88-F-18

History of Economic Theory (7)

by

Takashi Negishi
The University of Tokyo

October 1988

PREPARED UNDER
THE PROJECT ON MACROECONOMICS
RESEARCH INSTITUTE FOR THE JAPANESE ECONOMY

Walras and the General Equilibrium Theory

1. Augustin Cournot

Of the four greatest economists in the world, according to Schumpeter, three were French, i.e., Francois Quesnay, Augstin Cournot and Léon Walras (Samuelson[26], pp. 1501 1502, 1556). This is, however, not the only reason why we sould consider Cournot in this section before our discussion, in the rest of this chapter, of Walras, one of the three big stars of the marginal revolution, the initiator of the general equilibrium theory and the author of Eléments d'économie politique pure, the Bible of the modern neo-classical economics. The so-called Walras' law, which signifies the self-compactness or closedness of an economic system, was already suggested in Cournot's discussion of the exchanges. Walrasian view of the competitive market prices, quite distinguished from those of Menger and jevons, can be considered as developed under a strong influences of Cournot's view.

Antoine Augustin Cournot was born in 1801 at Gray, in Haute-Saone, France. He studied mathematics in the Ecole Normale at Paris, and became Professor of Mathematics at Lyons in 1834, by the help of the great physicist and statistician Poisson. He served also as Rector of the Academy at Grenoble, Inspecteur General des Etudes, and Rector of the Academy at Dijon. Cournot published extensively on mathematics, phylosophy and economics, before he died in 1877. The most important book on economics, is, of course, Recherches sur les principes mathématiques de la théorie des richesses(1838), in which he applied differential calculus for the first time in economics. Although this book is usually regarded as a classical treatise on such topics as monopoly and oligopoly, its significance is by no means limited to them, from the point of view of modern economic theory as well as from that of the history of economic thoughts.

Cournot first considers the mutual interdependence of countries in a world economy in his theory of exchanges (Chapter 3). " To find the equations of exchange, we will suppose, to begin with, that the cost of exchange is less than the cost of transportation, or that the exchange takes place without any real transportation of money, without any change in the distribution of the precious metals between the to commercial centers " (Cournot[5], p. 30). The rate of exchange of currencies of different countries (commercial centres) should be adjustable freely so that there is no transportation of money. If there are r countries, the number of the rate of exchange can be considered as r(r - 1). But it can be reduced to r - 1, if arbitrages between two and three countries are carried out freely. The equations to determine (r - 1) unknown rates of exchange are obtained from conditions that what one country owes to all the others is of precisely the same value as what all the other owe to it, since there is no actual transportation of money. These equations are in number r, but one of them is not independent from others. " Adding all these equations together except the first, and eliminating from each member the terms which cancel, we obtain again the first equation. Thus there are only just as many distinct equations as there are independent variables " (Cournot[5], p. 34).

This is essentially identical to Walras's discussion of the existence of a general equilibrium in a r good economy to the effect that the number of unknown relative prices is r - 1 while the number of independent conditions of the equality of demand and supply is also r - 1, in view of the so-called Walras's law that the value of aggregate demand for all goods is identically equal to the of the aggregate supply of all goods.

As for the theory of monopoly, Cournot (Chapter 5) obtained the condition for the maximization of profit, which is essentially identical to the equality of marginal revenue and marginal cost. Assuming the law of indifference (the law of the price for one good), Cournot considers that the demand F(p) is a continuous function of the market price p. To Maximize the net receipts which is obtained by subtracting the cost of production from the gross receipts, Cournot maximizes $p\,F(p) - \phi[D(p)]$ with respect to p, where the cost of production is considered as a continuous function $\phi(D)$ of the quantity to be supplied D and the demand and supply equilibrium $D = F(p)$ is assumed. The condition for the maximization of the net receipts is

(1) $$D + (p - d\phi/dD)\, dD/dp = 0.$$

Cournot considered the duopoly next (Chapter 7). The condition for the quality of demand and suuply is $D_1 + D_2 = F(p)$, where D_1 and D_2 are respectively quantities to be supplies to be supplied from two firms. It is conbenient to consider the inverse demand function $p = f(D_1 + D_2)$. The net receipts of the first firm is

(2) $$f(D_1 + D_2)D_1 - \phi(D_1)$$

and that of the second firm is

(3) $$f(D_1 + D_2)D_2 - \phi(D_2).$$

In words, the net receipts of a firm is a function not only of its own supply but also of the supply of the other firm. Cournot considers that each firm will independently maximized its net receipts, assuming that the supply of the other is unchanged. The equilibrium of duopoly is therefore a Nash solution of non-cooperative game, and the conditions for it are obtained by the differentiation of (2) and (3) respectively with respect to D_1 and D_2,

(4) $$f(D_1 + D_2) + D_1 f'(D_1 + D_2) - \phi'(D_1) = 0$$

and

(5) $$f(D_1 + D_2) + D_2 f'(D_1 + D_2) - \phi (D_2) = 0$$

where f' and ϕ' denote the derivatives of f and ϕ .

Since two firms are enjoying the entirely identical conditions, we should have $D_1 = D_2$ at equilibrium. By adding (4) and (5) together, therefore, we have

(6) $$2 f(D) + Df'(D) - 2 \phi (D/2) = 0$$

where $D = D_1 + D_2$.

Similarly, in the case of oligopoly [1)] of n firms, the i-th firms's net receipts is

(7) $$f(\sum_i D_i)D_i - \phi (D_i), \qquad i = 1, ---, n,$$

where D_i, $i = 1, ---, n$, is the supply of the i-th firm. The condition to maximize (7) with respect to D_i is given as

(8) $$f(\sum_i D_i) D_i f'(\sum_i D_i) - \phi'(D_i) = 0, \quad i = 1, ---, n,$$

if the other firms' supplies D_j, $j = i$, are considered unchanged. Conditions are identical for all firms, and we should have $D_1 = D_2 = --- = D_n$ in (8). Adding n conditions together in (8), then, we have

(9) $$nf(D) = Df'(D) - n \phi'(D/n) = 0,$$

where $D = \sum_i D_i$.

Being based on (9), Counot insists that the price p is equalized to the marginal cost ϕ' when the number of firms is sufficiently large and the competition is unlimited, i.e., the case of the perfect competition (Chapter 8). To see this, let us simplify the story by assuming that the demand function is linear and that the marginal cost is constant. By substituting $p = f(D) = a - bD$ and $'(D/n) = c$, where a, b and c are positive constants, into (9), we obtain

(10) $D = n (a - c)/b(n + 1)$.

The substitution of (10) into $p = a - bD$ yields

(11) $p = (a + nc)/(n + 1)$.

If n gets sufficiently larger, then, the price p approaches to the marginal cost c. The competitive price is a limit of oligopoly prices when the number of oligopoly firms gets infinitely large.

This limit theorem of Cournot is a pionnering contribution to the theory of a large economy (an economy with intinitely large number of participants) which is a central topit of modern mathematical economics. It is very interesting to compare this with another classical limit theorem, Edgeworth's limit theorem, which we shall argue in Capther 9. In the latter theorem, the law of indifference is established only at the limit where the allocation of goods is that of the perfect competition. As we saw, Cournot assumed the law from the begining so that there exists a unique market price p even when the number of firms is small and p is not a perfectly competitive price. This defference may be worth recognizing, since Cournot with respect to this assumption is a pioneer to Walras and therefore it is also the difference between Walras on one hand and Edgeworth and his pionner Jevons on the other.

In view of Cournot's limit theorem, we have to admit that the existence of infinitely many firms is a sufficient condition for the perfect competition in which a single firm perceives an infinitely elastic demand curve. It is, however, by no means a necessary condition. Even for the case of n = 2, that is, a duopoly, Bertrand and Fellner argued that the price will be equalized to the marginal cost if each duopolist assumes that the other will keep his price (not supply) unchanged and indentical average as well as marginal costs are constant.[2)] If the price is higher than the cost, each firm will undercut its rival by a very small margin because it will monopolize the market and obtain maximum profits by undercutting infinitesimally. This process continues until the price is equalized to the marginal cost. In other words, a Bertrand-type duopolist behaves as if he perceived an infinitely elastic demand curve. Bertrand's assumption can be criticized, of course since duopolists will know, when they are out of equilibrium or when they decide to test their assumption, that their assumption are incorrect ; their rival do not keep their prices constant. But Cournot's assumption is also subject to the same criticism, and we cannot accept Cournot and at the same time reject Bertrand.

The basic assumption of Cournot's theory of duopoly (and oligopoly) is that each firm changes its supply assuming that the supply of the other firm is unchanged, In other words, each firm adjusts its supply to the given supply of the other. In Bowley's terminoloby, conjectual variations are both zero. Stackelberg called such a behavior of firms as followership. If a firm acts as a follower, however, the other firms can make a larger profit (net receipts) by taking advantage of it. Suppose the first firm chooses D_1 in a accordance with (4) when D_2 is given. The second firm can make its conjectual variation

$$(12) \qquad dD_1/dD_2 = (f' + D_1 f'')/(\phi'' - D_1 f'' - 2 f')$$

by differentiating (4), where f'' and ϕ'' are second derivatives of f and ϕ. The second firm's supply to maximize its profit is then obtained by the substitution of (12) into

$$(13) \qquad f(D_1 + D_2) + D_2 f'(D_1 + D_2) + D_2 f'(D_1 + D_2)\, dD_1/dD_2 - \phi'(D_2) = 0$$

and by the replacement of (5) with (13). Stackelberg called such an active behavior of a firm as leadership. While Cournot's duopoly equilibrium is follower-follower equilibrium, a leader-follower equilibrium is certainly a possible alternative. If both firms act as leader, however, there is no equilibrium. This is the problem of Stackelberg disequilibrium.[3)]

A recent development is to consider a multi-period model of duopoly which allows for collusion without requiring the firms to make binding agreements (Friedman [9], pp. 123 - 124). Consider the maximization of the joint profit of two identical firms. The condition for it is

(14) $$f(2D) + 2Df'(2D) - \phi'(D) = 0$$

where D signifies the supply of a single firm, f is the inverse demand function and ϕ is a common cost function. Let us denote the value of D which satisfies (14) by D^m and the maximized joint profit by π^m. Similarly, let us denote Cournot solution which satisfies (4) and (5) by $D_1 = D_2 = D^c$ and the corresponding profit of each single firm by π^c. Certainly $\pi^m/2 > \pi^c$.

Suppose the time horizon of two firms is infinite and the second firm openly announces the following strategy. In the first period, the second firm supplies D^m, and it continues to do so in the future provided that the first firm also supplies D^m in each period. If the first firm does not supply D^m in a period, the second firm supplies D^c in the future. For the first firm, there are two alternative reactions possible. Firstly, it can supply D^m in the first period and also in the future, so that it can enjoy profit $\pi^m/2$ in all periods. Secondly, given the supply D^m of the second firm in the first period, the first firm may maximizes its profit in the first period, the condition for which is

$$(15) \qquad f(D_1 + D^m) + D_1 f'(D_1 + D^m) - \phi'(D_1) = 0.$$

The profit of the first firm corresponding to D_1 which satisfies (15) is larger than $\pi^m/2$. The first firm can expect, however, no more profit than π^c in the future since the second firm supplies D^c in this case. Unless the rate of the time preference of the first firm is very high, therefore, it is possible to have the case of tacit collusion of duopoly firms or self-enforcing agreements between them.

2. Éléments d'économie politique pure.

