94-F-9

Asymmetry in Economic Time Series and Simultaneous Switching Autoregressive Model

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January 1994

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Abstract

One important characteristic of many economic time series, which has been often ignored, is the asymmetrical movement between in the downward phase and in the up-ward phase in their sample paths. Since this feature cannot be described by the standard ARMA time series models, we introduce a new class of the Simultaneous Switching Autoregressive (SSAR) model, which is a non-linear Markovian switching time series model. We discuss the problems of coherency, ergodicity, the stationary distribution and its moments, and the estimation method for its unknown parameters. We also give a simple empirical example of an agricultural market model with price and quantity in the context of disequilibrium econometrics.

Key Words

Asymmetry, Non-linear Time Series, Simultaneous Switching Autoregressive Model, Disequilibrium Econometric Model, Coherency, Ergodicity, Stationary Distribution

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1. Introduction

In the past decade, several non-linear time series models have been proposed by statisticians and econometricians. For instance, Granger and Andersen (1978) has introduced the bilinear time series models. Also Ozaki and Oda (1978), and Tong (1983) have proposed the exponential autoregressive (EXPAR) model and the threshold autoregressive (TAR) model, respectively, in the field of statistical time series analysis. In particular, a considerable attention has been paid on the TAR model in the past decade by statistician and econometricians and several related applications have been reported. The statistical details of many non-linear time series models have been discussed by Tong (1990). In the econometric analyses several non-linear time series models have been proposed and used in some empirical studies. For instance, Hamilton (1989) has introduced a Markovian switching time series model and applied it to some macro-economic data. Also McCulloch and Tsay (1992) have proposed to use the Bayesian approach to analyze another type of Markovian switching time series model.

The main purpose of this paper is to introduce an alternative non-linear time series model, which is called the Simultaneous Switching Autoregressive (SSAR) time series model. This model is a kind of Markovian switching time series model with a quite distinctive structure of simultaneity. We shall propose this statistical model because we have a conviction that the standard Autoregressive Moving-average (ARMA) time series model cannot describe one important aspect in many economic time series data, that is, the asymmetrical movement in the up-ward phase (or regime) and in the down-ward phase (or regime). In this paper we introduce a particular version of the SSAR model as the reduced form equations of the disequilibrium econometric models proposed by a number of econometricians including Fair and Jaffee (1972), and Laffont and Garcia (1977). We shall show that the class of the SSAR model proposed gives us some explanations and descriptions to very important aspect of the asymmetrical movement of time series in two different phases. This characterestic of economic time series has been observed by a number of economists. But there has not been any useful time series model incorporating this feature as far as we know in the econometric literature.

In Section 2, we shall discuss a wide-spread observation among economists on the asymmetrical movement of economic time series data, which has been often ignored in the recent econometric analyses, and its implications to the statistical time series anaysis. In Section 3, we shall re-consider a simple two equation disequilibrium econometric model originally proposed by Laffont and Garcia (1977) and introduce the simple SSAR model. Then in Section 4, we define the multivariate Simultaneous Switching Autoregressive (SSAR) model with exogenous variables and discuss its distinct feature of simultaneity. In Section 5, we shall discuss the conditions for the coherency and the ergodicity of the SSAR model, and the stationary distribution and its moments. We shall also give some results on the asymptotic properties of the maximum likelihood estimators of unknown parameters. Then in Section 6, we shall give an empirical application of the SSAR model to Japanese agricultural markets. Finally, some concluding remarks on the related issues will be given in Section 7. The proof of theorems and some algebraic details will be gathered in Appendix.

2. Asymmetry in Economic Time Series

It has been sometimes (or often) argued that major cyclical variables such as the unemployment rate and the growth rate of GDP display some kind of asymmetrical movements over various phases of the business cycle. In particular, many economists have observed that the length of the up-ward phase has been often longer than that of the down-ward phase. Neftici (1984) argued that there is enough evidence in the U.S. time series data to warrant a serious consideration of whether economic time series indeed go through two different regimes in the business cycle. According to another interesting paper by Neftici (1993), the earlier work by Burns and Mitchell (1946) has already devoted a considerable effort to investigate the asymmetrical V-shaped patterns on the sample paths of major economic time series data.

Turning to Japanese economic time series data, a number of leading economists in the past have mentioned to similar observations. Among many of them, we present Table 2.1 [1] which is taken from the published (or official) statistics by the Japanese Government. It gives the official estimate of the turning points and the cycles periods of the business cycles during the past 40 years. From Table 2.1, we find that the lengths of expanding periods are longer than those of contracting periods in most cases. This reflects a common view among economists in the Japanese Government that the down-ward phase of business cycle occurs abruptly or rapidly developed. Although this observation could not be justified in the narrow sense of statistical reasoning, it is difficult to be denied because a number of leading economists have observed this kind of asymmetrical aspect in many economic time series data.

Table 2.1: Official Dates of Japanese Business Cycles

cycle	*trough*	*peak*	*trough*	*expanding period*	*contracting period*
1*st*		1951.6	1951.10		4*months*
2*nd*	1951.10	1954.1	1954.11	27*months*	10*months*
3*rd*	1954.11	1957.6	1958.6	31*months*	12*months*
4*th*	1958.6	1961.12	1962.10	42*months*	10*months*
5*th*	1962.10	1964.10	1965.10	24*months*	12*months*
6*th*	1965.10	1970.7	1971.12	57*months*	17*months*
7*th*	1971.12	1973.11	1975.3	23*months*	16*months*
8*th*	1975.3	1977.1	1977.10	22*months*	9*months*
9*th*	1977.10	1980.2	1983.2	28*months*	36*months*
10*th*	1983.2	1985.6	1986.11	28*months*	17*months*

[1] We constructed Table 2.1 from Indexes of Business Conditions (July 1991) published by the Research Bureau, Economic Planning Agency, Japanese Government.

Now we consider an interesting question, namely, is it possible to use the standard autoregressive moving average (ARMA) time series model to capture this asymmetrical feature in economic time series data ? Our tentative answer to this question is quite negative. For the sake of simplicity, we take the first order autoregressive (AR(1)) model and illustrate some symmetrical properties of the standard ARMA time series model, which has been often used in recent econometric studies.

Let a sequence of observable random variables $\{ y_t \}$ follow

$$y_t = \phi_1 y_{t-1} + v_t \ , \quad t = 1, 2, \cdots \quad , \tag{2.1}$$

where ϕ_1 is the unknown coefficient and $\{v_t\}$ are independently and identically distributed random variables with $E(v_t) = 0$ and $E(v_t)^2 = \sigma^2$. In the AR(1) model, the necessary and sufficient condition for stationarity has been well-known and is given by

$$|\phi_1| < 1 \quad . \tag{2.2}$$

The question we have is whether the sample paths of the stochastic process $\{y_t\}$ could exhibit some kind of asymmetrical pattern in the time domain or not. First, we take the minus of y_{t-1} and y_t in (2.1). Then we have the new stochastic process $y_t^* = -y_t$, which has an AR(1) representation

$$y_t^* = \phi_1 y_{t-1}^* - v_t \quad . \tag{2.3}$$

If we assume that the disturbance terms $\{v_t\}$ are symmetrically distributed around the origin as we usually do, the distribution of $\{y_t^*\}$ is exactly the same as that of $\{y_t\}$. Hence we conclude that the stochastic process $\{y_t^*\}$ is symmetrical around the axis $y = 0$. Second, let us consider the reversed AR(1) model defined by

$$y_t^{**} = \phi_1^* y_{t-1}^{**} + v_t^{**} \ , \quad t = T-1, T-2, \cdots \quad . \tag{2.4}$$

We assume that the disturbance terms $\{v_t^{**}\}$ in (2.4) are independently and identically distributed with $E(v_t^{**}) = 0$ and $E(v_t^{**})^2 = \sigma_*^2$. If we further assume the stationarity condition on $\{y_t^{**}\}$

$$|\phi_1^*| < 1 \quad , \tag{2.5}$$

the autocorrelation function for $\{y_t^{**}\}$ is exactly the same as that for $\{y_t\}$, provided that $\phi_1 = \phi_1^*$. Hence this consideration leads to the second conclusion that given the specific autocorrelation function of the AR(1) model, there are two alternative representations in the time domain. In one representation given by (2.1), the present value of the observed y_t can be an infinite moving average of the past disturbance terms, while in the other representation given by (2.4), y_t can be expressed as an infinite moving average of future disturbance terms. In this sense, we conclude that the standard AR(1) model has a kind of symmetric structure in the time domain. Also the above arguments can be generalized to the more general case of the standard ARMA models by using some results given in Section 3 of Brockwell and Davis (1991).

The most important and distinct characteristic of the stationary ARMA model as time series model is its linearity. In the linear time series models the present value of the variable y_t can be expressed as a linear combination of uncorrelated disturbance terms

$$y_t = \sum_{s=-\infty}^{\infty} c_s v_{t-s} \ , \tag{2.6}$$

where

$$\sum_{s=-\infty}^{\infty} |c_s| < +\infty \ , \tag{2.7}$$

and the coefficients $\{c_s\}$ in this moving average representation are independent of t. It is known that the linear time series model given by (2.6) can be approximated accurately by some ARMA model. Hence we should expect that it is quite difficult to describe the asymmetrical patterns of economic time series data between in the down-ward phase and in the up-ward phase.

Next, from the purely data analytic point of view, we examine the asymmetrical property in the economic time series data. For this purpose, we use some time series data on agricultural products traded in Tokyo. By modifying the AR(1) model with some exogenous variables, we can construct a non-linear model

$$y_t = \beta_0 + \beta_1^+ \Delta y_{t-1}^+ + \beta_1^- \Delta y_{t-1}^- + \gamma' z_t^* + v_t \ , \quad t = 1, 2, \cdots \ , \tag{2.8}$$

where y_t is the explained variable, z_t^* is a vector of exogenous variables, and $\{v_t\}$ are the disturbance terms with $E(v_t) = 0$ and $E(v_t^2) = \sigma^2 > 0$. As the explanatory variables in (2.8), we have $\{z_t^*\}$ and the signed lagged explained variables defined by

$$\Delta y_{t-1}^+ = \begin{cases} \Delta y_{t-1} & \text{if } \Delta y_{t-1} \geq 0 \\ 0 & \text{if } \Delta y_{t-1} < 0 \end{cases} \ , \tag{2.9}$$

and

$$\Delta y_{t-1}^- = \begin{cases} 0 & \text{if } \Delta y_{t-1} \geq 0 \\ \Delta y_{t-1} & \text{if } \Delta y_{t-1} < 0 \end{cases} \ , \tag{2.10}$$

where the difference operator Δ is meant by $\Delta y_{t-1} = y_{t-1} - y_{t-2}$. In (2.8), β_0 , β_1^+ and β_1^- are scalar coefficients, while γ' is a vector of coefficients. In particular, if $\beta_1^+ = \beta_1^-$ in (2.8), the asymmetric terms in the model disappear and we have an AR(2) model with some exogenous variables.

As the preliminary data analysis we have used the time series data on prices and quantities of the pork market and the hen eggs market traded in Tokyo. As the exogenous variables we used the disposable income of consumers and the price index of feeds which may appear in the demand function and the supply function, respectively. These data will be further investigated in Section 6. Using these time series data, we have estimated (2.8) by the standard least squares method.

The result of estimation are summarized in Table 2.2. From this table, we find that the estimated unknown coefficients β_1^+ and β_1^- are considerably different both in the price equation and in the quantity equation.

Table 2.2: Estimated Results of (2.8)

(i) Price of Pork

	β_1^+	β_1^-
Estimate	0.6162	0.5300
S.D.	0.2204	0.2459

(iii) Price of Egg

	β_1^+	β_1^-
Estimate	0.4796	-0.0114
S.D.	0.2977	0.2592

(ii) Quantity of Pork

	β_1^+	β_1^-
Estimate	-0.3062	0.3362
S.D.	0.1851	0.1800

(iv) Quantity of Egg

	β_1^+	β_1^-
Estimate	0.2683	-0.0095
S.D.	0.3061	0.2087

In the price and quantity equations for the egg market, the estimated coefficients of β_1^+ and β_1^- are substantially different, but many of them are not significantly different from zero. On the other hand, all of the estimated coefficients β_1^+ and β_1^- in the price and quantity equations for the pork market are significantly different from zero, but the difference of the estimated coefficients β_1^+ and β_1^- are smaller than those in the egg market. In order to see the degree of model fitting, we have calculated the value of AIC(Akaike's information criterion) in each case. We have found that the estimated models with Δy_{t-1}^+ and Δy_{t-1}^- are better than those with only Δy_{t-1} in all cases by the minimum AIC. Therefore we tentatively conclude that by using the model given by (2.8), (2.9) and (2.10), we have picked up some kind of asymmetrical aspect in the time series data we analyzed.

