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Long-Memory and Geometric Brownian Motion in Security Market Models

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Abstract

We first give a set of sufficient conditions on the rate of returns of securities with some autocorrelations structures for that the discrete security price process converges to the geometric Brownian motion. We also give a set of sufficient conditions on the rate of returns of securities with some long-memory properties for that the discrete security price process converges to the fractional geometric Brownian motion. In the fractional geometric Brownian motion case, however, we show that there does not exist any continuous martingale measure, which is equivalent to the fractional Brownian motion. This suggests that many statistical time series models with strong dependent characteristics are not consistent with the standard no-arbitrage condition in finance. We also discuss the estimation methods for the volatility parameter in the geometric Brownian motion model. We argue that the standard estimation methods discussed in textbooks are not appropriate in the general situation and suggest some other estimation methods.

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1 Introduction

In recent financial economics it has been often assumed that the security prices follow the continuous diffusion stochastic processes. In particular, the geometric Brownian motion process has been often used in many theoretical studies. Also there has been a growing interest in estimating the parameter values of the continuous stochastic processes governing the historical security price movements. Especially, since the option price formulae are some nonlinear functions of the volatility parameter on the underlying geometric Brownian motion process, some special attention has been paid to estimate its numerical value.

On the other hand, econometricians and statisticians have been mainly interested in the econometric and statistical methods for estimating the unknown parameters in the discrete time series models. The main reason for this is that we usually have a set of finite number of observations on a particular realization of the underlying stochastic process for statistical inferences.

In this paper we shall first investigate the relationship between the discrete time series models often used by econometricians and the geometric Brownian motion process often used by financial economists. We shall give a set of sufficient conditions that the discrete price processes implied by the standard time series models for the rate of returns of securities converge to the geometric Brownian motion process as the unit of time interval goes to zero. Since in financial markets we can have monthly, weekly, daily, and minutes by minutes observations on the underlying processes, this asymptotic theory, which has been called the continuous record asymptotics according to Phillips (1987), may have some value to justify the method of approximation to the distributions in continuous stochastic processes.

Second, there have been recent empirical studies on the financial markets suggesting that there may be some autocorrelations on the historical data sets of the returns on securities. This finding has been often interpreted as some evidence against the geometric Brownian motion model for the continuous process of security prices because its logarithmic transformation has independent increments. Furthermore, some empirical studies on financial markets have suggested there may be some autorerrelations on the rate of returns in the long-run. For instance, Fama and French (1988) and Poterba and Summers (1988) have given some evidence of this kind on the U.S. stock markets and the foreign exchange markets. In the recent econometric as well as statistical terminology, this type of phenomena on the autocorrelations in the long-run have been investigated and often called the long- memory or the "strong" dependence. We shall also give a set of sufficient conditions that the price processes implied by the strong dependent time series models for the rate of returns on securities converge to the geometric "fractional" Brownian motion process as the unit of time interval goes to zero. In this context, the convergence result for the standard time series models can be regarded as that of the "weak" dependent case. However, in this case we shall show that there does not exist any martingale measure, which is equivalent to the continuous stochastic processes based on the fractional Brownian motion. Then by using the results of Harrison and Kreps (1979), this leads to the rather strong conclusion that the strong dependent time series models are inconsistent with the standard no-arbitrage condition in finance. This finding has some theoretical as well as practical consequences. For instance, the statistical models with strong dependent characteristics cannot be used for the valuation of option contracts. Also our theoretical result in this paper is consistent with the recent empirical result by Lo (1991).

The third purpose of this paper is to investigate the estimation methods of the volatility parameter in the continuous security price models. In this respect, several methods have been proposed to estimate the volatility parameter of the underlying stochastic process from a set of discrete observations on the asset prices. Surprisingly, we shall show that the standard variance estimator frequently explained by the finance textbooks and commonly used in practice tends to be inconsistent as the unit of interval goes to zero in the general case. This finding lead to the natural question: how to measure the volatility parameter and estimate its numerical value by using the discrete data set on the asset prices. In this paper we discuss some new and old estimation methods for the volatility parameter. Under a set of fairly general conditions the estimation methods in this paper yield consistent estimators, which are also asymptotically normally distributed with some additonal conditions in the sense of the above asymptotics.

As for the estimation of the volatility parameter under the assumption that the underlying security prices follow the continuous geometric Brownian motion model, several methods have been proposed in finance. For instance, Parkinson (1980) proposed the extreme value method. Kunitomo (1992a) also has improved the method by Parkinson. We shall discuss these estimation procedures of the volatility parameter in relation to the estimation methods proposed in this paper.

In Section 2, we shall give a set of sufficient conditions on the rate of return process for the convergence to the geometric Brownian motion process and the geometric fractional Brownian process. Then we shall give a theorem on the non-existence of martingales for the the fractional Brownian motion in Section 3. Then in Section 4, we shall give some new estimation methods for the volatility parameter and discuss on the relation between the methods in this paper and some estimation methods of the volatility parameter already known in finance. The proofs of Theorems are given in Section 5.