Marie-Esprit Léon Walras was born in 1834 at Evreux, in Normandy, France. His father was Antoine Auguste Walras, an economist who suggested rareté as the true source of value, rejecting utility and labor. His mother was Louise Aline of Sainte-Beuve. Having twice failed in the examination to enter the Ecole Polytechinique, Leon Walras was admitted to the Ecole des Mines. Since he was not interested in enginneering, however, he turned to literature, philosophy and history. In 1858 he published his principal novel, Francis Sauveur, which enjoyed, however, no real success. He was then pursuaded by his father to devote himself to the development of economics, and in 1860 wrote his first work on economics, L'économie politiques et la justice, Examen critique et réfutation des doctrines économiques de M. P.-J. Proudhon. Before he obtained an academic position, however, he had to work as a journalist, a clerk in a railway office, a managing director of a bank for co-operatives, a newspaper editor, etc. Walras's interest in taxation and social justice led him to participate in an international taxation congress at Lausanne in Switzerland in 1860. Because he made an excellent impression there, he was appointed professeur extraordinaire at the University of Lausanne when a chair in political economy was established in 1870.

According to his " Notice autobiographique " (Walras[32], p. 5), the selection committee consisted three notable persons of the district and four professors of economics. While the former group was for Walras, three of the latter were against. The fourth professor, Professor Dameth of Geneve, voted for Walras, considering that is useful to have Walras for the development of economics, even though he himself is against Walras's ideas. In 1870, Walras was promoted to professeur ordinaire After published " Principe d'une théorie mathématique de l'échange " (1873), " Equations de l'échange " (1875), " Equations de la production " (1876) and " Equations de la capitalisation et du crédit " (1876) in a rapid succession, Walras completed the first edition of Eléments d'économie politique pure (1874 - 1877). It was followed by Théorie mathématique de la richesse sociale (1883) which contains mathematical discussion of bimetallism, and Théorie de la monnaie (1886). Walras retired from the University of Lausanne in 1892 and was succeeded by his disciple, Vifredo Pareto. Walras continued, however, his research and showed his system of economics by publishing Etudes d'economie social (1896) and Etudes d'economie politique appliquee (1898). In 1896, Walras published the third edition of Eléments which contains his article on the marginal productivity, " Note sur la réfutation de la théorie anglaise du fermage de M. Wicksteed." And after " Equations de la circulation " (1899), it was followed by the fourth edition of Eléments, which is actually the Edition definitive. It was in 1910 that Walras died at Clarens, near Lausanne.

Walras's system of economics consists of pure economics, applied economics and social economics. This is based on the fact that social wealth is defined as all things, material or immaterial, that are scarce, that is to say, useful to us and only available to us in limited quantity and that such useful things limited in quantity are valuable and exchangeable, can be produced and multiplied by industry, and are appropriable (Walras [31], pp. 65 - 67). " From what point of view shall we study it [social wealth] ? Shall we do it from the point of view of value in exchange, that is, from the point of view of the influences of purchase and sale to which social wealth is subject ? Or shall we do it from the point of view of industrial production, that is, from the point of view of the conditions which favour or hinder the increase in quantity of social wealth ? Or, finally, shall we do it from the point of view of property, the object of which is social wealth, that is to say, from the point of view of the conditions which render the appropriation of social wealth legitimate or illegitimate ?" (Walras[31], p. 68). Pure economics, applied economics and social economics study social wealth respectively from the point of view of value in exchange, industry and property.

Walras insist that " wheat is worth 24 francs a hectolitre " is a natural phenomenon which " does not result either from the will of the buyer or from the will of the seller or from any agreement between the two " (Walras[31], p. 69). If two things have a definite value in exchange with respect to each other, it is because they are more or less scarce, that is, more or less useful and more or less limited in quantity, but both of these conditions are natural phenomena. Value in exchange is also a mathematical fact. Pure economics, that is, the theory of exchange and value in exchange is, therefore, a physico-mathematical science which uses mathematical method. Since " the mathematical method is not an experimental method ; it is a rational method," " the pure science of economics should then abstract and define ideal-type concepts in terms of which it carries on its reasoning. The return to reality should not take place until the science is completed and then only with a view to practical applications " (Walras[31], p. 71). Pure economic must precede applied economics, just as pure mechanics ought to precede applied machanics.

Industry is defined " as the sum total of relations between persons and things designed to subordinate the purpose of things to the purpose of persons " (Walras[31], p. 73). Industrial production pursues a twofold aim, firstly, to increase social wealth, that is, useful things limited in quantity, and secondly, to transform it, that is, to transform things with indirect utility like wool into things with direct utility like cloth. This twofold aim is pursued through two distinct classes of operations, technical operations and the economic organization of industry under a system of the division of labor. The two phenomena are both humand and not natural, and both industrial and not social. The theory of the economic production of social wealth, that is, of the organization of industry under a system of the devision of labor is thus an applied science or a theory of policy. Therefore, walras called it applied science or a theory of policy. Therefore, Walras called it applied economics.

" The appropriation of scarce things or of social wealth is a phenomenon of human contrivance and not a natural phenomenon. It has its origins in the exercise of the human will and in human behavior and not in the play of natural forces." " Moreover, the appropriation of things by persons or the distribution of social wealth among men in society is a moral and not an industrial phenomenon. It is a relationship among persons " (Walras[31], pp. 76 - 77). In other words, it is within our power to determine a way in which the appropriation is carried on, though this power does not belong to each of us individually but to all of us taken collectively. If the mode of appropriation is good, justice will rule and there will be a mutual coordination of human destinies. If the mode of appropriation is bad, injustice will prevail and the destiny of some will be subordinated to that of others. What mode of appropriation is compatible with the requirement of moral personality is the problem of property which Walras defined consists in fair and rational or rightful appropriation. While appropriation itself is a pure and simple objective fact, property is a right which is a phenomenon involving the concept of justice. " Between the objective fact and the right, there is a place for moral theory " (Walras[31], p. 78). While appropriation is a moral phenomenon, the theory of property must be a moral science. Since justice is defined as rendering to each that which is properly his, the science of the distribution of social wealth must espouse justice as its guiding principle. Walras designate this science social economics.

In the terminology of modern economics, pure economics is a postive science of market mechanism, applied economics is a normative science of optimal allocation of resources, and social economics is a normative sceince of optimal distribution of income. It should be emphasized that the economics of Walras consists not only of pure economics but also of applied and social economics and that Walras made a very clear distinction of these three branches of economics.

Although its implications are not necessarily clear, Walras insisted that pure economics proved the principle of laisser-faire, laisser-passer, that is, the attainment of maximum utility through free competition.[4)] The principle of free competition must be applied, therefore, in applied economics which considers from the point of view of material well-being the relation between persons and things in the organization of agriculture, industry and trade. Walras warned, however, that the application should be limited to the cases where his proof is established. " The principle of free competition, which is applicable to the production of things for private demand, is not applicable to the production of things where public interest is involved. Are there not economists, however, who have fallen into the error of advocating that public services be brought within the fold of free competition by turning these services over to private industry ?" (Walras[31], p 257).

Walras argued also that the principle of free competition can be applied in applied economics which is the economics of the industry, but that it cannot be applied in social economics which is the economics of the property right. " Though our description of tree competition emphasizes the problem of utility, it leaves the question of justice entirely to one side, since our sole object has been to show how a certain distribution of services gives rise to a certain distribution of products. The question of the [original] distibution of services remains open, however. And yet, are there not economicsts who, not content with exaggerating the applicability of laisser-faire, laisser-passer to industry, even extend it to the completely extraneous question of property ? Such are pitfalls into which a science stumbles when treated as literature " (Walras[31], p. 257). As for the problem of the property rights, it is well known that Walras insisted the nationalization of the landed property.

Only pure economics, however, was developed sistematically by Walras, i.e., in his Elément d'économie politique pure. As for applied economics and social economics, Walras gave up his plan to develop them systematically and left intead Etudes d'économie politique appliquée and Etudes d'économie social, which are merely collection of independent essays. From the point of view of influences Walras gave to later-day economists, only Elément is important. Furthermore, we can see in Eléments what Walras had in mind on the relation between pure economics and applied and social economics, on the limit of pure economics, and on the whole system of economics he was planning. The first part of Eléments is entitled as Object and Divisions of Plitical and Social Economy, and gives a bird's-eye view of Valras's system of economics, which consists of pure economics, applied economics and social economics.

The second part of Eléments, Theory of Exchange of Two Commodities for Each Other, is based on Walras's article, " Principe d'une théorie mathématique de l'échange " (1873). To develop his theory of pure economics mathematically, Leon Walras' starts with the concept of demand curves suggested by Cournot, an old friend of Leon's father, Auguste Walras. Firstly, the supply curve of one commodity is derived from the demand curve of the other commodity. Equilibrium price ratio is then determined at the intersection of demand and supply curves of a commodity. Secondly, demand curves of commodities are shown to be derived from each party's utility or want curves for these commotities and the given initial stock which each party possesses, through the law of the equality of the ratio of marginal utility to price. In other words, by the use of the maximization of utility Walras theoretically explained demand curves which Cournot assumed to be given empirically. As for the marginal utility, Leon Walras followed his father Auguste to use the term raretē and sharpened its definition as " the intensity of the last want satisfied by any given quantity consumed of a commodity " (Walras[31], p. 119).

One of the contributions made by Walras in Element is certainly the discovery or rediscovery of the principles of marginal utility, which he shares with Menger and Jevons. It is, however, by no means the greatest contribution of Walras. What made the fame of Walras eternal is " to establish for the first time general conditions of the economic equilibrium," as is written in the bronze commemoration medal in the corridor of the University of Lausanne.[5)] In Eléments, the theory of general equilibrium is developed through the successive solutions of four major problems of pure economics i.e., the general equilibrium of exchange, the general equilibrium of production, the general equilibrium of capitalization and credit, and the general equilibrium of circulation and money.

Being based on the article " Equations de l'echange ' (1875), the theory of the general equilibrium of exchange, i.e., Theory of Exchange of Several Commodities for the One Another, is studied in the third part of Eléments. Equations to determine the equilibrium of exchange are, firstly, equations which show that individuals' demand and supply of commodities depends on market prices of all commodities, and, secondly, equations which require that the equality of demand and supply in all markets. The former is a pioneering study of modern theory of consumers' behavior. Commodities are simply assumed to exist and there is no problem of production in the theory of exchange. Nor is any special commoditiy called money which serves as a medium of exchange. Instead, an arbitrary ordinary commodity is chosen as the numeraire, i.e., " the commodity in terms of which the prices of all the others are expressed " (Walras[31], p. 161).

The general equilibrium of produciton was first studied in the article " Equations de la production " (1876) and then developed detailedly in the fourth part of Eléments, Theory of Production. While the law of supply (offer) and demand is formulate in the theory of exchange, the law of the cost of production or of cost price is to be added in the theory of production. Capital in general is defined as " all durable goods, all forms of social wealth which are not used up at all or are used up only after a lapse of time, i.e., every utility limited in quantity which outlasts its first use, or which, in a ward, can be used more than once, like a house or a piece of furniture " (Walras[31], p. 212). It consists, therefore, landed capital, personal capital and capital goods proper. The entrepreneur " leases land from land owners on payment of a rent, hires the personal faculties of workers on payment of wages, borrows capital from capitalsits on payment of interest charges " (Walras[31], p. 227) and combines productive services of landed capital, personal capital and capital goods proper in a certain ratio to produce consumers' goods. " In a state of equilibrium in production, entrepreneurs make neither profit nor loss " (Walras[31], p. 225). The conditions for the equilibrium of production are, therefore, (1) the equality of demand from entrepreneurs and supply from land owners, workers and capitalists in the markets of factors of production, (2) the quality of demand from landowners, workers and capitalists and Supply from entrepreneurs in the markets of consumers' goods, and (3) the equality of the selling prices of the products and the cost of the services employed in making them.

The ratios of factors of production to be combined, or the coefficients of production are determined by the marginal productivity of factors. The theory of marginal productivity shows the underlying motive of the demand for services and the offer of products by entrepreneurs, just as the theory of final utility shows the underlying motive of the demand for products and offer of services by landowners, workers and capitalists " (Walras[31], p. 385). Walras's consideration of the theory of marginal productivity starts with " Note sur la réfutation de la théorie anglaise du fermage de M. Wicksteed " (1896) and is developed into Lesson 36 in the seventh part of Eléments.