3. Re-interpretation of a Disequilibrium Econometric Model

We have argued in the previous section that we often observed some kind of asymmetrical pattern in many economic time series data. The question we now have is to find an economic reasoning in some generality to lead the asymmetrical pattern we have discussed in economic time series. Generally speaking, there could be many ways to solve this problem. In this section we shall start our discussion by investigating a very specific econometric model which may give one affirmative answer to the theoretical question we have raised in the previous section. We shall re-consider a version of the disequilibrium econometric model originally investigated by Laffont and Garcia (1977). The disequilibrium econometric modelling has been developed initially by Fair and Jaffee (1972). Since then a number of different econometric models have been proposed. Some of them have been surveyed and explained in Chapter 10 of Maddala (1983) or Quandt (1988).

For the illustrative purpose mainly, we first make a brief review on the particular version of the disequilibrium model discussed by Laffont and Garcia (1977). We first set up the standard system consisting of the demand function and the supply function in a small market. Let D_t and S_t be the demand and supply of a commodity at time t. By assuming that they are linear for the sake of simplicity, these two equations are written as

$$\text{(3.1)} \qquad \begin{cases} D_t &= \beta_1 p_t + \boldsymbol{\gamma}_1' \boldsymbol{z}_{1t}^* + u_{1t} \\ S_t &= \beta_2 p_t + \boldsymbol{\gamma}_2' \boldsymbol{z}_{2t}^* + u_{2t} \end{cases} ,$$

where p_t is the price level, $\boldsymbol{z}_{1t}^*$ and $\boldsymbol{z}_{2t}^*$ are the strictly exogenous variables appeared in the demand and supply equations, respectively [2]. The demand shocks and the supply shocks are described by the disturbance terms u_{1t} and u_{2t}, respectively. The coefficients β_1, β_2, $\boldsymbol{\gamma}_1'$ (a $K_1 \times 1$ vector), and $\boldsymbol{\gamma}_2'$ (a $K_2 \times 1$ vector) are unknown.

The equilibrium condition explained by textbooks is given by $q_t = D_t = S_t$, where q_t is the quantity of the commodity traded in the market at t. Using this condition with (3.1), the observed quantity q_t and price p_t are simultaneously determined at each t given the exogenous variables and the disturbance terms. Instead of the equilibrium condition, Fair and Jaffee (1972) introduced a disequilibrium condition

$$\text{(3.2)} \qquad q_t = \min(D_t, S_t) \quad .$$

We note that when we replace (3.2) instead of the equilibrium condition, the econometric model consisting of (3.1) and (3.2) is not complete in the proper statistical sense. It is because the quantity variable q_t is determined by (3.1) and (3.2) once the price variable p_t is given. There have been several formulations to make the disequilibrium econometric model complete. In this section we shall adopt one simple formulation by Laffont and Garcia (1977). If $D_t > S_t$ at t in the market, there is an excess demand, which leads to make the price variable p_t go up. On the other hand, if $S_t > D_t$ at t in the market, there is an excess supply, which leads to make p_t go down. This consideration leads to the linearized price adjustment process

$$\text{(3.3)} \qquad \Delta p_{t+1} = \begin{cases} \delta_1 (D_t - S_t) & \text{if } D_t \geq S_t \\ \delta_2 (D_t - S_t) & \text{if } D_t < S_t \end{cases} ,$$

where $\Delta p_{t+1} = p_{t+1} - p_t$. [3] Since the coefficients δ_1 and δ_2 represent the adjustment speeds in the up-ward phase (or regime) and in the down-ward phase (or

[2] When there are some lagged prices and quantities as well as common exogenous variables, it is straight-forward to modify the following analyses.

[3] Kunitomo and Sato (1993) has used $\Delta p_t = p_t - p_{t-1}$ instead of Δp_{t+1}, but the following derivations are similar.

regime), $\delta_i > 0 (i = 1, 2)$ and they may not necessarily take the same value. Also there could be some economic justifications that they are different.

We now consider the disequilibrium econometric model consisting of (3.1), (3.2), and (3.3). The new aspect in our investigation here is to shed some light on the time series aspect of this type of disequilibrium econometric models. It seems that enough attention has not been paid on this problem in the disequilibrium literature, as far as we know. Let the 1×2 vector of endogenous variables $\boldsymbol{y}_t' = (q_t, p_{t+1})$ and the $1 \times K$ vector of exogenous variables $\boldsymbol{z}_t^{*'} = (\boldsymbol{z}_{1t}^{*'}, \boldsymbol{z}_{2t}^{*'})$. If the price variable p_t is in the up-ward phase, then $\Delta p_{t+1} \geq 0$, $q_t = S_t$ and

$$(3.4) \qquad D_t = q_t + (D_t - S_t) = q_t + \frac{1}{\delta_1}\Delta p_{t+1} \quad ,$$

provided that $\delta_1 > 0$. Hence the system of demand and supply functions can be rewritten as

$$(3.5) \qquad \begin{pmatrix} 1 & \delta_1^{-1} \\ 1 & 0 \end{pmatrix} \boldsymbol{y}_t = \begin{pmatrix} 0 & \delta_1^{-1} + \beta_1 \\ 0 & \beta_2 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} \gamma_1' & 0 \\ 0 & \gamma_2' \end{pmatrix} \boldsymbol{z}_t^* + \boldsymbol{u}_t \quad ,$$

where $\boldsymbol{u}_t' = (u_{t1}, u_{t2})$ is a 1×2 vector of the disturbance terms. If we assume $\delta_1 > 0$, we can solve (3.5) with respect to $\boldsymbol{y}_t$. The reduced form equations for (3.5) become

$$(3.6) \qquad \boldsymbol{y}_t = \begin{pmatrix} 0 & \beta_2 \\ 0 & 1 + \delta_1(\beta_1 - \beta_2) \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} 0 & \gamma_2' \\ \delta_1\gamma_1' & -\delta_1\gamma_2' \end{pmatrix} \boldsymbol{z}_t^* + \boldsymbol{v}_t^{(1)} \quad ,$$

where the disturbance vector of the reduced form equation is given by

$$(3.7) \qquad \boldsymbol{v}_t^{(1)} = \begin{pmatrix} 0 & 1 \\ \delta_1 & -\delta_1 \end{pmatrix} \boldsymbol{u}_t \quad .$$

We denote the matrix coefficients in (3.6) by $\boldsymbol{\Pi}_1^{(1)}$ and $\boldsymbol{\Pi}_*^{(1)}$, respectively. Then we may rewrite (3.6) as

$$(3.8) \qquad \boldsymbol{y}_t = \boldsymbol{\Pi}_1^{(1)} \boldsymbol{y}_{t-1} + \boldsymbol{\Pi}_*^{(1)} \boldsymbol{z}_t^* + \boldsymbol{v}_t^{(1)} \quad .$$

Similarly, if the price variable p_t is in the down-ward phase, $\Delta p_{t+1} < 0$, $q_t = D_t$ and

$$(3.9) \qquad S_t = q_t - \frac{1}{\delta_2}\Delta p_{t+1} \quad ,$$

provided that $\delta_2 > 0$. Hence the system of the demand and supply functions can be written as

$$(3.10) \qquad \begin{pmatrix} 1 & 0 \\ 1 & -\delta_2^{-1} \end{pmatrix} \boldsymbol{y}_t = \begin{pmatrix} 0 & \beta_1 \\ 0 & \beta_2 - \delta_2^{-1} \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} \gamma_1' & 0 \\ 0 & \gamma_2' \end{pmatrix} \boldsymbol{z}_t^* + \boldsymbol{u}_t \quad .$$

By the same arguments to (3.6) and (3.7), the reduced form equations are given by

$$(3.11) \qquad \boldsymbol{y}_t = \begin{pmatrix} 0 & \beta_1 \\ 0 & 1 + \delta_2(\beta_1 - \beta_2) \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} \gamma_1' & 0 \\ \delta_2\gamma_1' & -\delta_2\gamma_2' \end{pmatrix} \boldsymbol{z}_t^* + \boldsymbol{v}_t^{(2)} \quad ,$$

where the disturbance vector of the reduced form in this case is given by

$$v_t^{(2)} = \begin{pmatrix} 0 & 1 \\ \delta_2 & -\delta_2 \end{pmatrix} u_t \quad . \tag{3.12}$$

We also denote the matrix coefficients in (3.11) by $\boldsymbol{\Pi}_1^{(2)}$ and $\boldsymbol{\Pi}_*^{(2)}$, respectively. Then we may rewrite (3.11) as

$$y_t = \boldsymbol{\Pi}_1^{(2)} y_{t-1} + \boldsymbol{\Pi}_*^{(2)} z_t^* + v_t^{(2)} \quad . \tag{3.13}$$

Let also a 1×2 vector $e_2' = (0,1)$. Then the condition of $\Delta p_{t+1} \geq 0$ is equivalent to

$$e_2' y_t \geq e_2' y_{t-1} \quad . \tag{3.14}$$

We now summarize the reduced form equations for the disequilibrium econometric model consisting of (3.1), (3.2), and (3.3). They can be rewritten as

$$y_t = \begin{cases} \boldsymbol{\Pi}_1^{(1)} y_{t-1} + \boldsymbol{\Pi}_*^{(1)} z_t^* + v_t^{(1)} & (\text{if } e_2' y_t \geq e_2' y_{t-1}) \\ \boldsymbol{\Pi}_1^{(2)} y_{t-1} + \boldsymbol{\Pi}_*^{(2)} z_t^* + v_t^{(2)} & (\text{if } e_2' y_t < e_2' y_{t-1}) \end{cases} \quad . \tag{3.15}$$

The most important feature of this representation is that the endogenous variables may take quite different values in two different phases or regimes. This type of statistical time series models could be termed as the threshold models in the recent time series literature. However, since the endogenous variables and two phases at time t are determined simultaneously, we shall call this type of time series models as the simultaneous switching time series model. It will turn out that this simultaneity has not only an important economic interpretation, but also a new aspect in the non-linear time series modelling. This finding leads to some interesting problems in econometrics.

4. Simultaneous Switching Time Series Model

4.1 Some Non-linear Time Series Models

In the simultaneous switching time series model given by (3.15), the vector of the endogenous variables y_t can not be written as a linear combination of the disturbance terms $v_s^{(j)}$ $(j = 1, 2, s \leq t)$ in the past, and it is a non-linear Markovian switching time series model. Generally speaking, it may be possible to generalize this type of non-linear time series model in a variety of different ways. In this paper, however, we shall consider only a simple way of extending the model given by (3.15).

Let an $m \times 1$ vector of the endogenous variables y_t. Then we consider a set of equations

$$
(4.1) \qquad \boldsymbol{y}_t = \begin{cases} \sum_{i=1}^{p} \boldsymbol{\Pi}_i^{(1)} \boldsymbol{y}_{t-i} + \boldsymbol{\Pi}_*^{(1)} \boldsymbol{z}_t^* + \boldsymbol{v}_t^{(1)} & \text{(in the phase } S_t^{(1)}) \\ \sum_{i=1}^{p} \boldsymbol{\Pi}_i^{(2)} \boldsymbol{y}_{t-i} + \boldsymbol{\Pi}_*^{(2)} \boldsymbol{z}_t^* + \boldsymbol{v}_t^{(2)} & \text{(in the phase } S_t^{(2)}) \end{cases} ,
$$

where $\boldsymbol{z}_t^*$ is a $K \times 1$ vector of the strictly exogenous variables, $\boldsymbol{\Pi}_i^{(j)}$ $(i = 1, \ldots, p;\ j = 1,2)$ and $\boldsymbol{\Pi}_*^{(j)} (j = 1,2)$ are $m \times m$ and $m \times k$ matrices of unknown coefficients, respectively. Without loss of generality, we assume that the first component of $\boldsymbol{z}_t^*$ is the constant term. The two phases for the endogenous variables are determined by the states $S_t^{(j)} (j = 1,2)$ defined by

$$
(4.2) \qquad \begin{cases} S_t^{(1)} = \{\boldsymbol{c}_0' \boldsymbol{y}_t \geq \boldsymbol{c}_1' \boldsymbol{y}_{t-1} + c_*\} \\ S_t^{(2)} = \{\boldsymbol{c}_0' \boldsymbol{y}_t < \boldsymbol{c}_1' \boldsymbol{y}_{t-1} + c_*\} \end{cases} ,
$$

where $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ are $m \times 1$ vectors of known constants, and c_* is a known scalar constant.

