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(1)

In the Preface to the second eddition (1879) of The Theory of Political Economy, Jevons described the emergence of mathematical econpomics very detailedly. He spent enormous energy in the search of predecessors of mathematical economics and discussed their contribution in the light of his own mathematical economics. Our aim in this essay is, therefore, to succeed Jevons and to discuss the significance of mathematical theories of Jevons and his contemporaries in the light of the development of modern mathematical economics.

Firstly, Jevons's contribution to the theory of exchange must be highly evaluated. Although Jevons himself did not developed it fully, his non-Walrasian concept of competitive markets, which was succeeded and developed further by Edgeworth, supplements nicely Cournot-Walras theory in the modern mathematical theory of microeconomics. In sections (2) - (5) below, we shall discuss the significance of Jevons's equations of exchange from this point of view.

Jevons's contribution to the economic science is, however, not limited to the theory of exchange. In addition to his famous equations of exchange, Jevons's general expression for the rate of interest is also well known in the history of economic thoughts. Unfortunately, his theory of capital is much more sketchy and informal than his theory of exchange. Secondly, therefore, in sections (6) - (8), we shall try to see whether Jevons's suggestion of a non-Ricardian theory of the falling rate of interest can be vindicated by the use of a mathematical model which was developed later than Jevons's capital theory.

As Fisher pointed out, Jevons's initial model, from which a series of other models are derived, is the model in the chapter on theory of utility of <u>The Theory of political Economy</u>, which is used to discuss the distribution of a commodity in different uses. Jevons considers a commodity, which is capable of two distinct uses. Let the two quantities appropriated to these two uses be represented by x and y, and the whole stock of the commodity, by s, so that $x + y = s$. Let Δu_1 and Δu_2 be the increments of utility, which might arise respectively from consuming an increment of the commodity in the two different ways. When the distribution is completed, Jevons insists, we ought to have $\Delta u_1 = \Delta u_2$, or at the limit we have the equation

(1) $du_1/dx = du_2/dy$.

We must, in other words, have the final degree of utility in the two different uses equal.

This is nothing but the so-called Gossen's second law of utility. While Jevons insists that his theory was developed independently from that of Gossen, Jevons frankly admitted that Gossen is one of his predecessors. Though highly incomplete, Gossen tried to solve the problem of exchange on the basis of his theory of utility. As we shall see, it is Jevons who completed this plan of Gossen, since his model in the chapter on theory of exchange of <u>The Theory of Political Economy</u> is derived from his initial model, i.e., that of the theory of utility. Before to consider Jevons's theory of exchange, however, we have to discuss two concepts, i.e., the trading body and the law of indifference, both of which are very important if we are to understand the true implications of Jevons's theory of exchange.

By a trading body Jevons means any body either of buyers or sellers. England and North America may be trading bodies, if we are considering the corn England receives from America in exchange for iron and other goods. The continent of Europe is a trading body as purchasing coal from England. The farmers of England are a trading body when they sell corn to the millers, and the millers both when they buy corn from the farmers and sell flour to the bakers. The reason why Jevons consider a trading bodies rather than an individual in his theory of exchange is that the behavior of the aggregate or average person is much more stable than that of an individual person. In other words, differential calculus can only be used for the case of the aggregate or average person.

The law of indifference, which insists that there is only one price for each commodity in equilibrium, is necessary, as we shall see, to derive Jevons's equation of exchange from his initial model of distribution of commodity in different uses. Jevons explains that the law is established only at the equilibrium through the arbitrage behavior of sellers and buyers. In other words, it is not to be presupposed, in the case of markets he has in mind, unlike in the case of the well-organized markets considered by Cournot and Walras. If two commodities are bartered in the ratio of x for y, Jevons argues, an infinitely small part of x must be exchanged for an infinitely small part of y, in the same ratio as the whole quantities. The increments concerned in the process of exchange must obay the equation

(2) $dy/dx = y/x$.

This is the equation of the law of indifference, which is established only when an equilibrium of exchange is attained.

