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Term Structure of Interest Rates and the Pricing of Fixed Income Claims and Bonds

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1 Introduction

The *term structure of interest rates* is an array ("structure") of price or yields on bonds with different terms to maturity. The relationship among the yields—the term structure— varies over time. For example, if the yields on long-term bonds are above those on short term bonds, as they were in the U.S. in mid-1993, the term structure is said to be *upward* sloping. By contrast, the term structure was *downward sloping* in 1973 and the early 1980s—short-term yields were above long-term yields. The term structure is usually computed from the observed prices of Government bonds which are typically regarded as default-free in developed countries, or the prices of securities which depend upon the bonds, such as interest-rate swaps.

Until the mid- to late-1970s, term structure analysis and related research on fixed income management more or less existed as a stand-alone field in finance. Basic default-free bond concepts and terminology such as *yields*, *spot* and *forward* rates, *immunization*, *duration*, and the *liquidity preference* and *expectations* hypotheses concerning *term premiums* were neither influenced, nor had much influence on, developments in the asset pricing models. The latter dealt with the *price of risk* whereas term structure models dealt with *the price of time* alone[1]—in the one-period and partial equilibrium portfolio allocation and asset pricing models prior to the mid-1970s,

[1]This is not to say that portfolio theory and asset pricing concepts were *never* used to study the term structure of interest rates. For example, Stiglitz (1970) derived the relative demands for one-period ("short" term) and two-period ("long" term) bonds from a model in which investors make investment and consumption decisions to maximize expected lifetime utility. Long (1974) derived investor demand functions and prices for stocks, bonds, and commodity "futures" contracts in the presence of multiperiod uncertainty about consumption good prices, changes in the investment opportunity set, and (nominal) wealth. Long's paper, along with various sections of Merton's (1971)(1975a) papers, foreshadowed much of the subsequent general equilibrium term structure analysis. On the empirical side, Roll (1970)(1971) applied a recursive version of the Sharpe-Lintner capital asset pricing model to monthly returns on longer-term bonds. Marsh (1985) tested whether consumption-based and one-period asset pricing model constraints across the monthly returns on bonds with different maturities were satisfied.

one-period bonds were no more than numeraire.

In the last ten years, however, fixed income models have become increasingly integrated with mainstream models of asset allocation and asset pricing. The "enabling" technology for this integration has been the intertemporal equilibrium asset pricing theory and risk-neutral valuation techniques developed in the mid- to late-1970s. The extension of this new technology to bonds has undoubtedly been stimulated by the increased volatility of interest rates, along with the rapid expansion of the cash market (e.g. the issuance of Government bonds in the U.S. and Japan),[2] and derivative markets (e.g. mortgage-backed securities, bond futures, interest-rate swaps, and the like) in fixed income securities. Continual improvement in the technology has, in turn, stimulated further growth of the derivatives market.

The primary objective of this chapter is to explain how the intertemporal and risk neutral pricing techniques have been applied in the fixed income area. The exposition is in essentially reverse chronological order. It begins with an explanation of how the risk-neutral approach has been adapted from the options pricing literature. It then turns to the equilibrium bond pricing models which were the focus of research in the late 1970s and early 1980s. Both the risk-neutral and equilibrium approaches are initially presented under the assumption that term structure movements depend upon only a single factor—the short rate of interest. This factor, and thus movements in bond prices, can then be depicted in the familiar "tree" diagram used to exposit contingent claim pricing methods. Once the arbitrage-free and equilbrium approaches to modelling the term structure have been introduced, the extent to which the approaches "incorporate" the information in the current term structure, and the number

[2] In 1990 fixed income securities account for about half of the estimated $22 trillion worth of outstanding world stocks, bonds, and cash, though not all of these securities are freely traded; some 30%-40% of Government bonds are held by central banks, for example.

of factors which might be needed to adequately describe term structure movements, are discussed. Finally, some issues in fitting the fixed income models and applying them in the valuation of interest rate derivative claims are addressed.

For the most part, the paper deals with the term structure of default-free *zero coupon* or *discount* bond prices, or equivalently, the *spot* and *forward* rates of interest implied by those prices. In principle, a coupon bond can be decomposed into a portfolio of zero-coupon bonds, each with a face value equal to the coupon, as if it were stripped from the coupon bond and priced separately. For example, a two-year 8% coupon bond with face value $1,000 makes a $40 coupon payment at the end of 6, 12, and 18 months, and a $1,040 payment at the end of two years—this final payment is the face value of principal plus the final coupon payment. As such, the 8% coupon bond notionally consists of a portfolio of three zero-coupon bonds paying $40 at the end of of 6 months, 12 months, and eighteen months respectively, together with a $1,040 zero with a maturity of 2 years.[3]

In practice, it is well-known that "coupon" effects, tax effects, "on-the-run" effects, and various other idiosyncracies are important in pricing Treasury securities. Data errors, stale quotes, and the like must also be taken into account when fitting a term structure to bond prices, i.e. in deriving *implied zero coupon* yields. These important practical issues cannot be examined here. Typically, spline fitting techniques are used to (cross-sectionally) "smooth out" the pricing idiosyncracies in the fitted term structure at each point in time[4] (cf. Nelson and Siegel (1987), Beim (1992) and

[3]In 1984, the U.S. Treasury allowed coupons which are stripped like this from its outstanding obligations to be held as Strips, an acronym for Separate Trading of Registered Interest and Principal of Securities, whereby stripped securities can be held in separate book-entry form at the Federal Reserve. After 1987, bonds could also be reconstructed from Strips.

[4]"Smoothing out" the effects doesn't necessarily mean "smoothing away," e.g. if there are no zero-coupon bonds traded, it is not possible to isolate all coupon effects, and they will be smoothed into the fitted yield curve.

Diament (1993) for discussion and references). Unfortunately, the design and pricing of various coupon and payout provisions on debt cannot be covered here either. Cox, Ingersoll, and Ross (1980) and Ramaswamy and Sundaresan (1986) are examples of this work. Finally, by ignoring coupons, duration and immunization measures are not *per se* discussed here, though the models of interest rate movements on which their validity depends are; Ingersoll, Skelton, and Weil (1978) contains a good overview of the duration literature; see also Boyle (1978).

The focus of the paper is on dynamic models for the time series of term structures of implied zero-coupon bond yields after the idiosyncracies have been smoothed out. The smoothing is assumed to have been done in a first step; the dynamic modelling is then a second step. In a *state-space* framework, these two steps could be *integrated.* In such a framework, the discount function would be defined as a state variable whose dynamics are modelled in a *state equation.* Differences between the observed term structure and the discount function—the differences which are "smoothed" away in a separate step before fitting dynamic models, would then be simultaneously handled in a measurement equation.[5]

Whether obtained in a state-space framework or by the two step procedure of smoothing and then fitting a dynamic model to the smoothed term structure, the fitted discount function ends up being "smoothed" in *two* ways relative to the term structures of actual discount bond prices at successive points in time. First, estimated idiosyncracies such as coupon effects, on-the-run effects, data errors, stale

[5]As an example of why it is useful to integrate the measurement and state equations, consider liquidity effects on bond prices. One could possibly choose to model liquidity, or some systematic factors responsible for it, as something priced in the state equation, or alternatively relegate it to the measurement equation. Integration should also force the user to think about the consequences of estimation error—presumably the loss function for errors in measuring the term structure at the long end does is not the same as at the short end.

bond quotes, etc.—the type of effects which the spline fits attempt to smooth out, are eliminated as residuals or "model measurement errors" in the measurement equation. Second, if the dynamic model is to have "teeth," it must have a small number of factors (potentially the prices of a subset of bonds themselves) to explain movements in the smoothed discount function—the state equation.

In Section 2 a brief explanation of equilibrium and risk-neutral valuation of assets, and their application to bonds, is given. In Section 3, a detailed exposition is presented of the risk-neutral method in a simple "tree-diagram" context. In Section 4, the calibration of the risk-neutral model with the observed term structure is discussed. Equilibrium models of the term structure are introduced in more detail in Section 5, where the relative degree to which equilibrium and risk-neutral models fit the observed term structure is also discussed. The issue of the number of factors underlying shifts in the term structure is discussed in Section 6. Finally, Section 7 contains a brief summary and discussion of issues for future research.

2 Brief Overview of Asset Valuation Methods Applied to Bonds

2.1 Introduction

The analysis here will deal with the pricing of zero-coupon bonds which pay, with certainty, a standardized face value of 100 at maturity date T, $T = 1, ...\overline{T}$, where $\overline{T}$ is the longest maturity bond.[6] The payoff 100 is in units of a given country's currency, e.g. \$100 in the U.S. or Y100 in Japan. Given *covered interest rate parity*, a U.S. investor will be indifferent between a U.S. Government bond or a fully hedged investment in Japanese Government bonds.[7]

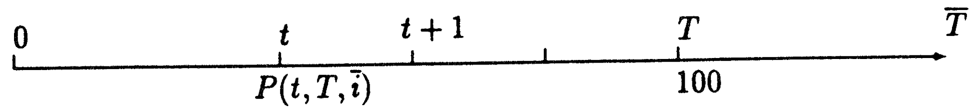
Current time will be denoted by t. "States" $i = 1, ...S$ are used to represent uncertainty: today's state $\bar{i}$ is known, but future states are unknown. The abstract-sounding states will become nodes in "tree diagrams" used to illustrate the uncertainty in bond prices. To fix ideas, a state could refer to a level of production technology on which payoffs on some of the investments in individuals' portfolios depend, or to something that influences enjoyment from consumption—a "consumption technol-

[6]Interestingly, the maturity range $\overline{T}$ seems to differ substantially across countries—30 year Government bonds and now 50 year corporates in the U.S., but rarely more than 10 year maturities with any liquidity outside the U.S. Further, the maturity structure of debt (especially corporate debt) seems to shift substantially over time within countries. There does not seem to be much analysis of the economic and institutional factors responsible for these differences.

[7]Let $S(t)$ be the spot exchange rate at time t in dollars per yen. Let $F(t, t+1)$ be the one-period forward exchange price, at time t, of dollars per yen. A U.S. investor who invests one dollar to buy a (t+1) maturity U.S. Government bond for $P(t, t+1, \bar{i})$ earns a known return of $R(t, t+1, \bar{i})$. Alternatively, the investor can convert the dollar to $\frac{1}{S(t)}$ yen, which when invested in Japanese Government bonds, will earn the yen-denominated riskless return $R^*(t, t+1, \bar{i})$. This payoff can be sold forward for $\frac{(1+R^*)F(t,t+1)}{S(t)}$ dollars at time t (assuming no default risk on the forward market). If *covered interest rate parity* $(1+R) = \frac{F}{S}(1+R^*)$ obtains, as it must to rule out the possibility of arbitrage profits, then the U.S. investor will be indifferent between the two alternatives. (Actually, the location of the investor becomes unimportant—the "U.S. investor" is one who purchases consumption goods in the U.S.). Interestingly, ***uncovered interest rate parity***, in which the U.S. investor does ***not*** hedge the yen investment and thus expects to receive $\frac{1}{S(t)}(1 + R^*)E_t[S(t+1)]$, where $E_t[S(t+1)]$ is the time t expectation of the time $t+1$ price of dollars in yen, does not seem to hold very well historically.

ogy."

The price of a *discount bond* will in general be a function of the current time t, the bond's maturity T, and the current state $\bar{i}$. The following time line illustrates the time sequence:



The *term structure* of bond prices at time t and state $\bar{i}$ is the shape of $P(t, T, \bar{i})$ for increasing T.