2 Convergence to Geometric and Fractional Geometric Brownian Motions

We first start with the weak dependent time series models for the rate of returns on the security prices. We divide the interval $[0, T]$ into n intervals of $[t_{i-1}, t_i]$, where $t_i = iT/n \, (i = 0, 1, \cdots, n)$. Let $S_n(t_i)$ be the i-th security price at $t_i = iT/n \, (i = 0, 1, \cdots, n)$ and we have $n+1$ observations. We define the rate of returns on $S_n(t)$ in the i-th interval as

$$(2.1) \qquad X_n(t_i) = \frac{S_n(t_i) - S_n(t_{i-1})}{S_n(t_{i-1})}.$$

In the above definition the order of variance of $X_n(t_i)$ should be $1/n$ because the length of time unit is $1/n$. Since the security prices $S_n(t_i)$ are non-negative, we have a natural restriction on the range of $X_n(t_i)$:

$$(2.2) \qquad X_n(t_i) \geq -1.$$

We denote the expected rate of returns $X_n(t_i)$ and its variance in the interval $[t_{i-1}, t_i]$ by $E[X_n(t_i)] = \mu_n(t_i)/n$ and $V[X_n(t_i)] = \sigma_n^2(t_i)/n$, respectively. We also denote the excess returns from the expected rate of returns and the (normalized) cumulative excess rate of returns by

$$(2.3) \qquad Y_n(t_i) = \sqrt{n}[X_n(t_i) - \mu_n(t_i)/n],$$

and

$$(2.4) \qquad Z_n(k) = \sum_{i=1}^{k} Y_n(t_i) \quad (k = 1, \cdots, n),$$

respectively. Then by construction we have $E[Y_n(t_i)] = 0$ and $E[Z_n(k)] = 0$. We make the following conditions on the generating process of $\{Y_n(t)\}$.

Assumption A: (i) There exist μ and σ (constants) such that

$$(2.5) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mu_n(t_i) = \mu,$$

$$(2.6) \qquad \max_{1\leq i\leq n} \frac{1}{n} |\mu_n(t_i)| \to 0,$$

and

$$(2.7) \qquad \lim_{n\to\infty} \frac{1}{n} E[Z_n(n)^2] = \sigma^2 > 0.$$

(ii) The sequence $\{Y_n(t_i)^2, (i = 1, \cdots, n)\}$ is uniformly integrable and

$$(2.8) \qquad \| E[Y_n(t_i) | \mathcal{F}_{n,j-k}] \|_2 \leq \eta_k \sigma_n(t_i),$$

$$(2.9) \qquad \| Y_n(t_i) - E[Y_n(t_i) | \mathcal{F}_{n,j+k}] \|_2 \leq \eta_{k+1} \sigma_n(t_i),$$

and

$$(2.10) \qquad \sum_{k=1}^{\infty} [\sum_{n=0}^{k} \eta_n^{-2}]^{-1/2} < \infty.$$

where $\|\cdot\|_2$ is the L_2-norm and $\mathcal{F}_{n,k}$ is a double array of the σ-field generated by $\{Y_n(t_i), (1 \leq i \leq k)\}$.

We have adopted the conditions (ii) from McLeish (1977), which has given a set of sufficient conditions for the weak convergence (i.e. the functional central limit theorem). It is important to note that the condition (2.10) is automatically satisfied by many statistical time series models. The condition (2.10) implies the condition

$$\sum_{k=1}^{\infty} \eta_n^{\alpha} < \infty \tag{2.11}$$

with $\alpha = 2$ and is implied by (2.11) with $\alpha < 2$. Hence immediately we know that the above conditions are satisfied by the stationary autoregressive moving-average (ARMA) models with non-Gaussian innovations, which have been often used in the empirical studies of financial markets. The conditions in Assumption I are different from the conditions used by Phillips (1987) for the weak convergence (i.e. the functional central limit theorem) of the discrete time series models in econometrics.

We have divided a fixed interval $[0, T]$ into n intervals with each length T/n. In financial markets, it becomes possible to have data with many different frequencies, i.e. monthly, weekly, daily, hourly or even minute by minute data. Hence it may be interesting to see how we can approximate the distribution of the discrete security price $S_n(t)$ by a continuous process when n is large. More precisely, as n goes to infinity we expect that the discrete stochastic process $S_n(t)$ converges to a continuous stochastic process based on the Brownian motion.

Theorem 1 *Suppose the conditions in Assumption A hold. As $n \to \infty$, the discrete security price process $S_n(t)$ converges weakly to a continuous stochastic process $S(t)$, which satisfies*

$$dS = \mu' S dt + \sigma S dB, \tag{2.12}$$

and $\mu' = \mu - \left(\sigma_y^2 - \sigma^2\right)/2$, provided that

$$\sigma_y^2 = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E[Y_n(t_i)^2] < +\infty \tag{2.13}$$

and $B(t)$ is the standard Brownian motion in $[0,T]$.

We notice that the drift parameter μ' in (2.12) is different from μ because of the presence of autocorrelations in the rate of return series. Also this theorem means that although the discrete rate of return process has substantial autocorrelations we have the geometric Brownian motion in the limit. Hence the continuous geometric Brownian motion model for the security price process is a good approximation to a broad range of statistical time series models for the discrete security price processes.

However, it is important to note that the volatility parameter σ is not necessarily equal to σ_y. To illustrate this issue, suppose that the generating process for the discrete return process $Y_n(t)$ is the autoregressive process AR(1):

$$Y_n(t_i) = \beta Y_n(t_{i-1}) + u_n(t_i), \quad i = 1, \cdots \tag{2.14}$$

where $u_n(t_i)$ are independently, identically, and normally distributed random variables with zero means and σ_u^2 variances. Then we have

$$\sigma_y^2 = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_n(t_i)^2 = \frac{\sigma_u^2}{(1-\beta^2)}, \tag{2.15}$$

and

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} E[Z_n(n)^2] = \frac{\sigma_u^2}{(1-\beta)^2}. \tag{2.16}$$

If the generating process for the excess rate of return $Y_n(t)$ is a sequence of independently and identically distributed random variables random variables with zero means and σ_y^2 variances, we immediately have $\sigma^2 = \sigma_y^2$. This is an important special case of Theorem 1.