(3)

Now we are ready to consider the proposition which contains, according to Jevons, the keystone of the whole theory of exchange, and of the principal problems of economics. " The ratio of exchange of any two commodities will be the reciprocal of the ratio of the final degrees of utility of commodity available for consumption after the exchange is completed." Consider that the first trading body A (the representative individual in the body) originally possessed the quantity a of corn, and the second trading body B (the representative individual in the body) possessed the quantity b of beef. If the exchange consists in giving x of corn for y of beef, the quantities exchanged satisfy two equations,

(3) $$F_1(a - x)/G_1(y) = y/x = F_2(x)/G_2(b - y)$$

where F_1 and G_1, and F_2 and G_2), respectively denote A's (i.e., its representative individual's) final degree ofutility of corn and beef, and B's (i.e., its representative individual's) final degree of corn and beef. These two equations are sufficient to determine two unknown quantities, x and y, the quantities given and received.

Two equations (3) are derived from the equations of Gossen's second law of utility (1), which can be rewritten in this case as

(4) $$F_1(a - x)/G_1(y) = dy/dx$$

for the trading body A and

(5) $$F_2(x)/G_2(b - y) = dy/dx$$

for the trading body B, since Jevons considers that different uses in the distribution of a commodity can be expanded to include exchange as a possibility. For example, (4) requires that A's final degree of utility from the direct consumption of corn must be equal to its final degree of utility from an indirect consumption of corn, i.e., utility from the consumption of beef received from B in exchange for corn. Equations (4) and (5) are conditions on x and y for Pareto optimality of the resulting allocation of commodities between trading bodies, since they require the equal marginal rate of subsitution between trading bodies. Jevons's equations of exchange (3) are, then, derived from these conditions and the law of indifference (2), and determine an equilibrium point (x, y) among Pareto optimal allocations.

Walras's regard for Jevons's equations of exchange (3) was not high, since Walras pointed out that Jevons had failed to derive equation of demand as a function of price, which is indispensable for the solution of the problem of the determination of equilibrium price. This is not surprising, since the view of market of Walras, who followed Cournot, is different from that of Jevons, who followed Gossen. Walras presupposed the existence of market prices which competitive traders always accept as data, while Jevons tried to justify this supposition by explaining market prices as the equilibrium ratio of exchange resulting from a process of free exchanges and arbitrage among competitive traders.

Walras defines the equilibrium as the equality of demand and supply, both of which are functions of the given market price. Since the law of indifference is simply presupposed, all the individual traders in the market take the identical prices, even in disequilibrium situations. One might suppose a well-organized, highly institutionalized market in which the specialized auctioneer determines the market price and changes it according to the excess demand or supply generated by price-taking traders, as the incarnation of the law of supply and demand. For Jevons, however, demand and supply are trivially equal even in disequilibrium situations. In his explanation of the exchange, x is the quantity of corn demanded by trading body B and at the same time the quantity of corn supplied by trading body A. This equality must necessarily exist if any exchange takes place at all. Instead, equilibrium is defined by Jevons by the law of indifference, which is established by arbitrage of different exchange ratios, which exist at disequilibrium.

Although Walras and Jevons viewed the process of market differently, the resulting equilibrium is idenical, since the identical set of x and y is determined both by Walras's demand and supply equality and by Jevons's equations of exchange (3). In other words, in spite of Walras, Jevons arrived at the identical equilibrium which Walras cpnsidered, even though Javons failed to derive the equation of the demand as a function of price, which is, therefore, not necessarily indispensable for the solution of the equilibrium price or equilibrium exchange ratio y/x. While Walras discussed in detail the process of tâtonnement by which the equilibrium is established, however, Jevons unfortunately did not make clear how the equilibrium (3) is established by the process of arbitrage among traders. The problem is left to Edgeworth.

(4)

To consider " Professor Jevons's example," Edgeworth first introduced the now famous concepts of the contract curve and indifference curves in his Mathematical Psychics. Since Edgeworth's own presentation is not easy to follow, however, let us rather consider the problem by using the so-called Edgeworth's box diagram, which was adumbrated by Pareto twelve years after the publication of Mathematical Psychics.