If a bond matures at time T=t+1, then its price at time t+1 must, with certainty, equal 100. At time t, its price in state $i = \bar{i}$ can be set equal to:

$$P(t, t+1, \bar{i}) = \frac{100}{R(t, t+1, \bar{i})} \tag{1}$$

which defines the one-period *riskless return* $R(t, t+1, \bar{i})$ as the return that an investor would receive *with certainty* by buying the bond at time t for $P(t, t+1, \bar{i})$ and redeeming it at time t+1 for 100. Next, consider bonds which mature at time t+2 and beyond. Even though these bonds have a sure payoff at maturity, their time t+1 prices are *uncertain* at time t—future states are uncertain, and thus so are future bond prices and interest rates. The following diagram, reproduced from Ho and Lee (1986), shows term structures at time 0, 1, and 2. In the diagram, as time changes from 0 to 1, the bond which is specifically highlighted as a 3 year bond at time 0 becomes a two year bond at time 2, and its time 1 price depends upon whether state 1 or state 0 is realized. At time 2, it will be a one-year bond, and its price will depend upon which of the three time 2 states is realized. In general, as time passes, the bond's price approaches its par value of 100 and price volatility decreases.

Fig. 1 here

2.2 Equilibrium Pricing Models

How are the time t prices of bonds determined, given the uncertainty in $t+1$ prices? In an ***equilibrium*** approach to asset pricing, bonds are treated as just one asset in an investor's portfolio.[8] In the standard analysis, representative investors are assumed to make their consumption and portfolio decisions so as to maximize the sum of their (additively separable) expected utilities of consumption at each time τ in state $\tilde{i}$, $u\left[C(\tau,\tilde{i})\right]$. A necessary condition for an interior maximum is that, at any time t, the increase in an investor's satisfaction from selling a marginal dollar's worth of assets at t, and thus giving up expected future consumption, must just balance the marginal utility of consuming the proceeds at t. Selling the time-T maturity bond at time t at price $P(t,T,\bar{i})$ and consuming the proceeds gives a *marginal* increase in satisfaction of $P(t,T,\bar{i})u_C\left[C(t,i)\right]$. The expected loss in utility from not holding onto the bond is the expectation of its uncertain liquidation value at t+1, $P(t+1,T,i)$ times the marginal utility from consuming these (uncertain) proceeds at $t+1$, which is $u_C\left[C(t+1,i)\right]$. That is, in intertemporal equilibrium:

$$P(t,T,\bar{i})u_C\left[C(t,\bar{i})\right] = \Sigma_{i=1}^{S} Q_i P(t+1,T,i)u_C\left[C(t+1,i)\right] \tag{2}$$

where Q_i is the probability of state i at time t+1.[9] It is convenient to define $m_i(t,t+1,\bar{i}) \equiv \frac{u_C[C(t+1,i)]}{u_C[C(t,i)]}$; in words, $m_i(t,t+1,\bar{i})$ is the price at which, at the margin, time-t consumption in state $\bar{i}$ is traded off against (uncertain) time-$(t+1)$ consumption in

[8]If the bonds are in zero gross supply, the bond prices will be shadow prices.

[9]This transition probability could depend upon the state at time t, and it will definitely depend upon the time-to-maturity of the bond, but the dependence is suppressed for the time being to simplify notation.

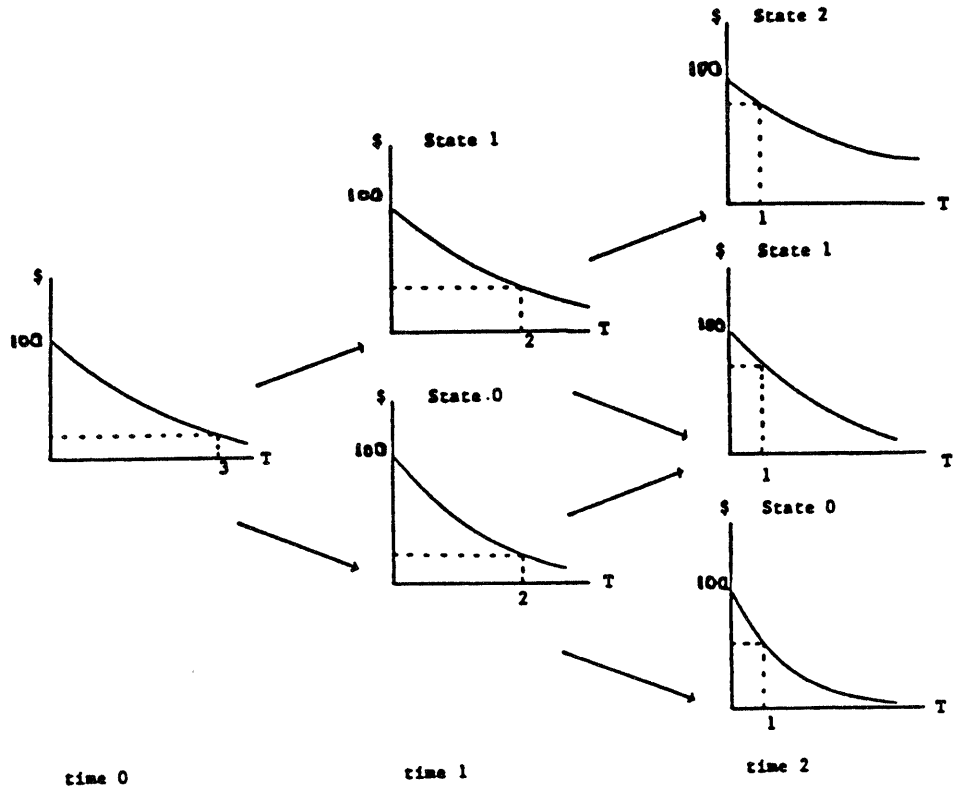


Fig. 1: Example of dynamics in the discount function (Reproduced from T. Ho and S. Lee, "Term structure movements and pricing interest rate contingent claims," Journal of Finance 51(5), Fig. 1)

state $i = 1, ..., S$. The restriction implied by the equilibrium in (2) is that m_i does not depend upon the specific asset which is used as the investment vehicle to transfer a dollar to time $t+1$: all assets are equally desirable at the margin. Using the definition of m_i and rearranging (2) gives:

$$P(t,T,i=\bar{i}) = \Sigma_{i=1}^{S} Q_i \left[m_i(t,t+1,\bar{i}) P(t+1,T,i) \right] \tag{3}$$

The Euler equation (2) has to hold between time t and any future point in time, not just between t and $t+1$. In particular, it must hold between t and the maturity time T of each bond. Thus, (3) can be rewritten as:

$$P(t,T,\bar{i}) = 100 \Sigma_{i=1}^{S} Q_i m_i \equiv \frac{100}{\left[R(t,T,\bar{i})\right]^{T-t}} \tag{4}$$

where $R(t,T,\bar{i})$ is defined as the $(T-t)$-period *spot rate of interest* or *yield-to-maturity.* Some typical yield curve shapes are illustrated in Fig. 2.

Fig. 2 here

The Euler equation (2) is a necessary condition on consumption and investment for individuals to maximize their expected utility of lifetime consumption, e.g. Samuelson and Merton (1969), Merton (1971), Rubinstein(1976), and Lucas(1978). LeRoy(1982) and Breeden (1986) specifically discuss the implications of the Euler equation for bond pricing. The one-period capital asset pricing model, the intertemporal capital asset pricing model, and the arbitrage pricing theory can all be derived from (2), so bond pricing[10] in this framework makes it consistent with these models.

[10]The pricing formula can be derived in terms of either real consumption or nominal consumption expenditures, so it can be considered an equilibrium model for either the *real* or *nominal* term structure and bond prices. (If the investor's known consumption bundle consists of, say, a home in the U.S., a Japanese car, and vacations in Paris, the known payoffs would be converted using *forward* currency rates which are known at time t. Uncertainty about the relative prices of consumption goods can affect real spot rates of interest, however.

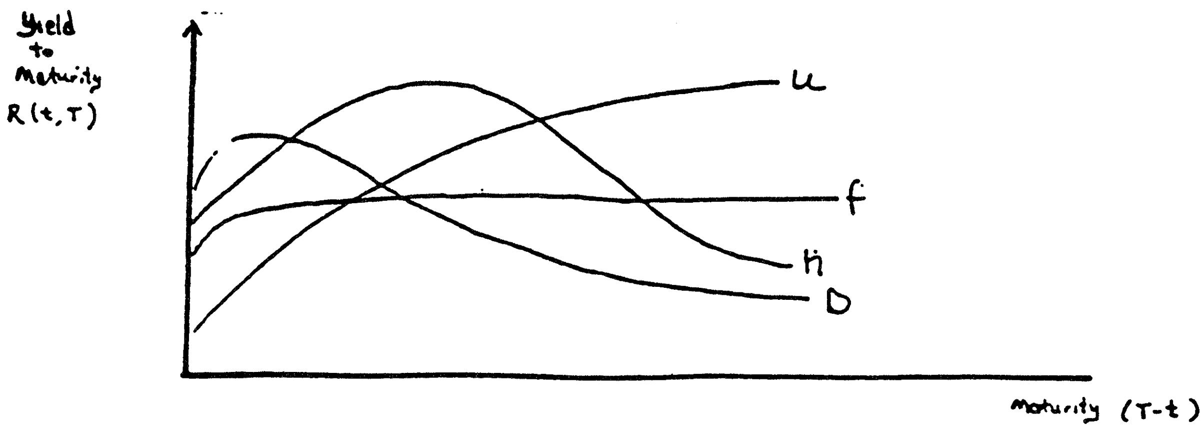


Fig. 2: Alternative potential shapes of the yield curve at time t; u = upward sloping yield curve; D = downward sloping yield curve; f = flat yield curve; and h = humped yield curve

The concept of states is not fully exploited in the development of (3). The most primitive securities in the state pricing framework are those that pay a dollar in a given state and zero otherwise. Any asset can be "assembled" from these primitive securities and priced *relative to* them by arbitrage-free techniques. For example, since the $(T-t)$ maturity bond pays off in all states at time T, its price $P(t,T,\bar{i})$ in (4) must be equal to the sum of the prices of the primitive securities paying off in the different states at time T. If it doesn't, arbitrage profits could be earned by making a portfolio of all the primitive securities which, like the bond, would be guaranteed to payoff in every state at time T, but would have a time t price which is different from the bond. Breeden and Litzenberger (1978) and Banz and Miller (1978) show that the prices of the primitive securities—state prices—can be obtained as the prices of options on aggregate consumption and the market index, respectively. The prices of bonds can then be calculated in terms of the state prices. These state prices, and thus the bond prices, will still reflect the same probablilities of consumption payoffs and their marginal utility in the payoff states, just as in (2) above.

Many empirical tests find that the Euler condition fails to hold as a restriction across the period-by-period returns on assets, including default-free Treasury securities (e.g. Hansen and Singleton (1982)). However, it is difficult to know how to interpret these test results. The generality of the Euler condition as a null hypothesis regarding asset returns is, *ipso facto*, a weakness insofar as its rejection for any particular specification of the utility function doesn't tell us much—the generality admits a plethora of alternatives, one of which will eventually fit the data. In Stigler's words, "it takes a model to beat a model," which can only be done by putting more flesh around the Euler equation by developing more detailed models of the the pricing operator $m_i(t,t+1,\bar{i})$. This is the subject of Section 5. An alternative approach, which *does* accomplish the objective which seems to underlie the Euler equation tests, is to formulate the conditions on prices which are necessary to rule out arbitrage opportunities. Such opportunities must, of course, be absent in any equilibrium.[11] Assuming that arbitrage opportunities can't exist, then these will be minimal conditions which must hold on prices. From a practical point of view, if the conditions are not satisfied, "yield curve arbitrage" profits are possible.[12] In addition, following Black and Scholes (1973), Merton (1973), and Cox and Ross (1976), the no-arbitrage conditions are all that will be needed for pricing contingent claims. The no-arbitrage conditions on bond prices are now derived.

[11]Parital equilibrium models, in which the $m_i(t, t+1, \bar{\imath})$ is partly given exogenously (e.g. exogenous risk premiums) could be considered another "less ambitious" way to go. However, as Cox, Ingersoll, and Ross (1985) emphasized, arbitrarily specifying some components of the equilibrium is a dangerous business, since there is no longer any guarantee that the equilibrium does indeed imply the absence of arbitrage opportunities. Constantinides (1992) suggests that the $m_i(t, t+1)$ be represented as a reduced form statistical process which would satisfy restrictions on bond prices.