In the theory of production only the production of consumers' goods is explicitly considered, since the case of the application of raw materials can be reduced to the case of the direct combination of productive services alone (Walras[31], p. 240). Capital goods proper are simply assumed to exist, as in the case of landed capital. In the article " Equations de la capitalisation et du crédit " (1876) and the fifth part of Elément, Theory of Capital Formation [Capitalisation] and Credit, however, Walras takes the production of new capital goods and saving into consideration. Walras's theory of capital is, in short, a theory of fixed capital, though he tries to consider circulating capital in this theory of circulation and money. It stands in sharp contrast to the theory of circulating capital of classical economics, Marxian economics and Austrian economics.[6)] As we shall see in section 5, howver, modern theory of economic dynamics and economic growth has been developed, not on the basis of Austrian theory but on the basis of Walrasian theory of capital.

As Walras himself indicated in his Preface of the fourth edition of Eléments ([31], pp. 38 - 39), his theory of money underwent several important changes between 1876 and 1899. In the first edition of Eléments, it was a " Fisherian " equation of exchange based on the concept of the aggregate demand for money required to subserve the circulation of goods. It was replaced by a " Cambridge " equation based on the concepts of the individual demand for desired cash balance in Théorie de la monnaie (1886). Then, it was further developed, through " Equations de la circulation " (1899), into the sixth part of the fourth edition of Eléments, Theory of Circulation and Money. In this last theory of the general equilibrium, a Special commodity called money is finally introduced and its value is explained by the application of the theory of the marginal utility. In other words, the theory of moeney is now combined with the theory of relative prices. This is the final contribution mede by Walras in his more than fifty years' research on economics.

In each of his consideration of the problems of equilibirum of exchange, production, capitalization and credit, and circulation and money, Walras repeatedly tries two different solutions, i.e., theoretical or mathematical solution and empireical or practical solution.

The former solution is to confirm the equility of the nubmer of unknowns, like the equilibirum prices, and the number of equations, like the conditions of equilibrium of supply and demand. For example, suppose the number of consumers' goods is m and the number of factors of production is n in the the theory of production. The number of unkowns is 2 n + 2 m - 1, i.e., (1) n equilibrium quantities of supply of factors, (2) n equilibrium prices of factors in terms of the m-th consumers' good, numéraire, (3) m equilibrium quantities of demand for consumers' goods, and (4) m - 1 equilibrium prices of consumers' goods in terms of the m-th. On the other hand,there seem to be 2 n + 2 m equations, i.e., (1) n supply equations of facotrs, (2) m demand equations for consumers' goods, (3) n conditions of the quality of the quantity of factors used in the production and the quantity of factors supplied, and (4) m conditions that the selling prices of the products are equal to the cost of the factors employed in making them. In view of the so-called Walras law, howver, one of these equations is not independent from the others. Therefore, we have 2 m + 2 n - 1 equations to determine 2 m + 2 n - 1 unknowns. Of course, this is by no means sufficient to prove the existence of the general equilibrium, which we shall consider in the next section.

Walras's second solution of the equilibrium problems is the famous theory fo tâtonnement, which explains how the problems of equilibium is empirically or practically solved in the markets by the mechanism of competition. Walras simplified the problem by assuming that exchange and therefore production can take place only when the equilibrium is established, and that productive service have to be transformed into products instantaneously once the equilibrium has been established. This assumption is not a mere simplifying one, but reflects the basic attitude of Walras who supposes " that the market is perfectly competitive, just as in pure mechanics we suppose, to start with, that machine are perfectly frictionless " (Walras[31], p. 84). It was clearly suggested to Walras from the observation of some operations in the Paris Stock Exchange where disequilibrium transactions actually did not occur (Jaffé[15], p. 247). However, it is very unrealistic to apply such a model of special markets to the whole economy.. Thus the significance of exchange and production taken place in disequilibrium is entirely disregarded in Walras's economics and therefore in modern neo-classical economics. Even with this stringent assumption, furthermore, Walras's demonstration of the second solution was not perfect. He could not prove rigorously that the general equilibrium can be approached by the preliminary adjustments made in markets before actual exchange transactions and productions take place. We shall discuss this problem in section 4.[7)]

3. The Existence of a General Equilibrium.

Walras's theoretical or mathematical solution is not sufficient to assure the existence of a general equilibrium, since it is merely to count the number of unknowns and that of equations. If these numbers are identical, a theoretical model of an economy may be called complete or consistent, but it does not quarantee that there exists an economically meaningful solution. For example, equilibrium prices have to be real, and, in general, positive. Apart from some pioneering attempts, proofs of the existence of an economically meaningfull solution in a general equilibrium model were given in 1950's by such people as K.J.Arrow, G.Debreu, L.W.McKenzie, D.Gale and F.Nikaido.[8] Mathematically, they are applications of the fixed point theorem of topology. It must be recognized, however, important roles are played in the process of proofs by the bomogeneity of demand and supply functions with respect to prices and the so called Walras' law that the value of aggregate demand is identically equal to that of aggregate supply, both of which Walras discussed in his use of numeraire and to show the dependency of n equations.

Unfortunately, such proofs which deal with a general case are highly technical and cannot be reproduced here in full details. We have to be satisfied with a discussion of a vary simple case like an equilibrium of exchange in two commodity two individual model of an economy. Let us consider a proof which is based on the Pareto optimality of a competitive equilibrium.[9] It is well known that resource allocation is optimal in the sense of Pareto, the successor of Walras, in a perfectly competitive equilibrium. It implies that the utility of no individual can be increased without reducing that of some others. The condition that assures the Pareto optimality is the equality of marginal rates of substitution among all individuals and all firms. This condition is satisfied in a perfectly competitive equilibrium, since marginal rates of substitution are equalized to price ratios which are common to all individuals and firms. In addition to this condition, furthermore, budget constaints of all individuals have to be satisfied in a perfectly competitive equilibrium. To prove the existence of such an equiliblium, therefore, it is sufficient to show the existence of a Pareto optimal situation in which budget constraints are satisfied when marginal rates of substitutions are interpreted as competitive price ratios.

Since one might wonder whether we can assume perfect competition in the case of two individual economy, let us suppose that two commodities are exchanged among 2 n individuals for sufficiently large $n > 0$. Individuals are assumed to be grouped into two homogeneous groups of n individuals in the sense that individua groups of n individuals in the sense that individuals in the same group are completely idential each other, having the same taste (utility function, indifference map) and the same initial holdings of commodities. Equilibrium of exchange among 2 n individuals, then, can be described by the equilibrium of exchange between the representative individuals of two groups. In other words, we consider in the below the equilibrium of exchange between two individuals, not isolated two individuals, but the two representative individuals arbitrarily chosen respectively from different homogeneous groups of n individuals. Our model is a reduced one with the scale of one to n of the original 2 n individual model of a competitive economy.

To consider two individual two commodity exchange equilibrium, it is convenient to use the so-called Edgeworth box diagram. Let us suppose that the first individual has an intial endowment of $\bar{X}_{11}$ units of commodity 1 and $\bar{X}_{12}$ units of commodity 2. Similarly, suppose that the second individual has an initial endowment of $\bar{X}_{21}$ units of commodity 1 and X_{22} units of commodity 2. Since there is no production, the total quantity of commodity 1 in the economy before and after exchange is given by $\bar{X}_1 = \bar{X}_{11} + \bar{X}_{21}$, and the total quantity of commodity 2 is given by $\bar{X}_2 = \bar{X}_{12} + \bar{X}_{22}$. It follows that the only possible states of the economy are those represented by a set of points contained in a rectangle having dimensions $\bar{X}_1$ by $\bar{X}_2$ (Figure 7.1), where the quantity of commodity 1 is measured horizontally, and that of commodity 2, vertically. Any point in the box represents a particular distribution of the commodities between the two individuals. For example, if the distribution of commoditeis is given by point M, the quantities of commodity 1 and commodity 2 obtained by the first individual, x_{11} and x_{12} respectively, are measured by the coordinates of M, using the south-west corner 0 as the orogin ; the quantities obtained by the second individual, x_{21} and x_{22} respectively, are measured by the coordinates of point M, using the north-east corner 0' as the origin.

The indifference map of the first individual is drawn, using 0 as the origin, and the indifference map of the second individual, using 0' as the origin. The marginal rates of substitution of two individuals are equal where an indifference curve of the first individual is tangent to an indifference curve of the second individual. The locus of all such points is the contranct curve CC'. The marginal rates of substitution are unequal at points not on the contract curve, say point A of the initial endowments, and it is possible to increase the utility level of both individuals by changing the existing distribution of commodities. For example, if the final position after a redistribution of commodities is is in the inside of the area surrounded by two indifference curves passing A, both individuals would have gained, since both would be on higher indifference curves than at A. It a point on the contract curve is reached, it is not possible to improve further the position of either person without a deterioration in the position of the other. In other words, points on the contract curve are Pareto optimal.

Suppose that two individuals are acting price takers or quantity adjusters to the given market prices. Denoting the price ratio, i.e., the price of commdotity 1 in terms of commodity 2 by p, the first individual will choose the best combination of two commodities (X_{11}, X_{12}) being subject to his budget constraint, $p X_{11} + X_{12} = p \bar{X}_{11} + \bar{X}_{12}$. Similarly the second individual will choose (X_{21}, X_{22}) being subject to $p X_{21} + X_{22} = p \bar{X}_{21} + \bar{X}_{22}$. Equilibrium conditions for each individual are (1) the equality of the marginal rate of substitution and the price ratio (i.e., the tangency of indifference curbe and price line) and (2) the budget constraints. The market equilibrium is attained when demand and supply of two commodities are equal, i.e., $X_{11} + X_{21} = \bar{X}_1$, $X_{12} + X_{22} = \bar{X}_2$. It should be noted that among two budget constraints and two market clearing conditions any one condition is implied by the remaining three conditions. In other words the so-called Walras' law holds.

At any point in the box diagram market equilibrium conditions are satisfied, If the common tangent of two indivverence curves is considered as the price line, the condition (1) of the equilibrium of each individual is realized at any point on the contract curve CC'. Therefore, all the conditions for the competitive eqilibrium are satisfied at such a point as point B on the contract curve, where the price line BA passee the initial endowment point A, i.e., the budget constraint is satisfied for each individual.

Since markets are always cleared in the box to prove the existence of a competitive equilibrium, we have only to show that there always exists a point like B on the contract curve CC' at which a budget constraint is satisfied. Let us first consider in Figure 7.1 the point M, i.e., the point of intersection of contract curve and the indifference curve of the first individual passing through point A. If indifference curves are strictly convex to the origin, we have

$$(16) \qquad F(X_{11}, X_{12}) = p(X_{11}, X_{12}) X_{11} + X_{12} - p(X_{11}, X_{12}) \bar{X}_{11} - \bar{X}_{12} < 0$$

at point M, where p is the marginal rate of substitution of the first individual. Budget constrint is not satisfied and M is not a competitive equilibrium. Similarly we have at point N, i.e., the point of intersection of the contract curve and the indifference curve of the second individual passing point A,

$$(17) \qquad F(X_{11}, X_{12}) = p(X_{11}, X_{12}) X_{11} + X_{12} - p(X_{11}, X_{12}) \bar{X}_{11} - \bar{X}_{12} > 0.$$

Budget constraint is not satisfied, and N is again not a competitive equilibrium.