In Figure 1, where the quantity of corn is measured horizontally, and that of beef vertically, the quantities of commodity available to the representative individual in the trading body A are measured with the origin at A, those available to the representative individual in the trading body B with the origin at B, and point C denotes the initial allocation of commodities. Curves I, II, etc. are indifference curves of the individual in the trading body A, and curves 1, 2, etc. are those of the individual in the trading body B. The common tangent to two indifference curves at E passes to point C, so that Jevons's equations of exchange are satisfied at E. The curve DEF is the contract curve which is a locus of points where indifference curves of two individuals are tangent to each other. In other words, on these points allocation of commodities is Pareto optimal and Gossen's second law of utility is satisfied for both individuals between different uses of a commodity, direct consumption and indirect consumption through exchange.

If each trading body consists of only a single individual, Edgeworth insisted that there is no reason why only E can be an equilibrium and Jevons's equations of exchange cannot always be satisfied. In the case of isolated exchange or bilateral monopoly, the equilibrium point is indeterminate in the sense that any point on the contract curve between D and F is a stable outcome of exchange. Although the allocation at such a point is Pareto optimal, the law of indifference may not be satisfied. Edgeworth interpreted, however, Jevons's theory correctly that Jevons's trading bodies consist of infinitely many individuals and his equations of exchange takes place only in the case of the perfect competition.[1] To make the story clearer, Edgeworth assumed that each trading body is homogeneous and consists of infinitely many individuals who have the identical taste and identical stock of commodities.

Edgeworth first demonstrated that the outcome of exchange is identical for any couple of an individual of trading body A and an individual of trading body B. If an exchange between an A individual A_1 and a B individual B_1 ends up at D and an exchange between an A individual A_2 and a B individual B_2, at F, the coalition of A_1 and B_2 can block such contracts, since they can be better off by themselves at any points between D and F.

Suppose, then, in Figure 2, the outcome of exchange is at point H for any couple of individuals A and B. This contract can be blocked by a coalition of all the A individuals and some of B individuals, since they can be better off by themselves, i.e., by trading only among themselves. Some A individuals continue trade with B individuals in the coalition and are located at H, while the rest of the A individuals who cancelled trade with non-coalition B individuals are at C. If the number of B individuals joining the coalition is properly chosen, the average allocation for A individuals (some at H, the rest at C) can be at point like J which is preferred for A individuals to the indifference curve passing through H. By reallocating among themselves, therefore, all the individuals joining the coalition can be better off than they are at H.

Similarly, it can be shown that any point between D and F, where the common tangent to two indifference curves does not pass through the point C, cannot be a stable outcome of exchange, if necessary, by changing the role of A individuals and B individuals from those in the case of the allocation H. Obviously, only the point E, where the common tangent passes through the point C, can be a stable outcome of exchange. In other words, Edgeworth thus demonstrated that an equilibrium which satisfies the law of indifference and therefore Jevons's equations of exchange can be established by the process of exchange and arbitrage among inmdividuals.

(5)

The point E in Figure 1 is also an equilibrium in the sense of Walras, since demand and supply are equalized at the price shown by the slope of the line EC, which all the individuals take as given. What Edgeworth demonstrated is that Jevons's equilibrium and Walras's equilibrium are equivalent in the limiting case of infinitely many individuals. This equivalence theorem or limiting theorem can now be demonstrated by the modern mathematical economics under much weaker assumptions than those made by Edgeworth.[2)]