Corollary 1 *If $Y_n(t)$ are independently and identically distributed random variables with zero means and σ_y^2 variances, then $\sigma^2 = \sigma_y^2$ and the continuous stochastic process $S(t)$ satisfies*

$$dS = \mu S dt + \sigma S dB. \tag{2.17}$$

This corollary has been stated and could be standard in financial economics. The conditions in Corollary 1 can be further weaken to allow some conditional as well as unconditional heteroscedasticity in the rate of returns process. For this purpose we need other type of conditions for the central limit theorems based on the martingale differences and heteroscedastic time series. The assumptions for this type of theorems have been discussed by Kunitomo (1992b), for instance.

Next, we deal with the strong dependent time series models for the rate of return on the security prices. To illustrate the strong dependent case, suppose that the generating process for $\{Y_n(t)\}$ is the fractional autoregressive (FA) time series model:

$$(1 - L)^d Y_n(t_i) = u_n(t_i), \quad i = 1, \cdots \tag{2.18}$$

where $0 < d < 1/2$, L is the lag operator, and $\{u_n(t_i)\}$ are independently, identically, and normally distributed random variables with zero means and σ_u^2 variances. (See Yajima (1989), for instance.) If we denote the h-th autocovariance function by $\gamma(h)$, the summability condition

$$\sum_{h=0}^{\infty} |\gamma(h)| < +\infty \tag{2.19}$$

cannot hold in this model. Hence the spectral density function diverges at the origin, which implies that there are some strong autocorrelations in the long-run frequency range.

The strong dependent time series models have some distinct differences from the weak dependent case in the standard statistical time series models. In the latter case we have obtained the convergence to the geometric Brownian motion given by Theorem 1. Instead of the the conditions (ii) in Assumption I, we need another set of conditions because they are not satisfied by the storngly dependent time series models. We assume that the discrete stochastic process $\{Y_n(t_i)\}$ is constructed by

$$Y_n(t_i) = \sum_{k=-\infty}^{\infty} \alpha_k u_n(t_{i-k}), \quad i = 1, \cdots \tag{2.20}$$

where

$$\sum_{k=-\infty}^{\infty} \alpha_k^2 < +\infty, \tag{2.21}$$

and $\{u_n(t_i)\}$ are independently distributed random variables with zero means and σ_u^2 variances. The following conditions for the functional central limit theorem (i.e. the weak convergence) have been adopted from Davydov (1970).

Assumption B: Assume the conditions (i) on $\{\mu_n(t_i)\}$ in Assumption A and there exists H such that $1/2 < H < 1$, $V[X_n(t_i)] = \sigma_n^2(t_i)/n^{2H}$, $Y_n(t_i) = n^H[X_n(t_i) - \mu_n(t_i)/n]$, and

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{2H} E[Z_n(n)^2] = \sigma^2 > 0. \tag{2.22}$$

(ii) For some $\delta > 0$,

$$E[u_n(t_i)^{2+\delta}] < \infty. \tag{2.23}$$

In the above formulation we have restricted our attention into the linear time series model given by (2.20). Another set of conditions with stationary Gaussian time series sequence, which are not necessarily linear with respect to the innovations and $0 < H < 1$, have been given by Taqqu (1975). For the strong dependent case with Assumption B, we have a weak convergence result to the fractional geometric Brownian motion.

Theorem 2 *Suppose the conditions in Assumption B hold. As $n \to +\infty$, the discrete security price process $S_n(t)$ converges weakly to a continuous stochastic process $S(t)$, which satisfies*

$$dS = \mu S dt + \sigma S dB_H, \tag{2.24}$$

where $B_H(t)$ is the fractional Brownian motion in $[0, T]$.

3 Non-existence of Equivalent Martingale for Fractional Brownian Motion Model

When there are some strongly dependent autocorrelations in the rate of returns, the standard methods in finance such as the option pricing theory based on the geometric Brownian motion cannot be applied. Hence, there is an interesting question how the standard methods in finance such as the Black=Sholes theory (Black and Scholes (1973)) should be modified. It turns out that we shall show a kind of negative result in this line of extending the standard derivative security theory in finance.

By the result of Mandelbrot and Van Ness (1968), the fractional Brownian motion $B_H(t)$ has a representation

$$(3.1)\qquad \begin{aligned} B_H(t) &= \frac{1}{\Gamma(H+1/2)}\Big(\int_{-\infty}^{0}\left[(t-s)^{H-1/2}-(-s)^{H-1/2}\right]dB(s) \\ &\quad + \int_0^t (t-s)^{H-1/2}\,dB(s)\Big). \end{aligned}$$

Also it has been known that $B_H(t)$ is a continuous Gaussian process with the covariance property

$$(3.2)\qquad Cov\,(B_H(s), B_H(t)) = \frac{1}{2}\{s^{2H}+t^{2H}-|t-s|^{2H}\}.$$

Furthermore, the fractional Brownian motion $B_H(t)$ reduces to the standard Brownian motion when $H = 1/2$.