If the marginal rate of substitution of the first individual p is assumed to be a continuous function of the quantity of two commodities obtained by the same individual, $F(X_{11}, X_{12})$ is also a continuous function. The movement on the contract curve from M to N can be expressed as $X_{11} = X_{11}(t)$, $X_{12} = X_{12}(t)$, $0 \leqq t \leqq 1$, where $[X_{11}(0), X_{12}(0)]$ is the coordinates of M, $[X_{11}(1), X_{12}(1)]$ is the coordinates of N and X_{11} and X_{12} are continuous function of t. Then, $F[X_{11}(t), X_{12}(t)]$ is a continuous function of t such that $F < 0$ at $t = 0$ and $F > 0$ at $t = 1$. By the theorem of intermediate values of a continuous function (Figure 7.2), we are sure that $F = 0$ at some t (possibly not unique) such that $0 < \bar{t} < 1$. The point whose coordinates from the origin 0 is [$X_{11}(\bar{t})$, $X_{12}(\bar{t})$] is a competitive equilibrium, i.e., point B in the box diagram.

If indifference curves are not strictly convex to the origin, however, there may not exist any equilibrium. Figure 7.3 shows the case of the so-called Arrow anomaly, wehre indiffernce curves of the second individual are not strictly convex and the point A of the initial endowment is located on the boundary of the Edgeworth box. Curves I_1 and I_2 are indifference curves of the first individual with the origin at 0 and curves J_1 and J_2 are those of the second individual with the origin at 0'. Note that the utility of the commodity 1 is satiated for the second individual at points between 0 and A (including 0 and A) while it is not so for the first individual. Point A is clearly Pareto optimal and on the contract cirve but not a competitive equilibrium as is pointed out by Arrow [1], since there can be no price line which is tangent to both I_1 and J_2 at A. Suppose price of commodity 1 (in terms of commodity 2) is zero. Budget line through A is then horizontal and the first individual demands indefinitely large quantity of commodity 1. Suppose the price of commodity 1 is positive. The second individual demands, then, larger quantity of the commodity 2 than at A, since his budget line is not tangential to indifference curve J_2 at A, so that it is again impossible to stay in the box. Since A is also the point of the initial endowments, it is evident that there is no equilibrium, if A cannot be one. To avoid such an anomalous case, we have to assume either that indifference curves are strictly convex (utility of no commodity is satiated) or that the point of initial endowments is not located on the boundary of box (every individual has strictly positive quantity of every commodity).

The gist of the proof of the above for the case of two commodity two individual (or two homogeneous group of individuals of the same size) exchange economy is to find a point with budget constraints satisfied (point B) among Pareto optimal pointed (curve CC') by changing the weights of individuals in Pareto optimal distribution (the movement from M to N). The essentially same method of proof can be applied to the general case of m commodity n individual exchange economy, though more advanced mathematical therems, fixed point theorems instead of the simple theorem of intermediate values of a continuous function. Consider the maximization of a weightdd sum of utilities of n individuals,

$$(18) \qquad \sum_i a_i U_i (X_{i1}, ---. X_{im})$$

being subject to conditions of the equality of demand and supply,

$$(19) \qquad \sum_i X_{ij} = \sum_i \bar{X}_{ij} \qquad j = 1, ---, m,$$

where a_i is the given positive constant, X_{ij} is the quantity of the j-th commodity to be given to the i-th individual, and $\bar{X}_{ij}$ is the i-th individual's given initial holding of the j-th good commodity. If the budget constraints

$$(20) \qquad \sum_j p_j X_{ij} = \sum_j p_j \bar{X}_{ij} \qquad i = 1, ---, n,$$

where p_j is the Lagrangean multiplier corresponding to the j-th condition in (19) which is interpreted as the price of the j-th commodity, are satisfied, the maximum of (18) corresponds to the perfectly competitive exchange equilibrium. To prove the existence of an equilibrium, therefore, one must find such a proper sets of weights of individuals in Pareto optimal distribution (i.e., a_i's in (18)) that budget constraints are satisfied.

Similarly, the extension to the case of a production economy is also straightforward, provided that the possibility of the economy of scale is ruled out. The difficulty of the case with the economy of scale can be seen by the consideration of two commodity one individual one firm (or n identical individual n identical firm) production economy. In Figure 7.4, the quantity of one of two commodities, the service of labor (time) is measured vertically, and that of the other commodity, a labor product horizontally. The curves I_1 and I_2 are indifference curves of an individual. The initial quantity of labor service held by an individual is represented by OA. It is assumed that an overhead cost is incurred to produce the product (measured horizontally), which is represented by the input AB of labor service for a firm. On the other hand, the variable cost of production of the firm is shown by the curve BC, expressing the relation between input of labor service measured downword from B and the output of the product measured to the right from 0. Since the variable cost is increasing, the average cost (measured in terms of labor input) is expressed by a typical U-shaped cirve. If a positive quantity of the product is produced in a competitive equilibrium, the marginal rates of substitution of individual and firm must be equal, as they are at D, to the price ratio. The price ratio of the produced commodity and labor service at D is equal to the slope of the commom tangent DE of curves BC and I_1. The profit of the firm is negative, however, since the value, interms of labor, of product DH produced from the inputs AH of labor is merely EH. If the point of no production A is a trivial equilibrium on the other hand, the price ratio is equal to the slope of the tangent AF to the indifference curve I_2 at A. Then A cannot be an equilibrium, since it is profitable for the firm to produce a positive quantity under the price ratio AF.

Such a difficulty does not exist if we ruled out the existence of the economy of scale by assuming away the overhead cost so that the average as well as marginal costs are increasing for any level of output. In Figure 7.4, points A and B coincide. Suppose the initial holding of labor is OB for an individual. Point D represents a competitive equilibrium in which the profit, in terms of labor, EB of a firm is distributed to an individual so that the product DH of the firm is bought by the individual with the wage income BH and the profit distributed EB.

If we suppose the profit of a firm is distributed equally to individuals, furthermore, our model can be considered a reduced scale model of an economy consist of n identical individuals and n identical firms. If the number n is infinitely large, then, a competitive equilibrium can exists, even if there is an economy of scale due to the existence of the overhead cost in each firm. Suppose, again, that the initial holding of labor is OA for an individual and that an overhead cost AB is incurred for a firs, in Figure 7.5, which is a reproduction of Figure 7.4. Since the number of firms is infinite, the input of labor and the output of product per firm can be at any point on the line AJ which is tangent to the curve BC representing the variable cost, if we suppose some firms are at A (no production) and the rest of firms are at J. Then, an equilibrium exists at K, where indifference curve I_3 of every individual is tangent to line AJ. Since the price ratio of the product and labor service is equal to the slope of AJ, the profit of firsm at J is also zero. The income of each individual is wage income only, so that AJ is his bhdget line. Each individual's utility is maximized at J, and demand and supply of the product as well as of the labor service are equalized, since the number of individuals is equal to that of firms.[10)]

4. The Stability of a General Equilibrium.

Consider a two commodity exchange economy. The theoretical or mathematical solution of an exchange equilibrium is to find the equilibrium price p of the first commodity (in terms of the second commodity) which satisfies

(21) $E_1(p) = 0$

where E_1 denote the excess demand (demand - supply) for the first good. If p satisfies (21), the excess demand for the second commodity E_2 is also zero, since Walras' law holds, i.s., $p_1 E_1(p) + E_2(p) \equiv 0$. The empirical or practical solution is, then, to approarch to such an equilibrium price through the law of supply and demand, " the commodity having an effective demand greater than its effective offer must rise in price, and the commodity having an effective offer greater than its effective demand must fall in price " (Walras [31], p. 106). To make this solution possible, E_1 has to be a decreasing function of p (the excess demand curve is downward sloping) so that it is negative for any p higher than the equilibrium p which satisfies (21) and positive for any p lower than the equilibrium p, which is the Walrasian stability condition for a single market or two commodity economy.

In the empirical or practical solution, Walras made an implicit assumption that no actual exchange transactions take place at disequilibria where price are being changed according to the law of supply and demand.[11)] In other words, all the contracts made at disequilibrium prices can be cancelled so that recontracting is possible at new price. Walras's process of tâtonnement (a French word meaning groping) is therefore a process of preliminary adjustment in prices (and level of production, etc.) which is made before actual transactions are carried out at equilibrium prices. The reason why we need tâtonnement or recontract assumption is that otherwise the empirical solution is not consistent with the mathematical solution. In figure 7.6, the first commodity is measured horizontally, and the second, vertically. The equilibrium price p, i.e., the mathematical solution of (21), is indicated by the slope of the budget line AB of an individual who has initially AO of the second commodity and demands OD_1, of the first commodity at p. Suppose a purchase of OD_2 of the first commodity is actually carried out at a disequilibrium price higher than p, which is indicated by the slope of AC. If this purchase cannot be cancelled, the demand for the first commodity at the equilibrium price p obtained from the mathematical solution decreases from OD_1 to OD_3, while on the other hand there may also be a decrease in the supply of the first commodity at p from an individual who sold OD_2 at the disequilibrium price. Since there is in general so assurance that the decrease in demand is cancelled by that of the supply, the excess demand is not zero at p. In other words, the original excess demand curve is shifted by the disequilibrium transactions. Without tâtonnement assumption, the empirical solution depends on the route followed by disequilibrium prices and on the extent of transactions made at such prices, and does not coincide with the mathematical solution.[12)]

Let us now proceed to an m commodity economy. The original form of Walrasian tâtonnement is the process of successive adjustment in each single market (Walras[31], pp. 170-172). Suppose the initial set of prices (p_1, ---, p_{m-1}) does not satisfy the condition of general equilibrium

(22) $E_j(p_1, ---, p_{m-1}) = 0, \quad j = 1, ---, m-1,$

where p_j and E_j denote respectively the price of the j-th commodity (in terms of the m th commodity, numéraire) and the excess demand in the market of the j-th commodity. We are, for example, in a situation described by

$$(23)\qquad \begin{aligned} &E_1(\, p_1,\ ---,\ p_{m-1}\,) > 0 \\ &E_2(\, p_1,\ --,\ p_{m-1}\,) < 0 \\ &---------------------- \\ &E_{m-1}(\, p_1,\ ---,\ p_{m-1}\,) > 0 \end{aligned}$$

The price of the first commodity p_1 is now adjusted by reference to its excess demand E_1, and increased in the situation (23) until an equilibrium in the first market is established, i.e.,

$$(24)\qquad E_1(\, p_1',\ p_2,\ ---,\ p_{m-1}\,) = 0.$$

Here Walrasian assumption of the stability in the single market is assumed, so that E_1 is an decreasing function of p_1, or $E_{11} < 0$, if we write the partial derivative of the excess demand function for the j-th commodity with respect to the k-th price by as E_{jk}.

Under the new price system (p_1', p_2, ---, p_{m-1}) the remaining m - 1 markets may or may not be in equilibrium. If the second market is out of equilibrium, again under the assumeption that $E_{22} < 0$, the price of the second commodity is changed from p_2 to p_2' so as to satisfy

$$(25)\qquad E_2(\, p_1',\ p_2',\ p_3,\ ---,\ p_{m-1}\,) = 0.$$

Generally, this will upset the equilibrium is the first market (24). Under the price system (p_1', p_2', p_3, ---, p_{m-1}), then, the price of the third commodity p_3 is adjusted if the third market, where $E_{33} < 0$, is out of equilibrium, upsetting the equilibrium in the second market (25) just established. In this way the last, m - 1 -th msrket, where $E_{m-1,m-1} < 0$, is eventually cleared by changing the price system from (p_1, ---, p_{m-2}', p_{m-1}) into (p_1', ---, p_{m-2}', p_{m-1}') so as to satisfy

$$(26)\qquad E_{m-1}(\, p_1,\ ---,\ p_{m-2}, p_{m-1}\,) = 0.$$

By this time all the markets except the last, which were once cleared successively, have generally been thrown out of their respective equilibria again. Neither the price system we have just arrived at, (p'_1, ---, p'_{m-1}), nor the initial system (p_1, ---, p_{m-1}), is a general equilibrium one. The question is then which of the systems is closer to a general equilibrium that satisfies (22). Walras ([31], p. 172) argued that the former price system is closer to equilibrium than the latter since, for example, $E_1(p'_1, ---, p'_{m-1}) \neq 0$ is closer to 0 than $E_1(p_1, --, p_{m-1}) \neq 0$. The reason for this, according to Walras is that the change from p_1 to p'_1 which established (24) exerted a direct influence that was invariably in the direction of zero excess demand so far as the first commodity is concerned. but the subsequent changes from p_2 to p'_2, ---, p_{m-1} to p'_{m-1}, which jointly moved the first excess demand way from zero ezerted only indirect influences, some in the direction of equilibrium and some in the opposite direction, at least so far as the excess demand for the first commodity is concerned. So up to a certain point they cancelled each other out. Hence, Walras concluded, by repeating the successive adjustment of m - 1 markets along the same lines, i.e., changing prices according to the law of supply and demand, we can move closer and closer to a general equilibrium.