For Walras, who followed Cournot, the law of indifference, i.e., the existence of the uniform market price, is presupposed, and competitive individuals are assumed to be price takers. In the Jevons-Edgeworth approach, however, the law of indifference is established only at the equilibrium, through arbitrage activities of individuals who try to take advantage of the existence of different prices in the same market. Individuals are not price takers there, and are free to make contract at whatever price they like, to cancel it to make recontract at more favorable terms, and to organize coalitions to block existing contracts. The assumptions made in the Cournot-Walras approach are not realistic, unless there is an auctioneer, as in the case of well-organized markets. Edgeworth's equivalent theorem justifies Walrasian assumption, however, since it is not necessarily the assumption but the outcome that matters for a theory, and we can assume that individuals are price takers, even though they are actually not, provided that we have the same outcome as assured by Edgeworth. Walrasian theory based on the demand and supply functions can be safely applied to situations where individuals are not price takers and are fre to form and break coalitions in the process of bargaining as in the Jevons-Edgeworth approach, so that demand and supply functions do not, strictly speaking, make sense. Walras, who criticized Jevons for the lack of a clear concept of demand functions in the latter's equations of exchange, is thus helped by Edgeworth, who followed Jevons, to increase the relevancy of his theory.

(6)

In the chapter on theory of capital of The Theory of Political Economy, Jevons pointed out that Ricardo attributed the fall of interest to the rise in the cost of labor. With the given real natural wage, an increase in capital leads to a larger labor population in the Ricardian theory. Agricultural production must be increased and the wage cost in terms of embodied labor is raised by the diminishing returns in agriculture. At the margin of cultivation, where there is no rent, then, we can see that the rate of interest falls. Jevons was against Ricardo, since Ricardo's view is not in agreement with the view which Jevons has ventured to take concerning the origin of interest. According to Jevons, the rate of interest depends on the advantage of the last increment of capital, and the advantages of previous increments may be greater. In other words, Jevons attributed the fall of interest to the use of more roundabout method of production which is caused by the accumulation of capital. The fall of interest must be true, therefore, even if wage changes proportionally to changes in labor productivity so that there is no rise in the cost of labor.

Jevons obtained his general expression for the rate of interest as follows, by supposing that the produce for the same amount of labor is an increasing function of the time elapsing between the expenditure of the labor and the enjoyment of the result. Let the time in question be t, and the produce for the same amount of labor be denoted by F(t). If we extend the time to t + Δt, the produce will be F(t + Δt), and the increment of produce F(t + Δt) - F(t). The ratio which this increment bears to the increment of investment of capital will determine the rate of interest. The amount of increased investment of capital is F(t)Δt, since at the end of the time t we might receive the product F(t) which is the amount of capital left invested when we extend the time by Δt. When we reduce the magnitude of Δt infinitely, then, we find the rate of interest to be represented by dF/dt•1/F(t). The interest of capital is, in other words, the rate of increase of the produce divided by the whole produce.

From this general expression for the rate of interest, Jevons argued that the rate of interest falls if the time t between the expenditure of labor and the enjoyment of the result is increased. For example, he considered the case of F(t) = αt, in which α is an unkown positive constant. The rate of interest is, then, α/αt or 1/t so that it varies inversely as the time of investment.

Jevons defined free capital as the wages of labor, either in its transitory form of money, or its real form of food and other necessaries of life. As for the fixed capital, he would not say that a railway is fixed capital, but that capital is fixed in the railway. The time of investment t, then, can be increased if the supply of free capital is abundant. Abundance of free capital in an econoy means, for Jevons, that there are ample stock of food, clothing, and every other article which people may insist upon having. Abundant subsistence and conveniences of every kind are forthcoming without the labor of the economy being much used to provide them. Therefore, it is possible that a part of the laborers of the economy can be employed on works of which the utility is distant, and yet no one will feel scarcity in the present.

Thus, there is a " Tendency of Profits to a Minimum." Supposing accumulation of capital to go on, Jevons argues, the formula for the rate of interest shows that the rate must tend to sink towards zero, unless there be constant progress in technology. It is apparent that here Jevons assumes the constant labor population, since he states in the concluding chapter of The Theory of Political Economy that the doctrine of population forms no part of the direct problem of economics. America and British Colonies are examples, for Jevons, of high rate of interest economies, where there is not sufficient capital accumulated to meet all the demands of the population. Examples of low rate of interest economies are, of course, England and other old countries, where there is abundance of capital and the urgent need of more is not actually felt.