[12]Checking the non-arbitrage conditions is not completely mechanical in practice due to the potential effects on bond prices of illiquidity, high or low coupon rates, and other idiosyncracies like those discussed in Section 1. It is also not mechanical in the sense that there are only potential arbitrage opportunities among N points on the yield curve if they are spanned by k¡¡N risk factors. This is a matter of risk model specification, e.g. early writers such as Culbertson (1957) believed that the market for bonds of different maturities was partly segmented, i.e. there were in principle as many factors as maturities.

3 Arbitrage-Free Restrictions on Default-Free Bond Prices

[13]

3.1 Introduction

Since the one-period riskless return $R(t,\bar{\imath})$ is a known constant conditional on the state at time t, the equilibrium pricing equation (3) can be rewritten:

$$P(t,T,i=\bar{\imath}) = \frac{1}{R(t,\bar{\imath})}\Sigma_{i=1}^{S} Q_i \left[m_i R(t,\bar{\imath}) P(t+1,T,i)\right] \quad (5)$$

Cox and Ross (1976), Ross (1978), Garman (1978), and Harrison and Kreps (1979) pointed out that, *if* arbitrage opportunities do not exist, then (5) can be formulated as:

$$P(t,T,i=\bar{\imath}) = \frac{\Sigma_{i=1}^{S}\theta_i P(t+1,T,i)}{R(t,\bar{\imath})} \quad (6)$$

where the nonnegative numbers θ_i are valid probabilities:

$$\Sigma_{i=1}^{S}\theta_i = 1 \quad (7)$$

The θ_i are called *pseudo* probabilities or *risk neutral* probabilities. The reason for

[13]The exposition in this section and the next most closely resembles that in Cox (1986) and Black, Derman, and Toy (1990). The basics are due to Black and Scholes' (1974) and Merton's (1973) ideas on the basic role of replication, and to Cox, Ross, and Rubinstein's (1979) paper in which these ideas were cast in the "tree diagram" context used here. (Mark Rubinstein credits William Sharpe with suggesting the binomial approach during a break at a 1975 conference in Din Bokek, Israel.) In addition, Vasicek (1977), Richard (1978) and Dothan (1978) derived closed-form expressions for the term structure in continuous-time under the assumption that bond prices are a function of the (instantaneous) short nominal or real rate of interest, as is the case in discrete-time-space in this section. If their models are solved "as if" investors are risk-neutral, their approach is the continuous-time analog of that used here; Indeed, this interpretation of their models is perhaps most appropriate since the partial-equilibrium dynamics they used were not intended to be endogenous to some full-fledged representation of the actual economy.

this terminology is readily seen if Equation 6 is rearranged:

$$E_\theta \left[P(t+1,T,.)\right] = P(t,T,\bar{\imath}).R(t,\bar{\imath}) \tag{8}$$

where E_θ is the expectation of the one-period-ahead bond price $P(t+1,T,.)$ under the risk-neutral probabilities $\{\theta_i\}$. That is, the bond's expected return computed using the risk neutral probabilities $\{\theta_i\}$, $E_{\theta_i}\left[P(t+1,T,.)\right]/P(t,T,\bar{\imath})$ equals the riskless return, just as it would if investors were risk-neutral.[14] Since the $\{\theta_i\}$ are non-negative and sum to unity, (7) says that non-arbitrage implies the existence of a linear pricing operator which can be used to value assets (e.g. Ross (1978)).

Garman derived (6) and (7) by using Farkas' lemma. Harrison and Kreps (1979) provided a recipe for calculating the risk neutral probabilities $\{\theta_i\}_i$ in (5) from the probabilities $\{Q_i\}_i$ in (6). They showed that the probability measure for the risk-neutral probability distribution, which here makes the discounted bond prices[15] a martingale, is a transformation of the original probability distribution $\{Q\}$. If investors have von Neumann-Morgenstern utilities, then this transformation impounds any aversion to risk in the probability distribution $\{Q\}$ by weighting the probability of each outcome by the marginal utility of wealth of that outcome—the $m_i(t,t+1,\bar{\imath})$ in (5).Samuelson and Merton (1969) referred to these transformed probabilities as "effective probabilities" and "util probs." Sharpe (1993) points out that the transformed probabilities can also be interpreted as *forward prices*, and suggests that it is confusing to refer to them as "probabilities." Cheng (1991) gives some examples of the transformation in the fixed income context, where $\{Q\}$ describes bond price move-

[14]The hypothesis that investors are risk-neutral is typically referred to as the *expectations* or *local expectations* hypothesis in the term structure literature. The actual probabilities of bond price movements will not equal the risk-neutral probabilities unless this local expectations hypothesis holds.

[15]More precisely, the discounted pricing functional. The discounting operation is the same as making one-period bonds the numeraire in pricing.

ments. If markets are complete, the transformation is unique, i.e. the risk neutral probabilities $\{\theta_i\}$ are unique (Harrison and Pliska (1981), Huang and Litzenberger (1988)).

The remainder of this section is devoted to a further explanation of this risk-neutral valuation formula (6). It is important to realize that the formula is not necessarily a model of changes in bond prices *per se*; rather it is the valuation formula that applies when we treat the economy *as if* it is risk neutral and arbitrage opportunities are ruled out in the pricing of fixed income claims.

3.2 Bond Pricing with Binomial Interest Rate Movements

Suppose that there is only a single "state variable," the one-period rate of interest, forcing movements in bond prices (e.g. Cox, Ingersoll, and Ross (1985)). In this case, the uncertainty in (say) two-period discount bond prices can be spanned by constructing a portfolio containing the three period bond and riskless borrowing and lending. The proportion of three-period bonds which must be held in the spanning portfolio will depend upon the sensitivity or *delta* of that bond's price, relative to the two-period bond's price, with respect to movements in the interest rate, which is the single source of uncertainty in bond prices. Spanning the two period bond with a portfolio of the three period and one period bonds is analogous to replicating a stock option's payoff with a portfolio of the underlying stock and riskless borrowing and lending.[16]

[16]Note that, so long as stock prices follow a geometric Brownian motion, it can be shown that the no-arbitrage price of the stock option *must* depend upon the price of the stock, and *only the* price of the stock (Merton(1977)). In the bond example here, it was necessary to make a *structural assumption* that a single state variable underlies bond price movements. However, it can also only be a *structural* assumption that stock prices are a geometric Brownian motion; if stock price volatilities are not constant, for example, option prices can depend upon those volatilities as a second state variable. Generally, there are several important assumptions that are built into stock price trees,

To explain this parallel between arbitrage-free or risk-neutral bond pricing and arbitrage-free option pricing more fully, we first simplify notation by referring to the price now (time t=0) of the two period bond, $P(0, T = 2, i = \bar{i})$ as G, the present price of the three period bond, $P(0, T = 3, i = \bar{i})$, as H, and the riskless return (one plus the rate of interest) $R(t, \bar{i})$ simply as R. Next, assume that at time t=1, interest rates, and thus bond prices G and H, can move to one of only two possible states—that is, prices can only move either up or down, to H_U or H_D and G_U or G_D respectively, when one plus the interest rate moves down to R_D or up to R_U. The step sizes $(\frac{H_U}{H}, \frac{H_D}{H})$ and $(\frac{G_U}{G}, \frac{G_D}{G})$ will differ because the maturities of these bonds differ. Specifically, the spread $\frac{G_U - G_D}{G}$ is 0.095, which is smaller than $\frac{H_U - H_D}{H}$=0.232 because bond G's time-to-maturity is shorter, and all default-free bond prices must approach their face values as maturity approaches[17]

These binomial movements in R, G, and H are plotted on a "tree diagram" in Fig. 3(a). At the end of next period, at time 1, bond G will become a one-period bond whose default-free time 2 payoff is known with certainty because the period 2 interest rate will then be known with certainty. In the case of bond H, its price can again move up or down in the second period in response to the change in one-period interest rates between time 1 and time 2, after which time it also becomes a one-period bond. In Fig. 3, the movements in interest rates and the price of bond H are assumed to be *path independent*. That is, the bond price at time 2 is the same irrespective of whether it went up in the first period and down in the second period, or down in the first period and up in the second period. Here, path independence requires that the *volatility* of percentage changes in the bond price do not depend upon the level of the

particularly path independence (or the process which is transformed to path independent).

[17]This could be regarded as a minimum condition on the behavior of a bond's volatility over time.

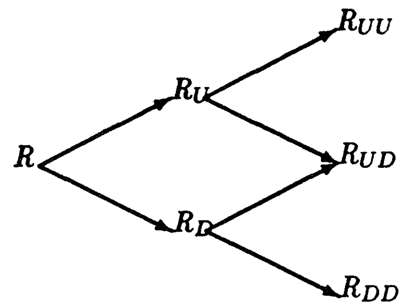
bond price.

In Fig. 3(b), numerical values have been attached to the branches of the tree, and these will be used in order to illustrate results as they are developed. The first period interest rate is assumed to be 8%. It is assumed that the interest rate will go up or down by 50% in the second period. That is, it will go up to 8% x (1+0.5) = 12% or down to 8% x (1-0.5) = 4%.[18] Bond G will then be worth either \$96.15 (if R_2=1.04) or \$89.29 (if R_2=1.12). Bond H's price is known at time 2, when it becomes a one-period bond.

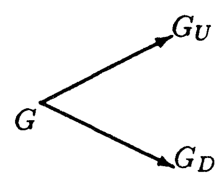
[18]Although this specification is for illustration only, the empirical results in Marsh and Rosenfeld (1983) and Chan, Karolyi, Longstaff, and Sanders (1990) suggest that such a model of proportional interest rate changes is not wildly unreasonable for nominal short interest rates measured over short intervals.

Figure 3(a)

One-Period Interest Rate



Two-Period Bond G



Three-Period Bond H

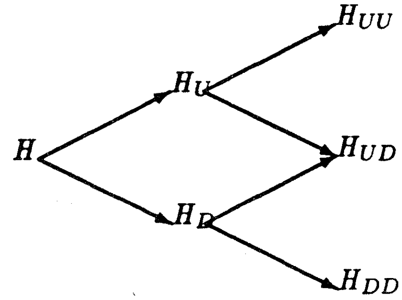
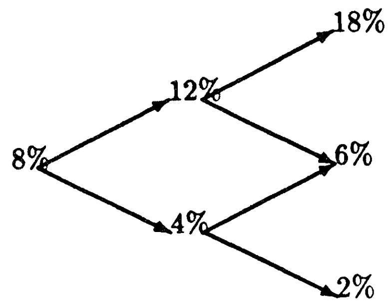
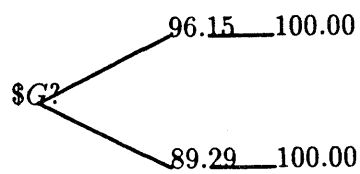


Figure 3(b)

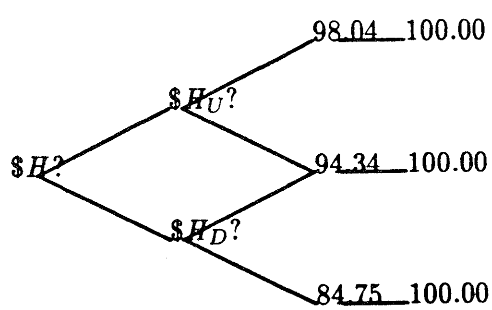
One-Period Interest Rate



Two-Period Bond G



Three-Period Bond H



Equation 7 tells us that, if arbitrage opportunities are ruled out, the time 0 value of bond G in Fig. 1(b) is some weighted average of $G_U = \$96.15$ and $G_D = \$89.29$, discounted at the period 1 riskless rate of interest of 8%. The weights, which are the risk neutral probability weights in Equation (7), are now derived.

