately, the end of the story, as long as the people's initial optimism was restricted to the j-holders' demand for good m. For, although there is an induced increase in the demand for good m from the original value of f_{im} to a higher value of $f_{im}+f_{ij}$ for any i ($\neq$ j and m), this is in general not large enough to upset the barter condition for the non-j consumers; that is, in general we still have

$$(22) \quad c/\hat{q}_{im}(t+1) + c/\hat{q}_{mk}(t+1) + b = c/(f_{im}+f_{ij}) + c/f_{mk} + b \geq c/\hat{q}_{ik}(t+1) = c/f_{ik}$$

for all k ($\neq$ j and m). Most of the consumers of non-j goods still seek to barter for them, and the use of good m as a medium of exchange is confined chiefly to the holders as well as the consumers of good j. It is, in other words, functioning merely as a local money circulating among a limited group of people.[30]

It then follows that, if there would be an evolution from a barter to a monetary equilibrium, the initial disturbance must have been not only large enough but also wide enough to involve most individuals' demands for good m.

Accordingly, suppose now that the initial disturbance was in fact so large and so wide-spread that the reversal of the conditions for barter exchange, as is given by (19), was not confined to a particular good j but extended to the whole array of goods from k = 1 to N, except of course for good m itself, or we have

$$(23) \quad c/\hat{q}_{km}(t) + c/\hat{q}_{mi}(t) + b < c/\hat{q}_{ki}(t) .$$

for all k ($\neq$ m) and i($\neq$ k and m). Then, all the i-supplying, k-consumers, except the consumers and owners of good m, begin to demand good m as a medium of exchange, so that $q_{im}(t)$, which used to be equal to f_{im}, now becomes equal to $\Sigma_k f_{ik}$, and all the $q_{ih}(t)$'s, which used to be equal to f_{ih}, now becomes equal to 0 for i($\neq$ m) and h($\neq$ i and m). Under the hypothesis of static expectations, we then have:

$$(24) \quad c/\hat{q}_{im}(t+1) + c/\hat{q}_{mh}(t+1) + b = c/\Sigma_k f_{ik} + c/f_{mh} + b < c/\hat{q}_{ih}(t+1) = c/0 = \infty$$

for all i($\neq$ m) and h($\neq$ i and m). Hence, all the h-supplying, i-consumers begin to switch their demands to good m in order to use it as a medium of exchange, so that $q_{hm}(t+1)$ becomes equal to $\Sigma_i f_{hi}$ and $q_{hl}(t+1)$ becomes equal to 0 for all l ($\neq$ m) and h($\neq$ l and m). Under static expectations, people's subjective frequencies in period t+2 are then adjusted to these values, and we finally have

$$(25) \quad c/\hat{q}_{hm}(t+2) + c/\hat{q}_{ml}(t+2) + b = c/\Sigma_i f_{hi} + c/f_{ml} + b < c/\hat{q}_{hl}(t+2) = c/0 = \infty ,$$

for all h ($\neq$ m) and l($\neq$ h and m)! However unfounded on the "real" conditions of the economy might be, a large enough and wide enough optimism about general demand for a particular good would trigger the bootstraps mechanism and create the conditions for its own fulfillment by transforming that particular good into the exclusively used and universally accepted medium of exchange -- money. A full-fledged monetary economy thus emerges out of nothingness with a large enough and wide enough disturbance which breaks the intrinsic symmetry of the barter economy "in the beginning".

15. The irreversible evolution of monetary equilibrium -- the case of the totally symmetric endowment-need distribution.

In the case of totally symmetric endowment-need distribution, represented by (7) in which $e_{ij} = 1/N(N-1)$ for any i and j ($\neq$ i), we can illustrate the above informal discussions by means of a simple diagram.

For this purpose, let us divide the frequencies of individual types into three groups, and set $q_{ij}(t) = x(t)$ for $i \neq m$ and $j \neq i$ and m ; $q_{mj}(t) = y(t)$ for $j \neq m$; and $q_{im}(t) = z(t)$ for $i \neq m$. Here, x(t) can be regarded as the representative supply-demand frequency of the individuals who neither hold nor demand good m, y(t) as the representative supply-demand frequency of individuals who currently hold good m, and z(t) as the representative supply-demand frequency who currently demand good m. Next let us also set $\hat{q}_{ij}(t) = \hat{x}(t)$ for $i \neq m$ and $j \neq i$ and m ; $\hat{q}_{mj}(t) = \hat{y}(t)$ for $j \neq m$; and $\hat{q}_{im}(t) = \hat{z}(t)$ for $i \neq m$. Their interpretations are obvious. Such bundling of objective as well as subjective supply-demand frequencies into three groups is tantamount to ignoring local disturbances which only affect the subjective supply-demand frequencies within each group.[31]

In the first place, let us note that these new variables have to satisfy the following adding-up equations:

(26a) $$\sum_i \sum_j q_{ij}(t) = (N-1)(N-2)x(t) + (N-1)y(t) + (N-1)z(t) = 1 \text{ ; and}$$

(26b) $$\sum_i \sum_j \hat{q}_{ij}(t) = (N-1)(N-2)\hat{x}(t) + (N-1)\hat{y}(t) + (N-1)\hat{z}(t) = 1.$$

Note also that in this case of totally symmetric endowment-need distribution (7), we have by assumption the following initial conditions:

(27) $$x(0) = y(0) = z(0) = \hat{x}(0) = \hat{y}(0) = \hat{z}(0) = \frac{1}{N(N-1)} .$$

As we saw in section 8, they satisfy all the conditions for the existence of a barter equilibrium (5).

Now suppose that in period t people become suddenly optimistic about $\hat{z}(t)$ at the expense of $\hat{x}(t)$, while keeping $\hat{y}(t)$ constant. If this disturbance is not large enough to upset the following inequality (which is in fact a restatement of the condition (5) for not using good m as a medium of exchange in the present context):

$$(28) \quad \frac{c}{\hat{y}(t)} + \frac{c}{\hat{z}(t)} + b \geqq \frac{c}{\hat{x}(t)} ,$$

then people still find it less costly to barter with each other, and we have

$$(29) \quad z(t) = x(t) = \frac{1}{N(N-1)} .$$

Under the hypothesis of static expectations, we then have

$$(30) \quad \hat{z}(t+1) = z(t) = \frac{1}{N(N-1)} ; \; \hat{x}(t+1) = x(t) = \frac{1}{N(N-1)} ,$$

and the economy find itself back to the initial barter equilibrium. This is what Proposition 9 meant.

