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Abstract

We shall propose a new method for estimating the volatility parameters of security prices, which is an improvement of the estimation method by Parkinson (1980). We assume that the security prices follow the geometric Brownian Motion. However, contrary to the setting of Parkinson (1980), the geometric Brownian Motion can have drift terms. We show that the efficiency of our estimator is about 10 in comparison with the standard sample variance estimator. Since the efficiency of the estimator by Parkinson (1980) is about 4.91, our estimation method may considerably improve the estimation methods already known in financial economics.

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1. **Introduction**

In recent financial economics it has been usually assumed that security prices follow the diffusion stochastic processes. In particular, the geometric Brownian Motion process has been often used in the theoretical as well as the empirical studies. See the Back=Scholes option pricing model explained in textbooks such as Cox and Rubinstein (1985), for instance. Also there has been a growing interest in estimating the parameter values of the stochastic processes governing the historical security price movements. Especially, since the option pricing formulae are usually nonlinear functions of the volatility parameter in the original geometric Brownian Motion process, some special attention has been paid to estimate its numerical value.

In this respect, several methods have been proposed to estimate the volatility parameters in dynamic security price models. Parkinson (1980) proposed the extreme value method for estimating the variance parameter of the rate of returns. It has been known that the efficiency of his estimator is surprisingly high and about 4.91 in comparison with the standard sample variance estimator. Since then several methods have been proposed for estimating the volatility parameters including Garman and Klass (1980). See Cox and Rubinstein (1985) for some other estimation methods known in finance literatures.

In this paper we shall propose a new estimation method for estimating the volatility parameters, which is an improvement of the estimation method by Parkinson (1980). Contrary to the setting of Parkinson (1980), we assume that the geometric Brownian Motion can have drift terms. We shall show that the efficiency of our estimator is about 10 in comparison with the standard sample variance estimator. Since the efficiency of the estimator by

Parkinson (1980) is about 4.91, our estimation method may considerably improve the estimation methods already known in financial economics.

2. A New Estimation Method

Let the security price S(t) at t follow a geometric Brownian Motion process

(2.1) $$dS = \mu S dt + \sigma S dW ,$$

where W(t) stands for the standard Brownian Motion process. Then the transformation X(t) = ln(S(t)) follows a Brownian Motion with the drift parameter $\mu' = \mu - \sigma^2/2$ and variance parameter σ^2 by Ito's lemma. We allow that the drift parameter μ' is not necessarily zero. We assume that we have n observations of {S(t)} in n intervals. Define the rate of return in the i-th interval as $d_i = X_i(t) - X_{i-1}(t)$ (i = 1,...,n). The classical sample variance estimator of the parameter σ^2 is given by

(2.2) $$\hat{\sigma}^2(c) = \frac{1}{T(n-1)}\sum_{i=1}^{n} (d_i - \bar{d})^2 ,$$

where $\bar{d} = (1/n)\sum_{i=1}^{n} d_i$ and T is the length of each interval. On the other hand, Parkinson (1980) proposed to use the range of X(t) in n intervals. Let the range of X(t) in the i-th interval $\mathbf{I}_i$ (i=1,...,n) be

(2.3) $$\ell_i = \max_{t \in \mathbf{I}_i} X(t) - \min_{t \in \mathbf{I}_i} X(t) .$$

Then the unbiased estimator of σ^2 proposed by Parkinson (1980) is

(2.4) $$\hat{\sigma}^2(p) = \frac{1}{(4\ln 2)Tn}\sum_{i=1}^{n} \ell_i^2 .$$

When the drift parameter $\mu' = 0$, it has been known that

$$(2.5) \qquad \frac{\mathrm{var}\{\hat{\sigma}^2(c)\}}{\mathrm{var}\{\hat{\sigma}^2(p)\}} = \frac{32(\ln 2)^2}{9\zeta(3)-16(\ln 2)^2},$$

where $\zeta(p)$ is the Riemann's ζ function and $\zeta(3) \simeq 1.20206$. It implies that the efficiency of $\hat{\sigma}^2(p)$ against $\hat{\sigma}^2(c)$ is about 4.91. However, it is important to note that this result is valid only when there is no drift term in the underlying stochastic process. This is because Parkinson (1980) used the density function of the range in every interval originally derived by Feller (1951), which in turn does depend on the assumption of no drift terms.