In the present formulation it is noteworthy that two phases $S_t^{(j)}$ on the endogenous variables are dependent upon not only their past values but also the value of the disturbances at t. This simultaneity makes the model given by (4.1) interesting from the statistical point of view. In this sense we call the model defined by (4.1) as the simultaneous switching time series model. In particular, when $m = 2$, $p = 1$, $\boldsymbol{c}_0 = \boldsymbol{c}_1 = \boldsymbol{e}_2$, and $c_* = 0$ in (4.1) we have (3.15). Also when $m = 1$, $\boldsymbol{c}_0 = 0$, $\boldsymbol{c}_1 \neq 0$, and $\Pi_*^{(j)} = O$ $(j = 1,2)$, we have the Threshold Autoregressive (TAR) model. The TAR model is an extension of the standard autoregressive (AR) model and much attention has been recently paid as a non-linear Markovian swiching model by time series analysts. In the above formulation, we call (4.1) as the Simultaneous Switching Autoregressive (SSAR) model when we do not have $\boldsymbol{z}_t^*$ except a constant or $\{\boldsymbol{z}_t^*\}$ are independently and identically distributed random variables.

4.2 Coherency Condition

We have presented a new non-linear time series model represented by (4.1). However, there is a basic question if the model given by (4.1) does have the logical consistency as a statistical model. Our answer to this question in the general situation is negative and we shall show that we need some additional conditions on the unknown parameters in (4.1). The issue we have can be called the coherency problem. We should note that this problem has been already systematically investigated by Gourieroux et. al. (1980) in their analysis of disequilibrium econometric models.

At time t all information available from the past is described as the σ-field $\mathcal{F}_{t-1} = \{\boldsymbol{y}_s, \boldsymbol{z}_{s+1}^*, s \leq t-1\}$. Given $\mathcal{F}_{t-1}$, there are two phases $S_t^{(j)}$ $(j = 1,2)$ at t and we do not know from the past information which phase we shall be at t. Given $\mathcal{F}_{t-1}$, the probability that the phase $S_t^{(1)}$ would actually occur is given by

$$(4.3) P\{S_t^{(1)}|\mathcal{F}_{t-1}\}$$

$$= P\{\boldsymbol{c}_0'(\boldsymbol{v}_t^{(1)} + \sum_{i=1}^{p} \boldsymbol{\Pi}_i^{(1)} \boldsymbol{y}_{t-i} + \boldsymbol{\Pi}_*^{(1)} \boldsymbol{z}_t^*) \geq \boldsymbol{c}_1' \boldsymbol{y}_{t-1} + c_* | \mathcal{F}_{t-1}\}$$

$$= P\{\boldsymbol{c}_0' \boldsymbol{v}_t^{(1)} \geq (\boldsymbol{c}_1' - \boldsymbol{c}_0' \boldsymbol{\Pi}_1^{(1)}) \boldsymbol{y}_{t-1} - \boldsymbol{c}_0' \sum_{i=2}^{p} \boldsymbol{\Pi}_i^{(1)} \boldsymbol{y}_{t-i} - (\boldsymbol{c}_0' \boldsymbol{\Pi}_*^{(1)} - c_* \boldsymbol{e}_1') \boldsymbol{z}_t^* | \mathcal{F}_{t-1}\} .$$

By the same token, the probability that the phase $S_t^{(2)}$ would actually occur given $\mathcal{F}_{t-1}$ is also given by

$$(4.4) P\{S_t^{(2)}|\mathcal{F}_{t-1}\}$$

$$= P\{\boldsymbol{c}_0' \boldsymbol{v}_t^{(2)} < (\boldsymbol{c}_1' - \boldsymbol{c}_0' \boldsymbol{\Pi}_1^{(2)}) \boldsymbol{y}_{t-1} - \boldsymbol{c}_0' \sum_{i=2}^{p} \boldsymbol{\Pi}_i^{(2)} \boldsymbol{y}_{t-i} - (\boldsymbol{c}_0' \boldsymbol{\Pi}_*^{(2)} - c_* \boldsymbol{e}_1') \boldsymbol{z}_t^* | \mathcal{F}_{t-1}\} .$$

Since there are only two phases $S_t^{(1)}$ and $S_t^{(2)}$ at t, the sum of two probabilities given by (4.3) and (4.4) should be one for arbitrary values of past observations and unknown parameters in the parameter space appropriately defined. In order to state some conditions on the coherency problem for the model given by (4.1), we assume the following condition on the disturbance terms $\boldsymbol{v}_t^{(j)}$ $(j = 1, 2)$.

(A1) The random variables $\boldsymbol{c}_0' \boldsymbol{v}_t^{(j)}$ $(j = 1, 2)$ are absolutely continuous random variables, which are mutually independent with respect to t and their distribution functions are given by $F^{(j)}(\cdot)$ $(j = 1, 2)$.

Under this assumption, we immediately obtain the following result on the coherency problem.

Theorem 4.1 : *Suppose $F^{(1)} \neq F^{(2)}$ with a positive Lebesgue measure under the assumption of (A1) and $c_* = 0$. Then we have the following conditions* (i) *or* (ii) *as a set of necessary conditions for the logical coherency of* (4.1). (i) $\boldsymbol{c}_0 = 0$, (ii) $\boldsymbol{c}_0 \neq 0$ *and for* $\boldsymbol{e}_j' = (0, \ldots, 0, 1, 0, \ldots, 0)$ $(j = 1, \ldots, m)$,

$$(4.5) \quad F^{(1)}(\boldsymbol{e}_j'(\boldsymbol{c}_1 - \boldsymbol{\Pi}_1^{(1)'} \boldsymbol{c}_0)) = F^{(2)}(\boldsymbol{e}_j'(\boldsymbol{c}_1 - \boldsymbol{\Pi}_1^{(2)'} \boldsymbol{c}_0)) ,$$

$$(4.6) \quad F^{(1)}\left(\boldsymbol{e}_j'(\boldsymbol{\Pi}_i^{(1)'} \boldsymbol{c}_0)\right) = F^{(2)}\left(\boldsymbol{e}_j'(\boldsymbol{\Pi}_i^{(2)'} \boldsymbol{c}_0)\right) \quad (i = 2, \ldots, p) ,$$

$$(4.7) \quad F^{(1)}\left(\boldsymbol{e}_j'(\boldsymbol{\Pi}_*^{(1)'} \boldsymbol{c}_0 - \boldsymbol{e}_1 c_*)\right) = F^{(2)}\left(\boldsymbol{e}_j'(\boldsymbol{\Pi}_*^{(2)'} \boldsymbol{c}_0 - \boldsymbol{e}_1 c_*)\right) .$$

When the condition (i) holds, the problem of simultaneity on two phases disappears and the problem of coherency is a trivial one. Furthermore, if there is no exogenous variables $\{\boldsymbol{z}_t^*\}$, (4.1) is identical to the multivariate p-th order TAR model. Hence for the TAR models in the non-linear time series analysis there is no serious coherency problem in this respect.

Next, we consider the case when the disturbance terms in (4.1) have second order moments. Let the variance-covariance matrices of $\boldsymbol{v}_t^{(j)}$ $(j = 1, 2)$ be

$\boldsymbol{\Omega}^{(j)}$ $(j = 1, 2)$. Then the variances of $\boldsymbol{c}_0' \boldsymbol{v}_t^{(j)}$ are given by $\sigma_j^2 = \boldsymbol{c}_0' \boldsymbol{\Omega}^{(j)} \boldsymbol{c}_0$ $(j = 1, 2)$. Also we consider the situation when the distribution functions $F^{(j)}$ can be written as

$$G\left(\frac{x}{\sigma_j}\right) = F^{(j)}(x) \quad (j = 1, 2) \quad . \tag{4.8}$$

When $c_* = 0$ in this case, the condition (ii) in Theorem 4.1 can be simply rewritten as the condition (ii)$'$

$$\frac{1}{\sigma_1}(\boldsymbol{c}_1' - \boldsymbol{c}_0' \boldsymbol{\Pi}_1^{(1)}) = \frac{1}{\sigma_2}(\boldsymbol{c}_1' - \boldsymbol{c}_0' \boldsymbol{\Pi}_1^{(2)}) \ , \tag{4.9}$$

$$\frac{1}{\sigma_1}\boldsymbol{c}_0' \boldsymbol{\Pi}_j^{(1)} = \frac{1}{\sigma_2}\boldsymbol{c}_0' \boldsymbol{\Pi}_j^{(2)} \quad (j = 2, \ldots, p) \ , \tag{4.10}$$

$$\frac{1}{\sigma_1}\boldsymbol{c}_0' \boldsymbol{\Pi}_*^{(1)} = \frac{1}{\sigma_2}\boldsymbol{c}_0' \boldsymbol{\Pi}_*^{(2)} \ . \tag{4.11}$$

We notice that the disturbance terms $\{\boldsymbol{v}_t^{(j)}, j = 1, 2\}$ in the reduced form equations for the disequilibrium econometric model in Section 3 are linear combinations of the disturbance term $\{\boldsymbol{u}_t\}$ in the structural form. Therefore we establish the following result from Theorem 4.1.

Corollary 4.1 : *The disequilibrium econometric model given by* (3.1), (3.2), *and* (3.3) *satisfies the condition* (ii)$'$ *for the coherency problem, provided that* $\delta_i > 0$ $(i = 1, 2)$ *and the disturbance terms* $\{\boldsymbol{u}_t\}$ *are mutually independent and absolutely continuous random variables with* $E[||\boldsymbol{u}_t||^2] < +\infty$.

Gourieroux et. al. (1980) has presented a general theorem for the coherency problem and the above results could be derived directly from it. However, we have presented the above propositions because the coherency problem has not been discussed in the statistical time series analysis.

4.3 Simulations of SSAR(1)

The simultaneous switching time series model given by (4.1) is a quite complicated stochastic process in the general case. In order to get some idea on its statistical properties, we first consider the simplest case. If we look at the price equation in (3.6) and (3.11), and we assume that there is only a constant term in $\{\boldsymbol{z}_t^*\}$, we have the first order SSAR model. Then this model is given by

$$y_t - \mu = \begin{cases} \Pi_1^{(1)}(y_{t-1} - \mu) + \sigma_1 v_t & (\text{if} \quad y_t \geq y_{t-1}) \\ \Pi_1^{(2)}(y_{t-1} - \mu) + \sigma_2 v_t & (\text{if} \quad y_t < y_{t-1}) \end{cases} \tag{4.12}$$

which is denoted by SSAR(1). This model can also be obtained if the exogenous variables except a constant term are independently and identically distributed random variables. The condition on coherency in this case is given by

$$\frac{1 - \Pi_1^{(1)}}{\sigma_1} = \frac{1 - \Pi_1^{(2)}}{\sigma_2} \quad . \tag{4.13}$$

If we further assume that the disturbance terms $\{v_t\}$ are independently and identically distributed with $N(0,1)$, the stochastic process $\{y_t\}$ is determined. Although there are four unknown parameters $\Pi_1^{(j)}, \sigma_j^2$ $(j = 1, 2)$ in (4.12), there are only three free parameters $a = \Pi_1^{(1)}$, $b = \Pi_1^{(2)}$ and $(1-a)/\sigma_1 = r$.

We took several sets of values of these parameters and did a large number of simulations. Among them we only present three cases in Figure 4.1. The middle one shows the sample path when $a = b = 0.5$, which means that the SSAR(1) model is actually the standard AR(1) model. When $a \neq b$, we can notice some asymmetrical patterns in the sample paths of the simulated time series. For economic time series, the case when $a = 0.8$ and $b = 0.2$ may be the most interesting one. Even though we use a very simple SSAR model, we found that we can get very interesting asymmetrical sample paths of simulated time series. This aspect can not be realized by the standard linear time series models such as the stationary ARMA model.