So long as there is a single interest rate factor which causes (perfectly correlated) shifts in bond prices of all maturities, the end-of-period 1 value of bond G can be replicated by creating a portfolio of bond H and riskless borrowing at time 0. In particular, the portfolio must contain Δ units of bond H and a one-period default-free loan of B, where Δ and B are chosen such that:

$$\Delta H_U + RB = G_U \tag{9}$$

$$\Delta H_D + RB = G_D \tag{10}$$

Solving these equations gives:

$$\Delta = \frac{G_U - G_D}{H_U - H_D} \tag{11}$$

$$B = \frac{H_U G_D - G_U H_D}{(H_U - H_D)R} \tag{12}$$

Since the portfolio has the same payoff as bond G, the time 0 value of bond G must be:

$$G = \Delta H + B \tag{13}$$

To see the restriction which this no-arbitrage constraint implies across the term structure of bond prices at a point in time, it is necessary to introduce probabilities for interest rate, and thus bond price, changes. We let Q be the probability of a downward movement in interest rates, and thus the probability of an upward movement in bond prices. Then, multiplying (9) by Q and (10) by $(1-Q)$ and adding them together:

$$QG_U + (1-Q)G_D = \Delta\left[QH_U + (1-Q)H_D\right] + RB \tag{14}$$

Imposing the no-arbitrage condition $G = \Delta H + B$, Equation 14 becomes:

$$QG_U + (1-Q)G_D - RG = \Delta\left[QH_U + (1-Q)H_D - RH\right] \tag{15}$$

Further, the *delta* of the no-arbitrage portfolio is $\Delta = (G_U - G_D) \div (H_U - H_D)$, so Equation 15 becomes:

$$\frac{QG_U + (1-Q)G_D - RG}{G_U - G_D} = \frac{QH_U + (1-Q)H_D - RH}{H_U - H_D} \equiv \lambda \tag{16}$$

Equation 16 has an easy economic interpretation which is the same as that of its counterpart in option pricing. It says that the *return premium* on all bonds, per unit of risk measured by the *volatility* of the bond prices, must be identical across bonds. That is, a bond's premium per unit of risk cannot depend upon its maturity, though it could change as a function of the level of interest rates or calendar time. In general, it will also be a function of the length of the time interval in the tree diagram.

Rewriting Equation 16 for bond G, the no-arbitrage Equation 6 becomes, in this binomial context:

$$G = \frac{[(Q-\lambda)G_U + [1-(Q-\lambda)]G_D]}{R} \tag{17}$$

where $(Q-\lambda)$ is the *risk aversion adjusted*, *risk neutral*, or *pseudo* probability θ in Equation 7. We will refer to Equation 17 as the *risk neutral valuation* formula, though as Sharpe (1993) emphasizes, it is really just a forward valuation equation in which the risk-neutral "probabilities" are Arrow-Debreu forward prices of the payouts G_U and G_D.

Since the derivation of the non-arbitrage constraint on bond prices is identical to that of the binomial option pricing formula, it is not suprising that $(Q - \lambda)$ is analogous to the risk-neutral probability in that formula. For example, using the standard terminology in Cox and Rubinstein(1985, p. 173), if the interest rate becomes uR in the upstate (u=1.5 in the numerical example in Fig. 3(b)), and dR in the downstate (d=0.5), then $p' \equiv \frac{u-R}{u-d}$ is the risk-neutral probability of interest rates going up. Thus, $p \equiv (1 - p')$ is the risk-neutral probability of an increase in bond prices.[19] In the numerical example, $p = \frac{1.5-0.8}{1.5-0.5} = 0.3$.

3.3 Risk (Liquidity) Premiums on Bonds

If the *local expectations* hypothesis holds, then $\lambda \equiv 0$. In this case, the *risk neutral* probabilities are the *actual probabilities* of bond price movements. If $\lambda(R_U - R_D) > 0$, the holding period returns on a long-term bond exceed the riskless return on a one-period bond. So long as the variation in interest rates, $R_U - R_D$, reflects the variation in the representative investor's marginal utility of wealth, then the risk premium on bonds is compensation for the *systematic risk* of bond price movements. It is unclear whether this explanation for the risk premium is consistent

[19]If λ is interpreted as an expected stock price change in excess of the return on a riskless bond position, and Q the probability that the stock's price moves up, then it may be verified that the binomial call option pricing formula can be written (using the terminology of Cox and Rubinstein, pp. 172-173), as:

$$C = \frac{(Q-\lambda)C_U + [1-(Q-\lambda)]\,C_D}{R}$$

When written this way, the call option price C depends upon the risk premium on the stock, as well as on the probability of stock price movements. However, C can be rewritten (Cox and Rubinstein (p.173, (3)) in terms of risk-neutral probabilities which depend only upon parameters defining movements in the stock's price, u and d (as well as the interest rate):

$$C = [pC_U + (1-p)C_D]\,/R$$

where:

$$p \equiv \frac{(R-d)}{(u-d)}$$
$$(1-p) \equiv \frac{(u-R)}{(u-d)}$$

In like fashion, the formula for the price of bond G in (12) can be rewritten as:

$$G = [pG_U + (1-p)G_D] \div R$$

where:

$$p = \left[\frac{(RH - H_D)}{(H_U - H_D)}\right]$$

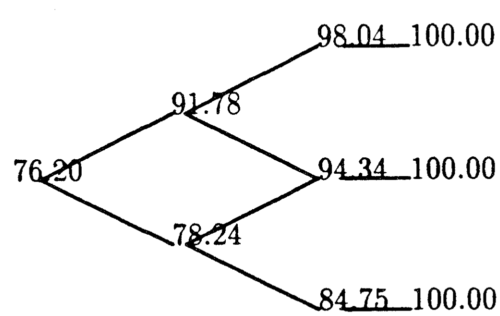
Now the parameters for the interest rate probability and bond risk premium are impounded in the price of bond H, whose price movements can be used to price bond G.

with Keynes' argument that long-term bonds are priced to yield a premium because investors have a *liquidity preference*—he proposed (Keynes (1935, p. 169)) that one source of liquidity preference is the "...risk of a loss being incurred in purchasing a long-term debt and subsequently turning it into cash" because "the future rate of interest is uncertain"; if the uncertainty in interest rates is due to the same factors that cause uncertainty in investors' well-diversified portfolios, Keynes' liquidity premium is essentially the same as the risk premium here.

3.4 Numerical Example (contd.)

To illustrate the risk-neutral valuation result (17), we complete the numerical example in Fig. 1(b). Since only the risk-neutral probability $(Q - \lambda)$ is relevant in determining the risk-neutral price, we arbitrarily set $Q = 0.5$. We assume for the purposes of the example that the risk premium $\lambda = 0.2$ in all periods. Then the risk-neutral prices of bond H at times 0 and 1 are given in Fig. 4.

Figure 4: Bond H



For example, in Fig. 4, the value of bond H at time 1 if interest rates had gone up in the first period is:

$$H_D = [(Q-\lambda)H_{UD} + [(1-(Q-\lambda))H_{DD}] \div R_{2,U} \quad (18)$$

$$= [0.3(94.34) + 0.7(84.75)] \div 1.12 \quad (19)$$

$$= 78.24 \quad (20)$$

and similarly for H_U. H is then:

$$H = [(Q-\lambda)H_U + [1-(Q-\lambda)H_D] \div R \quad (21)$$

$$= [0.3(91.78) + 0.7(78.24)] \div 1.08 \quad (22)$$

$$= 76.20 \quad (23)$$

The *dispersion* of possible percentage price changes over the first period is greater for the three-year bond H than for the two-year bond G. Bond H's price increases from 76.20 at the beginning of the period to 91.78 if interest rates go down, but increases to only 78.24 if interest rates go up. These are percentage changes of 20.4% or 2.68% respectively; the volatility of the change is 11.54%. Bond G's price increases by 13.67% or 5.56% in these same two interest rate scenarios; the volatility is 9.61%. This greater range of uncertainty in the returns on the longer-term bond is not specific to the illustration. Long-term bond prices fluctuate more, over short holding periods, than short-term bond prices. The illustration makes clear that the reason for this is the one usually given in practice—if interest rates go up or down, there are more periods over which the long-term bond's payoff must be discounted by the higher interest rates along those branches of the tree.

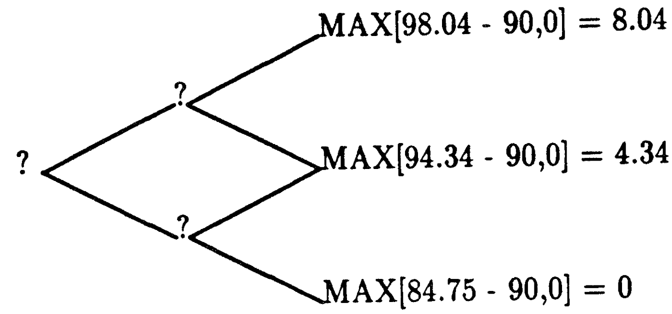
3.5 Pricing Fixed Income Contingent Claims

Contingent claims on fixed income securities can all be priced, and hedges computed, from the same lattice, like that plotted in Figure 3, on which interest rate and non-arbitrage bond price movements are drawn.[20]

To illustrate the pricing of claims, suppose that we want to find the time 0 value of a European call option on the three-year bond H, with a strike price of $90.00 and a maturity of two years. The terminal payoffs on the bond, which are given in Fig. 5, can be used to calculate the option payoffs.

[20] Using the same lattice for *all* contingent claims will ensure that they are priced consistently, that hedged positions can be aggregated for risk management, etc.. On the other hand, the one lattice won't generally be the most precise possible for any one security.

Figure 5



For example, if the period-3 interest rate turns out to be 2%, then the value of bond H at the beginning of the third period—the time at which the call option matures—is $98.04, and the option payoff is $8.04. We can then follow the familiar backward recursion along the tree diagram, using the risk-neutral valuation formula, to find the time 0 value of the option. Letting C_U and C_D refer to the time 1 value of the option if bond prices go up or down respectively in period 1:

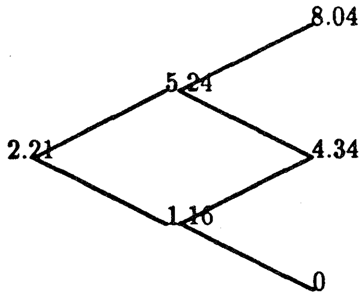
$$C_U = \{0.3(8.04) + 0.7(4.34)\} \div 1.04 = 5.24 \tag{24}$$

$$C_D = \{0.3(4.34) + 0.7(0)\} \div 1.12 = 1.16 \tag{25}$$

Finally, at time 0, the value of the call option C is:

$$C = [0.3(5.24) + 0.7(1.16)] \div 1.08 = 2.21 \tag{26}$$

The tree diagram for the call option values is:



The delta for the call option can be calculated in the standard way. For example, if the interest rate moves to 12% at the end of period 1,$\Delta = \frac{4.34-0}{94.34-84.75} = 0.4526$, while if the interest rate moves to 4%, $\Delta = \frac{8.04-4.34}{98.04-94.34} = 1.0$. (Note that, in this example, if the period-2 interest rate is 4%, the option will finish in- the-money whatever the outcome for the period-3 interest rate). The delta at time 0 is: $\Delta = \frac{5.24-1.16}{91.78-78.24} = 0.3013$.

In calculating the risk-neutral value of the call option to be 2.21, the risk-neutral probabilities $[Q-\lambda]$ and $[1-(Q-\lambda)]$ were required. However, $(Q-\lambda)$ can be re-stated so that it depends only upon the price of bond H, the volatility of bond H price movements, and the riskless return, i.e. $(Q-\lambda) = \frac{RH-H_D}{H_U-H_D} = \frac{1.08(76.20)-78.24}{91.78-78.24} = 0.3$.

3.6 Tree Diagrams in Spot Rates, Forward Rates, or Bond Discount Functions?

For the purposes of pricing contingent claims on bonds, Heath, Jarrow and Morton (HJM)(1989)(1990)(1991) have suggested transforming the no-arbitrage restrictions on bond prices into equivalent restrictions on the forward rates implied by those bond prices. To explain this forward rate approach, it is simplest to express bond prices in terms of their continuously compounded forward rates of return:

$$P(t,T) \equiv e^{-\int_t^T f(t,\tau)d\tau} \tag{27}$$

where the *forward rate* $f(t, \tau)$ is the rate of return, established at time t, on a default-free loan to be made at time τ in the future for an instant's duration. By construction, $f(t, t)$ is equal to the instantaneous riskless rate of interest at time t.