Now we introduce the transformation

$$(2.6) \qquad Y(t) = \frac{1}{\sqrt{T}}\{X(t) - \frac{t}{T} X(T)\}$$

for $0 \leqq t \leqq T$. Then if we take $X(0) = 0$ for normalization, we have $\mathbf{E}[Y(t)] = 0$ and

$$(2.7) \qquad \mathbf{E}[Y(s)Y(t)] = \sigma^2(\frac{s}{T})(1 - \frac{t}{T})$$

for $0 \leqq s \leqq t \leqq T$. Since $\{Y(t)\}$ is the Brownian Bridge process with $Y(0) = 0$ (a.s.), $Y(s)$ is independent of the drift terms in the original stochastic process $\{X(t)\}$. Let the range of $\{Y(t)\}$ in the i-th interval $\mathbf{I}_i$ be

$$(2.8) \qquad R_i = \max_{t \varepsilon \mathbf{I}_i} Y(t) - \min_{t \varepsilon \mathbf{I}_i} Y(t),$$

which is called the adjusted range by Feller (1951). Then our basic intuition can be simply explained by Figure 1.(See Figure 1.) The original range for $\{X(t)\}$ corresponds to the difference of $X(t_3)$ and $X(t_1)$ while the adjusted range corresponds to the difference of $X(t_2)$ and $X(t_4)$. To estimate the variance parameter σ^2 we should use the adjusted range instead of the original range because the former is free from the drift terms and it is expected to be more stable than the latter. Then we have the density function of R:

$$(2.9) \qquad f(r) = re''(r) + \sum_{k=2}^{\infty} \{2k(k-1)[e'((k-1)r) - e'(kr)] + (k-1)^2 re''((k-1)r) + k^2 re''(kr)\}.$$

where $e(x) = \exp(-2x^2/T)$. This equation has been obtained by Feller (1951), but for the sake of completeness we give its simpler derivation in Appendix. From this density we have the simple moment formula of R in the following.

Lemma: For $p \geqq 2$,

$$(2.10) \qquad \mathbf{E}(R^p) = 2\sigma^p(p-1)\Gamma(\frac{p+2}{2})(\frac{T}{2})^{p/2}\zeta(p) ,$$

where $\zeta(p)$ is the Riemann's ζ function. For p=1,

$$(2.11) \qquad \mathbf{E}(R) = \sigma\sqrt{\frac{T\pi}{2}} .$$

The proof of Lemma is given in Appendix. Using this lemma for p=2 and $\zeta(2) = \pi^2/6$, we have

$$(2.12) \qquad \mathbf{E}(R^2) = \sigma^2 T \frac{\pi^2}{6} .$$

Hence when we have n observations R_i (i=1,...,n) in n different intervals, we define an unbiased estimator of σ^2 by

$$(2.13) \qquad \hat{\sigma}^2(k) = \frac{1}{nT}\left(\frac{6}{\pi^2}\right)\sum_{i=1}^{n} R_i^2 .$$

Next we compute the efficiency of this estimator in comparison with the classical estimator $\hat{\sigma}^2(c)$. Using Lemma for p=4 and $\zeta(4) = \pi^4/90$, we have

$$(2.14) \qquad \mathrm{Var}(\hat{\sigma}^2(k)) = \frac{\sigma^4}{nT^2}\left(\frac{6}{\pi^2}\right)^2 \mathbf{E}\{R_i^4 - (\mathbf{E}(R_i^2))^2\}$$

$$= \frac{\sigma^4}{n}\left(\frac{6}{\pi^2}\right)^2\left\{\frac{\pi^4}{30} - \left(\frac{\pi^2}{6}\right)^2\right\} = \frac{\sigma^4}{5n} .$$

Since it is well-known that $\mathrm{Var}(\hat{\sigma}^2) = 2\sigma^4/(n-1)$, we have

$$(2.15) \qquad \frac{\mathrm{var}\{\hat{\sigma}^2(c)\}}{\mathrm{var}\{\hat{\sigma}^2(k)\}} = 10\,\frac{n}{n-1} .$$

Hence we summarize the main result in this paper.

Proposition: The efficiency of the unbiased estimator $\hat{\sigma}^2(k)$ given by (2.13) of σ^2 is 10n/(n-1) against the classical estimator $\hat{\sigma}^2(c)$.