4.4 A Relation Between SSAR(1) and TAR(1)

The SSAR(1) model given in (4.12) is different from the first order threshold autoregressive model, which is often denoted by TAR(1). However, there is a close connection between these two switching models. Let a random variable w_t be defined by

$$w_t = v_t - r(y_{t-1} - \mu). \tag{4.14}$$

Then using (4.12), the stochastic process $\{w_t\}$ follows

$$w_t = \begin{cases} \Pi_1^{(1)} w_{t-1} + v_t - v_{t-1} & (\text{if} \quad w_{t-1} \geq 0\ ) \\ \Pi_1^{(2)} w_{t-1} + v_t - v_{t-1} & (\text{if} \quad w_{t-1} < 0\ )\ . \end{cases} \tag{4.15}$$

By this stochastic process $\{w_t\}$, the first difference of the observed process $\{y_t\}$ can be written as

$$\Delta y_t = \begin{cases} \sigma_1 w_t & (\text{if} \quad w_t \geq 0) \\ \sigma_2 w_t & (\text{if} \quad w_t < 0)\ , \end{cases} \tag{4.16}$$

where $\Delta y_t = y_t - y_{t-1}$. From this representation we find an interesting relation between the SSAR(1) and the TAR(1) models. In more general cases, however, their relations become more complicated.

5. Statistical Properties of the SSAR model

5.1 A Multivariate SSAR model

In this section we shall investigate some basic statistical properties of a version of the multivariate SSAR model. Let $\boldsymbol{y}_t$ be an $m \times 1$ vector of the endogenous variables. The model we consider in this section is represented by

$$(5.1) \qquad \boldsymbol{y}_t = \begin{cases} \boldsymbol{\mu}_1 + \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{\Sigma}_1^{1/2}\boldsymbol{u}_t & \text{if} \quad \boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1} \\ \boldsymbol{\mu}_2 + \boldsymbol{B}\boldsymbol{y}_{t-1} + \boldsymbol{\Sigma}_2^{1/2}\boldsymbol{u}_t & \text{if} \quad \boldsymbol{e}_m'\boldsymbol{y}_t < \boldsymbol{e}_m'\boldsymbol{y}_{t-1} \end{cases},$$

where $\boldsymbol{e}_m' = (0, \cdots, 0, 1)$ and $\boldsymbol{\mu}_i'$ $(i = 1, 2)$ are $1 \times m$ vectors of constants, and $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{\Sigma}_i^{1/2}$ $(i = 1, 2)$ are $m \times m$ matrices. The disturbance terms $\{\boldsymbol{u}_t\}$ are independently and identically distributed, and they are absolutely continuous random variables with the density function $g(\boldsymbol{u})$. We denote this model as $\mathrm{SSAR}_m(1)$ and also we simply denote $\mathrm{SSAR}_1(1)$ as SSAR(1). By using the standard Markovian representation the p-th order SSAR model can be reduced to the $\mathrm{SSAR}_m(1)$ model. Hence without loss of generality we shall consider the $\mathrm{SSAR}_m(1)$ model given by (5.1).

The conditions of $\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}$ and $\boldsymbol{e}_m'\boldsymbol{y}_t < \boldsymbol{e}_m'\boldsymbol{y}_{t-1}$ can be rewritten as

$$(5.2) \qquad \boldsymbol{e}_m'\boldsymbol{\Sigma}_1^{1/2}\boldsymbol{u}_t \geq \boldsymbol{e}_m'(\boldsymbol{I}_m - \boldsymbol{A})\boldsymbol{y}_{t-1} - \boldsymbol{e}_m'\boldsymbol{\mu}_1 \quad ,$$

and

$$(5.3) \qquad \boldsymbol{e}_m'\boldsymbol{\Sigma}_2^{1/2}\boldsymbol{u}_t < \boldsymbol{e}_m'(\boldsymbol{I}_m - \boldsymbol{B})\boldsymbol{y}_{t-1} - \boldsymbol{e}_m'\boldsymbol{\mu}_2 \quad ,$$

respectively. When $\boldsymbol{\Sigma}_i^{1/2}$ $(i = 1, 2)$ are positive definite, the necessary and sufficient conditions on the coherency problem for (5.1) can be summarized by a $1 \times (m+1)$ vector

$$(5.4) \qquad \begin{aligned} \frac{1}{\sigma_1}[\boldsymbol{e}_m'(\boldsymbol{I}_m - \boldsymbol{A}),\ \boldsymbol{e}_m'\boldsymbol{\mu}_1] &= \frac{1}{\sigma_2}[\boldsymbol{e}_m'(\boldsymbol{I}_m - \boldsymbol{B}),\ \boldsymbol{e}_m'\boldsymbol{\mu}_2] \\ &= \boldsymbol{r}' \ , \end{aligned}$$

where $\sigma_j^2 = \boldsymbol{e}_m'\boldsymbol{\Sigma}_j\boldsymbol{e}_m = \boldsymbol{e}_m'\boldsymbol{\Sigma}_j^{1/2}\boldsymbol{\Sigma}_j^{1/2\prime}\boldsymbol{e}_m$ $(j = 1, 2)$.

5.2 Ergodicity

The first question on (5.1) is its ergodicity. The SSAR model defined by (5.1) is a Markovian time series model, which is a non-linear in the state variables. It is important to notice that this model has a representation

$$(5.5) \qquad \boldsymbol{y}_t = T(\boldsymbol{y}_{t-1}, \boldsymbol{u}_t) \quad ,$$

where $T(\cdot)$ is a mapping from $\boldsymbol{R}^{2m} \longrightarrow \boldsymbol{R}^m$, and it cannot be written as a sum of two components involving $\boldsymbol{y}_{t-1}$ and $\boldsymbol{u}_t$, separately. This is because the phase of the state variables at t depends on not only the past state variables $\boldsymbol{y}_{t-1}$, but also $\boldsymbol{u}_t$ at t.

Since (5.1) is a non-linear Markovian switching process in the discrete time, it has been generally known in the statistical time series anaysis that it is difficult to deal with its ergodicity. In this respect, there is a sharp difference between linear time series models and non-linear time series models. For the ergodicity of the Markov chain defined by (5.1), we have the following characterization result if we assume the strong condition for coherency given by (5.4). The proof is given in Appendix.

Theorem 5.1 : *Suppose (5.4) hold. Then the $SSAR_m(1)$ model given by (5.1) is strongly continuous with respect to the state variables $\boldsymbol{y}_{t-1} = x$.*

There are several different concepts of continuity on Markov chain with general states. Since the model we are dealing with is a Markovian process with uncountable states, we need some kind of continuity. The discrete Markovian process $\{\boldsymbol{y}_t\}$ is strongly continuous iff

$$E[h(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = x] \tag{5.6}$$

is continuous with respect to the vector of state variables x for every bounded continuous function $h(\cdot)$. Establishing the strong continuity by Theorem 5.1, it is possible to use a well-known theorem on the sufficient condition of the geometrical ergodicity on Markovian process due to Tweedie (1975). However, it is difficult to deal with the problem of ergodicity in the general case. We first state the result when $m = 1$. The proof is given in Appendix.

Theorem 5.2 : *Suppose the density function of $\{u_t\}$ satisfies the condition*

$$\lim_{u \to \pm\infty} u^2 g(u) = 0 \quad , \tag{5.7}$$

and $E[|u_t|] < +\infty$. Then the Markov chain defined by (5.1) with $m = 1$ is ergodic iff

$$A < 1,\ B < 1,\ AB < 1 \ . \tag{5.8}$$

The condition (5.7) is satisfied by the normal distribution. Figure 5.1 gives the region of the ergodicity for SSAR(1). It is interesting to see the conditions $|A| < 1$ and $|B| < 1$ are too strong for the ergodicity. When $A < 0, B < 0$, and $AB < 1$, we have the sample paths, which are interesting in many respects. In Figure 5.2, we present some sample paths of simulated time series in this case. In the context of disequilibrium econometric models, there could be some interesting interpretations if we assume that $\beta_1 < 0 < \beta_2$ and $\delta_i > 0$ $(i = 1, 2)$. This case corresponds to the situation when the absolute values of β_1, β_2 and δ_i $(i = 1, 2)$ are large and there are some over-reactions or over-adjustments in the market. Also we should note that the ergodic region shown by Figure 5.1 is the same as that for the following TAR(1) model,

$$(5.9) \qquad y_t = \begin{cases} Ay_{t-1} + u_t & \text{if} \quad y_{t-1} \geq 0 \\ By_{t-1} + u_t & \text{if} \quad y_{t-1} < 0 \end{cases} .$$

The conditions on the ergodicity for the TAR(1) model have been fully investigated by Petrucelli and Woolford (1984). In the more general case when $m > 1$ in (5.1), we first present some sufficient conditions on the ergodicity. The proof is similar to the first part of the proof of Theorem 5.2 and it is omitted.

Theorem 5.3 : *When $m \geq 1$, suppose that $E[|u_{it}|] < +\infty$ ($\boldsymbol{u}_t = (u_{it})$) for $i = 1, \cdots, m$. The sufficient conditions for the ergodicity of the model (5.1) are given by either (5.10) or (5.11),*

$$(5.10) \qquad \max_j \left\{ \sum_{i=1}^{m} |a_{ij}| \, , \; \sum_{i=1}^{m} |b_{ij}| \right\} < 1 \; ,$$

$$(5.11) \qquad \max_i \left\{ \sum_{j=1}^{m} |a_{ij}| \, , \; \sum_{j=1}^{m} |b_{ij}| \right\} < 1 \; ,$$

where $A = (a_{ij})$ and $B = (b_{ij})$.

The conditions given in Theorem 5.3 are very restrictive and not satisfactory in many applications. The p-th order univariate SSAR model is given by

$$(5.12) \qquad y_t = \begin{cases} a_0 + \sum_{j=1}^{p} a_j y_{t-j} + \sigma_1 u_t & \text{if} \quad y_t \geq y_{t-1} \\ b_0 + \sum_{j=1}^{p} b_j y_{t-j} + \sigma_2 u_t & \text{if} \quad y_t < y_{t-1} \; , \end{cases}$$

where $\{a_j\}$ and $\{b_j\}$ ($j = 0, \cdots, p$) are unknown coefficients. This model can be written as (5.1). Let define $p \times 1$ vectors $\boldsymbol{y}_t$ and $\boldsymbol{\mu}_i$ ($i = 1, 2$) by

$$(5.13) \qquad \boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix} , \quad \boldsymbol{\mu}_1 = \begin{pmatrix} a_0 \\ \vdots \\ 0 \end{pmatrix} , \quad \boldsymbol{\mu}_2 = \begin{pmatrix} b_0 \\ \vdots \\ 0 \end{pmatrix} ,$$

and also define $p \times p$ matrices

$$(5.14) \qquad \boldsymbol{A} = \begin{pmatrix} a_1 & \cdots & \cdots & a_p \\ 1 & & \mathbf{0} & \\ & \ddots & & \\ \mathbf{0} & & 1 & 0 \end{pmatrix} , \quad \boldsymbol{B} = \begin{pmatrix} b_1 & \cdots & \cdots & b_p \\ 1 & & \mathbf{0} & \\ & \ddots & & \\ \mathbf{0} & & 1 & 0 \end{pmatrix} .$$

Then (5.12) can be regarded as (5.1) by the standard Markovian representation. In this case, however, the conditions given in Theorem 5.3 cannot be satisfied.

Hence in the univariate SSAR processes we need alternative set of sufficient conditions for the ergodicity.

Theorem 5.4 : *When $m = 1$ and $p \geq 1$, assume that $E[|u_t|] < +\infty$. Then the sufficient condition for the ergodicity of (5.12) is given by*

$$\max\left\{\sum_{i=1}^{m}|a_i| \ , \ \sum_{i=1}^{m}|b_i|\right\} < 1 \quad . \tag{5.15}$$

The above condition is the same as that given by Lemma 3.1 of Chan and Tong (1985) for the p-th order univariate TAR model without constant term. The proof is similar to that given by Chan and Tong (1985) and so it is omitted. More recently, Brockwell, Liu, and Tweedie (1992) have presented alternative sufficient conditions for the multivariate TAR models with some restrictions on their parameters. It seems that those sufficient conditions are related to the conditions for the $\text{SSAR}_m(1)$ model.

5.3 Stationary Distribution and its Moments

When the model (5.1) satisfies the sufficient conditions for the ergodicity, then the next question would be the properties of the stationary distribution. Unlike the standard ARMA models, the problem of stationary distribution is also not a trivial one. In order to investigate the stationary distribution of the vector process $\{\boldsymbol{y}_t\}$, we assume that the disturbance terms $\{\boldsymbol{u}_t\}$ are independently and identically distributed random variables with the density function $g(\boldsymbol{u})$.