Walras's argument for the convergence of the tâtonnement process to a general equilibrium was, as we just saw merely an argument for the plausibility of such convergence and cannot be considered as a rigorous demonstration of the stability of a general equilibrium. Whether indirect influence of the prices of other commodities on the excess demand of a given commodity cancel each other out will certainly depend on substitutability and complementarity between commodities. For example, indirect influences are not cancelled out and the excess demand of a commodity is increased if the prices of all gross substitutes are raised and the prices of all gross complements are lowered. In addition to the Walrasian stability condition for a single market, i.e., $E_{jj} < 0$ for all j, therefore, some conditions on the cross-effects of prices on excess demands, i.e., on E_{jk}, $j \neq k$, have to be imposed so as to demonstrate the convergence of tâtonnement or the stability of a general equilibrium.

It was Allais ([2], pp. 486 - 489) who first demonstrated the convergence of Walrasian tâtonnement by assuming gross substitutability, i.e., $E_{jk} > 0$ for all $j \neq k$. To see whether the price system moves closer and closer to a general equilibrium, which he assumes to be at least locally unique, Allais defines the distance D of a price system from the equilibrium price system as the sum of the absolute values of the value of excess demand for all commodities, including the numéraire. The convergence of tâtonnement is then demonstrated by showing that this distance D is always decreased by changes in prices that are made in accordance with the law of supply and demand. His demonstrarion may be reformulated in our notation as follows.

The distance to the general equilibrium is defined as

(27) $$D = \sum_j | p_j E_j |$$

where the summation runs from $j = 1$ to $j = m$, and E_j is defined as a function of p_1, ---, p_{m-1} as in (22). In view of Walras'law

(27) $$\sum_j p_j E_j \equiv 0,$$

D can be replaced either by the summation of positive excess demands

(28) $$D^+ = \sum_j p_j \max(0, E_j)$$

or by the summation of negative excess demands

(29) $$D^- = - \sum_j p_j \min(0, E_j)$$

where max (0, E_j) denotes E_j if it is positive and 0 if E_j is negative, and min(0, E_j) denotes E_j if it is negative and 0 if E_j is positive.

From (27), i.e., $D^+ - D^- = 0$, it is evident that

(30) $\quad D = 2\, D^+ = 2\, D^-$

so that whether D is increasing or decreasing can be seen by checking whether D^+ or D^- (whichever is more convenient) is increasing or decreasing.

Suppose E_1 to be positive as in (23) and that p_1 is raised following the law of supply and demand. From (29), we have

(31) $\quad \partial D^- / \partial p_1 < 0$

since $E_{j1} > 0$ for any j such that $E_j < 0$, from gross substitutability. In other words, a change in the price of the first commodity from p_1 to p_1' so as to satisfy (24) decreases the sum of negative excess demands D^- and therefore the distance D to the general equilibrium. Suppose next that $E_2(p_1', p_2, ---, p_{m-1})$ is negative and p_2 is lowered to p_2' so as to satisfy (25). From (28) this time, we have

(32) $\quad \partial D^+ / \partial p_2 > 0$

since $E_{j2} > 0$ for any j such that $E_j > 0$ from gross substitutability. In other words, a decrease in the price of the second commodity from p_2 to p_2' decreases the sum of positive excess demand D^+ and therefore the distance D to the general equilibrium.

Generally, if E_j is positive and p_j is raised, D is decreased which can be seen from the fact that D^- is decreased. Similarly, if E_j is negative and p_j is lowered, again D is decreased, which can be seen from the consideration of the behavior of D^+. Out of the general equilibrium, D remains positive and there exists at least one non-numēraire commodity with non-zero excess demand, so that its price is changing. The distance to general equilibrium alwaus decreases out of equilibrium, and therefore we can move closer and closer to that equilibrium by changing prices according to the law of supply and demand, provided that gross substitutability is assumed. The process terminates only when all the excess demands are zero. i.e., at the general equilibrium.

Though Walras discussed the behavior of the process of successive adjustment, he was not against the consideration of simultaneous adjustment processes in all markets (Uzawa[28]m, Jaffe[15], p. 253). If we assume that adjustments take place not only simultaneously but also continuously, the tâtonnement process that the rate of change of price is governed by excess demand can be described by a set of differential equations

(33) $d\, p_j / d\, t = a_j E_j(\, p_1, ---, p_{m-1} \,), \quad j = i, ---, m - 1,$

where t denote time and the a_j's are positive constants signifying the speed of adjustment in the j-th market. The study of the behavior of the solutions of (33), i.e., prices as functions of t, which was initiated by Samuelson ([26], p. 551), has been extensively carried out by many modern mathematical economists (Arrow and Hahn[3], pp. 263 - 323, Negishi[20], pp. 191 - 206). It is well known that gross substitutability is also a sufficient condition for the convergence of adjustment process like (33) and that the homogeneity of excess demands with respect to prices of all commodities and Walras'law play essential roles in the proof of the stability of equilibrium through the process(33).

5. Capital and money.

1. The time element is ignored by Walras in the theory of production by the assumption of the instantaneous production. In the theory of capitalization and credit, however, it has to play an essential role. There are two alternative ways to introduce the time element into the Walrasian theory of general equilibrium. One is to consider Hicks's temporary equilibrium, as was done by Moriahima ([18], [19], pp. 70 - 81). The other is, like Yasui ([35], pp. 173 - 278) to insist that the Walrasian equilibrium should be a stationary equilibrium. The latter approach can be developed into the theory of balanced growth based on von-Neumann's model

As a matter of fact, Walras himself declared that the he shall consider only the case of a progressive economy in which net investment is positive (Walras[31], p. 269). Walras called an economy static if, whthin a given period of time, no change is allowed in the data like the propensities to save and to consume and the new capital goods play still no part in the production, even though the economy is progressive (Walras[31], pp. 269, 283).13) Since Walras's static economy implies that it remains unchaged only in a single period under consideration, it is similar to the concept of Hicks's temporary equilibrium. It is not easy to interprete Walras's equilibrium as the stationary equilibrium in which consumption, saving, the productivity of capital, etc. remain unchanged through periods, so that the economy's rate of growth is zero.

Yasui insists, however that the equilibrium in Walras's theory of capitalization and creadit should be the equilibrium of stationary state. The reason is that only in such a stationary state the price of the service of capital goods remain unchanged indefinitely into the future, which walras assumed in his equations of the equality between the selling price of consumers' and new capital goods and their costs of production, and equations of the uniformity of the rate of net income for all capital goods proper. Of course, the factors of production can have the same relative values or prices in the future as they have at the present, not only in a stationary state but also in a progressive economy of balanced growth. As wicksell ([34], pp. 226 - 227) pointed out, however, the latter case is inconceivable, " as the sum of natural forces cannot be increased." The stationary state is, therefore, the only remaining possibility. Incidentally, to have an equilibrium of the stationary state possible, we cannot regard the existing quantity of capital goods as data, and cannot consider the supply of their services as functions of prices based only on the given taste of their owners. These are unkowns to be determined so that the value of depreciation of capital goods are equalized to the gross saving. As Yasui pointed out therefore, Walras's original theory of capitalization and credit has to be modified so as to be interpreted as the theory of a stationary economy.[14)]

We have to note that the assumption of the perfect foresight on prices in the future is responsible for the conclusion that the only logical possibility is the stationary state. Without such an assumption, therefore, we can interprete that Walrasian equilibrium is a temporary equilibrium which is based on the given arbitrary subjective expectations of individuals and firms. Following Morishima, we can consider a temporary equilibrium, assuming that expectations on the future prices are static, i.e., the elasticity of expectation is 1. The original Walrasian idea of a progressive economy can be revived, since the quantity of capital goods can be increased through the changing temporary equilibria of successive periods.

Since the original Walrasian system of equations of capitalization and credit is too complicated to describe, let us consider a drastically simplified version of a two good (consumers' and capital goods) two factor (labor and capital) economy.[15)] Two goods are produced from the input of labor service and the service of capital goods under constant returns to scale. Labor is the sole primary factor of production and there is no inventory investment, nor is money.

Let X_1 and X_2 be the level of output of the consumers' and new capital goods, respectively. The aggregate income of laborers and capitalists is

(34) $$Y = w(a_1 X_1 + a_2 X_2) + q(b_1 X_1 + b_2 X_2)$$

where w denotes the rate of wage, q denotes the price of the service of capital goods, a_1 and a_2 are the labor input coefficients in the production of the conumsers' and capital goods and b_1 and b_2 are the capital input coefficients in the production of consuemrs' and capital goods respectively.[16)]

At the equilibrium, there is no profit for entrepreneurs, so that

(35) $$p_1 + w\, a_1 + q\, b_1$$

(36) $$p_2 = w\, a_2 + q\, b_2$$

where p_1 and p_2 are respectively the price of the consumers' and capital goods. Since markets for two goods have to be cleared,

(37) $$D(p_1, p_2, w, q, Y) = X_1$$

(38) $$H = X_2$$

where D denotes the demand for consumers' goods and H stands for the demand for new capital goods. Factor markets have also to be cleared so that

(39) $$a_1 X_1 + a_2 X_2 = L$$

(40) $$b_1 X_1 + b_2 X_2 = K$$

where L and K are respectively the given existing lavor force and the given existing stock of capital goods. Since there is no money, suppose capitalists own capital goods and lend them to entrepreneurs or sell the service of capital goods to them. If gross saving is defined as the excess of income over consumption, then, capitalists save in kind or purchase new capital goods with saving, so that

(41) $p_2 H = S(p_1, p_2, w, q, Y)$

where S denotes the aggregate gross saving.

If the equation (34) - (41) are interpreted as the description of a temporary equilibrium, there are 8 equations to determine 7 uncnowns, Y w, q, X_1, X_2, p_2, H, since we can choose the consumers' goods as numeraire so that $p_1 = 1$. Eight equations are not indepemdent, however, and one of equations can be derived from other equations and Walras' law

(42) $Y \equiv p_1 D + S.$

In the determination of consumption and saving, capitalists assume that goods and service of factors have the same prices in the future as they have at the present monent, and the difference between resultant gross saving and the value of the depreciation of capital goods, i.e., the net saving can be either positive or negative. If it is positive, we have the case of a progressive economy which Walras wished to consider. The capital stock K is larger in the next period than in the current period so that temporary equilibrium prices the former are in general different from those in the latter, even though capitalists in the current period expected uncahnged prices through periods.

The assumption of the saving in kind is not necessary if we follow Walras to introduce a commodity E consisting of perpetual net income of a unit of numeraire, the price of which is the inverse of the rate of perpetual net income or the rate of interest i (Walras[31], p. 274).

If this commodity is sold by intrepreneurs or firms wishing to buy new capital goods, and is purchased by capitalists wishing to save, the clearance of the market of this commodity through changes in i implies that aggregate gross saving = " aggregate excess of income over consumption = aggregate demand for (E) X price of (E) = aggregate demand for new capital goods X price of capital goods " (Walras[31], p. 21) Therefore,

(41)' $p_2 H = S(p_1 w, i, Y)$

instead of (41) since capitalists are now concerned, not with p_2 and q, but with i in the determination of consumption and saving. Similarly, (37) may be replaced by

(37)' $D(p_1, w, i, Y) = X_1.$

At equilibrium, the rate of net income for capital goods has to be equalized for the rate of net income for the commodity E,

(43) $(q/p_2) - d = i$

where d denotes the technically given rate of depreciation of capital goods. Since the introduction of a new unknown i is matched by the introduction of additional equation (43), we still have the equality between the number of unknowns and that of equations.