Theorem 3 *Let the continuous stochastic process $S(t)$ be a function of the fractinal Brownian motion with $H \neq 1/2$ under the probability measure P. Then there does not exist any probability measure P^* which is absolutely continuous to P and $S(t)$ is a martingale with respect to P^*.*

The proof of Theorem 3 relies on a simplified version of Theorem 2 by Kôno (1969). However, since the original proof of Kôno (1969) contains an error, we give its detailed proof of the simplified version as a lemma in Section 5. We notice that if there does not exist any equivalent martingale measure P^*, it may not be consistent to the no-arbitrage condition in finance. The fundamental result in finance that the exisitence of equivalent martingale measure implies the absense of arbitrage condition has been obtained by Harrison and Kreps (1979). Kusuoka (1992) also has given some conditions that the absence of arbitrage implies the existence of continuous martingale measures. Thus we have the following proposition from Theorem 3.

Corollary 2 *The frictionless continuous time security markets represented by some continuous process based on the fractional Brownian motion having an equivalent martingale measure imply that* $H = 1/2$.

From this proposition we have a conclusion that the strong dependent time series models are usually not consistent with the standard condition of no-arbitrage in dynamic security markets. In the class of continuous fractional Gaussian proces, the only process which can be used for describing security movements is the process associated with the Brownian motion, that is , the diffusion process or the Ito process. Hence this finding has some important theoretical as well as practical consequences. For instance, the strong dependent statistical time series models cannot be used for the valuation of derivative securities such as option contracts. Lo (1991) recently has found that there is no strong empirical evidence for long-memory characteristics in the U.S. stock markets. In this respect, our theoretical result in this section could be interpreted as a theoretical support for the empirical results by Lo (1991).

4 Estimation of Volatility

In the continuous security price model given by (2.12), the parameter σ has been called the volatility parameter in finance. In many literatures in finance, it has been explained that this parameter can be estimated by the standard variance estimation method from a set of discrete observations (see Ingersol (1987), for instance). When we have n observations on the rate of return $Y_n(t)$ in $[0, T]$, the classical estimator of σ_y^2 is given by

$$\hat{\sigma}_y^2 = \frac{1}{n} \sum_{i=1}^{n} \left[Y_n(t_i) - \bar{Y}_n \right]^2 , \tag{4.1}$$

where $\bar{Y}_n = (1/n)\sum_{i=1}^{n} Y_n(t_i)$. As we have illustrated in Section 2 using the AR(1) example, this estimator is a consistent estimator of σ_y^2 but it is not necessarily a consisitent estimator of σ^2 as n goes to infinity. This is because we have allowed substantial autocorrelations on the rate of return in the formulation of discrete statistical time series models.

In order to investigate the estimation problem of the volatility parameter σ from the discrete observations, we first assume that the discrete excess rate of return process is weakly stationary. We denote the k-th autocorrelation of $Y_n(t)$ by

$$\sigma_n(k) = Cov\left(Y_n(t_i), Y_n(t_{i+k})\right) \tag{4.2}$$

and we assume that $\sigma_n(k) \longrightarrow \sigma(k)$ as $n \longrightarrow +\infty$. Then $\sigma(0) = \sigma_y^2$ and

$$\sigma^2 = \sigma(0) + 2\sum_{k=1}^{\infty} \sigma(k), \tag{4.3}$$

provided that $\sigma^2 < +\infty$. The last condition is implied by the standard condition in the weakly dependent time series

$$\sum_{k=-\infty}^{\infty} |\sigma(k)| = \sigma(0) + 2\sum_{k=1}^{\infty} |\sigma(k)| < +\infty. \tag{4.4}$$

Under this condition, the spectral density function for the weakly stationary stochastic process $Y_n(t)$ can be defined as

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} \sigma(h)\cos(\lambda h). \tag{4.5}$$

From (4.4) and (4.5), we have a simple but key relationship between the spectral density and the volatility parameter:

$$\sigma^2 = 2\pi f(0). \tag{4.6}$$

Now we consider the estimation problem of the parameter σ^2 in (2.12) from a set of discrete observations. In order to solve this problem, we first consider the non-parametric estimation problem of the spectral density $f(\lambda)$ from a finite number of observations. Let the kernel estimator of $f(\lambda)$ be

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{h=-K_n}^{K_n} k\left(\frac{h}{K_n}\right) c(h) \cos(\lambda h), \tag{4.7}$$

where $c(h)$ is the h-th sample autocorrelation function

$$c(h) = c(-h) = \frac{1}{n} \sum_{i=|h|+1}^{n} \left(Y_n(t_i) - \bar{Y}_n\right)\left(Y_n(t_{i-|h|}) - \bar{Y}_n\right). \tag{4.8}$$

We confine our discussion into the case when the kernel function $k(x)$ satisfies the following three conditions: (i) $k(x) = k(-x)$, (ii) $k(x)$ is continuous and bounded in $[-1, 1]$, (iii) there exist $q > 0$ and $k > 0$ such that

$$\lim_{x \to 0} \frac{1 - k(x)}{|x|^q} = k > 0. \tag{4.9}$$

Hence we have an idea that we could use the relation (4.6) for estimation. It may be natural to construct a class of estimators on σ by making use of the kernel type estimation methods for the spectral density $f(\lambda)$. The resulting estimator is given by

$$\hat{\sigma}_n^2 = 2\pi \hat{f}_n(0). \tag{4.10}$$

In order to state the asymptotic properties of this type of estimators, we further make the following assumptions on $\{Y_n(t_i)\}$.