Let $f(\boldsymbol{y})$ be the density function of the stationary distribution of the process $\{\boldsymbol{y}_t\}$. Then we have the following representation.

Theorem 5.5 : *Suppose the ergodicity conditions and the assumptions we made for (5.1) hold. Then the density function of the stationary distribution for $\{f(\boldsymbol{y}_t)\}$ satisfies the equation*

$$\begin{aligned} f(\boldsymbol{y}) &= \int_{y_m \geq z_m} \boldsymbol{\Sigma}_1^{-1/2} g\left[\boldsymbol{\Sigma}_1^{-1/2}(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{z})\right] f(\boldsymbol{z})\, d\boldsymbol{z} \\ &\quad + \int_{y_m < z_m} \boldsymbol{\Sigma}_2^{-1/2} g\left[\boldsymbol{\Sigma}_2^{-1/2}(\boldsymbol{y} - \boldsymbol{B}\boldsymbol{z})\right] f(\boldsymbol{z})\, d\boldsymbol{z} \quad , \end{aligned} \tag{5.16}$$

where $\boldsymbol{y}' = (y_1, \cdots, y_m)$ *and* $\boldsymbol{z}' = (z_1, \cdots, z_m)$.

In the general case, (5.16) is an integral equation and we could not have obtained an explicit formula for the stationary distribution even if we assume that $g(\boldsymbol{u})$ is the multivariate normal density function. Since it is not easy to obtain the explicit form of the stationary density function, we have investigated it by the numerical method. Let $f(\boldsymbol{y}_t)$ be the conditional density function of $\boldsymbol{y}_t$ given the σ-field $\mathcal{F}_{t-1}$. When $m = 1$, the ergodicity of the stochastic process $\{\boldsymbol{y}_t\}$ implies that the conditional density function $f_t(y)$ converges to $f(y)$ as $t \rightarrow +\infty$, where $f_t(y)$ is defined successively by

$$f_t(y) = \int_{y \geq z} \frac{1}{\sigma_1} g\left[\frac{y - Az}{\sigma_1}\right] f_{t-1}(z)\, dz + \int_{y < z} \frac{1}{\sigma_2} g\left[\frac{y - Bz}{\sigma_2}\right] f_{t-1}(z)\, dz \quad . \tag{5.17}$$

Hence starting from a reasonable density function $f_0(y)$, we expect to have the stationary density function after a sufficiently large number of iterations using the numerical integration method. In our numerical analysis, we have used the standard normal density for $g(x)$. Among many numerical examples, we shall present only one case in Figure 5.1.

In Figure 5.3, the initial distribution $f_0(y) = n(0, 1)$ is shown by the dotted line. We show the density functions for the stationary distributions when $a < b$. The conditional density function $f_t(y)$ moves to the right and converges to $f(y)$. The center of $f(y)$ is positive and the variance is smaller than 1 in this case. Although the density function of the stationary distribution is symmetric around zero when $a = b$, they are not when $a \neq b$. When $a < 0$, the stationary density is considerably skewed.

We also have investigated the moments of the stationary distribution. The sufficient conditions for the existence of moments can be directly derived by a general theorem due to Tweedie (1983). (See the proof in Appendix.) We summarize our result in the following proposition.

Theorem 5.6 : *In the model (5.1), suppose that (5.7),* $E[|u_{it}|^j] < +\infty$ *for* $j = 1, \cdots, k$*, and the sufficient conditions for the ergodicity hold . Then the moments of the stationary distribution exist up to the k-th order.*

In our numerical computations, we have calculated the first four moments assuming that $m = 1, r = 1$, $\mu_1 = \mu_2 = 0$, and the normality on the disturbance terms. The numerical values of the mean, variance, skewness, and kurtosis have been summarized in Table 5.1. From Table 5.1 we immediately notice that the mean and skewness of the stationary distribution are different from zero when $A \neq B$. It has been well-known that the mean and skewness are zero and the kurtosis is 3 when $A = B$. The variance of the stationary distribution depends on the values of both A and B. When $|A| < 1$ and $|B| < 1$, the kurtosis of stationary distribution is not far from 3. However, when $A < 0 < B$, it can be large and the stationary distribution is considerably different from the normal distribution. Hence we have found that the moment properties of stationary distribution are quite different from those for the standard Gaussian ARMA processes.

Table 5.1 : Moments of SSAR(1) [4]

(i) Mean

	$B=0.8$	$B=0.5$	$B=0.2$	$B=-0.2$	$B=-2$
$A=0.8$	0.000	-0.399	-0.622	-0.837	-1.484
$A=0.5$	0.399	0.000	-0.227	-0.449	-1.147
$A=0.2$	0.624	0.227	0.000	-0.227	-0.990
$A=-0.2$	0.840	0.450	0.227	0.000	-1.016
$A=-2$	1.489	1.147	0.990	1.018	NA

(ii) Variance

	$B=0.8$	$B=0.5$	$B=0.2$	$B=-0.2$	$B=-2$
$A=0.8$	0.111	0.201	0.298	0.449	1.520
$A=0.5$	0.201	0.333	0.475	0.700	2.406
$A=0.2$	0.298	0.475	0.667	0.985	3.702
$A=-0.2$	0.449	0.700	0.985	1.500	8.570
$A=-2$	1.513	2.408	3.708	8.593	NA

(iii) Skewness

	$B=0.8$	$B=0.5$	$B=0.2$	$B=-0.2$	$B=-2$
$A=0.8$	0.000	-0.210	-0.409	-0.668	-1.487
$A=0.5$	0.210	0.000	-0.212	-0.513	-1.535
$A=0.2$	0.410	0.212	0.000	-0.330	-1.493
$A=-0.2$	0.669	0.513	0.330	0.000	-1.117
$A=-2$	1.490	1.537	1.495	1.117	NA

(iv) Kurtosis

	$B=0.8$	$B=0.5$	$B=0.2$	$B=-0.2$	$B=-2$
$A=0.8$	3.000	3.080	3.313	3.820	6.477
$A=0.5$	3.079	3.000	3.090	3.498	6.503
$A=0.2$	3.311	3.089	3.000	3.195	5.842
$A=-0.2$	3.816	3.496	3.194	3.000	4.097
$A=-2$	6.481	6.501	5.839	4.089	NA

5.4 Asymptotic Properties of the Maximum Likelihood Estimator of SSAR

The SSAR model in the general case is quite complex as a statistical model in two respects. The first aspect is that it is a non-linear Markovian switching time series model. The other aspect is due to the fact that the conditional distribution

[4] We calculated the first four moments by integrating the stationary distribution numerically. In Table 5.1 NA stands for the case when the SSAR(1) model is non-ergodic and there is no stationary distribution.

is a kind of mixture distribution at t given the information available in the past. Thus there are some difficulties to investigate fully the statistical properties of the estimation methods in the general case.

The standard estimation method for this kind of non-linear model is the maximum likelihood (ML) estimation. In this section we shall report some asymptotic properties of the ML estimator for (5.1). Under the assumption of normal disturbances and $|\boldsymbol{\Sigma}_i| \neq 0$ $(i = 1, 2)$, the log-likelihood function is given by

$$
\begin{aligned}
\log L_T(\boldsymbol{\theta}) = & -\frac{Tm}{2}\log 2\pi \\
& -\frac{1}{2}\log|\boldsymbol{\Sigma}_1|\sum_{t=1}^{T} I(\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}) \\
& -\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{y}_t - \boldsymbol{A}\boldsymbol{y}_{t-1}\right)'\boldsymbol{\Sigma}_1^{-1}\left(\boldsymbol{y}_t - \boldsymbol{A}\boldsymbol{y}_{t-1}\right) I(\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}) \\
& -\frac{1}{2}\log|\boldsymbol{\Sigma}_2|\sum_{t=1}^{T} I(\boldsymbol{e}_m'\boldsymbol{y}_t < \boldsymbol{e}_m'\boldsymbol{y}_{t-1}) \\
& -\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{y}_t - \boldsymbol{B}\boldsymbol{y}_{t-1}\right)'\boldsymbol{\Sigma}_2^{-1}\left(\boldsymbol{y}_t - \boldsymbol{B}\boldsymbol{y}_{t-1}\right) I(\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}),
\end{aligned}
\tag{5.18}
$$

where $\boldsymbol{\theta}' = \left(\boldsymbol{r}', (vech(\boldsymbol{\Sigma}_1))', (vech(\boldsymbol{\Sigma}_2))', \boldsymbol{e}_i'\boldsymbol{A}, \boldsymbol{e}_i'\boldsymbol{B}, \boldsymbol{e}_i'\boldsymbol{\mu}_j \ (i = 1, \cdots, m-1, j = 1, 2)\right)$ denotes the vector of unknown parameters in $\mathrm{SSAR}_m(1)$ and the parameter space $\boldsymbol{\Theta}$ is defined correspondingly. In the present case, it is possible to show directly that the partial derivatives of the log-likelihood function is a martingale process (see Appendix for its derivation),

$$
E\left[\frac{\partial \log L_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} | \mathcal{F}_{t-1}\right] = \frac{\partial \log L_{t-1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \quad , \tag{5.19}
$$

where $\mathcal{F}_{t-1}$ is the $\sigma-$field generated by $\{\boldsymbol{y}_s, s \leq t-1\}$. From this relation we expect that it is possible to use the martingale central limit theorems (MCLT) for the $\mathrm{SSAR}_m(1)$ model. By using the MCLT developed by Dvoretzsky (1972) (see also Hall and Heyde (1980), for instance) and the standard method in the asymptotic theory (see Section 4 of Amemiya (1985), for instance), we can claim that the following asymptotic properties of the ML estimators hold.

Theorem 5.7 : *For the $SSAR_m(1)$ model given by (5.1), suppose the sufficient conditions for the coherency and ergodicity hold and the disturbances terms $\{\boldsymbol{u}_t\}$ are independently distributed as $N(\mathbf{o}, \boldsymbol{I}_m)$ with $|\boldsymbol{\Sigma}_i| \neq 0$ $(i = 1, 2)$. Also suppose that the true parameter vector $\boldsymbol{\theta}$ is an interior point of the parameter space $\boldsymbol{\Theta}$. Then the ML estimators $\hat{\boldsymbol{\theta}}_{ML}$ of unknown parameters $\boldsymbol{\theta}$ are consistent and asymptotically normally distributed as*

$$
\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left[0, \ I(\boldsymbol{\theta})^{-1}\right] \quad , \tag{5.20}
$$

where

$$I(\boldsymbol{\theta}) = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\left[-\frac{\partial^2 \log L_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] \quad . \tag{5.21}$$

This result may lead to the conclusion that we can use the standard asymptotic method based on the limiting normal distribution although we are dealing with a complicated non-linear time series model for practical purposes. Strictly speaking, however, the statements in Theorem 5.7 are not precise. As Kiefer (1978), Amemiya (1985), and Quandt (1988) have pointed out in their discussions on the switching regression models, there can be several solutions for the likelihoood equations in the non-linear models as (5.1). The precise meaning of the ML estimator and its asymptotic properties under non-standard situations has been discussed in Section 4 of Amemiya (1985). Whether this problem of non-standard situation causes serious troubles or not in the analysis of time series data has not been fully investigated yet.

6. An Empirical Example

In this section we shall report one empirical example using some time series data in Japanese agricultural markets. The data we used are the same as we mentioned to in Section 2, and those observed in the pork market and the hen eggs market in Tokyo. They are monthly data from January of 1982 to December of 1991. As the endogenous variables we took the quantities traded and prices, and as the exogenous variables, we have included the disposable income of households in the demand side and also the price index of feeds, the lagged quantity in the supply side. All variables we used are the logarithms of the original data. Using these data, we have estimated the disequilibrium econometric model discussed in Section 3. Most data exhibit strong seasonal patterns and so all data are seasonally adjusted by the dummy variables method.

The estimation of unknown parameters has been done by the ML method under the assumption of normal disturbances. Since we cannot obtain an explicit formula for the ML estimators of unknown parameters, we have used a numerical non-linear optimization technique. In the numerical optimization, the restrictions on the parameter space

$$\beta_1 < 0 < \beta_2,\ \delta_i > 0\ (i = 1, 2) \tag{6.1}$$

have been imposed.