Entrepreneurs and capitalists fail to expect correctly the prices in the future in a progressive economy, since changes in prices are induced by changes in K in a series of successive temporary equilibria. The expectation of unchanged prices can be correct only in the case of stationary state in which K remains through periods. The condition for the stationary state is that the aggregate gross saving is equal to the value of depreciation of capital goods, or

(44) $H = d K$

in view of (41) or (41)'. Since the number of equations is increased by the addition of (44), then, we should have one more unknown introduced. The existing stock of capital goods K is, therefore, no longer an arbitrary given quantity, and has to be solved jointly with other unknowns from equations of general equilibrium. For example we have 9 unkowns, K, i, Y, w, q, X_1 X_2, p_2 and H to be solved from any 9 equations from 10 equations (34) - (36), (37)', (38) - (40), (41)', (43), and (44), since $p_1 = 1$ and one of equations is not independent in view of (42).

Two alternative interpretations of Walras's theory of capitalization and credit, that is, temporary equilibrium and stationary state, corresponds to two methods of economic dynamics in the modern economic theory, that is, the temporary equilibrium method and the growth equilibrium method, distinguished in Hicks's Capital and Growth (Hicks [13], p. 28). Also it is well known that Walras's theory of capital gives the micro economic foundation to the so-called neo-classical macro growth theory developed by Solow, Swan, Meade and Uzawa. The criticism given by Cambridge, England against the macro production function and macro theory of marginal productivity is, however, not effective to the original Walrasian general equilibrium theory of capital, though it made cirtain points to the neo-classical macro growth theory.

2. In his theory of circulation and money, Walras tried to develop the general equilibrium theory of a cash-balance equation from the point of view of the marginal utility theory, though Patinkin ([25], pp. 541 - 572) criticezed Walras that Walras cannot be credited with having presented a cash-balance theory, which is different from a cash-balance equation itself. Walras introduced money in his thoery of general equilibrium as circulating capital rendering a service of storage.

People demand for money, therefore, for the reduction of costs of transaction and inconveniences in the process of circulating. This has to require, strictly speaking, the existence of a lack of synchronization between the receipt of income and its outlay, and the existence of some uncertainties in the process of transactions and/or some imperfectness in credit markets. While cirticizing Walras's mechanical application of marginal utility theory, Patinkin ([25], chapters V - VII) developed Walrasian theories of money by explicitly indtroducing such imperfections. Such attempts seem to succeed, however, to the extent that they are in conflict with basic structure of Walras's general equilibrium theories. It is very diffeicult to introduce money into the original Walrasian system of general equilibria in a satisfactory way.

The reason is that Walras's theory of money is heavily handicapped by the place it occupies in the system of general equilibria. Walras insisted that complicated phenomena can be studied only if the rule of proceeding from the simple to the complex is always observed. He first decomposes a complicated economy of the real world into several fundamental components like consumer-traders, entrepreneurs, consumers' goods, factors of production, newly produced capital goods, and money. A very simple model of a pure exchange economy is then composed from a very limited number of such components, i.e., individual consumer-traders and consumers' goods, where the existence of all other components are simply disregarded. Travel from this simple model to the complex proceeds by adding one by one those components so far excluded, i.e., entrepreneurs and factors of production first in the theory of production, then newly produced capital goods in the theory of capitalization, and finally money in the theory of circulation and money.

From our standpoint we must emphasize that all exchanges have to be non-monetary (i.e., direct exchanges of goods for goods) in all the Walrasian theories of exchange, production and capitalization and credit, since money has not yet been introduced. Relative prices (including the rate of interest) and hence consumption and production activities are determined in non-monetary real models without using money, while the role of the model of circulation and money lies only in the determination of the level of absolute prices by the use of money (Morishima[19], pp. 170 - 184, Negishi[21], pp. 9 - 35). Thus Walrasian economics is completely dichotomized between non-monetary real theories and monetary theory, in the sense that all non monetary real variables are determined in the former and money is neutral, i.e., it does not matter for the determination of such variables.. " That being the case, the equation of monetary circulation, when money is not a commodity, comes very close, in reality, to falling outside the system of equations of (general) economic equilibrium " (Walras[31], pp. 326 - 327). This dichotomy is, of course designed to show the fundamental significance of non-monetary, real mechanism of the economy, which underlie the behavior of a modern monetary economy.

In each of his non-monetary theories Walras tried to show the establishment of a general equilibrium in its corresponding self-compact closed model. General equilibrium is of course a state of the economy in which not only each individual consumer-trader (entrepreneur) achieves the maximum obtainable satisfaction (profit) under given conditions but also demand and supply are equalized in all markets. In a large economy, how can we make such a situation possible without introducing money ? Even in the most simple case of an exchange economy, it seems in general almost impossible to satisfy all individual traders by barter exchanges, unless mutual coincidence of wants accidentally prevails everywhere. Walras ingeneously solved this difficult problem by his tâtonnement, a preliminary process of price (and quantity) adjustment which precedes actual exchange transactions and/or the making of effective contracts.

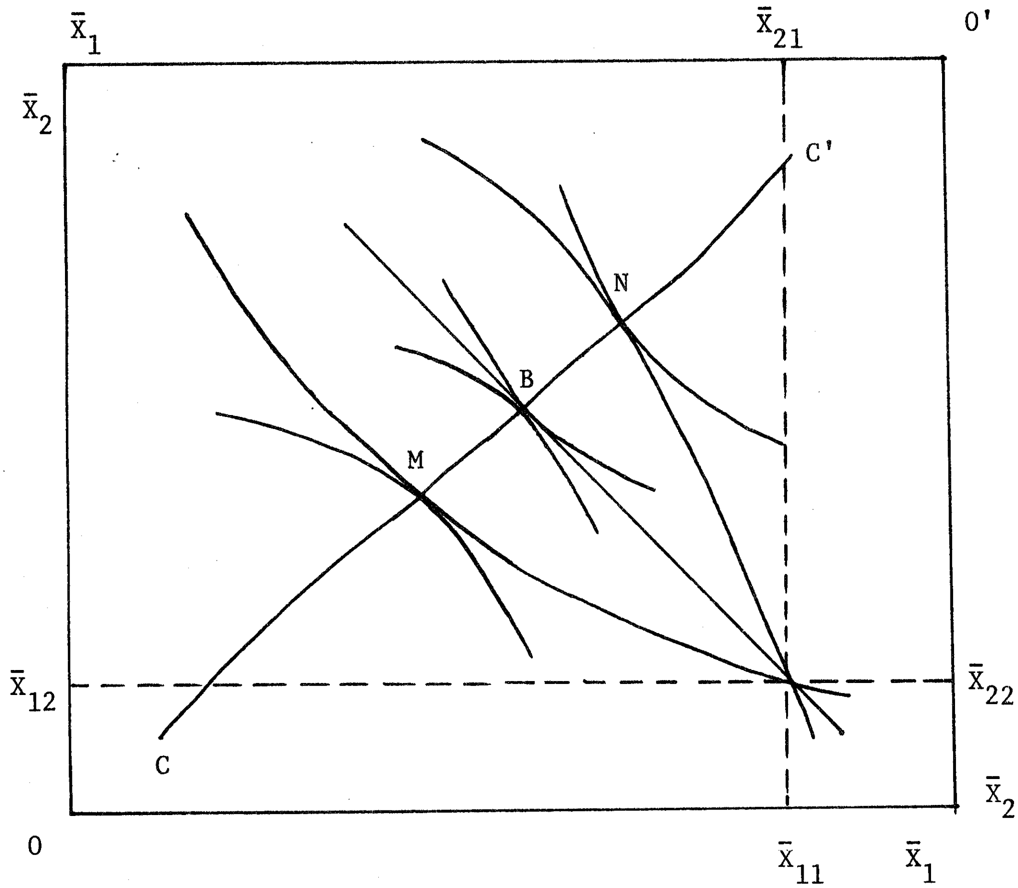
Prices change in the process of tâtonnement of a competitive economy and it is generally impossible for a single trader to purchase or sell whatever amount he wishes at going prices. Nevertheless, each trader behaves on the assumption that prices are unchanged and that unlimited quantities of demand and supply can be realized at the current prices. This conjectures is justified by the very fact that no exchange transactions are made and no trade contracts are in effect during the tâtonnement, until general qequilibrium is established where prices are no longer changed, and every trader can purchase and sell exactly the amount he wishes at the going prices. The idea of tâtonnement was clearly suggested to Walras from the observation of how business is done in some well organized markets in the real world, like the stock exchanges, commercial markets, grain markets and fish markets. Tâtonnement is therefore not entirely unrealistic as a model of adjustment in such special markets.

However, it is sertainly very unrealistic to apply such a model of special markets to the whole economy of the real world, since preliminary adjustments are usually not made before exchange transactions and effective contracts take place, even in markets where competition, though not so well organized, functions fairly satisfactorily. In such a monetary economy of real world, where of course at least some exchange transactions actually take place before general equilibrium is established, even a competitive trader without power to control prices has to expect price changes and to try to sell when the price is high and to buy when the price is low though he may not always succeed in doing so. This leads to the separation of sales and purchases, a separation which is made possible only by the use of money as the unit of account, the medium of exchange and the store of value. Other commodities cannot be used as the medium of exchange, since their prices are in general changing in disequilibrium. In Walrasian non-monetary real models where the tâtonnement assumption is make, on the other hand, sales and purchases are synchronized when general equilibrium is established, so that there is no need for money, since only relative prices matter and any commodity can be used as numeraire. Even if sales and surchases are not synchronized in a single market period after the general equilibrium is established, furthermore, there is no reason why the role of medium of exchange should be exclusively assigned to a single item called money. Since equilibrium prices are already fixed and unchanged, almost any non-perishable commodity can be used if a medium of exchange is necessary.

Walras considered tâtonnement even in his final model of the general equilibrium of economy, i.e., that of circulation and money. Since disequilibrium transactions are thus excluded and there is no uncertainty, there is no room here for money as a store of value. We have to assume therefore that people demand for money only for the sake of convenience in transactions. Since all actual transactions are carried out at general equilibrium after the preliminary tâtonnement is over, however, this rationale for the demand for money is not at all convincing. The only role left for money is to determine its own price, i.e., the general level of absolute prices.

While money plays a limited role in Walras's tâtonnement economics, it is interesting to see that introduction of money is necessary in the so-called non-tatonnement models. When exchange transactions are carried out at disequilibrium, only the traders on the short side (i.e., suppliers if there is excess demand, demanders if there is excess supply) can realize their plan of demand or supply fully. This short-side principle is, however, not consistent withthe rule of voluntary exchange or no overfulfillment that no traders are forced to buy or sell any commodity more than they wish. Money can be, however, exempt from this rule, since, after the sale of a commodity and before the purchase of the other, people oftem have more money than they wanted to keep ultimately. The introduction of money is necessary, therefore, to reconcile the short-side principle with the rule of voluntary exchange (Negishi[21], p. 22 - 23). Money can play its essential role only in a non-Walrasian world where transactions take place out of equilibrium.

paradoxically, the significance of Walras's theory of money lies in its demonstration that Walrasian theory of general equilibrium is not almighty since it is based on the tâtonnement assumption. By absorbing various non-Walrasian theories, Walrasian economics developed into the modern neo-classical economic theory. The dichotomy between real and monetary aspects remains, however, intact through the development of Walrasian economics into neo-classical economics. In other words, the role of money in exchange transactions is not properly recognized in neo-Walrasian economics. Various attempts are being make to correct this defect of modern economic theory. We may call such attempts non-Walrasian economic theories. Incidentally, theory of exchange and money considered by C. Menger, who shares the fame of founding fathers of modern economic theory with Walras, is in sharp contrast with that of Walras, as we shall see in the next chapter. Non-Walrasian economics begins with Menger.