Given the relation (27) between bond prices and forward rates, probability models for bond prices can be transformed into probability models for the forward rate, and vice versa. In particular, given an assumed volatility of forward rates, the risk-neutral probability of bond price movements can be transformed into an adjustment to the drift of the stochastic process for forward rates.[21]

To illustrate the calculation of forward rate probabilities which are consistent with risk-neutral bond values, consider first the two-period bond G in our example. It has a current risk-neutral value of $84.58, which will rise to $96.15 if interest rates drop to 4% at the end of the year, or will rise to $89.29 if interest rates rise to 12%.[22] Then, using the terminology $f(0, 1)$) to refer to the (implied) forward rate *under risk-neutral valuation* at time 0, for a dollar loan from the end of year 1 to the end of year 2, we can solve:

$$84.58 = \frac{100}{(1.08)(f(0,1))} \tag{28}$$

to find $f(0, 1) \approx 9.47\%$ *under the risk-neutral valuation.* Similarly, using the relative prices of bonds G and H, we can calculate that $f(0, 2) \approx 11.00\%$ under risk-neutral valuation. Thus, under risk-neutral valuation, the forward rate curve at time 0 is: $[f(0, 0), f(0, 1), f(0, 2)] = [0.08, 0.0947, 0.11]$.

Now, what is the change in forward rates from time 0 to time 1, subject to the

[21]Note that the forward rate drift adjustment will reflect both the λ and the fact that the bond price is non-linear in the forward rate, e.g. if investors were risk-neutral, and thus risk-neutral probabilities and actual probabilities were equal, there would still be a drift adjustment in the forward rate to make it risk-neutral-consistent.

[22]Note that the beginning-of-year 2 values are trivially risk-neutral values, since at that time bond G will be a one-year bond whose payoff is known with certainty.

no-arbitrage restriction on bond prices (i.e. under risk-neutral valuation)? If the one-period spot interest rate decreases to 4%, it can be verified that: $[f(1,1), f(1,2)] = [0.04, 0.0476]$; if the rate goes up to 12%, $[f(1,1), f(1,2)] = [0.12, 0.229]$.[23] We cannot decompose this variation between time 0 and time 1 in forward rates into a "drift" component and a "volatility" component until we have some specification for the volatility, i.e. the adjustment to the drift of the forward rate process which is consistent with risk-neutral pricing depends upon the forward rate volatility.[24]

An interesting question, especially from a practical standpoint, is whether there is any reason to prefer that term structure uncertainty be specified in terms of the bond *discount function*, *forward rates*, or *spot rates of interest.* Equation (27) gives bond prices as a function of forward rates, and the "zero maturity" forward rate is just the instantaneous spot rate. So if it is assumed that forward rates are generated by, say, a one-factor diffusion process, then we can change variables from the forward rates to bond prices using (27) and Ito's lemma. In mapping from one variable to another, we can expect restrictions on functional forms (on the volatility structure) for one variable implied by the parameterization for a second. Nevertheless, in the "easiest" cases, it does seem possible to characterize these mappings and restrictions, and in principle it seems possible to make a computer program with a "front-end" which transforms derivative payoffs expressed in the most natural terms (e.g. interest rate caps in terms of forward or spot rates) to a desired common stochastic environment.[25] As long as

[23]If we added a new three year bond at time 1, there would, of course, still be a 3-dimensional vector of forward rates.

[24]See the discussion in HJM(1989,(11))(1989, Section 6)).

[25]In the risk neutral versions of models based on any of the three variables, risk premiums are not required; if forward rates are covariance stationary, then percentage changes in discount bond prices, which are just linear aggregations of forward rate changes, will also be stationary; the non-negativity restriction on forward rates can be applied directly to the discount function (shorter maturity pure discount bond values cannot be less than longer-maturity values); in the bond price formulation, the volatility parameter for a given bond must change over time as its maturity decreases, but the

things are done right, we will get identical contigent claim values in any of the three environments. Unfortunately, the transformations must usually be done numerically in the "extended" or multi-factor models since in most cases they are not analytically tractable. In these instances, it is difficult to obtain general expressions for the restrictions on a parameterization in forward rates implied by a parameterization in bond prices or interest rates, e.g. the restrictions on a forward rate process in HJM implied by, say, the extended CIR model in Hull and White (1990).

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cross-section of volatilities for the discount function need not change as long as the function spans the same range of maturities.

4 Calibrating Bond Pricing Dynamics with the Observed Term Structure

4.1 Introduction

The procedure in the previous section was to specify a lattice for interest rate or bond price movements, and then derive arbitrage-free constraints on bond prices. As Ho and Lee (1986) pointed out, this procedure can be reversed: bond prices can be used to calibrate or specify the lattice, which would then be used for pricing other bonds or interest rate derivatives. The logic parallels that in stock option pricing. There, the pricing function, e.g. the Black-Scholes formula, can be inverted to obtain the *implied volatility* of stock prices which is consistent with observed option and stock prices at a point in time.[26]

4.2 Calibration in the Binomial Example

To illustrate the calibration procedure using the numerical example of the previous section, recall that at time 0, the no-arbitrage prices of bonds G and H were calculated as \$84.58 and \$76.20 respectively. These prices were derived using risk-neutral probabilities and an assumed process for interest rate, or bond price, dynamics. Now, assume that we *don't* know the stochastic process for interest rates, but that bond prices are *observed*; we then want to back out the parameters of the interest rate or bond price process that are consistent with these prices. The local expecations hypothesis *does not* have to be correct in order for us to use observed prices to calibrate

[26] In practice, it is not often possible to find an observed term structure of bonds which are free of imbedded options. In principle, this means that if the risk-neutral model is being calibrated for the purpose of valuing such options, we are implicitly searching for a fixed point in the fit of the term structure which is both adjusted for imbedded options using the model and which serves as a basis for valuing non-nested options.

the risk-neutral model.

Suppose that the following term structure is *observed* at the beginning of period 1: a one-period spot interest rate of 8%, i.e. the price of a zero-coupon bond which matures at the end of period one is \$92.59; the price of the two-period bond is G=\$84.58, which implies a two-period spot rate (or yield) of 8.734%; and the price of a three-year bond H=\$76.20, implying a three-period yield of 9.483%. Suppose for the moment that the risk-neutral probability of a down-move in interest rates, $(Q-\lambda)$, is set at 0.3. Suppose that we also know that the interest rate changes geometrically, i.e. if i is the *rate* of interest $R-1$, then $i_U = i(1+k)$, $i_D = i(1-k)$. Then:

$$84.58 = \left[(Q-\lambda)\frac{100}{R_D} + [1-(Q-\lambda)]\frac{100}{R_U}\right] \div 1.08 \quad (29)$$

$$(30)$$

$$= \left[(Q-\lambda)\frac{100}{(1+i(1-k))} + [1-(Q-\lambda)]\frac{100}{(1+i(1+k))}\right] \div (1+i) \quad (31)$$

With $(Q-\lambda)=0.3$, this one equation can be solved for k; of course, $k=0.50$.

In general, this implied volatility will be approximately that of the actual probability distribution of bond price changes so long as the interval of time in each step is short.

What can we learn by bringing in the observed price of the three-year bond, bond H? If we continue to assume that $(Q-\lambda)=0.3$, then:

$$76.20 = [0.3H_U + 0.7H_D] \div 1.08 \quad (32)$$

$$H_U = [0.3H_{UU} + 0.7H_{UD}] \div 1.08(1-k) \quad (33)$$

$$(34)$$

$$= \left[0.3\frac{100}{1.08(1-k)(1+k)} + 0.7\frac{100}{1.08(1+k)(1+k)}\right] \div 1 + i(1-k) \quad (35)$$

$$H_D = [0.3H_{UD} + 0.7H_{DD}] \div 1.08(1+k) \quad (36)$$

$$(37)$$

$$= \left[0.3\frac{100}{1.08(1-k)(1-k)} + 0.7\frac{100}{1.08(1-k)(1+k)}\right] \div 1 + i(1+k) \quad (38)$$

If H_U and H_D are substituted into the equation for H, it may be verified that the positive root is $k = 0.50$. (we hope: verify)

Of course, under the assumption of *constant* proportional interest rate movements, the solution for k obtained from the three-year bond is redundant. However, still under the assumption that $(Q - \lambda)$ is known, the value of k could be allowed to be different in periods two and three: i.e. we could calibrate a model with a term structure of volatilities from the bond prices. In fact, it is very common for practitioners to calibrate bond pricing models using the volatility of estimated zero-coupon yields on bonds with different maturities. Empirically, the typical term structure of *yield volatility* is downward sloping—i.e. period-to-period variation in zero-coupon yields are a decreasing function of maturity (e.g. Murphy (1987, Ch.5)). (Since the yields are just geometric averages of spot and forward rates, this reflects the regressivity of those rates). A more detailed discussion of what features of the data can be matched with what models is given below.

It might also appear that we could use the information in a cross-section of two-period and three-period bond prices to solve jointly for $(Q - \lambda)$ and the constant interest rate volatility k under the assumption that the former is not known. However, the risk-neutral probability $(Q - \lambda)$ is *not* a free parameter, but changes as the size of the time grid changes. It is easy to see the reason by recalling that bond G's price was derived as:

$$84.58 = \left[(Q-\lambda)\frac{100}{R_D} + [1-(Q-\lambda)]\frac{100}{R_U}\right] \div R \quad (39)$$

Since the return on one-period bonds R is known, once a specification is made for the up and down interest rate movements, $(Q - \lambda)$ must be set so as to make the risk-neutral return on bond G equal to R. For example, the pseudo-probability (Q-λ) converges to 0.5 when the up and down interest rate movements in the binomial tree, R_U and R_D, are chosen to make the binomial process converge to a lognormal diffusion for the instantaneous interest rate as the step size goes to zero (as in Cox, Ross, and Rubinstein (1979)).

Perhaps the easiest way to see that the risk-neutral probability is not a "free parameter" is to remember that it is really just a forward price of the payouts on bond G in the up-state and down-state (see Sharpe (1993) for discussion). To prevent arbitrage, the value G of the bond must equal the discounted value of these forward prices multiplied by the payouts G_U and G_D; conversely, when we know G and the discount rate, we know the forward prices in the binomial case.

In practice, bond values are typically stated in terms of yield-to-maturity. The *yield-to-maturity* on a zero-coupon bond which has $T - t$ periods to maturity and a current price $P(t, T)$ per dollar of face value is defined as $y(t, T)$ where $P(t, T) = [1 + y(t, T)]^{-(T-t)}$. In the case of zero-coupon bonds, the $(T - t)$ yield-to-maturity is also the $(T - t)$ *spot rate* of interest. Turning to our example, the *actual* prices of bonds G and H are 84.58 and 76.20 respectively, implying yields of 8.734% and 9.483%, respectively. In a risk-neutral economy, the *term structure* is then (8.0%, 8.734%, 9.483%) for one, two, and three period maturities, i.e. it is upward-sloping.

The term structure of yields is, of course, related to assumed interest rate behavior. If investors are risk-neutral, then $\lambda = 0$, and the actual probability of the interest rate increasing from 8% to 12% in the next period is 0.7, while the probability of a decrease in the rate to 4% is 0.3. That is, the interest rate is expected to increase from 8% to 9.6% in period two and 9.82% in period three. However, the yields-to-maturity on the bonds are *lower* than their expected returns to maturity as implied by these interest rates (and risk-neutral or local expectations pricing). For example, the expected return on the two-period bond G is (1.08)(1.096) = 1.184, or 8.80% per period, while the two-period yield is $(1.08734)^2 = 1.182$, or 8.73% per period.