(7)

Let us confirm these assertions of Jevons, which are derived from his non-Ricardian theory of interest, by the use of a mathematical model developed later than Jevons's theory. For this purpose it seems natural to use the so-called Wicksellian model of a stationary aggregate economy with a point-input, point output production process, since Jevons's formula can be derived in such a model.

An aggregate point-input, point-output production function is

(1) $Y = f(t)L$

where Y is the volume of output of consumers' goods, L is labor input equal to the given labor population, and f(t) is an increasing concave function of t, i.e., the time elasping betwen the input and the output. Rate of interest r is implicitely defined by

(2) $Y = w\ e^{rt}$

where w is the real wage in terms of consumers' goods. Given w, we can solve (1) and (2) for r as a function of t. From the maximization of r with respect to t, then, we obtain

(3) $r = f'(t)/f(t)$

which is nothing but Jevons's general expression for the rate of interest. Finally, the value of the aggregate circulating capital K in terms of consumers' goods is given as

(4) $K = \int_0^t w\ Le^{ru}\ du = \int_0^t L\ f(u)\ du$

where the last equality follows from (1) and (2).

If we close the model by giving the value of K exogeneously, as Wicksell did reluctantly, four unknowns Y, t, r and w are determined by four equations (1) - (4). From (4) we have

(5) $d\ t\ /\ d\ K = 1/Lf(t) > 0,$

and from (3)

(6) $d\ r\ /\ d\ t = (\ f'' - rf'\ )/f < 0.$

In words, the rate of interest falls as the capital is accumulated, through the use of more roundabout method of production. There is, unfortunately, no certain relation between the cost of labor, w/f(t), and the value of capital K in this model.

It does not make sense, however, to give the value of capital K exogenously, since it is the value of heterogeneous capital, which depends, as is seen in (4), on such endogenous variables asd t, r and w. It should be considered as an endogenous variable, i.e., an unknown, defined by (4). The model should, then, be closed by the equality of investment and saving, so that the stationary state can be maintained,

(7) $sY = wL$

where s denotes the given rate of gross saving. Five unkowns Y, t, r, w and k are now determined by five equations (1) - (4) and (7). There are, unfortunately, no unambiguous relations between the exogenous parameter related to capital accumulation, i.e., s, on one hand, and the endogenous variables r, t, and K, on the other hand. The cost of labor, furthermore, increases with s, since $w/f(t) = s$.

(8)

Let us instead consider the so-called modern neo-classical macro model of a stationary economy, in which the output is assumed to be malleable so that it can either be consumed or be invested. The aggregate production function F is a linear homogeneous function of labor L and capital K, so that

(1) $Y = F(L, K) = Lf(a)$

where Y is the volume of output and a is the capital-labor ratio K/L, and f(a) is an increasing concave function of a. Capital K available in the t-th period consists of output of past periods up to the t-1-th period. The capital-labor ratio a plays, therefore, the role of Jevons's t, i.e., the time of investment.

In view of (1), the sum of labor inputs directly and indirectly necessary to produce Y is

(2) $W(Y, a) = L + L[\theta a/f(a)] + L[\theta a/f(a)]^2 + ----,$

where θ is the given rate of depreciation. Since the stationary state cannnot be maintained unless $f(a) > \theta a$,

(3) $W(Y, a) = Y/[f(a) - \theta a]$,

from which

(4) $dW/da = -Y[f'(a) - \theta]/[f(a) - \theta a]^2$.

Let us assume that W is decreasing with respect to a, which corresponds to Jevons's assumption that his F(t) is increasing with respect to t.

This assumption implies, from (4), that $[f'(a) - \theta] > 0$.

The rate of interest r is implicitely defined by

(5) $F(L, K) - w L = (\theta + r) K$

where w is the real wage. From the maximization of r (given w) with respect to K and L, respectively,

(6) $F_K = \theta + r$

and

(7) $F_L = w$,

where F_K and F_L denote, respectively, the partial derivative of F with respect to K and L. Finally, the model is closed by the equality of investment and saving,

(8) $s F(L, K) = \theta K$

where s is the given rate of gross saving. Four unknowns K, a, r and w are determined by four equations (1) , (6) - (8), if L is equalized to the given labor polpulation, since (5) is not independent in view of the linear homogeneity of production function.