Table 6.1 :Estimated Results of A Disequilibrium Model

(i) Pork Market

	Estimates	S.D	t-value
δ_1	1.008	0.1166	8.641
δ_2	0.9741	0.1291	7.545
β_1	-0.3857	0.03231	11.94
γ_{11}	17.57	0.9041	19.43
γ_{12}	-0.6016	0.1299	4.631
σ_{11}	0.00124	0.00021	***
β_2	0.00002	0.00012	0.1987
γ_{21}	10.37	1.490	6.958
γ_{22}	-0.2178	0.04310	5.055
γ_{23}	0.2015	0.1146	1.758
σ_{22}	0.00204	0.00037	***
		AIC	-837.346
		(by equilibrium)	-806.713

(ii) Egg Market

	Estimates	S.D	t-value
δ_1	2.568	0.2635	9.746
δ_2	4.603	0.6521	7.059
β_1	-0.1300	0.00946	13.73
γ_{11}	5.144	0.2606	19.74
γ_{12}	1.369	0.04300	31.84
σ_{11}	0.00036	0.00006	***
β_2	0.04073	0.01161	3.507
γ_{21}	1.758	0.6688	2.629
γ_{22}	-0.1046	0.02612	4.004
γ_{23}	0.8760	0.04750	18.44
σ_{22}	0.00031	0.00005	***
		AIC	-921.293
		(by equilibrium)	-909.777

The estimated parameters and their standard deviations have been reported in Table 6.1 [5]. Since we have calculated the estimated variances by the numerical evaluation based on

$$(6.2) \qquad -\frac{\partial^2 \log L_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\bigg|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_{\mathrm{ML}}} .$$

[5]In Table 6.1, γ_{11} and γ_{21} are the coefficients of the constant term, and also γ_{12},γ_{22}, and γ_{23} are the coefficients of the disposable income, the price index of feeds, and the lagged quantity, respectively.

We also have reported the values of AIC (Akaike's information criterion) for the equilibrium econometric model and the disequilibrium econometric model. From the data analytic view we may use the minimum AIC method. For both markets, the values of AIC for the disequilibrium econometric models are smaller than those for the corresponding equilibrium econometric models. This indicates that the former is better than the latter in a statistical sense. It should be noted that there is an estimated coefficient of β_2 for the pork market, which is small and statistically insignificant. One interpretation for this may be that the price and quantity have been determined mainly by the demand curve. On the other hand, the estimated results on the egg market are satisfactory in many respects. Hence we have presented the estimated phase diagram for this market in Figure 6.1. In Figure 6.1, we have shown the estimated demand function and the supply function as well as the data points and the unrealized demand (or supply) points at each period. This kind of the estimated phase diagram is potentially useful for interpreting the empirical results.

7. Concluding Remarks

In this paper, we have focused on one important aspect in many economic time series data, which has been often ignored in the past econometric studies. We have pointed out that many economists have observed the asymmetry or some kind of non-linearity in the sample paths of economic time series data. We have argued that such asymmetrical patterns can not be represented properly by linear time series models including the standard ARMA processes, which have been used in many econometric studies in the past.

Then we have introduced the simultaneous switching time series model, which is one type of Markovian switching non-linear time series models. In particular, we have investigated some important statistical properties of the simultaneous switching autoregressive (SSAR) model, namely, the conditions on coherency and ergodicity, the stationary distribution and its moments. We have also investigated some asymptotic properties of the maximum likelihood (ML) estimator for the vector SSAR model. We hope these results in this paper may shed some new light on the time series properties commonly observed by many economists.

However, there are still a number of problems unsolved. First, we could not have obtained the necessary and sufficient conditions for the ergodicity in the general case. From our limited number of simulations, there are some interesting cases when the sufficient conditions discussed in Section 5 are not necessarily satisfied in the multivariate SSAR models, which may lead to some interesting economic interpretations. Second, our results on the estimation of the SSAR models are preliminary. Since the SSAR model is a non-linear time series model, the asymptotic results on the ML estimation crucially depends on the assumption of the distribution for the disturbances terms. There could be some other estimation methods except the ML method. Also the finite sample properties of the ML estimation method for the SSAR model should be clarified. Third, there are some related interesting issues of the prediction problem in the non-linear switching time series models including the SSAR models discussed in this paper.

Finally, we should mention that there could be potentially several areas in economics using the simultaneous switching time series model including macroeconomics and financial economics, for instance. In the econometric analyses of time series data in these areas the linear times series models have been extensively used in the past decade. It is important to notice that the disequilibrium econometric model discussed in Section 3 is only one example to lead one version of the SSAR model with exogenous variables. These problems will be investigated in a subsequent study.

8. Appendix

In this appendix, we shall give the proofs of some theorems and some details of algebra used in the previous sections. We assume that there is no constant term in (5.1). This makes often our algebra simpler, while it does not change the essential parts of our derivations. Thus without loss of generality we assume $\boldsymbol{\mu}_i = \mathbf{o}$ $(i = 1, 2)$ in (5.1).

[A1] Proof of Theorem 5.1 : Let $\boldsymbol{v}_t^{(i)} = (1/\sigma_i)\, \boldsymbol{\Sigma}_i^{1/2} \boldsymbol{u}_t$ $(i = 1, 2)$. Then the condition $\boldsymbol{e}_m' \boldsymbol{y}_t \geq \boldsymbol{e}_m' \boldsymbol{y}_{t-1}$ is equivalent to

$$v_{mt}^{(1)} \equiv \frac{1}{\sigma_1} \boldsymbol{e}_m' \boldsymbol{\Sigma}_1^{1/2} \boldsymbol{u}_t \geq \boldsymbol{r}' \boldsymbol{y}_{t-1} \quad . \tag{A.1}$$

We take an arbitrary bounded continuous function $h(\cdot)$. Then for an $m \times 1$ vector $\boldsymbol{c}$,

$$E[h(\boldsymbol{y}_{t+1}) | \boldsymbol{y}_t = \boldsymbol{x} + \boldsymbol{c}] - E[h(\boldsymbol{y}_{t+1}) | \boldsymbol{y}_t = \boldsymbol{x}] \tag{A.2}$$

$$= \int_{v_m \geq \boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})} h[\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{c}) + \sigma_1 \boldsymbol{v}]\, dG(\boldsymbol{v}) - \int_{v_m \geq \boldsymbol{r}'\boldsymbol{x}} h[\boldsymbol{A}\boldsymbol{x} + \sigma_1 \boldsymbol{v}]\, dG(\boldsymbol{v})$$

$$+ \int_{v_m < \boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})} h[\boldsymbol{B}(\boldsymbol{x} + \boldsymbol{c}) + \sigma_2 \boldsymbol{v}]\, dG(\boldsymbol{v}) - \int_{v_m < \boldsymbol{r}'\boldsymbol{x}} h[\boldsymbol{B}\boldsymbol{x} + \sigma_2 \boldsymbol{v}]\, dG(\boldsymbol{v}) \ ,$$

where $G(\boldsymbol{v})$ is the distribution function of $\{\boldsymbol{v}_t\}$ and v_m is the $m-$th component of $\boldsymbol{v}$. The first two terms in (A.2) can be re-written as

$$\int_{v_m \geq \boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})} \{h[\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{c}) + \sigma_1 \boldsymbol{v}] - h[\boldsymbol{A}\boldsymbol{x} + \sigma_1 \boldsymbol{v}]\}\ dG(\boldsymbol{v}) \tag{A.3}$$

$$+ \int_{\boldsymbol{r}'\boldsymbol{x} \leq v_m < \boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})} h[\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{c}) + \sigma_1 \boldsymbol{v}]\, dG(\boldsymbol{v}) \ .$$

Then the absolute value of the last term of (A.3) is less than

$$\text{(A.4)} \qquad \int_{\boldsymbol{r}'\boldsymbol{x} \le v_m < \boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})} |\, h[\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{c}) + \sigma_1 \boldsymbol{v}] \,|\; dG(\boldsymbol{v})$$

$$\le \; M\,[G_m(\boldsymbol{r}'(\boldsymbol{x}+\boldsymbol{c})) - G_m(\boldsymbol{r}'\boldsymbol{x})] \;\longrightarrow\; 0$$

as $\boldsymbol{r}'\boldsymbol{c} \to 0$, where M is a finite constant and $G_m(v_m)$ is the distribution function of v_{mt}. For other terms in (A.2),

$$\text{(A.5)} \qquad \lim_{\boldsymbol{c}\to 0} |\; (\text{ 1st term of (A.3) })\; |$$

$$\le \int_{v_m \ge \boldsymbol{r}'\boldsymbol{x}} \lim_{\boldsymbol{c}\to 0} |\, h[\boldsymbol{A}\boldsymbol{x} + \sigma_1 \boldsymbol{v} + A\boldsymbol{c}] - h[\boldsymbol{A}\boldsymbol{x} + \sigma_1 \boldsymbol{v}] \,|\; dG(\boldsymbol{v})$$

$$= \; 0$$

by the Lebesgue's convergence theorem. By applying the same arguments to the third and fourth terms of (A.2), we have the continuity property

$$\text{(A.6)} \qquad \lim_{\boldsymbol{c}\to 0} E[h(\boldsymbol{y}_{t+1}) \,|\, \boldsymbol{y}_t = \boldsymbol{x} + \boldsymbol{c}] = E[h(\boldsymbol{y}_{t+1}) \,|\, \boldsymbol{y}_t = \boldsymbol{x}] \quad .$$

□

[A2] **Proof of Theorem 5.2 :** When $m = p = 1$, the condition $y_t \ge y_{t-1}$ is equivalent to the condition

$$\text{(A.7)} \qquad u_t \ge r y_{t-1} \quad ,$$

where $r = (1 - A)/\sigma_1 = (1 - B)/\sigma_2$. We set the transition density function

$$\text{(A.8)} \qquad p(x, y) = \frac{1}{\sigma_1} g\left(\frac{y - Ax}{\sigma_1}\right) I(y \ge x) + \frac{1}{\sigma_2} g\left(\frac{y - Bx}{\sigma_2}\right) I(y < x) \quad ,$$

where $I(\cdot)$ is the indicator function. Then we shall use a similar method to that used by Petruccelli and Woolford(1984) for the TAR(1) model without constant term. However, we note that some changes in their method are necessary because the SSAR(1) model with constant terms is different from the TAR(1) model without constant term.

(i)Sufficiency: In order to show the sufficiency, we take the criterion function

$$\text{(A.9)} \qquad h(y) = \begin{cases} k_1 y & \text{if } y \ge 0 \\ k_2 |y| & \text{if } y < 0 \end{cases} \quad ,$$

where we take $k_i > 0$ $(i = 1, 2)$. Then for any $x > 0$

$$\text{(A.10)} \quad \begin{aligned} Q &\equiv \int P(x,y)h(y)dy \\ &= \frac{k_1}{\sigma_1}\int_{y\geq x} g\left(\frac{y-Ax}{\sigma_1}\right) y\,dy + \frac{k_1}{\sigma_1}\int_{0\leq y<x} g\left(\frac{y-Bx}{\sigma_2}\right) y\,dy \\ &- \frac{k_2}{\sigma_2}\int_{y<0} g\left(\frac{y-Bx}{\sigma_2}\right) y\,dy \ . \end{aligned}$$

By transforming the variable y into u, the first term in Q, for instance, is rewritten as

$$\text{(A.11)} \qquad k_1 Ax \int_{u\geq rx} g(u)du + k_1\sigma_1 \int_{u\geq rx} ug(u)du \ .$$

Because we assumed $E[|u_t|] < +\infty$, the second term of (A.11) is bounded. Similarly, by transforming y into u and ignoring the terms of small orders, the dominant terms in Q can be summarized as

$$\text{(A.12)} \qquad k_1 Ax \int_{u\geq rx} dG(u) + k_1 Bx \int_{-\frac{Bx}{\sigma_2}\leq u<rx} dG(u) - k_2 Bx \int_{u<-\frac{Bx}{\sigma_2}} dG(u) \ ,$$

where $G(u)$ is the distribution function. Since $A < 1, B < 1$,and $AB < 1$, we can take $k_1 > 0$ and $k_2 > 0$ such that $1 > A > -k_1^{-1}k_2$ and $1 > B > -k_2^{-1}k_1$. Then (A.12) is less than

$$\text{(A.13)} \quad k_1\left\{Ax[1-G(rx)] + Bx\left[G(rx) - G(-\frac{Bx}{\sigma_2})\right] + \eta x G\left(-\frac{Bx}{\sigma_2}\right)\right\} \ ,$$

where we take $0 < \eta < 1$ and $r > 0$. By using the relation $\lim_{x\to\infty} x[1-G(x)] = 0$ because of the condition (5.7), we can show that there exists $M > 0$ such that for $x > M > 0$

$$\text{(A.14)} \qquad Q \leq k_1 x - 1 \ .$$

Similarly, we can take $M > 0$ such that for $x < -M < 0$,

$$\text{(A.15)} \qquad Q \leq k_2|x| - 1 \ .$$

Then by using Theorem 5.1 and applying Theorem 4.2 of Tweedie (1975), the Markov chain defined by (5.1) is ergodic.