This result that the yield-to-maturity is below the expected bond return under risk-neutrality (the expectations hypothesis) is well-known to be due to the convexity of bond prices as a function of interest rates. The price and yield of bond G, for example, was computed as:

$$G = \frac{1}{R_1} E\left[\frac{100}{\tilde{R}_2}\right] \equiv \frac{100}{(1+y(0,2))^2} \tag{40}$$

where $\tilde{R}_2 = \{1.04, 1.12\}$. Since $E\left(\frac{1}{X}\right) > \left(\frac{1}{E(X)}\right)$, $y(0,2) < R_1 E(\tilde{R}_2)$.[27]

The above procedure involves calibrating the proportional interest rate model with the prices of bonds at a point in time. However, if additional information about the distribution of bond prices is available, it can be used to obtain the values of the future possible interest rates, not just their spread (e.g. Black, Derman, and Toy (1990) use yield volatilities). In the binomial tree, we will in fact be able to infer the interest rates at different nodes in the tree. To see how this can be done, let's consider the observed price of the two-period bond in our example:

$$84.58 \quad = \quad \left[(Q-\lambda)\frac{100}{R_D} + [1-(Q-\lambda)]\frac{100}{R_U}\right] \div 1.08 \tag{41}$$

[27]The difference between yields and expected interest rates is perhaps easiest to see when the risk-neutral probabilities of up moves and down moves in the interest rate, $(Q-\lambda)$, and $1-(Q-\lambda)$, are equal to 0.5, so that the expected interest rate in our example is 8% in all periods. Under risk neutrality (the expectations hypothesis), 8% is then also the expected rate of return on bonds G and H in each period. However, it may be verified that bond G's yield is 7.93% (its price is $85.85 under these probabilities), and bond H's is 7.80% (with a price of $79.83). That is, the yields-to-maturity become increasingly lower with maturity relative to the expected spot rate of 8.0%.

With $(Q - \lambda)$=0.3, notice that this one equation cannot be solved uniquely for the two unknown period-2 interest rates R_U and R_D. But, *if* the *actual* probability Q of an up-move or a down-move in interest rates is the same, the expected interest rate in period 2, $[0.5(R_U) + 0.5(R_D)]$, is 0.08. The volatility of interest rates in period 2 is then $[0.5(R_U - 1.08)^2 + 0.5(R_D - 1.08)^2]^{0.5}$. Setting this expression for the volatility of period-2 rates equal to the observed volatility constitutes a second equation which can be used in solving for R_U and R_D. If, for example, we know that the volatility of the two-period spot rate $\left[(1.08)\tilde{R}_2\right]^{0.5}$, where $\tilde{R}_2$ is equal to R_U or R_D, equals 0.0432, then the volatility of the period-2 rate, $\tilde{R}_2$ is 0.04. In the special case of the proportional interest rate model used in the example, it may be verified that the only period 2 interest rates which are consistent with a time 0 price of \$84.58 for the two-period bond and a two-period zero-coupon yield volatility of 0.0432, are R_U=1.12 and R_D=1.04. Alternatively, we could infer that the volatility of the percentage *change* in interest rates between time 0 and time 1 is 50%.

Taking into account the three-period bond H, we have two more pieces of information—the price of the three-period bond and the volatility of the three-period yield. Repeating the procedure above to back out R_{UU}, R_{UD}, and R_{DD}, we will need to ensure that *both* the volatility of the period-3 rate conditional on R_U, and the volatility of the period-3 rate conditional on R_D, are consistent with these two additional pieces of information. In the special proportionality example used here, the previously calculated R_U and R_D can be used together with the observed three-period spot rate volatility to solve for R_{UU}=18%, R_{UD}=6%, and R_{DD}=2%. Alternatively, under the proportionality assumption, we could infer that the volatility of the percentage changes in interest rates from period 2 to period 3 is 50%, *irrespective* of whether the second period riskless return is R_U or R_D.

4.3 Calibration Techniques Generally

At this point, one might reasonably wonder whether there are any general principles to guide the calibration procedure. At one level, the answer is obvious—observable bond prices or yields plus moments of the distribution of those prices are "inputs," and the parameters or nodes of the lattice are the "outputs" (the pseudo-probabilities are not free parameters, and they should satisfy the constraint that they lie between zero and one as soon as the bond price (interest rate) dynamics are specified). Loosely, there need to be enough inputs to derive the outputs. Further, though the discussion here has always assumes that prices and volatilities of cash market prices (or swaps) are used as "inputs," contingent claim prices (e.g. caps, swaptions) could also be used.

In recent work, the tendency has been to generalize the models of bond price (interest rate) dynamics so that they have enough parameters to fit the observed term structure of bond prices and bond price volatilities. For example, Hull and White (1990) allow time dependence in the drift and diffusion parameters in the Vasicek (1977) and CIR (1985) interest rate models in such a way that they are exactly identified from the current term structure of interest rates, the current and future volatilities of the short-term interest rate, and the current term structure of spot or forward rate volatilites. Heath, Jarrow, and Morton (1992) introduced a multi-factor model for forward rates with a parameterization that will fit any existing term structure and any specified volatility structure. Finally, Rubinstein (1994) has recently studied the most general conditions on a binomial tree for which parameters can be calibrated with option prices, i.e. a recipe for what might be called the ultimate "extended" model.[28]

If we interpret the "tree" models of the term structure, or their continuous time counterparts, as essentially "trees" of Arrow-Debreu time-state-contingent prices, then perhaps it is natural to extend their parameterization so as to have the finest grid of states and state prices consistent with observed bond and claim prices. Yet the models and their associated state prices are only as good as the exogenous specification behind the treee diagram models. Since the extended models are of a reduced-form nature and are parameterized so as to fit the desired characteristics of the bond price and/or interest rate-contingent claim data, the exogenous specification can be evaluated only[29] by: (1) analyzing the properties of and restrictions on the future behavior of the term structure that are implied when the model parameters are calibrated with the desired features of the current term structure (e.g. Caverhill (1992b), Webber (1992)). For a *given* set of desired features of the data (moment conditions), the looser the parameterization, the less chance that fitted functions, such as the volatility of the short rate, will have to be "strange-looking" to make them consistent with the data.[30]; or (2) simulating their performance in pricing interest rate dependent contingent claims, which is essentially doing (1), but using the contingent claim prices as the loss function in terms of which to evaluate the models. For example, Hull and White (1990) compare the performance of their extended models in pricing bond options and interest rate caps, and also against the performance of selected

[28] His discussion is in terms of stock options.

[29] Perhaps a third criterion should be computational tractability, since extended models will typically not have closed form solutions (the lattice becomes path-dependent, rather than path-independent, as in the illustration in Section 3). But computer power keeps increasing.

[30] For example, in the Section 4 example (and Black, Derman, and Toy (1990)), a time dependent volatility parameterization was calibrated with an observed term structure of bond prices and term structure of volatilities. To "shoehorn" the observed data into this parameterization with a constant interest rate drift can produce an implausible fitted time dependence in volatility. Broadening the parameterization to include a mean-reverting drift in interest rates can alleviate this problem.

two-factor models. Flesaker (1990) and Amin and Morton (1993) fit various versions of the Heath, Jarrow, and Morton (1992) model to Eurodollar futures option prices. Amin and Morton find that put options tend to be overpriced relative to call options, and that there are significant biases as a function of strike price and maturity for all versions of the Heath, Jarrow, and Morton model that they study. They also find enough instability in their two parameter specifications for volatility in that model to suggest that a one parameter specification is preferable for valuing options (or at least options like Eurodollar futures options) with maturities less than one year—intuitively, a single short-term interest rate volatility explains most of the variation in the prices of short maturity bonds.

The above procedures involve the calibration of *prespecified* models for interest rate or bond price dynamics with the observed term structure (under the local expectations hypothesis). For example, in the numerical illustration, the assumption that movements in a single state variable—the one-period interest rate—are lognormal formed the basis for the arbitrage restrictions, not vice versa. In general, the interest rate tree must incorporate *all* possible future paths for interest rates (or the bond price tree must incorporate all future bond price paths, etc.). Next period, the realized interest rate must be one of the nodes contemplated on this period's tree; otherwise the tree has to be "redrawn." Bur redrawing the tree involves irrational model expectations, in the sense of Muth (1961)—changes in future spot rates as a function of the time 1 realization of the term structure are not incorporated in the typical tree-diagram dynamics when inferences are being made from the term structure at time 0. The inconsistency is similar to that which occurs in the case of stock options when (say) the Black-Scholes formula, which assumes a constant stock price volatility, is used to compute a new implied volatility each period.

There is no direct distinction between permanent and transitory components of interest rate change in the tree; if a transitory interest rate realization occurs, then in principle the *up* and *down* steps following that realization will have to reflect its transitory nature, e.g. a positive "transitory" will be followed by a down-step. Incorporating transitories in the tree is consistent with the use of the tree for pricing interest-rate contingent claims, where it is the *dispersion* of interest rates that matters. However, it is not clear that today's typical tree specification is so flexible, e.g. the proportional interest rate dynamics in the example earlier automatically assumes that all unexpected changes in interest rates are permanent.

When the tree specification is calibrated using all available data, errors in data are of especial concern—there is no "residual" in such a "fit" which can incorporate measurement error in, say, bond prices or volatility estimates. Further, since there are no "residuals" in the model fit, there are none of the other typical diagnostics for specification that econometricians use (typically in contexts where they are estimating conditional means).

Finally, it is sometimes believed that by calibrating the arbitrage-free values of bonds with the observed term structure, information can to be extracted from that term structure which will be useful in trading default-free bonds. However, assuming that the arbitrage-free restriction is satisfied, all the information is endogenous to the existing term structure; it is not possible to make superior "bets" on shifts in the yield curve *per se* using only the information in the yield curve itself. The information is useful *only* for pricing contingent claims on the term structure at a point in time, or of course for identifying direct yield-curve arbitrage opportunities if they are not assumed away.

5 Equilibrium Models

5.1 Introduction

When the pricing of bonds is integrated with that of other assets,[31] the pricing formula can be expressed as follows:

$$P(t,T,\bar{i}) = E_Q\left[m_i P(t+1,T,i)\right] \tag{42}$$

where $E_Q[.]$ is the (conditional) expectation at time t taken with respect to the probability distribution of states at time $t+1$, and m_i is the (Arrow-Debreu) price at which investors trade off consumption at times t and $t+1$, and which must therefore be the same across all assets at time t. Dividing both sides of (42) by the time t price $P(t,T,\bar{i})$ gives:

$$1 = E_Q\left[m_i Z_i\right] \tag{43}$$

where $Z_i \equiv P(t+1,T,i)/P(t,T,i=\bar{i})$, is the return earned by buying the time-T maturity bond at time t and reselling it at time $t+1$.

If the risk neutral pricing equation (6) is rearranged as:

$$1 = \Sigma_{i=1}^{S}\left[\frac{\theta_i}{R_i}\right]\frac{P(t+1,T,i)}{P(t,T,\bar{i})} \tag{44}$$

i.e.

$$1 = E_\Theta\left[\Theta_i Z_i\right] \tag{45}$$

[31] This integration should help in understanding equity pricing: Since default-free bonds have known payoffs, it follows that, commodity price level aside, the *only* reason that they have uncertain returns over their terms to maturity is because of technological uncertainty concerning the opportunities for investment of their payoffs and investor wealth at time of payoff. By contrast, the payoffs on, say, corporate stocks, are affected by substantial company-specific uncertainty. As a result, default-free bonds may be very good hedges for economy-wide technological uncertainty in production, and if so, bond returns will be good instruments in tests of the effect of this technological uncertainty on the pricing of equities.

then equation (43) can be considered a special case of equation (6), or perhaps vice-versa. In words, an equilibrium model (43) can be transformed into a risk neutral world in which arbitrage opportunities are absent.

Many authors, among them Beja (1979), Ferson (1981), Sundaresan (1984), Cox, Ingersoll, and Ross (1985), Breeden (1986), and Benninga and Protopapadakis (1986) have derived endogenous bond returns Z_i and optimal paths of consumption and m_i in models where a more detailed specification concerning exchange and production uncertainty is superimposed on (43).[32] In these models, real bond yields are typically positively related to production and consumption growth rates and negatively related to uncertainty about future real production opportunities. The model parameters can be identified by fitting bond return (and other asset return) data. Constantinides (1992) proposes that the m_i (or Θ_i) pricing kernel be modelled directly as a reduced form statistical process.