$\bar{X}_1$
$\bar{X}_{21}$
0'
$\bar{X}_2$
C'
N
B
M
$\bar{X}_{12}$
$\bar{X}_{22}$
C
$\bar{X}_2$
0
$\bar{X}_{11}$
$\bar{X}_1$

Figure 7. 1

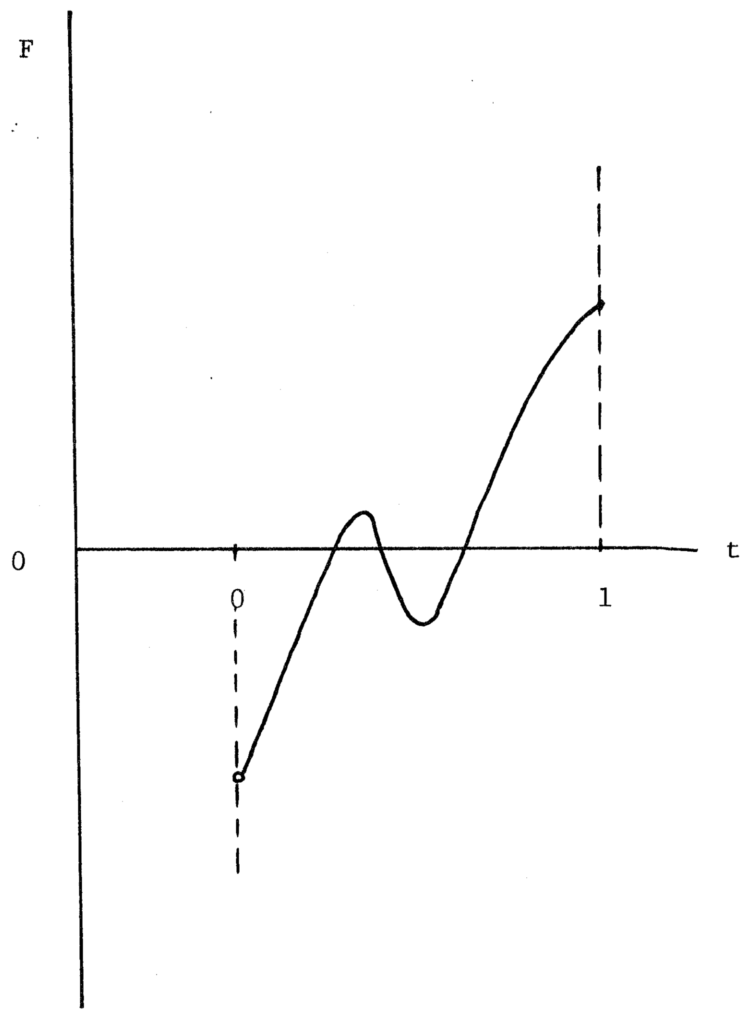


Figure 7. 2

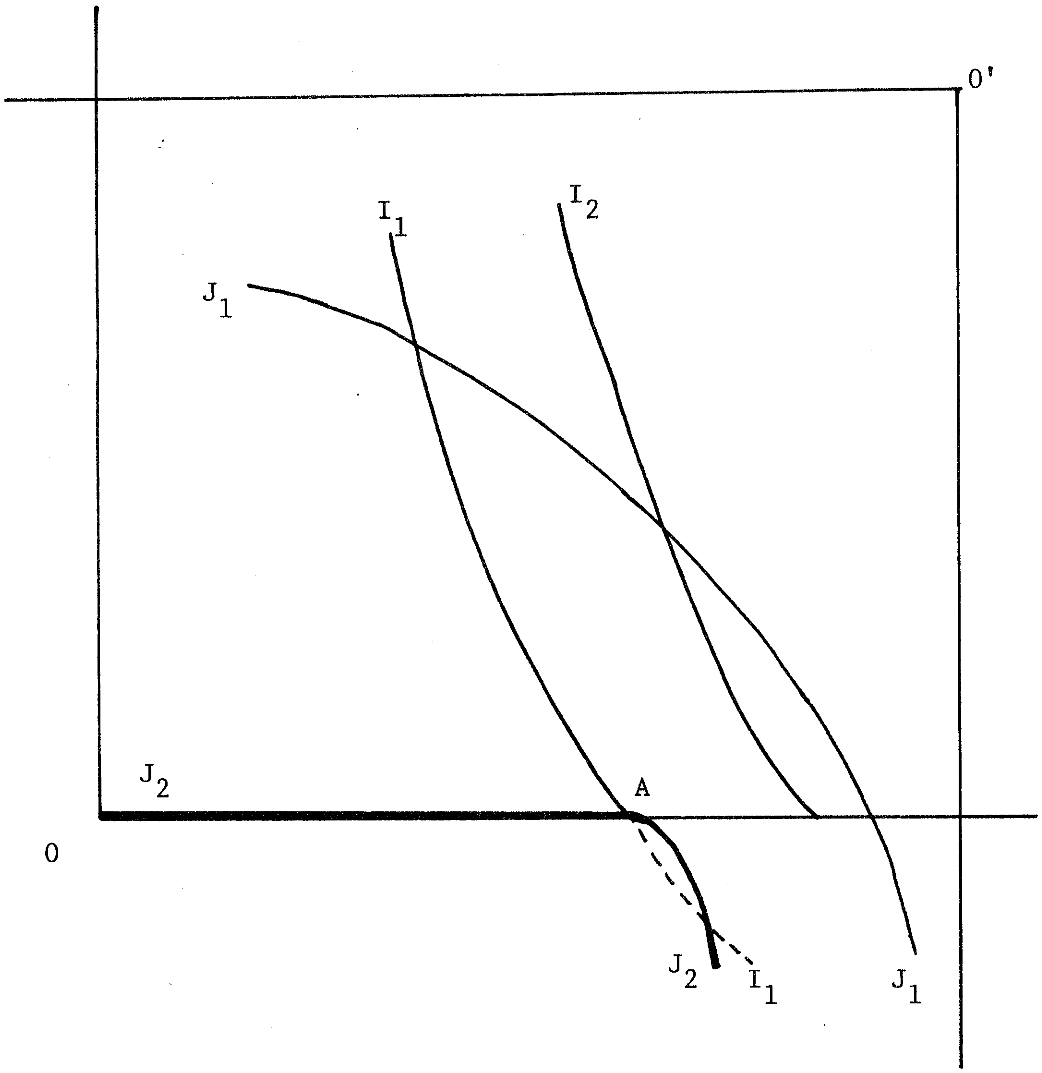


Figure 7. 3

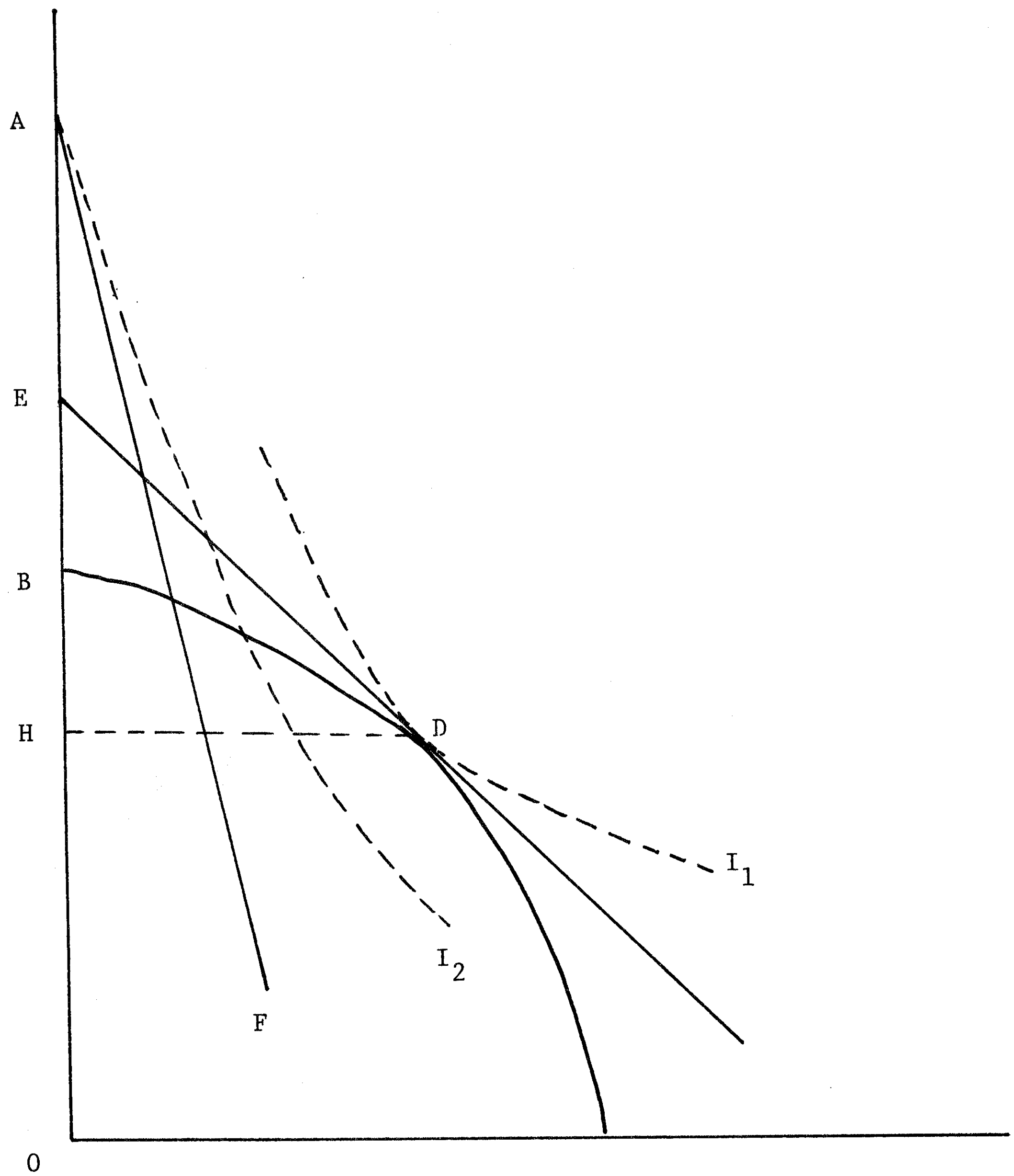


Figure 7.4

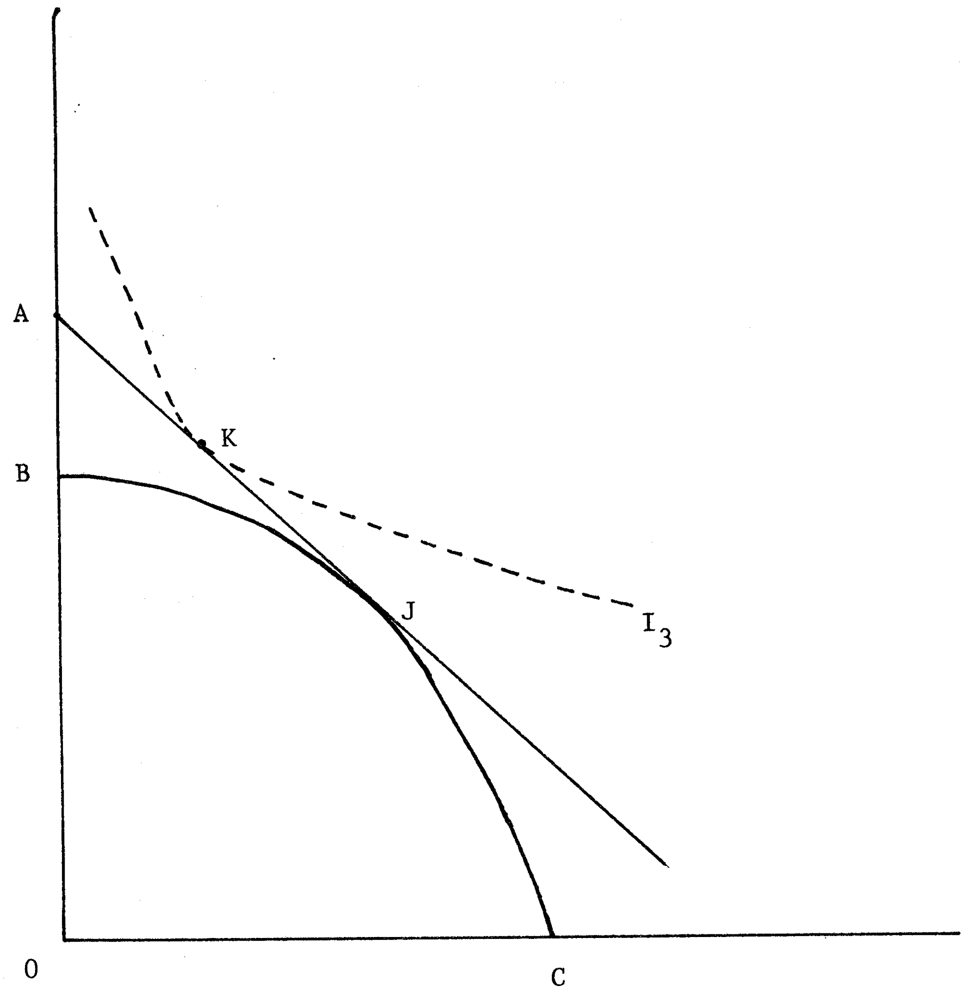


Figure 7. 5

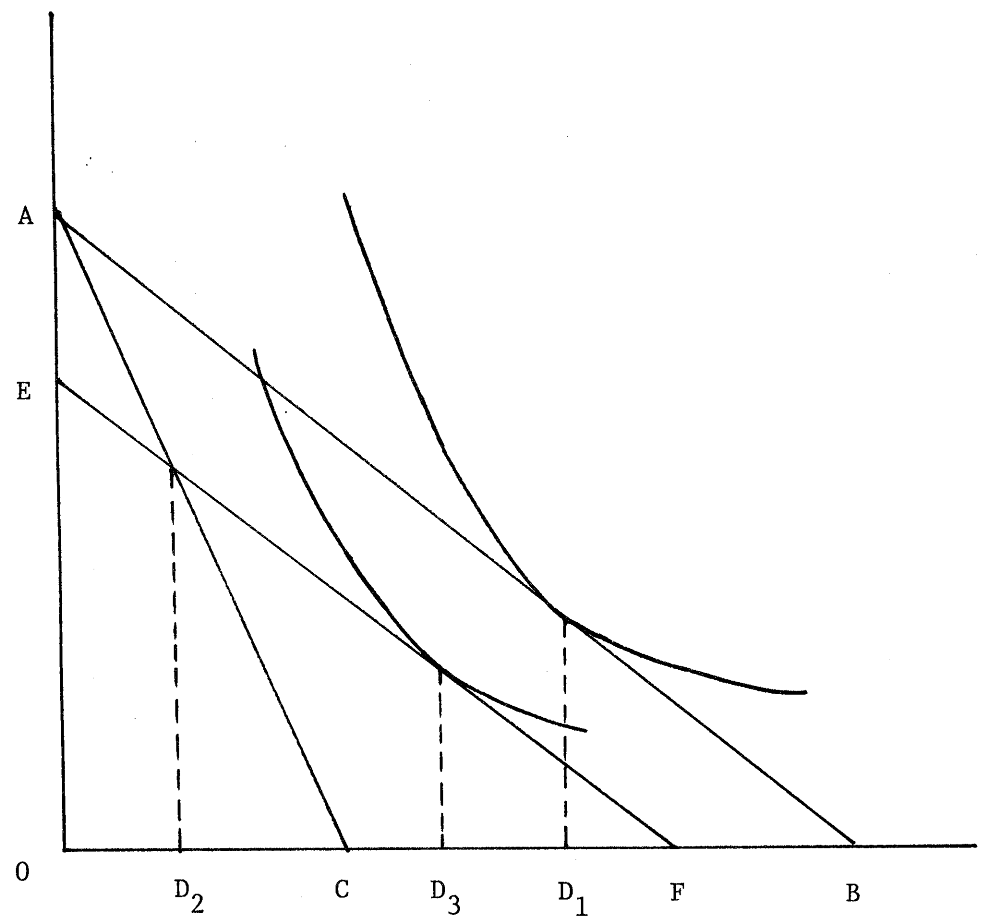


Figure 7. 6

Footnotes

1) Cournot did not use the term " oligopoly." According to Schumpeter ([27], p. 305), it was first used by Thomas More in his Utopia. See, however, Friedman [9], pp. 20 - 21, also. I owe this footnote to Mr. Jun-ishi Tominaga of University of Tokyo.

2) See Bertran and Fellner[8], pp. 77 - 86. Edgeworth[7], pp. 111 - 142 and Fellner[8], pp. 79 - 82 argued that price oscillation appears if cost functions are different for different firms and/or there is upper limit for the capacity of firms.

3) For conjectual veriation and leader-follower problem, see Hicks[11] and Fellner[8], pp. 71 - 72, 98 - 119. Ono[24] is an interesting recent contribution to this area.

4) Without the concept of Pareto optimality, it is impossible to demonstrate the possibility of optimal resource allocation through competition. See, however, Jaffe[15], pp. 326 - 335. See also Walker[29].

5) Jaffé insisted that Walras proceeded from the general equilibrium to the marginal utility, instead of climing up from marginal utility to the general euilibrium. See Jaffé[15], pp. 25, 312 - 313.

6) For the comparison of Walras's concept of capital with that of classical and Austrian schools, see Eagly[6], pp. 7 - 8, 127 - 131. See also Yasui[35], pp. 173 - 278.

7) For the life of Walras and Walras's economics in general, see Jaffé [15], Morishima[19], Walras[32], pp. 1 - 16, and Yasui[35].

8) For the history of the proofs of the existence of a general equilibrium, see Weintraub[33], pp. 59 - 107, and Arrow and Hahn[3], pp. 51, 127 - 128.

9) See Negishi[20], pp. 12 - 15. For the proofs based on the same

approach of a general case involving production, see Negishi[20], pp. 15 - 25, and Arrow and Hahn[3], pp. 107 - 128.

10) For the convexifying the economy in the case of a large economy, see Arrow and Hahn[3], pp. 188 - 195, and Hildenbrand and Kirman[14], pp. 165 - 169.

11) For the tâtonnement assumption, see Jaffé[15], pp. 221 - 266, especially p. 247, Newman[23], p. 102, and Patinkin[25], pp. 531 - 540, especially p. 533.

12) See Hicks[12], pp. 127 - 129, and Kaldor[16]. Japanese literature like Morishima[17] and Yasui[35], pp. 353 - 472, had also emphasized this possibility.

13) The same French word " statique " in pp. 244 and 260 in Walras[30] was translated into " stationary " in p. 269 and " static " in p. 283 in Walras[31].

14) As early as in 1986, Yasui pointed it out See Yasui[35], p. 248, and also Garegnani[10], part 2, Chapter 2. The fact that the stock of the existing capital goods cannot be arbitrarily given is, however, not the defect of Walrasian theory of capital. It is also the case with the classical theory of the stationary economy.

15) This is a simplified version of the model given in pp. 108 - 112 of Morishima[19]. We cannot, however, agree with the interpretation of the model in pp. 112 - 122 of Morishima[19]. See Negishi[22].

16) Input coefficients are functions of factor prices.

References