Assumption C: The stochastic process $Y_n(t_i)$ is a weakly linear stationary process

$$Y_n(t_i) = \sum_{k=-\infty}^{\infty} \alpha_k u_n(t_{i-k}), \tag{4.11}$$

where

$$(4.12) \qquad \sum_{k=-\infty}^{\infty} |\alpha_k| < +\infty,$$

and $u_n(t_i)$ are independently distributed random variables with $E[u_n(t_i)] = 0, E[u_n(t_i)^2] = \sigma_u^2$, and $E[u_n(t_i)^4] < +\infty$.

Then it can be shown that the above estimators on the volatility parameter σ^2 are consistent and asymptotically normal. The proof is similar to those of Theorems 9.4.3, 9.4.4, and 9.4.5 of Anderson (1971) and so omitted.

Theorem 4 *In addition to Assumption C, suppose (i)* $K_n \to +\infty$, $K_n/n \to 0$, *and* $K_n^{q+1}/n \to 0$ *as* $n \to \infty$ *for some* $q \geq 1$; *(ii)*

$$(4.13) \qquad \sum_{h=-\infty}^{\infty} |h|^q |\sigma(h)| < +\infty.$$

Then as $n \to \infty$

$$(4.14) \qquad \sqrt{\frac{n}{K_n}} \left(\hat{\sigma}_n^2 - \sigma^2\right) \to N\left(0, 2\sigma^4 \int_{-1}^{1} k(x)^2 \, dx\right).$$

For an illustration, we give an example of estimator on σ. The modified Bartlett estimator for the spectral density, which has been well-known in statistical time series analysis (see Chapter 9 of Anderson (1971)) is constructed by the kernel function

$$(4.15) \qquad k(x) = 1 - |x| \quad (|x| < 1).$$

The resulting estimator of σ based on this kernel function is given by

$$(4.16) \qquad \hat{\sigma}_n^2(B) = \sum_{h=-K_n}^{K_n} \left(1 - |\frac{h}{K_n}|\right) c(h),$$

which is incidentally identical to the estimation method proposed by Newey and West (1985) in econometrics. In this case, if we take $q = 1$, the conditions in Theorem 4 are such that $K_n/n \rightarrow 0$ and $K_n^2/n \rightarrow 0$ as $n \rightarrow +\infty$ and the asymptotic variance is given by

$$2\sigma^4 \int_{-1}^{1} k(x)^2 dx = \frac{4}{3}\sigma^4. \tag{4.17}$$

There are many possibilities to choose the kernel function $k(x)$ to estimate the spectral density. Some of them have been discussed in Chapter 9 of Anderson (1971) in some details, for instance. Hence it is also possible to construct consistent estimators of the volatility parameter in many ways.

Using the framework of the continuous stochastic process, Parkinson (1980) has proposed to use the range of the $L(t) = ln[S(t)]$. In order to explain his method, let the range of $L(t)$ in the i-th interval $I_i = [(i-1), i](i = 1, \cdots, T)$ be

$$l_i = \max_{t \in I_i} L(t) - \min_{t \in I_i} L(t). \tag{4.18}$$

Then the extreme value estimator proposed by Parkinson (1980) is given by

$$\hat{\sigma}^2(p) = \frac{1}{(4\ln(2))T} \sum_{i=1}^{T} l_i^2. \tag{4.19}$$

Instead of the original range of $L(t)$, Kunitomo (1992a) has proposed to use the adjusted range of $L(t)$. Let

$$L_i^*(t) = L(t) - L(i-1) - [t-(i-1)][L(i) - L(i-1)] \tag{4.20}$$

for $i - 1 \leq t \leq i (i = 1, \cdots, T)$ and define the adjusted range by

$$R_i = \max_{t \in I_i} L_i^*(t) - \min_{t \in I_i} L_i^*(t). \tag{4.21}$$

Since the stochastic process $\{L_i^*(t)\}$ is the continuous Brownian bridge process in each interval I_i, it is free from the drift term of the original stochastic process $S(t)$. For the method by Parkinson (1980), the range l_i does depend on the drift term of the original stochastic process. Then the estimator of σ proposed by Kunitomo (1992a) is given by

$$\hat{\sigma}^2(k) = \left(\frac{6}{\pi^2 T}\right) \sum_{i=1}^{T} R_i^2. \tag{4.22}$$

As we have proved in Theorem 1, the discrete stochastic process $S_n(t)$ converges weakly to the geometric Brownian motion process $S(t)$. Let further divide the interval $I_i (i = 1, \cdots, T)$ into n intervals and suppose we observe the weakly dependent discrete security prices at the time $t = ki/n (k = 0, \cdots, n)$. Then by the weak convergence of $S_n(t)$ in Theorem 1, we have

$$\max_{t \in I_i} S_n(t) \xrightarrow{d} \max_{t \in I_i} S(t) \tag{4.23}$$

and

$$\min_{t \in I_i} S_n(t) \xrightarrow{d} \min_{t \in I_i} S(t) \tag{4.24}$$

as $n \longrightarrow \infty$.

Hence we can approximate the distribution of $S_n(t)$ in each interval I_i by the corresponding continuous process $S(t)$. (See Billingsley (1968), for instance.) Kunitomo (1992a) has shown that the estimator $\hat{\sigma}^2(k)$ is unbiased and the efficiency is about 10 against the standard variance estimator based on the continuous Brownian Bridge process. The appealing feature in the estimation methods by Parkinson (1980) and Kunitomo (1992a) may be due to the fact that they are extremely simple and utilize only a small number of observations in each interval. On the other hand, the class of estimation methods discussed in this section utilize available information fully in each interval. The latter methods enjoy some desirable statistical properties as n tends to infinity because they are using more and more observations in each interval. On the contrary, the former type of estimation methods could be interpreted as some practical ways of estimating the volatility parameter σ, which can be implemented in the empirical studies easily.