The expectation in (42) is taken with respect to a discrete grid of states i=1,...,S. The probabilities $\{Q_i\}$ will depend upon the length of the interval of time $[t, t+1]$ over which the states change. Frequently, equilibrium models are developed where the interval of time $[t, t+1]$ is taken to a limit of zero, i.e. in continuous time. The continuous time environment is particularly useful if it produces analytically tractable results.[33] In the following, I briefly describe the widely-studied CIR(1985) continuous time equilibrium bond pricing model, and then discuss the relation between the

[32]Merton's (1975a) analysis, which "rolls back" the level of exogeneity to the growth model, rightly belongs in this list also, although that paper's purpose was not to draw out the term structure 'throwoffs' (to use Merton's own word).

[33]For a discussion of the merits of focusing on the continuous-time limit, see Merton (1975b). In general, any difference between the length of the interval over which price changes are observed and the length of the investor's decision interval in the model will have to be taken into account in the econometric technique to avoid biased estimates of model parameters.

continuous time model and the discrete time example presented in earlier sections.

5.2 The Cox, Ingersoll, and Ross One-Factor Model

On the demand side, CIR's non-monetary economy has identical log-utility investors, while on the supply side, changes in the economy's productive opportunities over time are represented by a single state variable following a "square root" diffusion process with mean reversion. Under these assumptions, the riskless *real* return, which we define as r, also follows a square-root process and reverts to a long-run rate of interest θ:[34]

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dZ \tag{46}$$

At time t when the interest rate is r, the price of a discount bond which matures at time T, P(r,t,T) is the solution to:

$$\frac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r + P_t - \lambda r P_r = 0 \tag{47}$$

with the boundary condition P(r,T,T) = 1. The solution is:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} \tag{48}$$

where:

$$A(t,T) \equiv \left[\frac{2\gamma e^{[(\kappa+\lambda+\gamma)(T-t)]/2}}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2}$$

$$B(t,T) \equiv \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa+\lambda)(e^{\lambda(T-t)}-1)+2\gamma}$$

[34]Longstaff (1989) examines the term structure implications of specifying that the regressivity in the short-term (instantaneous) rate is proportional to $(\theta - \sqrt{r})$ rather than $(\theta - r)$, as in (46).

$$\gamma \equiv \left[(\kappa+\lambda)^2 + 2\sigma^2\right]^{\frac{1}{2}}$$

The expected instantaneous real return on bond H is $r + \lambda r P_r / P$. In CIR's model, λr is the covariance between the instantaneous ("short") rate of interest r and investors' returns on their optimally invested portfolios of assets—the virtue of the equilibrium model is that, within the CIR economy, this risk measure is consistent across all assets. With $P_r/P < 0$, a *negative* covariance between interest rates and portfolio returns ($\lambda < 0$) results in a positive risk premium.

5.3 Estimation

If we assume that the CIR model (48) can be applied directly to nominal bonds and that the "short" rate is observable, or to real returns on bonds where the real short rate r is observable, there are four parameters—κ, θ, σ, and λ—to estimate. Notice that κ and λ always appear as a sum in (48) (because $\kappa + \lambda$ is the coefficient of rP_r in (47). Thus it is not possible to identify κ and λ separately without data on the time series of r_t, though even with this data, the estimates are typically very "noisy" in practice. Various parametric estimation procedures for the model in (48) are discussed in Brennan and Schwartz (1980), Marsh (1980), Brown and Dybvig (1986), Gibbons and Ramaswamy (1986), Brown and Schaefer (1988), and Pearson and Sun (1989).

Regarding the adequacy of the square root specification *per se* for short term interest rates in (46), results in Marsh and Rosenfeld (1983) and Chan, Karolyi, Longstaff, and Sanders (1990) suggest that nominal short-term intererst rates tend to display more heteroscedasticity than allowed for in the square root model. The same heteroscedasticity is picked up in the ARCH/GARCH models (see Bollerslev, Chou, and Kroner (1992) for a review of this evidence).[35] More recently, Ait-Sahalia (1992) has applied a nonparametric kernel estimator of both the marginal density of the spot interest rate and the transition density between successive interest rates in order to *identify* the diffusion function for interest rates (he parameterizes the drift, and the effects of this parameterization show up in the estimated diffusion function). Looking at Treasury Bill and Federal funds rates, he finds that interest rate diffusion is globally increasing as a function of the interest rate up to 18%, but that after the rate gets this high, there is so much mean regressivity coming from the drift that the diffusion function decreases sharply. Also recently, Hamilton (1988), Cai (1992), and Gray (1993) have proposed modelling interest rate movements as regime switches; although it is not clear now whether a regime switching model constitutes a good and parsimonious representation for continuous variables like interest rates, the potential isomorphism between regime switching and lattice movements seems worth exploring. Finally, it is worth noting that the moments used in both the parametric and nonparametric procedures include the volatility structure of bond prices (or interest rates); in this sense, equilibrium models are "calibrated" with the volatility structure of yields, just as is the risk-neutral model in Section 4.

There is also a large, more "traditional," literature concerned with hypotheses about expected returns and term- or risk-premiums on bonds with different maturities. The hypotheses are often framed in terms of the *expectations hypothesis* that: (i) the expected returns from holding a bond of any maturity for a specific period are equal; (ii) the (guaranteed) return from holding any discount bond to maturity

[35]The GARCH models seem more appropriate than the typical diffusion models like (46) because they allow (serially dependent) variation over time in the "amplitude" parameter σ and thus don't force all the variation in the volatility to be a function of the level of interest rates.

is equal to the return expected from rolling over a series of single-period bonds over the same period; or (iii) forward rates are unbiased predictors of future interest rates. Cox, Ingersoll, and Ross (1981) show that these three expressions of the ***expectations hypothesis*** are strictly (pairwise) inconsistent. Good surveys of the literature, which generally finds that the expectations hypothesis fails, can be found in Malkiel (1966) and Mellino (1988). The most recent tests in this literature, which remain primarily reduced form tests, seem to focus on shifts in the predictability in interest rates (when testing whether forward rates equal *expected* spot rates), and/or variation in the term premiums if they exist.

5.4 Binomial Trees and Continuous Time Models

We now return to the linkage between discrete-time equilibrium bond pricing models like (42) and continuous time models like CIR's. Ignoring estimation for a moment, the linkage can be studied by asking the conditions under which a time-state grid be constructed for the discrete-time model that will converge to a given continuous time model as a limit when the time interval shrinks to zero. He (1990) shows that, when a continuous-time equilibrium or no-arbitrage bond price depends upon an N-dimensional state variable vector that follows a diffusion process, it can be approximated by a N+1 dimensional multinomial process (thus the one-state variable model used in the example in Section 2 can be approximated by a binomial process). The multinomial "util probs" are Arrow-Debreu state prices. Willinger and Taqqu (1991) and Duffie and Protter (1992) also discuss the conditions under which finite state space, discrete -time models converge to continuous time limits.

In the one-state variable case, Nelson and Ramaswamy (1990) explore the conditions under which it is possible to construct computationally feasible "binomial" approximations for limiting diffusion processes. When the limiting process for the state variable does not have a constant volatility, such as the interest rate process in (46), they transform the state variable to make the volatility constant. Then the transformed variable can be approximated using a binomial tree in which the nodes are path-independent (like the tree in Figure 1). In path-independent trees, the number of nodes increases linearly in the number of time steps. To extend the Nelson and Ramaswamy results to the N-dimensional case using He's multinomial approach would require that each state variable's volatility depend only upon that variable.

5.5 Estimating Equilibrium Models versus Calibrating Lattice Models

It is sometimes argued that a major disadvantage of equilibrium term structure models is that they involve several unobservable parameters and do not provide a perfect fit to the term structure of interest rates at a point in time. By contrast, it is argued, the valuation of contingent claims in an *as if* risk-neutral economy does not involve unobservable parameters and can be calibrated with the observed term structure at a point in time.

However, the fact that risk premium and interest rate regressivity parameters are not identified in the risk-neutral distribution is an advantage or disadvantage, depending upon one's level of confidence in the volatility specification. Certainly the risk neutral approach is attractive, in that it has "fewer moving parts," as a tool for valuing derivatives. At the same time, fewer moving parts are not necessarily better at the *design* stage, where the jettisoned parts may help in understanding the remaining components. As an illustratration of the point, when Ait-Sahalia (1992) *identifies* the interest rate diffusion function using nonparametric techniques and a parameterization for the drift, he finds substantial interaction between the estimated diffusion function and the drift specification—as would be expected, the interaction is high when the interest rate is high and thus mean regressivity is strong. He and Leland (1992) also derive equilibrium asset price processes in which the expected return-risk premium and volatility specifications are internally consistent, and the former provides information about the latter. The tradeoff is really whether the risk premium and drift parameters in bond returns[36] are small and unstable enough relative to the conditional volatility of interest rates (this will depend in part on the periodicity of the returns),[37] that it is better to treat them as nuisance parameters, which they certainly become for pricing derivative assets once the stochastic process assumption for underlying bond prices or interest rates is given.

Whether the risk-neutral model or equilibrium model "perfectly" fits the observed term structure is not a point of difference between the models; rather it is simply an issue of how many bond price observations there are relative to parameters. If we have N parameters or jointly- identified parameters and N observations, then the equilibrium model will fit the obsevations just as "perfectly" (or imperfectly) as the risk-neutral model.

What happens when there are more bonds than parameters? One solution is to simply expand the parameterization of the equilibrium models by, say, replacing constants by deterministic functions of time (e.g. Jamshidian (1989) and Hull and White (1990)). This creates more parameters so that the enhanced model can be fit

[36]As we just saw for the CIR model, these parameters won't always be independently required.

[37]In Marsh (1985), I found that, using monthly data over the period 1958 - 1978, there was a peak in term premiums on bonds somewhere in the one to two year range, i.e. unconditional term premiums are "humped" as a function of maturity. McCulloch (1985) also reports low (negative) term premiums on bonds with maturities beyond two years over the period 1951-1982. Unfortunately, because of the variability of longer-term bond returns, it is difficult to be confident that the premiums on long-term bonds are significantly below those on one to two-year bonds.

exactly to the existing term structure. Alternatively, one can recognize that the existing parameters are overidentified. Unfortunately, ***neither the risk-neutral model nor the equilibrium model*** themselves contain information to help deal with this overidentification problem. As Professor A. Zellner often reminds his students, it is perilous to take care of the overidentification by, say, simply tacking on an additive error term which is outside the model.[38]

[38]The situation is exactly the same as that encountered when there are multiple options trading on a stock and Black-Scholes *implied volatilities* calculated from each option are not equal.

6 Sources of Interest Rate Uncertainty: How Many Factors?

6.1 Introduction

In the foregoing, it has been assumed that there is only one factor or state variable that causes movements in bond prices—the short-term rate of interest. Even fairly simple economic models and casual observation suggest that this is, in principle, a narrow assumption. In a real economy in which investor heterogeneity is not important, the assumption that real interest rates are the only determinant of real bond prices requires that investors are not uncertain about future investment opportunities, or don't care about these uncertainities. For example, in the CIR model, investors are assumed to have log utility, so they don't want to hedge against changes in investment opportunities—the interest rate is then proportional to a single state-variable in their model.

The "real" world is not the real economy. Default-free bonds are usually redeemed in units of currency[39]. Thus, it is reasonable to think that inflation and inflation uncertainty will be important factors in nominal bond prices. On the empirical side, it is easy to find time-periods with approximately the same short rate of interest but very different shapes of the term structure. In general, long-term bond yields tend to be more variable than predicted by the single factor model, where those yields converge to a constant at extreme (infinite) maturities.