If, on the other hand, the disturbance is large enough (and by construction wide enough) to reverse the above inequality, so that we obtain:

$$(31) \quad \frac{c}{\hat{y}(t)} + \frac{c}{\hat{z}(t)} + b < \frac{c}{\hat{x}(t)} ,$$

then every individual, except those who have real needs of good m and those who already have good m, find it less costly to use good m as a medium of exchange. (All the other inequalities in Proposition 3 are automatically satisfied in this situation.) This immediately implies that

(32) $z(t) = (N-1)\cdot\frac{1}{N(N-1)} = \frac{1}{N}$; and $x(t) = 0$.

Under the hypothesis of static expectations, we then have

(33) $\hat{z}(t+1) = z(t) = \frac{1}{N}$; and $\hat{x}(t+1) = x(t) = 0$,

thus justifying the initial large disturbance. The economy has now achieved a state of monetary equilibrium with good m used as money.

In order to visualize the above discussion, let us define the critical value $\hat{z}^c$ by

(34) $$\frac{c}{\hat{y}(t)} + \frac{c}{\hat{z}^c} + b = \frac{c}{\hat{x}(t)} .$$

If we substitute the relations: $\hat{y}(t) = 1/\{N(N-1)\}$ of (27) and $\hat{x}(t) = \{1/(N-2)\}\{1/(N-1) - \hat{y}(t) - \hat{z}(t)\}$ of (26b) into the above definition, we can solve it explicitly to obtain

(35) $$\hat{z}^c = \frac{b/c + \sqrt{(b/c)^2+4\{N(N-1)+b/c\}/N}}{2N\{N(N-1)+b/c\}} .$$

$\hat{z}^c$ just defined above is nothing but the water-shed value of $\hat{z}(t)$ between barter condition (28) and monetary condition (31). Then, we can restate the evolutionary dynamics of $\hat{z}(t)$ (under the hypothesis of static expectations) by the following simple rule:

(36)
When $\hat{z}(t) \leqq \hat{z}^c$, then $\hat{z}(t+1) = \frac{1}{N(N-1)}$,
or the economy attains a barter equilibrium;
When $\hat{z}(t) > \hat{z}^c$, then $\hat{z}(t+1) = \frac{1}{N}$,
or the economy attains a monetary equilibrium;

Fig. 2 illustrates this simple dynamics in a Cartesian diagram with $\hat{z}(t)$ as abscissa and $\hat{z}(t+1)$ as ordinate.[32] An exchange equilibrium is an intersection between the step function (36) relating $\hat{z}(t+1)$ to $\hat{z}(t)$ and the 45 degree line. There are indeed two such equilibria in this figure -- the lower one corresponding to the barter equilibrium with the value of $\hat{z} = 1/\{N(N-1)\}$ and the upper one corresponding to a monetary equilibrium with the value of $\hat{z} = 1/N$. The barter equilibrium is stable, so that if the economy will ever evolve from barter to monetary equilibrium the value of $\hat{z}(t)$ must increase beyond the critical level of $\hat{z}^c$. In the beginning, therefore, there must be a large disturbance which breaks the intrinsic symmetry of this locally stable barter equilibrium.

In fact, since the choice of monetary good m is totally arbitrary, we could have drawn an identical diagram for every one of N goods in the economy. And Fig. 2 tells us that each of these monetary equilibria is potentially very robust in the sense that in order for the economy to return to the original barter equilibrium the disturbance must be much larger than the one required for the original evolution, for the value of $\hat{z}(t)$ must now be lowered from 1/N below $\hat{z}^c$. There is thus a certain "irreversibility" in the dynamic evolution from barter to monetary equilibrium.

This is, however, not the end of the story. If the economy stays in one of monetary equilibria long enough, the process of monetary exchanges among individuals will gradually force the supply-need frequencies $\{f_{ij}\}$ to deviate from the original endowment-need frequencies $\{e_{ij}\}$ and will modify the very configuration of the monetary equilibrium. It will in the long-run approach a steady-state equilibrium.

Indeed, in the present example of totally symmetric distribution of endowment-need frequencies (7), a careful inspection of the set of equations (14a) -- (15c) in section 10 will allow us to calculate, after some toil and labor, the steady-state values of the endowment-need frequencies $\{f^*_{ij}\}$ and the supply-demand frequencies $\{q^*_{ij}\}$ in the following manner.[33]

(37) $f^*_{ij} = \dfrac{1}{2N(N-1)}$ for i ($\neq$ m) and j ($\neq$ i and m);

$f^*_{mj} = \dfrac{1}{2(N-1)}$ for j($\neq$ m); $f^*_{im} = \dfrac{1}{N(N-1)}$ for i($\neq$ m) ;

(38) $q^*_{mj} = q^*_{im} = \dfrac{1}{2(N-1)}$ for j($\neq$ m) and i($\neq$ m and j); and all the other $q^*_{ij} = 0$.

It then follows that the subjective frequencies of i-endowed, m-demanders $\hat{z} = \hat{q}_{im}$ is now reduced from the short-run equilibrium value of 1/N to the long-run value of $q^*_{im} = 1/\{2(N-1)\}$, and the subjective frequencies of m-supplying, m-consumers $\hat{y} = \hat{q}_{mj}$ is raised from $1/\{N(N-1)\}$ to $q^*_{mj} = 1/\{2(N-1)\}$. If we substitute this long-run value of $\hat{y}$ to definition (35) of the critical value of $\hat{z}^c$, we can calculate the latter's long-run value $\hat{z}^{*c}$ as

(39) $$\hat{z}^{*c} = \frac{4+(N-1)(b/c)-2N+\sqrt{\{4+(N-1)(b/c)-2N)^2+4+2(N-1)(b/c)}}{8+4(N-1)(b/c)}$$

It can be easily shown that $\hat{z}^{*c} < \hat{z}^c$.

Hence, under the hypothesis of static expectations the dynamics of $\hat{z}(t)$ after the economy has settled down to one of the possible monetary equilibria long enough is now written as:

When $\hat{z}(t) \leqq \hat{z}^{*c}$, then $\hat{z}(t+1) = e_{im} = \dfrac{1}{N(N-1)}$,

or the economy disintegrates into a barter equilibrium; and

(40) When $\hat{z}(t) > \hat{z}^{*c}$, then $\hat{z}(t+1) = q^*_{im} = \dfrac{1}{2(N-1)}$,

or the economy stays in a monetary equilibrium.