In view of (5), (6) and (8), we have

(9) $(\theta - s F_K) K = s w L > 0$,

while the differentiation of (8) withr espect to s and K gives

(10) $dK/ds = F(L, K)/(\theta - s F_K)$.

Since L is given, therefore, a higher rate of saving implies a larger supply of capital to labor, which, in view of (1) and (6), makes the rate of interest fall. The fall of interest rate due to capital accumulation continues unless the zero rate of interest is reached. Unlike the case of Ricardian process of capital accumulation, this fall of interest rate is not caused by the rise in the cost of labor, since any rise in wage is proportional to the rise in the labor productivity, as is seen in (7). The rate of interest falls through the use of more roundabout method of production (higher a), which is exactly the essence of a non-Ricardian theory Jevons suggested.

Footnotes

1) As a matter of fact, Jevons's equations of exchange can even be established in the case of duopoly. See Negishi(1989), pp. 339 - 341.

2) Generalizations of Edgeworth's theorem were made by Debreu, Scarf, Aumann, Hildenbrandt, Kirman, etc. See Hildenbrand and Kirman(1976).

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Figure 1

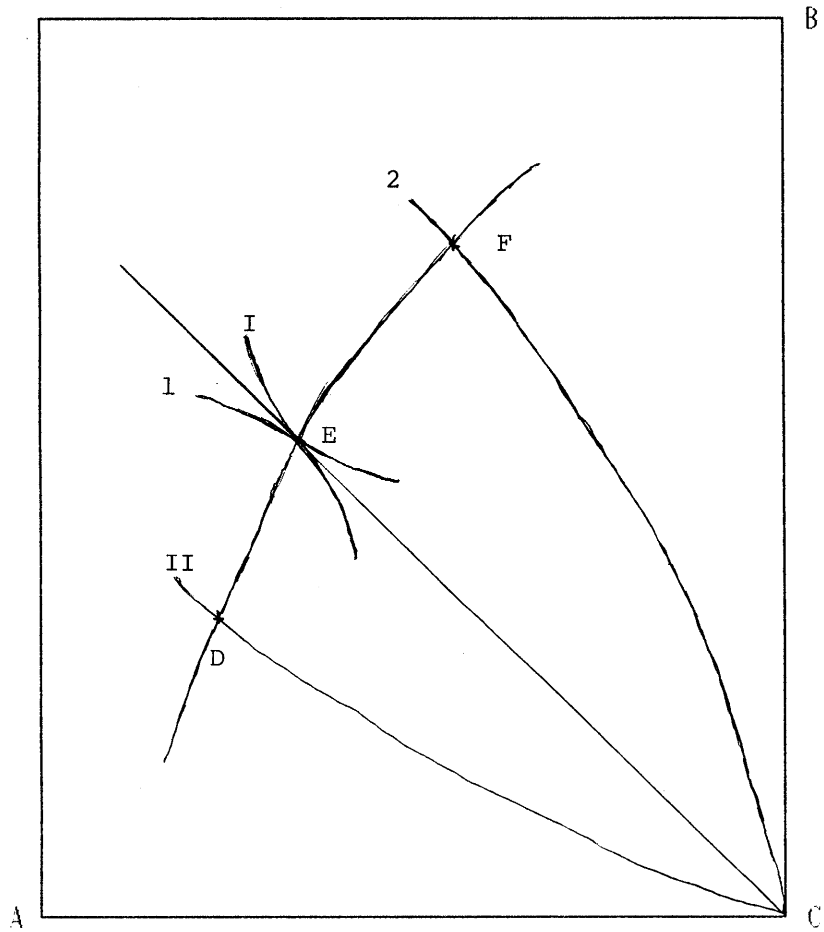

Figure 2