It is not difficult to come up with a "laundry list" of factors which could affect nominal bond prices. The list might include uncertainty about consumption

[39]Exceptions include Israeli index-linked bonds, the British Government Index-Linked bonds studied by Wilcox (1985) and Brown and Schaefer (1988), and the 1988 REALS (Real Yield Securities) underwritten by Morgan Stanley & Co. and issued by Franklin Savings (see Rogalski and Werlin (1988) for a description), all of which are indexed for *general* price level changes.

good prices, differences among investors in investment horizons and wealth, uncertainty about future production opportunities, illiquidity, and changes in regulation and taxes. The task is, of course, to come up with a parsimonious model which accounts for the "important" stochastic factors and either approximates the rest by time-varying deterministic functions or relegates them to a measurement equation.

6.2 Two-Factor Term Structure Models

The next most parsimonious stochastic model to the one-factor model is a two-factor model. The evidence, which is discussed below, suggests that two (linear) factors can account for much of the variation in bond prices, so the two-factor class of models (and sometimes a one-factor specialization) seems likely to be sufficient in many practical applications—especially when specification and estimation error are taken into account. Two-factor models that have been developed include: (i) the long rate - short rate model of Brennan and Schwartz (1982); (ii) models with one real factor and one nominal factor; and (iii) models in which the second factor is a measure of the stochastic volatility of interest rates.

If it is assumed that two unobservable factors cause variation in bond prices, one approach is to simply factor analyze bond price changes and estimate factor scores. Brennan and Schwartz (1982) suggest that if bond prices are a function of two unobervable state variables, then two combinations of observable but endogenous bond prices can be used as "instruments" to mimick the behavior of the factors, so long as the bond pricing function is invertible. Brennan and Schwartz choose a short rate of interest and a long rate of interest as the the two instruments. In a similar vein, Schaefer and Schwartz (1984) use the long rate and the *spread* between the long rate and the short rate as the two instruments. Effectively, the Brennan and Schwartz approach uses short rates and long rates to span the movements in intermediate maturity interest rates. If the mapping from underlying state variables to short and long rates is nonlinear, the length of the time interval over which bond prices are observed will be an important determinant of the adequacy of this spanning.

Canabarro (1993) compares the errors produced by various one- and two- factor models when they are used to compute deltas for constructing portfolios of bonds to replicate a given maturity bond. While the one- and two-factor models have roughly the same replication error when term structures are simulated from a two-factor CIR model, the two-factor models have a much lower replication error for long term bonds when actual term structure data is used. This is because the variability of long rates is higher than that which can be captured by the one-factor models, even the "extended" versions which are "re-calibrated" each period.[40]

The second two-factor formulation involves the nontrivial introduction of a monetary sector into the real equilibrium models, so that nominal bond prices can then be expressed as a function of a real factor and an inflation factor. To relate this to the first formulation, inflationary expectations might be identified with the short rate while the real factor is identified with the long rate or some transformation of the long-run and short run rates, e.g. the difference between them. The real factor - nominal factor route seems to be a promising direction to go if nominal bonds are the object of study: even if one subscribes to the "judge a model by its results" school, it does seem a bit incongruous to take our detailed derivations of general equilibrium real economies and then superimpose highly stylized inflation effects (e.g. price level homogeneity) in order to apply the models to nominal bonds.

[40]As noted earlier, the period-by-period recalibration of the deterministic time-dependent parameters in the extended models can't pick the dynamics of a higher dimensional stochastic process.

Casual empiricism suggests that commodity price level uncertainty could be at least as important as real interest rates or consumption growth rates in explaining movements in nominal bond prices, and the financial periodicals use a lot of ink in explaining how money supply changes affect interest rates. The difficulty, of course, is to account for interaction between inflation and real economic variables. CIR briefly consider general price level (single good price) uncertainty as a second factor with no real effects in their model. Richard (1978) derives closed form bond prices in a two-factor model with real interest rates and inflation which are assumed to move independently of each other and have constant prices of risk. Breeden (1986) introduces multiple good prices which can covary with instantaneous real consumption expenditures and thereby affect interest rates. Pennachi (1991) develops a two-factor nominal bond pricing model in which the two factors—real interest rates and expected inflation, are allowed to be interdependent. Given his identification assumptions, he reports that the two factors are significantly negatively correlated, and that the real interest rate is more volatile than the expected inflation rate.

Structural models to account for the covariation between a real factor and expected inflation could arise in at least two ways: (i) the "Mundell effect" where an increase in expected inflation and nominal rates causes substitution from money to capital which lowers the expected real rate of return on capital; (ii) the endogenity of commodity price levels to the same state variables (including economic uncertainty) that cause changes in real rates of interest (e.g. Black (1972), Shi (1994a,b)); (iii) imperfect indexation of private contracts; (iv) taxation of nominal interest income (e.g. Fischer and Modigliani (1978)).

There is also a substantial literature on co-movements in the term structure and business cycle variables like growth of consumption and GNP as well as stock prices and seasonality. In the absence of compelling structural macro-models of these co-movements, the results are usually reduced form. References include Friedman and Schwartz (1963), Kessel (1965), Keim and Stambaugh (1986), Harvey (1988), and Friedman and Kuttner (1992).[41]

A third two-factor formulation is to model the two factors affecting the nominal bond price as (say) the short term interest rate and a stochastic volatility of that rate. More generally, the first factor could be interpreted as "the level" of the term structure, while the second is the stochastic variation in "term structure uncertainty." Various formulations where stochastic volatility is introduced as a second factor are considered in Longstaff and Schwartz (1990), Fong and Vasicek (1991), and He and Marsh (1991).[42] The models have the advantage that they can usually be expressed so as to nest GARCH-like variation in interest rates, such as that reported by Engle and Ng (1991) for Treasury Bills. They are also consistent with various threads of evidence that the covariation between T-Bill returns and other asset returns is not very stable. Finally, they also accord roughly with the practice of many practitioners who fit, by one means or another, volatility structures to the data as a first step their analysis.

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[41]Romer and Romer (1990) present interesting evidence which they interpret as suggesting that the power of the term structure slope to predict subsequent economic activity is much stronger in episodes in which the Federal Reserve deliberately shifts its monetary policy.

[42]Gennotte and Marsh (1993) have a different model in which the volatility that affects interest rates is not that of interest rates directly, but rather that of aggregate cash flows; the latter then becomes a state variable underlying term structure movements.

6.3 Empirical Evidence on the Dimensionality of Term Structure Movements

On the empirical side, Garbade (1986) and Litterman and Scheinkman (1991) estimate an implicit linear factor model for implied zero coupon bond returns.[43] Both papers report that *three* implicit factors explain some 98% of the variation in returns. The first factor, called a *level* factor because the yields on bonds of all maturities have roughly the same loading on this factor, explains about 90% of the variation in returns. The second factor, called a *steepness* factor because it causes opposite changes in the yields of short-term and long-term bond yields, explains about 81% of the remaining variation.Garbade and Litterman and Scheinkman call their third factor a *curvature* factor because the estimated loadings for different bonds give it the effect of changing the curvature of the yield curve. This third factor never accounts for more than about 5% of the total explained variance of returns.

An interesting issue raised by the factor analysis concerns the importance and operation of the third implicit factor. Clearly it is not, *on average*, of great importance. It could reflect the presence of nonlinearity in the bond return dynamics when the linear factor model is applied. Alternatively, it could reflect an asymmetry in returns or instability in the parameters of the distribution of bond returns, which would also masquerade as an extra "factor."[44]In either case, the importance of the third factor in price changes on occasional days (or weeks) is potentially much greater than "just" an extra 5%. As a result, it might be very important in practice where it can cause

[43]In a series of papers, K. Garbade extended the analysis to: Treasury Strip and Federal Agency bonds, in Garbade (1987); to an international comparison of yield curve movements, in Garbade and Urich (1988); applied it in measuring the risk of portfolios or cross-sections of bonds, in Garbade (1989a,b); and allowed for shifts over time in the parameters of the factor model (Garbade (1990) and Baron (1989)).

[44]Litterman, Scheinkman, and Weiss (1991) consider one way in which volatility in the volatility of returns could cause the curvature effect.

an apparently perfect hedge to incur losses (or gains).

7 Summary and Discussion

The purpose of this chapter has been to outline how risk-neutral and equilibrium asset pricing techniques have come to be applied in the fixed income area over the last five to ten years. These techniques, together with the assumption that term structure movements can be attributed to a small number of factors, lead to pricing and risk hedging formulae for fixed income, and fixed income derivative, securities. The pricing formulae can be interpreted as restrictions across the prices of the fixed income securities. The arbitrage-free pricing restrictions provide a basis for "yield curve arbitrage" and for the pricing of fixed income derivatives. The equilibrium pricing restrictions are those which make the pricing of fixed income securities consistent with the pricing of all other assets, and a fortiori they imply the absence of arbitrage opportunities.

Leaving aside the general but obvious questions surrounding the dimensionality and specification of the factor structure of term structure movements, there are at least two broad areas which still seem to be unsettled, and thus to leave room for future research and industry development. First, many alternative parameterizations of term structure uncertainty have been proposed: in terms of *bond price* movements, as in Ho and Lee (1986); in *forward rate* formulations, as in Heath, Jarrow, and Morton (1992); and in terms of *interest rate* movements, as in the original (equilibrium model) formulation by Cox, Ingersoll, and Ross (1985) and the "extended" Cox, Ingersoll, and Ross (1985) and Vasicek (1977) models suggested by Hull and White (1990). When the parameterization of the term structure uncertainty in terms of the stochastic process for the selected one of these three variables becomes complicated, it can be difficult/impossible to transform among the three specifications. In practice, this often means that instead of using a common stochastic model for all interest rate-sensitive securities, different models are used for different securities. Even ignoring scientific niceties, this makes it hard to aggregate risk positions, etc. in practice.

Moreover, the tendency has been to increase the parameterization of the models, expressed in whichever of the three variables, so that they can be calibrated exactly to at least the observed term structure and term structure of volatilities for the purpose of pricing derivatives. The logic of calibrating a model with the observed moments of the term structure to price derivatives is appealing—when the model is represented as a binomial tree, it can easily be seen that the calibration procedure is just like inverting observed prices to obtain Arrow-Debreu state prices. In addition, the methodology of representing the stochastic process for term structure movments in a lattice and then calibrating parameters in the lattice with observed moments follows the recent trend in the statistics and econometrics literature toward use of non-parametric and empirically fitted models, and away from analytical parametric models (this trend undoubtedly follows the trend in computer power). But the calibrated model is only as good as its original specification; if the tree is wrong in the sense that over time it is found that future interest rate or bond price realizations are not on one of the paths in the tree, it will have to be "redrawn"; since the reconstruction of the tree in response to the realizations is not itself represented in tree, this is a form of Muthian irrationality. Use of "all the data" in calibrating tree models is, *ipso facto*, an advantage; but it leaves no "residuals" to absorb data errors and the like.

Second, it seems unlikely that there is a single term structure specification that is best for pricing all fixed income derivatives.[45] For example, a one-factor model

[45] This doesn't mean that the derivatives can't all be priced and hedged in terms of a common model which nests the specialized (restricted) versions best for various derivatives.

in which yields on bonds with different maturities are (instantaneously) perfectly correlated does not, *a priori*, seem a good candidate for valuing options to exchange one segment of the yield curve for a different segment (e.g. the SYCURVE options offered by Goldman Sachs),[46] though it might produce reasonable results in valuing some short-term options on bond prices. More generally, several of the most recent exotic interest rate options seem to "lever" volatility assumptions in the underlying tree, thus making their pricing and hedging much more sensitive to gyrations in the market which weren't contemplated in the tree. Unfortunately, there are currently few guidelines available to guide the selection of which model features are important for pricing and hedging which interest rate contingent claims. Kuwahara and Marsh (1994) have investigated whether bounds can be placed on the underlying interest rate uncertainty, and thus on the important features of uncertainty which must be modelled, by the various derivatives contracts.[47]

[46]Perhaps a one-factor model for the yield curve *spread* could work, but it would be specific to the points on the yield curve defining the spread, and could probably be dominated by a model using a second "curvature" type factor.

[47]The intuition is similar to that behind Grundy's (1991) derivation of bounds on asset price distributions implied by option prices.

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