Fig. 3 illustrates the above evolutionary dynamics of $\hat{z}(t)$ after it has settled down to one of N long-run monetary equilibria.

16. The "unnatural" evolution towards a long-run monetary equilibrium -- the case of the minimally connected endowment-need distribution.

Let us now turn to the examination of the evolution of monetary equilibrium in the second special case of minimally connected endowment-need distribution (8) where $e_{12} = e_{23} = \ldots = e_{N-1,N} = e_{N1} = 1/N$, and all the other $e_{ij} = 0$.

As we saw in section 8, this is the economy which has no barter equilibrium. Hence, our starting point should be the state of forced autarky where no individual is ever able to find a partner to barter. Of course, this economy is able to sustain any of the N monetary equilibria in the long-run, as any other economy is. But, it now has to overcome almost insurmountable obstacles in order to be able to approach one of them, as we shall now see.

Let us suppose, without loss of generality, that it would be the m-th good that is to be used as money if the economy could ever attain its monetary equilibrium in the long-run. Now, the most crucial condition for using good m as a medium of exchange is given by the inequality (6a) in Proposition 3, which can be restated in the present context as:

$$(41)\quad c/\hat{q}_{ji}(t) > b + c/\hat{q}_{jm}(t) + c/\hat{q}_{mi}(t),$$

for any $i(\neq m$ and $j)$ and $j(\neq m$ and $i)$. Now, we have as initial conditions $\hat{q}_{ji}(0) = q_{ji}(0) = f_{ji}(0)$ $(= e_{ji}) = 0$ for $i \neq j+1$ and $= 1/N$ for $i = j+1$, $\hat{q}_{jm}(0) = q_{jm}(0) = f_{jm}(0)$ $(= e_{jm}) = 0$ for $j \neq m-1$ and $= 1/N$ for $j = m-1$, and

$\hat{q}_{mi}(0) = q_{mi}(0) = f_{mi}(0)$ $(= e_{mi}) = 0$ for $i \neq m+1$ and $= 1/N$ for $i = m+1$. (Hence, $V_{i,i+1}(0) = u-b-c/\hat{q}_{i+1,i}(0) = u-b-c/0 = -\infty$.) It then follows that in order that the above set of inequalities be all satisfied the initial disturbance must be such that not only $\hat{q}_{jm}(t)$ increases at the expense of $\hat{q}_{ji}(t)$ but also $\hat{q}_{mi}(t)$ increases above zero (except for $i = m+1$). This fact should be contrasted with the case of totally symmetric endowment-need distribution, discussed in the preceding section, which "only" required large and wide-spread increases in $\hat{q}_{jm}$ at the expense of $\hat{q}_{ji}$. It is the first (extra) obstacle to the evolution of monetary equilibrium in this example. But, even if the economy could clear this first obstacle, it would still have to face other far more serious ones.

In order to see this, let us suppose that the disturbance in period t is sufficiently large and wide-spread, so that the above inequality (41) are satisfied for all $i(\neq m$ and $j)$ and $j(\neq m$ and $i)$. Then, all the i-endowed, (i+1)-consumers, except the holders of and consumers of good m, stop demanding good i+1 directly and begin to demand good m as a medium of exchange. Hence, $q_{im}(t)$, which used to be zero (except $q_{m-1,m}(t-1)$ which was equal to 1/N), now becomes equal to 1/N, and $q_{i,i+1}(t)$, which used to be 1/N, now becomes equal to zero (except for $q_{m-1,m}(t)$ which remains 1/N). They tend to justify the initially made unfounded optimism about $\hat{q}_{im}(t)$ as well as unfounded pessimism about $\hat{q}_{i,i+1}$.

But, how about the value of $q_{mj}(t)$, which used to be zero before the disturbance (except $q_{m,m+1}(t-1)$ which was equal to 1/N)? At least in period t, it still remains zero! For, although all the (j-1)-endowed, j-consumers switch their demand from good j to good m, they can actually transform themselves into m-supplying, j-consumers only one-period after they meet appropriate trading partners.

It then follows that under the hypothesis of static expectations (17) the economy with a minimally connected endowment-need distribution can never attain any of its potential monetary equilibria! Even if a large and wide-spread disturbance happens to create the situation which satisfies the inequality condition (41) for monetary equilibrium in period t, $q_{mj}(t)$ still remains zero for any j ($\neq$ m+1), and we shall have $\hat{q}_{mj}(t+1) = q_{mj}(t) = 0$. The inequality (41) will certainly reverse itself in period t+1.

This, however, does not exhaust all the obstacles to the evolutionary path to monetary equilibrium. Now the only m-holders who are searching in the economy in period t are m-endowed, (m+1)-consumers. This implies that within period t only (m+1)-supplying, (m+2)-consumers have a chance to meet their trading partners. Since the frequency of (m+1)-supplying, (m+2)-consumers in period t is 1/N and the frequency of m-supplying, (m+1)-consumer is also equal to 1/N, the expected number of their encounters is $1/N^2$. In period t+1 (not in period t) there will emerge $1/N^2$ frequency of m-supplying, (m+2)-consumers, or $q_{m,m+2}(t+1) = 1/N^2$. (We are using the so-called law of large numbers here.) All the other $q_{mj}(t+1)$'s, j $\neq$ m and m+1, still remain zero in period t+1, for their trading partners have not appeared in the economy! Then, passing to period t+1, the (m+2)-supplying, (m+3)-consumers now have a chance to meet those newly created m-supplying, (m+2)-consumers and, if successful, transform themselves into m-supplying, (m+3)-consumers. In period t+2, we will therefore have $q_{m,m+3}(t+2) = 1/N^3$. (At the same time, some of the remaining (m+1)-supplying, (m+2)-consumers also meet m-supplying, (m+1)-consumers and transform themselves into m-supplying, (m+2)-consumers, so that we will have $q_{m,m+2}(t+2) = 1/N^2 - 1/N^3 + (1/N - 1/N^2)^2$. But all the other $q_{mj}(t+2) = 0$, for j$\neq$ m, m+1 and m+2.) If we repeat the same kind of argument from j = m+4 through N, 1, to m-1, we can finally have $q_{m,m-1}(t+N-2) = 1/N^{(N-1)}$ in period t+N-2. In other words, it will take at least N-2 periods for all the q_{mj}'s to become strictly positive.