(ii) Necessity:

(ii-a) We first consider the case of $A \geq 1$. If $A = 1$, then $B = 1$ and y_t is a random walk process because we have the coherency condition (4.13). Thus without loss of generality we assume $A > 1$. The proof for the case $B > 1$ is similar to this case. First, we take η such that $A > \eta > 1$ and consider the situation when $y_{t-1} > M > 0$. Then we have the inequality

$$
\begin{aligned}
\text{(A.16)} \qquad & P\left\{y_t \le \frac{\eta+1}{2} y_{t-1} \middle| \, y_{t-1}\right\} \\
\le \;& P\left\{-u_t \ge \frac{1}{\sigma_1}\left(A - \frac{\eta+1}{2}\right) y_{t-1},\; -r y_{t-1} \ge -u_t\right\} + P\{u_t < r y_{t-1}\} \\
\le \;& P\left\{|u_t| \ge \frac{\eta-1}{2\sigma_1} y_{t-1} \middle| \, y_{t-1}\right\} + P\left\{|u_t| \ge -r y_{t-1} \middle| \, y_{t-1}\right\} \\
\le \;& 2P\left\{|u_t| \ge k_3 y_{t-1} \middle| \, y_{t-1}\right\} \;,
\end{aligned}
$$

where $k_3 = \min[\,\frac{\eta-1}{2\sigma_1}\,,\; -r\,] > 0$ and $r < 0$. Then by the Markov's inequality this probability is less than

$$
\text{(A.17)} \qquad 2\,\frac{E[|u_t|]}{k_3 y_{t-1}} \;\le\; \left[\frac{2E[|u_t|]}{k_3}\right]\frac{1}{M} \;=\; c \;<\; 1\,.
$$

If we take a large $M > 0$, we have

$$
\text{(A.18)} \qquad P\left\{y_t > \frac{\eta+1}{2} y_{t-1} \middle| \, y_{t-1}\right\} \ge 1 - c
$$

for an arbitrary small c. Next, by the use of (A.18) we have the inequality

$$
\begin{aligned}
\text{(A.19)} \qquad & P\left\{y_t > \frac{\eta+1}{2} y_{t-1},\, y_{t-1} > \frac{\eta+1}{2} y_{t-2} \middle| \, y_{t-2}\right\} \\
\ge \;& \int_{y_{t-1} > \frac{\eta+1}{2} y_{t-2}} \left[1 - \frac{2E[|u_t|]}{k_3 y_{t-1}}\right] P(y_{t-2}, dy_{t-1}) \\
\ge \;& (1 - \beta c) P\left\{y_{t-1} > \frac{\eta+1}{2} y_{t-2} \middle| y_{t-2}\right\} \\
\ge \;& (1 - \beta c)(1 - c)\;,
\end{aligned}
$$

where $\beta = 2/(\eta + 1) < 1$. Hence by repeating the above evaluation we have

$$
\begin{aligned}
\text{(A.20)} \qquad & P\left\{y_{i+1} > \frac{\eta+1}{2} y_i \; i = 1, \cdots, t \middle| \, y_1\right\} \\
\ge \;& \prod_{i=1}^{t} \left(1 - \beta^{i-1} c\right) \\
\ge \;& (1 - c)^{\frac{1}{1-\beta}} > 0\,.
\end{aligned}
$$

The last inequality for an arbitrary t implies that the Markov chain defined by the SSAR(1) model is non-ergodic.

(ii-b) We consider the case when $A < -1$ and $AB > 1$. The proof for the case $B < -1$ and $AB > 1$ is similar to this case. We take a large M such that $y_{t-2} < -M < 0$. Then there are four phases for $\{y_{t-1}, y_t\}$ given y_{t-2}. By using the Markov's inequality again, the probability

$$P\{u_{t-1} < ry_{t-2}\} \leq \frac{E[|u_{t-1}|]}{rM}\,, \tag{A.21}$$

which can be arbitrarily small. We can take η such that $A^2 > \eta > 1$ and $AB > \eta > 1$. Then

$$\begin{aligned} & P\left\{y_t \geq \frac{\eta+1}{2} y_{t-2} |\, y_{t-2}\right\} \\ \leq\ & P\left\{A^2 y_{t-2} + (A\sigma_1 u_{t-1} + \sigma_2 u_t) \geq \frac{\eta+1}{2} y_{t-2}, |\, y_{t-2}\right\} \\ +\ & P\left\{AB y_{t-2} + (B\sigma_1 u_{t-1} + \sigma_2 u_t) \geq \frac{\eta+1}{2} y_{t-2}, |\, y_{t-2}\right\} \\ +\ & P\{u_{t-1} < ry_{t-2} |\, y_{t-2}\}\ . \end{aligned} \tag{A.22}$$

The first term on the right hand side of (A.22) can be evaluated by the inequality

$$\begin{aligned} & P\left\{(A\sigma_1 u_{t-1} + \sigma_2 u_t) \geq \left(\frac{\eta+1}{2} - A^2\right) y_{t-2} |\, y_{t-2}\right\} \\ \leq\ & \frac{\sigma_1 |A| + \sigma_2}{k_4 M} E[|u_t|]\,, \end{aligned} \tag{A.23}$$

where $k_4 = A^2 - \frac{\eta+1}{2}$ is a positive constant. Then (A.23) can be arbitrarily small. By the same argument, the third term on the right hand side of (A.22) is arbitrarily small. Hence we have

$$P\left\{y_t < \frac{\eta+1}{2} y_{t-2} |\, y_{t-2}\right\} \geq 1 - c\,, \tag{A.24}$$

where c can be arbitrarily small. Then we can apply the similar arguments used in (ii-a).

(ii-c) Finally we have to deal with the case when $AB = 1$ and $A < 0$. In this case we should modify the method used in the proof of Lemma 2.3 by Chan et. al. (1985) to show that the process is non-ergodic. In their proof we treat as if $\phi(1,1) = A, \phi(1,l) = B, \phi(0,1) = \phi(0,l) = 0$, and use the similar arguments as they did for the multiple-threshhold AR(1) model. Since our evaluations of integrals are quite similar to theirs except using the condition (5.7) in our derivations as we did in (ii-a) and (ii-b), we omit the details. □

[A3] Proof of Theorem 5.6 : For illustration, we consider the case when $m = 1$. In order to show the existence of the k-th order moment for the stationary distribution, we take the criterion function

$$h(x) = \begin{cases} k_1^k x^k & + c \quad \text{if } x \geq 0 \\ k_2^k |x|^k & + c \quad \text{if } x < 0 \end{cases} , \tag{A.25}$$

where $k_1 > 0$, $k_2 > 0$, and $c > 0$. Then by the simple calculations we can show that

$$\int p(x,y)h(y)dy \leq (1-\epsilon)h(x) , \tag{A.26}$$

for some $\epsilon > 0$. Then we apply Theorem 3 of Tweedie(1983) to show the existence of moments in the present situation. □

[A4] Proof of Derivation of Equation (5.19) : Let the indicator functions be $I_t^{(1)} = I\left(v_{mt}^{(1)} \geq \boldsymbol{r}'\boldsymbol{y}_{t-1}\right)$ and $I_t^{(2)} = I\left(v_{mt}^{(2)} < \boldsymbol{r}'\boldsymbol{y}_{t-1}\right)$. By differentiating $\log L_T(\boldsymbol{\theta})$ with respect to the $j-$th component of $\boldsymbol{\theta}$, we have

$$\frac{\partial \log L_T(\boldsymbol{\theta})}{\partial \theta_j} = \sum_{t=1}^{T}\sum_{i=1}^{2} \partial L_{it}(\theta_j) , \tag{A.27}$$

where

$$\begin{aligned} \partial L_{it}(\theta_j) &= -\frac{1}{2}\frac{\partial \log|\boldsymbol{\Sigma}_i|}{\partial \theta_j} I_t^{(i)} + \sigma_i \boldsymbol{y}_{t-1}{}' \left(\frac{\partial \boldsymbol{A}_i{}'}{\partial \theta_j}\right) \boldsymbol{\Sigma}_i^{-1} \boldsymbol{v}_t^{(i)} I_t^{(i)} \\ &\quad -\frac{1}{2}\sigma_i^2 tr\left\{\frac{\partial \boldsymbol{\Sigma}_i{}^{-1}}{\partial \theta_j} \boldsymbol{v}_t^{(i)} \boldsymbol{v}_t{}^{(i)\prime} I_t^{(i)}\right\} , \end{aligned}$$

where we re-define $m \times m$ matrices $\boldsymbol{A}_1 = \boldsymbol{A}$ and $\boldsymbol{A}_2 = \boldsymbol{B}$. We partition $\boldsymbol{\Sigma}_i$ $(i = 1,2)$ into $((m-1)+1) \times ((m-1)+1)$ submatrices

$$\boldsymbol{\Sigma}_i = \begin{pmatrix} \boldsymbol{\Sigma}_{11}^{(i)} & \boldsymbol{\sigma}_{12}^{(i)} \\ \boldsymbol{\sigma}_{21}^{(i)} & \sigma_i^2 \end{pmatrix}$$

and $\boldsymbol{\Sigma}_{11.2}^{(i)} = \boldsymbol{\Sigma}_{11}^{(i)} - \boldsymbol{\sigma}_{12}^{(i)}\boldsymbol{\sigma}_{21}^{(i)}/\sigma_i^2$. Since $\{\boldsymbol{v}_t^{(i)}\}$ follows the multivariate normal distribution $N_m(\mathbf{o}, (1/\sigma_i^2)\boldsymbol{\Sigma}_i)$, we use the following formulas on the conditional expectations

$$E\left[\boldsymbol{v}_t^{(i)} \mid v_{mt}(i)\right] = \frac{1}{\sigma_i^2}\boldsymbol{\Sigma}_i \boldsymbol{e}_m v_{mt}^{(i)} , \tag{A.28}$$

$$E\left[\boldsymbol{v}_t^{(i)}\boldsymbol{v}_t^{(i)\prime} \mid v_{mt}^{(i)}\right] = \frac{1}{\sigma_i^2}\begin{pmatrix} \boldsymbol{\Sigma}_{11.2}^{(i)} & O \\ O & O \end{pmatrix} + \frac{1}{\sigma_i^4}\boldsymbol{\Sigma}_i \boldsymbol{e}_m \boldsymbol{e}_m' \boldsymbol{\Sigma}_i v_{mt}^{(i)2} . \tag{A.29}$$

We take the conditional expectation of $\partial L_{it}(\theta_j)$ given the $\sigma-$field $\mathcal{F}_{t-1}$. The second term of the conditional expectation of $\partial L_{it}(\theta_j)$ is given by

$$\text{(A.30)} \qquad \frac{1}{\sigma_i}\boldsymbol{y}'_{t-1}\frac{\partial \boldsymbol{A}'_i\boldsymbol{e}_m}{\partial \theta_j}E\left(v_{mt}^{(i)}I_t^{(i)} \mid \mathcal{F}_{t-1}\right) .$$

Also the third term of the conditional expectation of $\partial L_{it}(\theta_j)$ is given by

$$\begin{aligned}
\text{(A.31)} \qquad & \frac{1}{2}tr\left\{\frac{\partial \boldsymbol{\Sigma}_i}{\partial \theta_j}\boldsymbol{\Sigma}_i^{-1}\begin{pmatrix}\boldsymbol{\Sigma}_{11.2}^{(i)} & O \\ O & O\end{pmatrix}\boldsymbol{\Sigma}_i^{-1}\right\}E[I_t^{(i)} \mid \mathcal{F}_{t-1}] \\
& +\frac{1}{2\sigma_i^2}tr\left\{\frac{\partial \boldsymbol{\Sigma}_i}{\partial \theta_j}\boldsymbol{\Sigma}_i^{-1}\boldsymbol{e}_m\boldsymbol{e}'_m\boldsymbol{\Sigma}_i\boldsymbol{\Sigma}_i^{-1}\right\}E[v_{mt}^2I_t^{(i)} \mid \mathcal{F}_{t-1}] \\
= \; & \frac{1}{2}tr\left\{\frac{\partial \boldsymbol{\Sigma}_i}{\partial \theta_j}\boldsymbol{\Sigma}_i^{-1}\right\}E[I_t^{(i)} \mid \mathcal{F}_{t-1}] \\
& +\frac{1}{2\sigma_i^2}\frac{\partial \boldsymbol{e}'_m\boldsymbol{\Sigma}_i\boldsymbol{e}_m}{\partial \theta_j}E\left[\left(v_{mt}^{(i)2}-1\right)I_t^{(i)} \mid \mathcal{F}_{t-1}\right] .
\end{aligned}$$