5 Proof of Theorems

(i) **Proof of Theorem 1:** The key idea for the proof is to use the functional central theorem given by McLeish (1977). Without loss of generality, we take $T = 1$. Then for any fixed $t \in [0, 1]$, we have

$$
\begin{aligned}
\ln S_n\left(\frac{[nt]}{n}\right) - \ln S_n(0) &= \ln \prod_{i=1}^{[nt]} \left[\frac{S_n(t_i)}{S_n(t_{i-1})}\right] \\
&= \sum_{i=1}^{[nt]} \ln\left[1 + X_n(t_i)\right].
\end{aligned} \tag{5.1}
$$

By the Taylor expansion, we have

(5.2)

$$
\begin{aligned}
\sum_{i=1}^{[nt]} \ln\left[1 + X_n(t_i)\right] &= \sum_{i=1}^{[nt]} \ln\left[1 + \frac{1}{n}\mu_n(t_i)\right] \\
&+ \sum_{i=1}^{[nt]} \frac{1}{1+\frac{1}{n}\mu_n(t_i)} \left[\frac{Y_n(t_i)}{\sqrt{n}}\right] - \frac{1}{2}\sum_{i=1}^{[nt]} \frac{1}{\left[1+\frac{1}{n}\mu_n(t_i)\right]^2} \left[\frac{Y_n(t_i)}{\sqrt{n}}\right]^2 \\
&+ \frac{1}{3}\sum_{i=1}^{[nt]} \frac{1}{\left[1+\frac{1}{n}\theta_n(t_i)\right]^3} \left[\frac{Y_n(t_i)}{\sqrt{n}}\right]^3,
\end{aligned}
$$

where $|\theta_n(t_i)| \leq |\mu_n(t_i)|$. The first term of (5.2) is re-written as

$$
\frac{1}{n}\sum_{i=1}^{[nt]} \mu_n(t_i) + \sum_{i=1}^{[nt]} \left(\ln\left[1 + \frac{1}{n}\mu_n(t_i)\right] - \frac{1}{n}\mu_n(t_i)\right). \tag{5.3}
$$

We note that

$$
\frac{1}{n}\sum_{i=1}^{[nt]} \mu_n(t_i) = \left(\frac{[nt]}{n}\right) \frac{1}{[nt]} \sum_{i=1}^{[nt]} \mu_n(t_i) \to t\mu \tag{5.4}
$$

as $n \to +\infty$ because of (2.5). Also since

$$(5.5)\quad \sum_{i=1}^{[nt]} | \ln \left[1+\frac{1}{n}\mu_n(t_i)\right] - \frac{1}{n}\mu_n(t_i) | \le \frac{1}{2}\sum_{i=1}^{[nt]} \frac{1}{\left[1+\frac{1}{n}\mu_n(t_i)\right]^2}\left[\frac{\mu_n(t_i)}{\sqrt{n}}\right]^2,$$

and

$$(5.6)\qquad \sum_{i=1}^{[nt]} |\frac{1}{n}\mu_n(t_i)|^2 \le \left[\max_{1\le i\le [nt]} |\frac{1}{n}\mu_n(t_i)|\right] \sum_{i=1}^{[nt]} |\frac{1}{n}\mu_n(t_i)| \to 0$$

as $n \to \infty$, the first term of (5.2) converges to $t\mu$ as $n \to \infty$. Next, the second term of (5.2) is re-written as

$$(5.7)\qquad \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} Y_n(t_i) + \sum_{i=1}^{[nt]} \left(\frac{-\frac{1}{n}\mu_n(t_i)}{1+\frac{1}{n}\mu_n(t_i)}\right)\left[\frac{1}{\sqrt{n}}Y_n(t_i)\right].$$

The last term of (5.7) goes to zero as $n \to \infty$ because the coefficients of $Y_n(t_i)$ are bounded for sufficiently large n due to (5.5) and (5.6). Under the conditions (i) and (ii) in Assumption I, we have

$$(5.8)\qquad \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} Y_n(t_i) \xrightarrow{d} \sigma B(t)$$

as $n \to \infty$. This weak convergence result has been first proved by McLeish (1977). As for the third term of (5.2) is concerned,

$$(5.9)\qquad \frac{1}{n}\sum_{i=1}^{[nt]} Y_n(t_i)^2 \xrightarrow{p} t\sigma_y^2$$

as $n \to \infty$ because of the condition (i). Then by using (5.7) the third term of (5.2) converges to $-t\sigma_y^2/2$ as $n \to \infty$. As for the last term of (5.2) is concerned, it is less than

$$(5.10)\quad \left(\frac{1}{\sqrt{n}}\right)^3 \sum_{i=1}^{[nt]} |Y_n(t_i)|^3 \le \left[\frac{1}{\sqrt{n}}\max_{1\le i\le [nt]} |Y_n(t_i)|\right] \sum_{i=1}^{[nt]} |\frac{1}{\sqrt{n}}Y_n(t_i)|^2 \xrightarrow{p} 0$$

as $n \to \infty$. The convergence of the last term is due to the Lindeberge condition implied by the condition (i). Hence (5.1) converges to

$$t\left(\mu - \frac{\sigma_y^2}{2}\right) + \sigma B(t) \tag{5.11}$$

as $n \to \infty$. Finally, by using the Ito's formula, we have (2.12) in Theorem 1. □

(ii) **Proof of Theorem 2**: The proof is similar to that of Theorem 1 and we use the functional central theorem given by Davidov (1970) instead of McLeish (1977). Therefore, we omit its details. □

(iii) In order to give the proof of Theorem 3, we shall first prepare the following two lemmas. The first lemma is a simplified version of Theorem 2 of Kôno (1969).