The only hope for the emergence of a monetary equilibrium in this case of minimally connected economy, therefore, lies in the extreme slowness of the people's expectation-formation process, which works to sustain the value of $\hat{q}_{mj}$ above zero even if the value of the actual q_{mj} stays zero for a long period of time. An example of such slow expectation-formation process is that of adaptive expectations given by

$$(42) \quad \hat{q}_{ij}(t+1) - \hat{q}_{ij}(t) = \alpha \cdot [q_{ij}(t) - \hat{q}_{ij}(t)] \ ,$$

where $0 < \alpha < 1$. This rule implies that the value of $\hat{q}_{mj}(t+s)$ will decline in accordance with the formula: $(1-\alpha)^{s-1}\hat{q}_{mj}(t)$ until the actual value of $q_{mj}(t+s)$ for the first time takes the positive value of $1/N^{j-m-1}$ at $t+s = t+j-m-1$ (with a convention that if $j \leqq m$, $j-m-1 = N-m-1+j$). As long as the adaptive coefficient α is strictly smaller than 1, that is, except for the case of static expectations, $\hat{q}_{mj}(t+s)$ will forever remain positive once its value is raised above zero in period t, and the set of inequalities (41) will keep reproducing themselves. But this nice result may only highlight the unnaturalness of the hypothesis of adaptive expectations, especially of its infinitely long memory span, rather than to exhibit the possibility of the natural evolution of monetary equilibrium.

Of course, once the economy has miraculously succeeded in clearing all the above obstacles for the attainment of monetary equilibrium and has kept reproducing the set of inequalities (41) until period t+N-2, it will henceforth gain the momentum for the evolutionary approach to one of its long-run monetary equilibria. For all the supplies of goods by m-holders, q_{mj}, will become strictly positive in period t+N-2 (with $q_{m,m-1}(t+N-2)$ finally turning positive at the value of $1/N^{(N-1)}$), and the bootstraps mechanism of monetary equilibrium which can create the conditions for its own existence will begin to work itself out. The economy will then gradually approach one of its potential long-run monetary equilibria.

Let us therefore conclude the present section by presenting the steady-state supply-need frequencies $\{f^*_{ij}\}$ and supply-demand frequencies $\{q^*_{ij}\}$ for this minimally connected economy. Substituting (8) into (14a) -- (15c) of section 10, we can easily calculate them as follows:[34]

(43) $f^*_{mj} = \frac{1}{2N}$ for $j \neq m$ and $m+1$; $f^*_{i,i+1} = \frac{1}{2N}$ for $i \neq m-1$ and m;[34]

$f^*_{m-1,m} = f^*_{m,m+1} = \frac{1}{N}$; and all the other $f^*_{ij} = 0$.

(44) $q^*_{mj} = q^*_{im} = \frac{1}{2N}$ for i and j $\neq$ m-1, m, and m+1 ;

$q^*_{m-1,m} = q^*_{m,m+1} = \frac{1}{N}$; and all the other $q^*_{ij} = 0$.

17. Who gain from the transition to a monetary equilibrium?

We have thus seen the fundamental difficulty in the laissez-faire evolution of money and monetary equilibrium in our model of exchange economy. Is there any hope left for the establishment of monetary equilibrium as a cooperative solution of the society? Money is after all a social contrivance which enables the economy to overcome the difficulties of barter by making an exchange possible even between two people who have no double coincidence of wants. Doesn't this mean that money is welcomed by everybody in the economy?

In order to answer this question, we need some preparations. To begin with, let us suppose, as before, that it is the m-th good which is to become money in the economy. Then, we can classify the individuals into three different classes in accordance with their relationships to this particular good -- they are (i) those who happen to be born with an endowment of good m, (ii) those who happen to have real needs for good m, and (iii) those who have neither the endowment nor the needs of good m. It is necessary to follow their fates separately.

Now suppose that the economy initially was in a situation which used no medium of exchange. Since this situation is not necessarily a barter equilibrium (because a large number of economies do not have any barter equilibrium), we just call it a "no-medium" situation. As a matter of fact, when there is no barter equilibrium, some individuals in this situation may have no choice but to suffer the misery of autarky with the utility level of $-\infty$. In any case, if we denote the expected utility of an i-supplying, j-consumer in this no-medium situation by V^N_{ij}, we can express its value for each of the three different classes of individuals respectively as follows.

(45a) $V^N_{mj} = u - b - c/e_{jm}$, for $j \neq m$.

(45b) $V^N_{im} = u - b - c/e_{mi}$, for $i \neq m$.

(45c) $V^N_{ij} = u - b - c/e_{ji}$, for $j \neq m$ and $i \neq m$.

Suppose next that from the above no-medium situation everybody in this economy (except the one who was endowed with it and the who has real need of it) has suddenly made an attempt to use good m as money and switch the immediate demand from the needful good to this universal medium of exchange. The supply-need frequencies $\{f_{ij}\}$ have then no time to deviate from the original endowment-need frequencies $\{e_{ij}\}$, and the resulting supply-demand frequencies $\{q_{ij}\}$, given by (13a), (13b) and (13c) in section 9, are now expressed as $q_{jm} = \Sigma_k e_{jk}$ for $j \neq m$, $q_{mi} = e_{mi}$ for $i \neq m$ and $q_{ji} = 0$ for $i \neq m$ and $j \neq m$. If they satisfy the conditions for the existence of a monetary equilibrium, the economy can indeed attain a state of "short-run" monetary equilibrium in one period. Let us then denote the maximum expected utility of an i-supplying, j-consumer in this situation by $V^M{}_{ij}$. We can then calculate its value respectively for each of the three different classes as follows.

$$\text{(46a)} \quad V^M{}_{mj} = u - b - c/\{e_{jm} + \sum_{k \neq m} e_{jk}\} \qquad \text{for } j \neq m,$$

$$\text{(46b)} \quad V^M{}_{im} = u - b - c/e_{mi} \qquad \text{for } i \neq m,$$

$$\text{(46c)} \quad V^M{}_{ij} = u - 2b - c/\{e_{jm} + \sum_{k \neq m} e_{jk}\} - c/e_{mi} \qquad \text{for } j \neq m \text{ and } i \neq m.$$

Suppose finally that the economy has somehow managed to create the conditions for monetary equilibrium and after a long passage of time has reached a steady-state. We know from Proposition 7 of section 11 that such "long-run" monetary equilibrium exists in any economy. Let us then denote the maximum expected utility of an i-supplying, j-consumer in this situation by $V^{M*}{}_{ij}$. If we recall equations (15a), (15b) and (15c) of section 10, we can easily express its value respectively for each of the three different classes of individuals as follows.