[1] Arrow, K.J., " An Extension of the Basic Theorems of Classical Welfare Economics," J.Neyman ed., Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, 1951 pp. 507 - 532.

[2] Allais M., Traite d'economie pure, 2, Paris: Imprimerie Nationale, 1943.

[3] Arrow, K.J., and F.H.Hahn, General Competitive Analysis. Holden Day, 1971.

[4] Bertrand, J., Review of Walras: Theorie mathematique de la richesse sociale and Cournot: Recherches sur les principes mathematiques de la theorie des richesses, Journal des Savants, 1883, pp. 499 - 503.

[5] Cournot, A., Reseraches into the Mathematical Principles of the Theory of Wealth, N.T.Bacon tr. Macmillan, 1897.

[6] Eagly, R.V., The Structure of Classical Economic Theory, Oxford University Press, 1974.

[7] Edgeworth, F.Y., Papers Relating to Political Economy, 1, Macmillan, 1925.

[8] Fellner, W., Competition Among the Few, Kelley, 1965.

[9] Friedman, J., Oligopoly Theory, Cambridge University Press, 1983.

[10] Garegnani, P., Il Capitale nelle Teorie della Distribuzione, 1960

[11] Hicks, J.R., Annual Survey of Economic Theory : The Theory of Monopoly " Econometrica, 3(1935), pp. 1 - 20.

[12] Hicks, J R., Value and Capital, Oxford University Press, 1946.

[13] Hicks, J.R., Capital and Growth, Oxford University Press, 1965.

[14] Hildernbrand, W., and A.P.Kirman, Introduction to Equilibrium Analysis, North-Holland, 1976.

[15] Jaffé, W., Essay on Walras, D.A.Walker ed., Cambridge University Press, 1983.

[16] Kaldor, N., " A Classificatory Note on the Determinateness of Equilibrium," Review of Economic Studies, 1(1933-4), pp. 122 - 136.

[17] Morishima, M., Dogakuteki Keizai Riron (Dynamic Economic Theory), Kobundo, 1950.

[18] Morishima, M., " Existence of Solution to the Walrasian System of Capital Formation and Credit," Zeitschrift für Nationalokonomie, 20(1960), pp. 238 - 243.

[19] Morishima, M., Walras'Economics, Cambridge University Press, 1977.

[20] Negishi, T., General Equilibrium Theory and International Trade, North-Holland, 1972.

[21] Negishi, T., Microeconomic Foundations of Keynesian Macroeconomics, Noth-Holland, 1979.

[22] Negishi, T.,Review of Morishima: Walras' Economics, The Economic Review (Hitotsubashi University), 31(1980), pp. 89 - 91.

[23] Newman, P., The Theory of Exchange, Pretice Hall, 1965.

[24] Ono, Y., " The Equilibrium of Duopoly in a Market of Homogereous Goods," Economica, 45(1978), pp. 287 - 295.

[25] Patinkin, D., Money, Interest and Prices, Harper and Row, 1965.

[26] Samuelson, P.A., The Collected Scientific Paters, J.E.Stiglitz ed., M I T.Press, 1966.

[27] Schumpeter, J.A., History of Economic Analysis, Oxford University Press, 1954.

[28] Uzawa, H., " Walras' Tatonnement in the Theory of Exchange," Review of Economic Studies, 27(1960), pp. 182 - 194.

[29] Walker, D.A., " Walras and His Critics on the Maximum Utility of New Capital Goods, " History of Political Economy, 16(1984), pp. 529 - 544

[30] Walras, L., Eléments d'économie politique pure, R.Pichon and R. Durand-Auzias, 1926.

[31] Walras, L., Elements of Pure Economics, W. Jaffé tr. Irwin, 1954.

[32] Walras, L., Correspondence and Related Paters, 1, W. Jaffe ed., North Holland, 1965.

[33] Weintraub, E.R., General Equilibrium Analysis, Cambridge University Press, 1985.

[34] Wicksell, K., Lectures on Political Economy, 1, E. Classen tr., Kelly, 1967.

[35] Yasui, T., Walras o megutte (Essay on Walras), Sobunsha, 1970.