Lemma 1 *Let $X(t)$ be the continuous fractional Gaussian process with* $0 < H < 1$. *Then as $n \to \infty$,*

$$\sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{1/H} \to C = \int_{\mathcal{R}} |x|^{1/H} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad a.s.. \tag{5.12}$$

Proof of Lemma 1: Since this lemma is a special case of Theorem 2 of Kôno (1969), we can use the basic line of his proof and notations freely. We take $v(x) = x^H, P(s) = s^H$, and $Q(x) = x^{1/H}$. We also take $\|S_n\| = 2^{-n}, \Delta_i^n t = t_i^n - t_{i-1}^n = 2^{-n}$, $\Delta_i^n X = X(t_i^n) - X(t_{i-1}^n)$, then $\Delta_i^n X \sim N[0, 2^{-2H/n}]$ because $X(t)$ is a Gaussian process. Thus we have

$$E\left[\sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{1/H}\right] = 2^n \cdot \frac{1}{2^n} C = C. \tag{5.13}$$

Let

$$B_n = E\left[\sum_{k=1}^{2^n} Q\left(\Delta_i^n X(t)\right) - C\right]^2. \tag{5.14}$$

We take $\alpha = \beta = 1, 0 < b < 2(1-H)$, $0 < \mu < \frac{2(1-H)-b}{2(1-H)+b} < 1$. Then

(5.15)

$$\begin{aligned} B_n &= \left(\sum_{|t_{i-1}^n - t_{j-1}^n| < 2^{-\mu n}} + \sum_{|t_{i-1}^n - t_{j-1}^n| > 2^{-\mu n}}\right)\left(E\left[Q\left(|\Delta_i^n X|\right) Q\left(|\Delta_j^n X|\right)\right] - \left[\tfrac{1}{2^n} C\right]^2\right) \\ &= B_n(1) + B_n(2). \end{aligned}$$

Then by the Cauchy=Schwartz inequality, we have

(5.16)

$$\begin{aligned} |B_n(1)| &\leq \sum_{|t_{i-1}^n - t_{j-1}^n| < 2^{-\mu n}} \left(\sqrt{E\left[Q\left(|\Delta_i^n X|\right)^2\right] E\left[Q\left(|\Delta_j^n X|\right)^2\right]} - \left[\tfrac{1}{2^n} C\right]^2\right) \\ &= c\left(\tfrac{1}{2^n}\right)^{-\mu}, \end{aligned}$$

where c is a positive constant.

Next, we note that the second term of (5.16) is re-written as

(5.17)

$$\sum_{|t_{i-1}^n - t_{j-1}^n| > 2^{-\mu n}} \left[\frac{1}{2^n}\right]^2 \left(\int_{R^2} |x_1 x_2|^{1/H} n_2\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{ij}^n \\ \rho_{ij}^n & 1 \end{pmatrix}\right) dx_1 dx_2 - C^2\right),$$

where $n_2(\cdot, \Sigma)$ is the 2-dimensional normal density with zero means and the covariance matrix Σ, and

$$\rho_{ij}^n = \frac{E\left[\Delta_i^n X \Delta_j^n X\right]}{\sqrt{E\left[\Delta_i^n X\right]^2 E\left[\Delta_i^n X\right]^2}}. \qquad (5.18)$$

In order to evaluate $B_n(2)$, we notice that the integrand of $B_n(2)$ can be re-written as

$$|x_1 x_2|^{1/H} e^{-\frac{x_1^2 + x_2^2}{4}} B_n(x_1, x_2), \qquad (5.19)$$

where

(5.20)

$$B_n(x_1,x_2) = \left(\frac{1}{\sqrt{1-(\rho_{ij}^n)^2}} - 1\right)\exp\left(-\frac{\left(1+(\rho_{ij}^n)^2\right)(x_1^2+x_2^2)-2\rho_{ij}^n x_1x_2}{4\left(1-(\rho_{ij}^n)^2\right)}\right)$$
$$+\left(\exp\left(-\frac{\left(1+(\rho_{ij}^n)^2\right)(x_1^2+x_2^2)-2\rho_{ij}^n x_1x_2}{4\left(1-(\rho_{ij}^n)^2\right)}\right) - \exp\left(-\frac{x_1^2+x_2^2}{4}\right)\right)$$

Then

$$|B_n(x_1,x_2)| \leq \left(\left|\frac{(\rho_{ij}^n)^2}{\sqrt{1-(\rho_{ij}^n)^2}\left[1+\sqrt{1-(\rho_{ij}^n)^2}\right]}\right|\right)$$