$$\text{(47a)} \quad V^{M*}{}_{mj} = u - b - c/[e_{jm} + \sum_{k \neq m} \{\frac{q^*{}_{km}}{q^*{}_{km} + q^*{}_{mj}}\} e_{jk}] \qquad \text{for } j \neq m.$$

$$\text{(47b)} \quad V^{M*}{}_{im} = u - b - c/[e_{mi} + \sum_{k \neq m} \{\frac{q^*{}_{mk}}{q^*{}_{im} + q^*{}_{mk}}\} e_{ki}] \qquad \text{for } i \neq m.$$

$$(47c)\quad V^{M*}_{ij} = u - 2b - c/[e_{jm}+\sum_{k\neq m} \{\frac{q^*_{km}}{q^*_{km}+q^*_{mj}}\}e_{jk}] - c/[e_{mi}+\sum_{k\neq m} \{\frac{q^*_{mk}}{q^*_{im}+q^*_{mk}}\}e_{ki}],$$

for $j\neq m$ and $i\neq m$.

It is now time to compare the welfares of each of the three classes of individuals among the three different situations described above.

Let us begin with the class of fortunate individuals whose endowments are chosen to become money. If we compare V^N_{mj} in (45a), V^M_{mj} in (46a) and V^{M*}_{mj} in (47a), we can easily rank them as $V^N_{mj} < V^{M*}_{mj} < V^M_{mj}$. In words, if the economy is able to reach a state of monetary equilibrium in one period, individuals born with an endowment of the monetary good will benefit most from such a switch of exchange regime. The gain in expected utility, $V^M_{mj}-V^N_{mj} = c\sum_{k\neq m}e_{jk}/e_{jm}\{e_{jm}+\sum_{k\neq m}e_{jk}\}$, is nothing but the "seigniorage" accrued to those who have the fortune to be born with money. In the long run, however, the amount of seigniorage will gradually decline, as the increase in the fraction of individuals holding money will somewhat reduce the demand for money. And yet, the seigniorage will never disappear from the economy, and the individuals in this class will forever gain by the amount equal to $V^{M*}_{mj}-V^N_{mj} = c\sum_{k\neq m}\{q^*_{km}/(q^*_{km}+q^*_{mj})\}e_{jk}/e_{jm}[e_{jm}+\sum_{k\neq m}\{q^*_{km}/(q^*_{km}+q^*_{mj})\}e_{jk}]$.

Let us turn to the class of individuals who happen to have real needs of the good which is to be chosen as money. If we compare V^N_{im} in (45b), V^M_{im} in (46b) and V^{M*}_{im} in (47b), we can easily rank them as $V^N_{im} = V^M_{im} < V^{M*}_{im}$. The individuals in this class will not gain from the switch of exchange regime in the short-run. But, as more people have succeeded in acquiring money and begun to use it as the universal medium of exchange, the life of those money consumers becomes much easier. They are now able to meet the suppliers of the monetary good more frequently and will in the long-run gain from the saving of the search cost by the amount equal to $V^{M*}_{im}-V^{N}_{im} = c\Sigma_{k\neq m}\{q^*_{mk}/(q^*_{im}+q^*_{mk})\}e_{ki}/e_{mi}[e_{mi}+\Sigma_{k\neq m}\{q^*_{mk}/(q^*_{im}+q^*_{mk})\}e_{ki}]$.

So far so good. But how about the fates of the third class of individuals who have nothing to do with the monetary good either in their endowments or in their needs? In fact, it is not hard to see that a priori no definite ranking is possible among V^{N}_{ij} in (45c), V^{M}_{ij} in (46c) and V^{M*}_{ij} in (47c). We cannot in general exclude the possibility that a switch of the exchange regime to a monetary one will hurt some individuals who have neither endowments nor needs of the monetary good. If there were indeed such individuals, they would certainly resist the introduction of money into the economy.

This possibility can be most easily seen in the special example of totally symmetric endowment-need structure given in (7). In this case (which has both a barter equilibrium and a short-run monetary equilibrium), the expected utilities of the three classes of individuals in the above three different situations can be explicitly calculated as follows.

(48a, b & c) $V^{N}_{mj} = V^{N}_{im} = V^{N}_{ij} = u-b-cN(N-1)$.

(49a, b & c) $V^{M}_{mj} = u-b-cN$; $V^{M}_{im} = u-b-cN(N-1)$; $V^{M}_{ij} = u-2b-cN^2$.

(50a, b & c) $V^{M*}_{mj} = u-b-2c(N-1)$; $V^{M*}_{im} = u-b-2c(N-1)$; $V^{M*}_{ij} = u-2b-4c(N-1)$.

In the first place, we can confirm the general ranking orders: $V^{N}_{mj} < V^{M*}_{mj} < V^{M}_{mj}$ and $V^{N}_{im} = V^{M}_{im} < V^{M*}_{im}$ in the present example as well. What interests us here is, however, the ranking order among V^{N}_{ij}, V^{M}_{ij} and V^{M*}_{ij} for $i\neq m$ and

$j \neq m$. Indeed, we have $V^M{}_{ij} - V^N{}_{ij} = -b-cN < 0$, which implies that a switch from barter to monetary equilibrium will always hurt the welfares of those individuals who have neither endowments nor needs of the monetary good at least in the short-run. Will they gain in the long-run? The answer is, however, ambiguous, for we have $V^{M*}{}_{ij} - V^N{}_{ij} = -b-c(N-1)(4-N)$. This implies that as long as $N \leqq 4$ a switch to a monetary equilibrium will not improve their welfares even in the long-run. It is only when N becomes sufficiently large (relatively to b/c) that the possibility would arise that the individuals in this class will gain from such a regime shift after a sufficient long passage of time.

Note in passing that in the second special example of the minimally connected endowment-need frequencies given in (8), every individual will gain from a switch to a monetary equilibrium in the long-run, simply because neither barter equilibrium nor short-run monetary equilibrium is possible in this example.

In sum, we can state

<u>Proposition 10:</u> When an exchange economy transforms itself from the situation without money to the situation with money, (i) those individuals who were born with an endowment of the monetary good will gain most from such a regime change, though their gain (seigniorage) will have a tendency to decline in the long-run (but not to zero); and (ii) those who were born with the real need of the monetary good will be indifferent to such a regime change in the short-run but will have something to gain in the long-run. (iii)As for the welfares of those individuals who were born neither with the endowment nor with the needs of the monetary good, however, we cannot say anything <u>a priori</u> about their relative magnitudes. Indeed, there are many conceivable situations in

which such a regime change will not benefit those individuals either in the short- or long-run.