When n is large, the efficiency of $\hat{\sigma}^2(k)$ against the classical estimator $\hat{\sigma}^2(c)$ is about 10. In practice we are sometimes interested in the unbiased estimation of the standard deviation parameter σ instead of the variance parameter σ^2 in the geometric Brownian Motion. From the above lemma it is apparent that an unbiased estimator of σ is given by

$$(2.13) \qquad \hat{\sigma}(k) = \frac{1}{n\sqrt{T}}\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n} R_i .$$

3. **Some Concluding Remarks**

In this paper we propose a new estimation method based on the adjusted range, which may improve the method proposed by Parkinson (1980). Our estimation method allows the presence of the drift terms in stochastic processes. Also we have shown that the efficiency of our estimator is about 10 in comparison with the classical estimation method and it is more efficient than any other estimator already known in financial economics. Since we have daily observations on a large number of securities in financial markets, our method proposed in this paper may give a new useful estimation procedure on their volatility parameters in the underlying stochastic processes.

4. **Appendix**

In this Appendix we give some details of our derivations in Section 2. Although the results in the following may not be new in probability theory, I suspect that many researchers in finance are not familiar with them.

(i) **Density Function of R:**

Since Feller (1951) did not give the detailed derivations of the results, we first derive the density function of R. Using Theorem 2 in Page 286 of Gikhman and Skorokhod (1969), we start with the joint density function of the three random variables

$$\left(\min_{0 \leqq t \leqq T} X(t),\ \max_{0 \leqq t \leqq T} X(t),\ X(T) \right),$$

which is given by

$$\text{(A.1)} \qquad P\{a < \min_{0 \leqq t \leqq T} X(t) \leqq \max_{0 \leqq t \leqq T} X(t) < b,\ X(T) \in [c,\ d]\}$$

$$= \frac{1}{\sqrt{2\pi T}} \sum_{k=-\infty}^{\infty} \int_c^d [\exp\{-\frac{1}{2T}(x + 2k(b-a))^2\} - \exp\{-\frac{1}{2T}(x - 2b + 2k(b-a))\}]dx \ .$$

Let $e(x) = \exp(-2x^2/T)$. Then conditioning the terminal value of $X(t)$ by $X(T) = 0$, we have

$$(A.2) \qquad G(a,b) = P\{a < \min_{0 \leq t \leq T} X(t) \leq \max_{0 \leq t \leq T} X(t) < b \mid X(T) = 0\}$$

$$= \lim_{\Delta \to 0} \frac{P\{a < \min_{0 \leq t \leq T} X(t) \leq \max_{0 \leq t \leq T} X(t) < b,\ X(T) \varepsilon [0,\Delta]\}}{P\{X(T) \varepsilon [0,\Delta]\}}$$

$$= \sum_{k=-\infty}^{\infty} \{e(k(b-a)) - e(-b + k(b-a))\}$$

$$= 1 + e(b-a) - \sum_{k=1}^{\infty} \{e(-b + k(b-a)) + e(a + k(b-a))$$

$$- e(k(b-a)) - e((k+1)(b-a))\} \ ,$$

which is equivalent to (4.1) in Feller (1951). By transforming $S = \min X(t)$ and $R = \max X(t) - \min X(t)$, the conditional density function of R given $X(T) = 0$ is given by

$$(A.3) \qquad f(r) = \int_0^r \left[\frac{\partial^2 G(-a,b)}{\partial a \partial b} \right]_{\substack{a=s \\ b=r-s}} ds \quad .$$

We differentiate and integrate each term in the parenthesis of (A.3). For instance, we use the relations:

(A.4) $$\int_0^r \left[\frac{\partial^2 e(k(b+a))}{\partial a \partial b} \right]_{\substack{a=s \\ b=r-s}} ds = \int_0^r \frac{4k^2}{T} \{\frac{4k^2}{T} r^2 - 1\} e(kr) ds$$

$$= k^2 r e''(kr) ,$$

(A.5) $$\int_0^r \left[\frac{\partial^2 e(-b+k(b+a))}{\partial a \partial b} \right]_{\substack{a=s \\ b=r-s}} ds = \frac{4k(k-1)}{T} \int_0^R \{1 - \frac{4}{T}(s-(k-1)R)^2\} e(kr) ds$$

$$= k(k-1)\{e'((k-1)R) - e'(kR)\} .$$

Collecting each terms in (A.3), we have (2.10), which is the result obtained by Feller (1951). Rearranging each term in (2.9), we obtain

(A.6) $$f(r) = 2 \sum_{i=1}^{\infty} \{2ke'(kr) + rk^2 e''(kr)\} .$$

(ii) **Proof of Lemma**: Let

(A.7) $$H(T,p) = \int_0^\infty x^p e(x) dx$$

$$= \frac{1}{2}\left(\frac{T}{2}\right)^{p/2} \Gamma\left(\frac{p+1}{2}\right) .$$