(5.21)

$$+ \left|\frac{\left(1+(\rho_{ij}^n)^2\right)(x_1^2+x_2^2)-2(\rho_{ij}^n)x_1x_2}{4\left(1-(\rho_{ij}^2)^2\right)} - \frac{x_1^2+x_2^2}{4}\right|$$
$$\leq \left|\frac{\rho_{ij}^n}{1-(\rho_{ij}^n)^2}\right| + \left|\frac{\rho_{ij}^n}{1-(\rho_{ij}^n)^2}\left[\frac{x_1^2+x_2^2}{2} + \rho_{ij}^n x_1x_2\right]\right|$$
$$\leq \left|\frac{\rho_{ij}^n}{1-(\rho_{ij}^n)^2}\right|\left[1+x_1^2+x_2^2\right] \quad .$$

where we have used the inequality $|e^{-s}-e^{-t}| \leq |t-s|$ for any $s,t \geq 0$. Thus if there are positive constants c' and r such that

(5.22)
$$|B_n(2)| \leq c'\left(\frac{1}{2^n}\right)^r,$$

then

(5.23)
$$\sum_{n=1}^{\infty} |B_n| \leq \infty.$$

Let $k = |i-j|$. Then $|t_{i-1}^n - t_{j-1}^n| < \frac{1}{2^{\mu n}}$ implies that $k > 2^{(1-\mu)n} \to \infty$ as $n \to \infty$.

For the fractional Brownian motion, the $k-$th autocorrelation is given by

$$(5.24)\quad \begin{aligned}\rho^n_{ij} &= \tfrac{1}{2}k^{2H}\left[|1-\tfrac{1}{k}|^{2H}+|1+\tfrac{1}{k}|^{2H}-2\right]\\ &= O\left(\left(\tfrac{1}{2^n}\right)^{(1-H)(1-\mu)}\right).\end{aligned}$$

Finally, by the Borel=Cantelli lemma, we have

$$(5.25)\qquad \sum_{n=1}^{2^n} Q\left(|\Delta^n_i X(t)|\right) \to C \quad a.s..$$

□

Lemma 2 *(i) If* $H > \frac{1}{2}$*, then*

$$(5.26)\qquad \sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2 \to 0 \quad a.s..$$

(ii) If $H < \frac{1}{2}$*, then*

$$(5.27)\qquad \sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2 \to +\infty \quad a.s..$$

Proof of Lemma 2: If $H > \frac{1}{2}$, from Lemma 1 we have

$$(5.28)\quad \begin{aligned}\sum_{k=1}^{2^n} P\left(|X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)| > \epsilon\right) &\le \epsilon^{-2}\sum_{k=1}^{2^n} E\left[|X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2\right]\\ &\to 0 \quad a.s.\end{aligned}$$

as $n \to \infty$. By using the Borel=Cantelli lemma, we have

$$(5.29)\qquad \max_k |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)| \to 0 \quad a.s..$$

Then

(5.30)

$$\sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2$$
$$\leq \left[\max_k |X\left(\tfrac{k}{2^n}\right) - X\left(\tfrac{k-1}{2^n}\right)|\right]^{2-\frac{1}{H}} \sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{\frac{1}{H}}$$
$$\to 0 \quad a.s..$$

Similarly, if $H < \frac{1}{2}$, we take $1 > \beta' > \frac{1}{2} > \alpha'$ and

(5.31)

$$\begin{aligned}
\sum_{k=1}^{2^n} P\left(|X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{\beta'} > \epsilon\right) &\leq \epsilon^{-2} \sum_{k=1}^{2^n} E\left[|X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{2\beta'}\right] \\
&\leq \epsilon^{-2} \sum_{k=1}^{2^n} \left[E|X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2\right]^{\beta'} \\
&= \epsilon^{-2} \left[\tfrac{1}{2^n}\right]^{\frac{\beta'}{\alpha'}-1} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. From Theorem 1, the left-hand side of

(5.32)

$$\sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^{\frac{1}{H}}$$
$$\leq \left[\max_k |X\left(\tfrac{k}{2^n}\right) - X\left(\tfrac{k-1}{2^n}\right)|\right]^{\frac{1}{H}-2} \sum_{k=1}^{2^n} |X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)|^2$$

converges to C a.s.. But from (5.12), the first term of the right-hand side of (5.32) converges to zero a.s.. Then we have (5.27). □

Proof of Theorem 3: We notice that for any continuous martingale $M(t)$, there exists a quadratic variation process $\langle M\rangle(t)$ (see Ikeda=Watanabe (1989), for instance). Thus

(5.33) $$\sum_{k=1}^{2^n}\left[M\left(\frac{k}{2^n}\right)-M\left(\frac{k-1}{2^n}\right)\right]^2 \to \langle X\rangle(t) \quad a.s..$$

Also if $\langle X\rangle(t)=0 \quad a.s.$, then $M(t)=0$ for $t\in[0,1] \quad a.s.$. If the continuous process $X(t)$ is a martingale under a probability measure P^*, it should be a continuous martingale. Then

(5.34) $$\sum_{k=1}^{2^n}\left[X\left(\frac{k}{2^n}\right)-X\left(\frac{k-1}{2^n}\right)\right]^2 \to \langle X\rangle(t) \quad a.s.$$

under P^*. Hence it holds also a.s. under P because it is absolutely continuous to P^*. However, if $H<1/2$ then $\langle X\rangle(1)=+\infty$, which is a contradiction. On the other hand, if $H>1/2$ then $\langle X\rangle(1)=0$. This leads to the conclusion $X(t)=0 \quad a.s.$ for any $t\in[0,1]$ under P^*, which is also a contradiction. Therefore, there does not exist such P^* if $H\neq 1/2$. □

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