A transition to a monetary equilibrium may not be Pareto improving. If it is indeed the case, any attempt to create money will be strongly opposed by some member of the society. There is thus a fundamental difficulty in the establishment of money and monetary equilibrium even as a cooperative solution of the society.

18. A concluding remark.

No matter how difficult the evolution of money might be from the purely theoretical standpoint adopted in the present article, we cannot at the same time deny the unmistakable fact that we are actually living in a full-fledged monetary economy. What in the world was the large symmetry-breaking disturbance which historically created money and monetary economy "in the beginning"? But such question is certainly better put to the hands of historians, archaeologists and numismatists.[35] It is time to break off this already too long article on the theory of money.

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Footnotes

1. The best-known work on this problem is of course W. S. Jevons [1875]; more recent contributions include Starr [1972], Ostroy and Starr [1974] and Niehans [1978].
2. One such story is given by Adam Smith [1776] in pp. 22-23, but a similar story can be traced back at least to Aristotle' Politics, Chapter IX.
3. See Lippman and McCall [1976] for a useful survey of the theory of search. It was the work of George Stigler [1961] which opened up this theory and it was the publication of Phelps and others [1970] which established it as a well-defined discipline in economics. Lecture 2 of Diamond [1984] is a perceptive overview of its more recent development.
4. See Samuelson [1958]. The model of exchange economy to be developed in the present paper has a certain similarity to consumption-loan models which were first formulated by Samuelson in this seminal paper and now abound in literature on monetary theory and macroeconomics. There are, however, at least two distinguishing features in our model. In the first place, while consumption-loan models presuppose the use of some piece of paper as money from the beginning, our model explains that possibility endogenously. In the second place, while the medium of exchange function and the store of value function of money are inextricably intermingled in consumption-loan models, it is the medium of exchange function of money which is put full relief in our model.
5. Robert W. Clower [1967].
6. Carl Menger [1871]; pp. 257-262. See also his well-known paper [1892].
7. We can easily incorporate non-durable goods into our model.
8. Katsuhito Iwai [1988].

9. A formally similar model can be found in a thought-provoking paper by Robert A. Jones [1976].

10. The above assumption corresponds to what Diamond and Maskin [1979] called the case of "quadratic meeting technology" in their search model, for it implies that the aggregate number of meetings increases with the square of the frequency of searchers. See also Mortensen [1982] for a related discussion. We can in fact generalize it in such a way that the probability of meeting an i-holding, j-demander is proportional to $P(q_{ij})$, where $P(0) = 0$ and $dP(q)/dq > 0$, without losing any of the propositions that will follow. All we need for our theory is the assumption that an individual's probability of meeting a trading partner is positively related to the frequency of potential partners in each trading zone. Diamond later applied this search-theoretic framework to the analysis of barter and monetary exchange processes in Lecture 1 of [1984]. The prototype exchange model of Diamond, however, is a single-good barter exchange, and his "monetary economy" model presupposes a given structure of monetary transactions technology. It is one of the main purpose of the present paper to deduce the very structure of the monetary transactions technology on the basis of the search-theoretic analysis of individual exchange behaviors.

11. Carl Menger, [1871]; p. 259.

12. To be more precise, we can let the utility of consumption vary from individual to individual, for we can restate the following analysis of the individual search behavior in terms of expected cost minimization rather than of expected utility maximization. We have chosen the formulation of expected utility maximization simply for its potential generalizability.

13. If we wish to incorporate the time discount factor into our model, we have to reformulate it in the following manner. Let us suppose that the process of exchange is a random event that has the probability 1/b of being completed in one period. (Hence, b is the expected duration of an exchange process, which determines the cost of exchange implicitly.) If we denote by r (>0) the rate of time discount which determines the cost of search (that is, the cost of time) implicitly, the method of Dynamic Programming enables us to calculate V_{ij} in the following manner:

$$V_{ij} = \mathrm{Max}_k[V_{kj}\{(1+r)/(1+r/q_{ki})\}\{(1+r)/(1+rb)\}] .$$

Taking its logarithmic value, we have:

$$\log(V_{ij}) = \mathrm{Max}_k[\log(V_{kj})-\log\{(1+rb)/(1+r)\}-\log\{(1+r/q_{ki})/(1+r)\}] .$$

If we replace $\log(V_{ij})$, $\log\{(1+rb)/(1+r)\}$, and $\log\{(1+r/q_{ki})/(1+r)\}$ respectively by V_{ij}, b and c/q_{ji}, then these equations become formally equivalent to the functional equation (1) of the main text.

14. In the present paper we shall not suppose any continuity in their search programs. We, however, intend to write a sequel paper which deals with the model with infinite-horizon individuals (or families). We are in fact able to show that all the propositions established in the present paper remain intact even in the world of infinitely-lived individuals.

15. The notion of connectedness given above is closely related to that of "irreducibility" in the theory of Markov chains. See, for instance, Feller [1968] for the theory of Markov chains.

16. D. Cass and M. Yaari [1966]. To be precise, this is the static counterpart of the simpler of the two consumption-loan models developed in Samuelson [1958].

17. To be precise, the expression in Σ in (9) is true only when $V_{jk}-b-c/q_{ji}$ is the unique maximizer of V_{ik}. When two or more goods are

simultaneously maximizing V_{ik}, we should replace f_{ik} in Σ by $(s^j{}_{ik})f_{ik}$ where $s^j{}_{ik}$ takes any value in [0,1] as long as $\Sigma_h (s^h{}_{ik}) = 1$ with h being the index of good which maximizes V_{ik}. In such cases q_{ij} becomes set-valued. For the sake of expositional brevity (and for that sake only), we shall ignore this complication in the following presentation.

18. The definition of exchange equilibirum we have adopted here only stipulates the behaviors of the active searchers whose supply-demand frequencies are strictly positive. But these supply-demand frequencies $\{f_{ij}\}$ are not the "fundamentals" of the economy but the endogenous variables whose values are in general influenced by the ongoing exchange processes. Indeed, as time goes on, some of the zero frequencies may thus suddenly turn positive and upset the existing exchange equilibrium. In this sense the above equilibrium notion might be a little bit too fragile. In fact, in one of the earlier versions of this paper, we replaced eq. (9) of the text by the following equation:

$$q_{ij} = \sum_{\{k\neq j:\ \text{and}\ V_{jk}-b-c/q_{ji}=V_{ik}\}} f_{ik} + \begin{cases} f_{ij} & \text{if } u-b-c/q_{ji}=V_{ij}, \\ 0 & \text{otherwise,} \end{cases}$$

The notion of exchange equilibrium with equation (9) being replaced by the above equation is robust not only to a strategic variation of the active searchers in the economy but also to a sudden emergence of an individual type whose supply-demand frequencie has been long zero. Needless to say, under this stronger notion of exchange equilibrium we are able to obtain slightly stronger results than the ones to be presented in the present paper.