Then by integration we have

(A.8) $$\int_0^\infty x^p e''(kr) dr = \left(\frac{1}{k}\right)^{p+1} \{[x^p e'(x)]_0^\infty - p \int_0^\infty x^{p-1} e'(x) dx\}$$

$$= \left(\frac{1}{k}\right)^{p+1} (-p) \{[x^{p-1} e'(x)]_0^\infty - (p-1) \int_0^\infty x^{p-2} e(x) dx\}$$

$$= (\frac{1}{k})^{p+1} p(p-1)H(T,p-2) \ .$$

Similarly,

(A.8) $$\int_0^\infty x^p e'(kr)dr = (\frac{1}{k})^{p+1}\{[x^p e(x)]_0^\infty - p \int_0^\infty x^p e(x)dx\}$$

$$= (\frac{1}{k})^{p+1}(-p)H(T,p) \ . \quad \square$$

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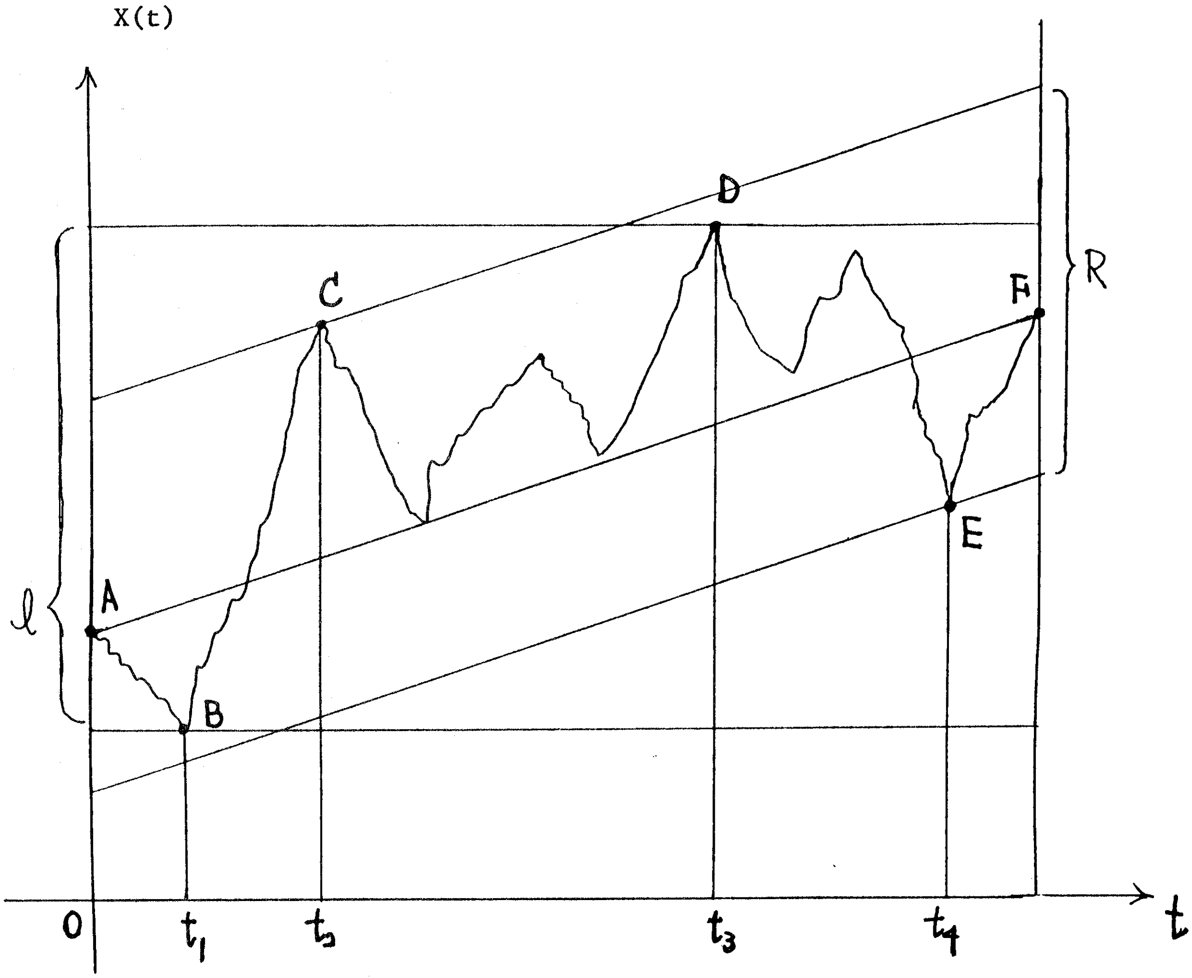

Figure 1