19. In section 14, however, we shall touch on the possibility of an exchange equilibrium with a local money.

20. It should be kept in mind that Fig. 1 is only one of N(N-1)(N-2) such diagrams, each for a different triplet of endowment-need frequencies.

21. If the economy is in a monetary equilibrium, we can at least write down the dynamic evolution of $\{f_{ij}\}$ as follows. (We use the law of large numbers below.)

$$\Delta f_{ij} = -q_{mi}f_{ij}+q_{jm}(e_{ij}-f_{ij}), \quad \text{for } i\neq m \text{ and } j\neq m;$$

$$\Delta f_{mj} = -q_{jm}f_{mj}+\sum_{k\neq m} q_{mk}f_{kj}, \quad \text{for } j\neq m;$$

$$\Delta f_{im} = 0 \quad \text{for } i\neq m;$$

or, substituting (13a)--(13c), we have

$$\Delta f_{ij} = -f_{mi}f_{ij}+(\sum_k f_{jk})(e_{ij}-f_{ij}), \quad \text{for } i\neq m \text{ and } j\neq m;$$

$$\Delta f_{mj} = -(\sum_k f_{jk})f_{mj}+\sum_{k\neq m} f_{mk}f_{kj}, \quad \text{for } j\neq m;$$

$$f_{im} = e_{im} \quad \text{for } i\neq m.$$

It appears difficult to solve the above set of difference equations explicitly. Its steady-state solution is, however, not hard to compute (by setting $\Delta f_{ij} = \Delta f_{mj} = \Delta f_{im} = 0$ and solving the resulting simultaneous equations) and is, not surprisingly, identical with the one obtained in the main text by the more pedestrian method. We have not yet been able to examine its stability.

22. See, for example, William Feller [1968].

23. This indeed suggests the possibility of a "fiat money" equilibrium which does not require any commodity substance for money. We intend to write a sequel paper to deal with this possibility more fully.

24. For the break-down of Say's law and its implications for the dynamics of macroeconomy, see Katsuhito Iwai [1981].

25. R. W. Clower, [1967].

26. This is perhaps what Frank Hahn had in mind when he wrote, "the Clower procedure assumes what should be explained," in p. 21 of [1981], a

booklet which contains many insightful discussions on what monetary economy is (or is not).

27. Karl Marx [1868].

28. The proviso about the possibility of multiple strategic choices, stated in footnote 17 also applies here.

29. Carl Menger [1871].

30. This discussion has incidentally demonstrated the possibility of an exchange equilibrium in which some goods are used as media of exchange with limited circulation. In fact, it is not hard to see that if the initial disturbance involved not only the j-holders' demand but also the h-holders' demand for good m such that the barter inequality (18) was reversed not only for j but also for h, good m in general becomes a local money circulating among the holders and consumers of both good j and h, We can of course extend this argument further to include many more goods. What we call monetary equilibirum can then be regarded as the limit case of such local money equilibria, which has with the widest area of circulation.

31. The discussion in the previous section, in particular in footnote 30, however, suggests the possibility of visualizing the evolutionary process of a local money in almost the same manner as below.

32. A similar diagram can be found in R. A. Jones [1976].

33. We have not checked their uniqueness as yet.

34. We have not checked their uniqueness as yet.

35. Philip Grierson [1978] has given us the most sophisticated account of the "origins of money". See also A. Quiggin [1949], G. Dalton [1965], P. Einzig [1966], H. Codere [1968] and many others.

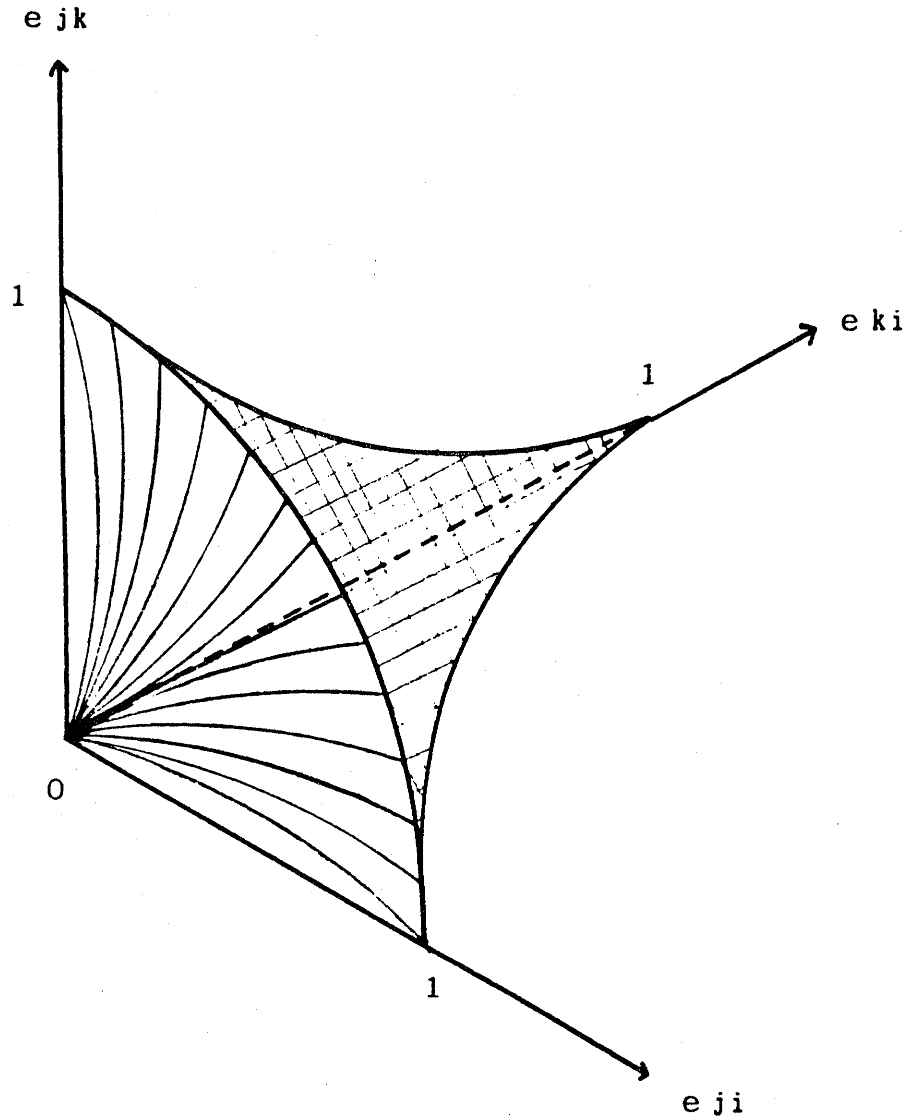

Fig.1: The allowable range of endowment-need frequencies for barter equilibrium. (The case of $b/c < 1$)

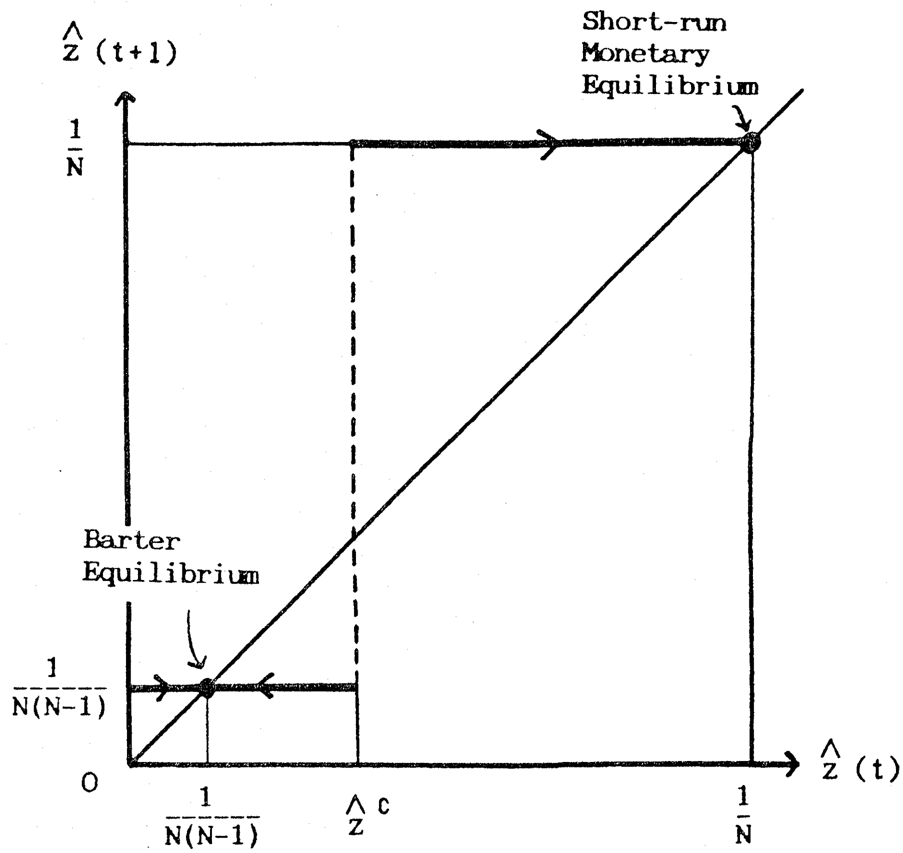

Fig.2: The evolutionary dynamics of $\hat{z}(t)$ (the subjective probability of meeting a money demander) from barter to one of short-run monetary equilibria. (The case of symmetric endowment-need distribution)

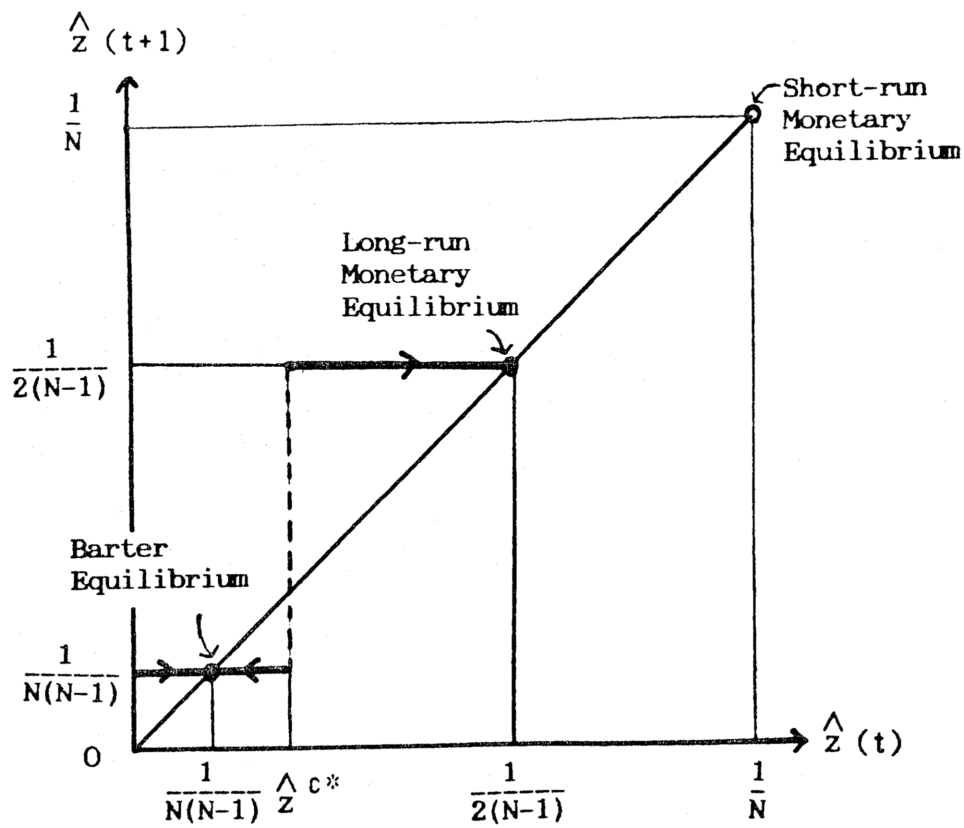

Fig.3: The evolutionary dynamics of $\hat{z}(t)$ (the subjective probability of meeting a money demander) from one of long-run monetary equilibria to barter equilibrium. (The case of symmetric endowment-need distribution)