Summarizing three terms in the conditional expectation of $\partial L_{it}(\theta_j)$, it can be written as

$$\begin{aligned}
\text{(A.32)}\; E\left[\partial L_{it}(\theta_j)|\mathcal{F}_{t-1}\right] = \; & \frac{1}{2}\left\{tr\left(\frac{\partial \boldsymbol{\Sigma}_i}{\partial \theta_j}\boldsymbol{\Sigma}_i^{-1}\right)-\frac{\partial log|\boldsymbol{\Sigma}_i|}{\partial \theta_j}\right\}E[I_t^{(i)} \mid \mathcal{F}_{t-1}] \\
& +\frac{1}{\sigma_i}\boldsymbol{y}'_{t-1}\frac{\partial \boldsymbol{A}'_i\boldsymbol{e}_m}{\partial \theta_j}E[v_{mt}^{(i)}I_t^{(i)} \mid \mathcal{F}_{t-1}] \\
& +\frac{1}{2\sigma_i^2}\frac{\partial \boldsymbol{e}'_m\boldsymbol{\Sigma}_i\boldsymbol{e}_m}{\partial \theta_j}E\left[\left(v_{mt}^{(i)2}-1\right)I_t^{(i)} \mid \mathcal{F}_{t-1}\right] .
\end{aligned}$$

By the assumption of normality on the disturbances, we also have the following relation

$$\text{(A.33)} \qquad \int_{\boldsymbol{r}'\boldsymbol{y}_{t-1}}^{\infty}\left[v_{mt}^{(i)2}-1-\left(\boldsymbol{r}'\boldsymbol{y}_{t-1}\right)v_{mt}^{(i)}\right]dG\left(v_{mt}^{(i)}\right)=0 .$$

If we take $\theta_j=\sigma_i^2$, we use the above relation and

$$\text{(A.34)} \qquad \frac{\partial \log|\boldsymbol{\Sigma}_i|}{\partial \sigma_j^2}=\boldsymbol{e}'_m\boldsymbol{\Sigma}_i^{-1}\boldsymbol{e}_m .$$

Then by the use of (A.33), we have

$$\text{(A.35)} \qquad E\left[\partial L_{it}(\theta_j) \mid \mathcal{F}_{t-1}\right]=0 .$$

Also for $\theta_j=r_j$, we use the relation $\boldsymbol{e}'_m\boldsymbol{A}_i=\boldsymbol{e}'_m-\sigma_i\boldsymbol{r}'$ $(i=1,2)$. Because the first and third terms in the conditional expectation of $\partial L_{it}(\theta_j)$ are zeros, we have

$$\text{(A.36)} \qquad E\left[\sum_{i=1}^{2} \partial L_{it}(\theta_j) \mid \mathcal{F}_{t-1}\right] = -\boldsymbol{y}'_{t-1}\boldsymbol{e}_j E\left[\sum_{i=1}^{2} v_{mt}^{(i)} I_t^{(i)} \mid \mathcal{F}_{t-1}\right] = 0 .$$

For other unknown parameters in $\boldsymbol{\theta}$, it is straightforward to see the conditional expectations of $\partial L_{it}(\theta_j)$ given $\mathcal{F}_{t-1}$ are zeros. □

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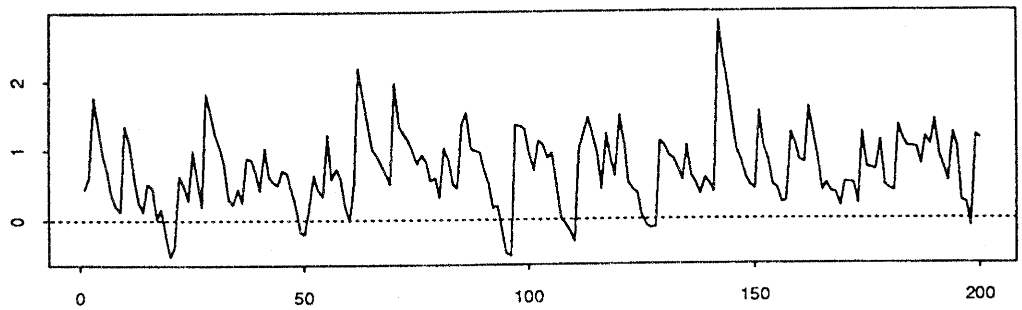

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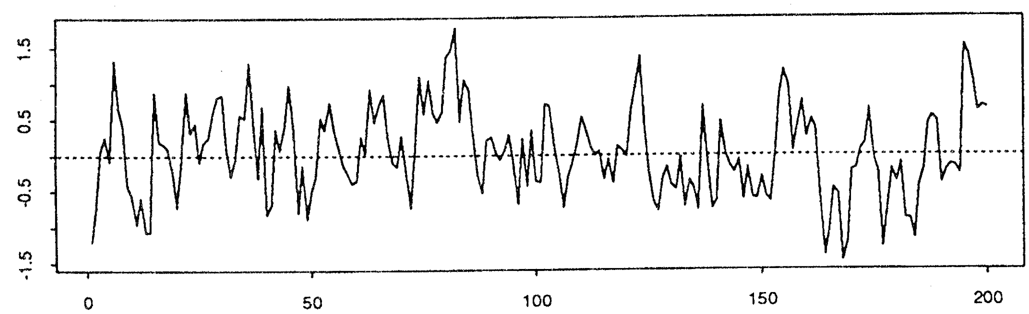

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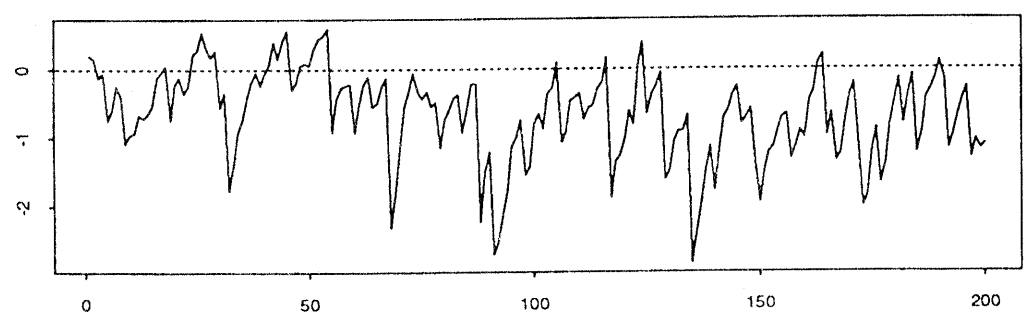


a = 0.8, b = 0.2

Figure 4.1: The sample paths of SSAR(1)
($r = 1$)

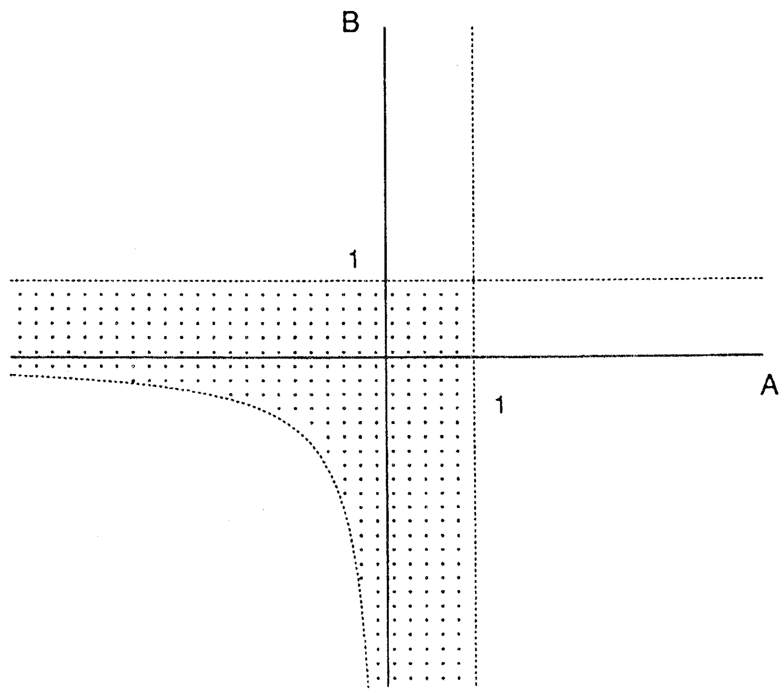
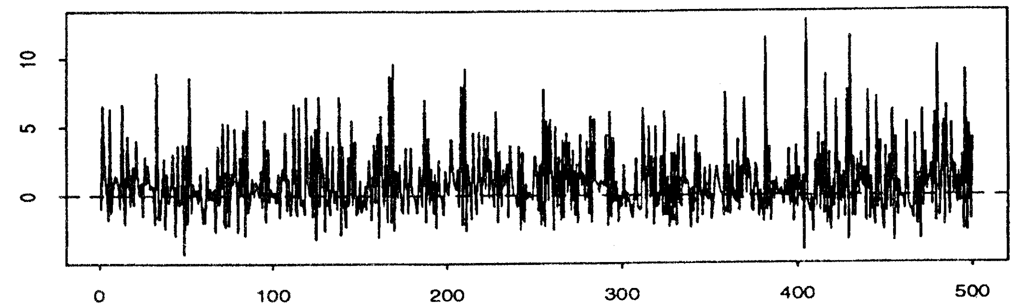
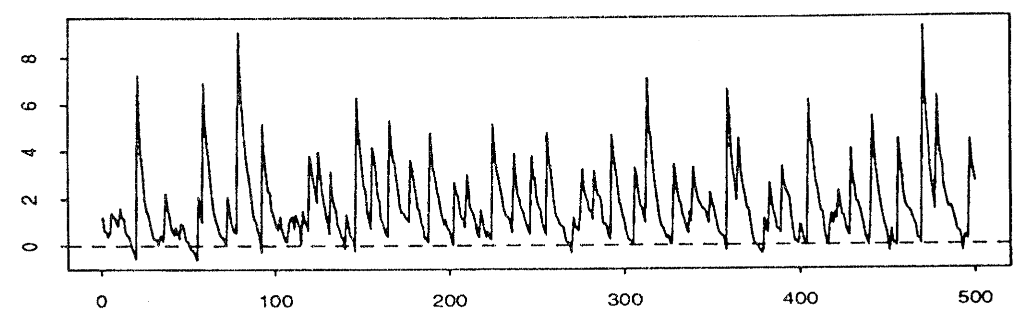


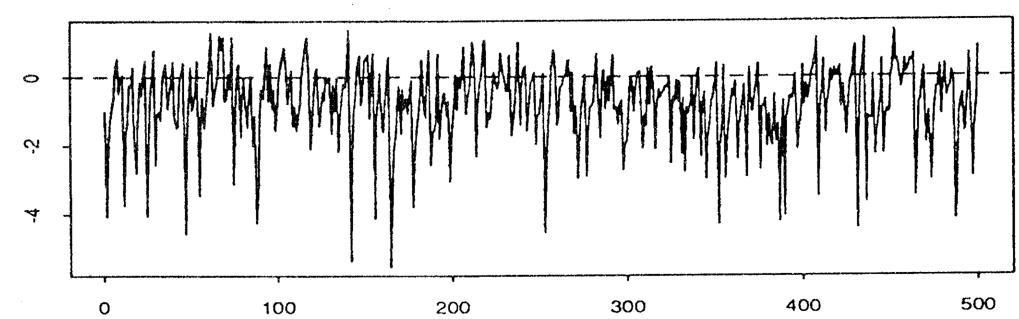
Figure 5.1: The region of ergodicity



a= -2 b= -0.2 r= 1



a= -3 b= 0.8 r= 1



a= 0.5 b= -1 r= 1

Figure 5.2: Other sample paths of SSAR(1)
($r = 1$)

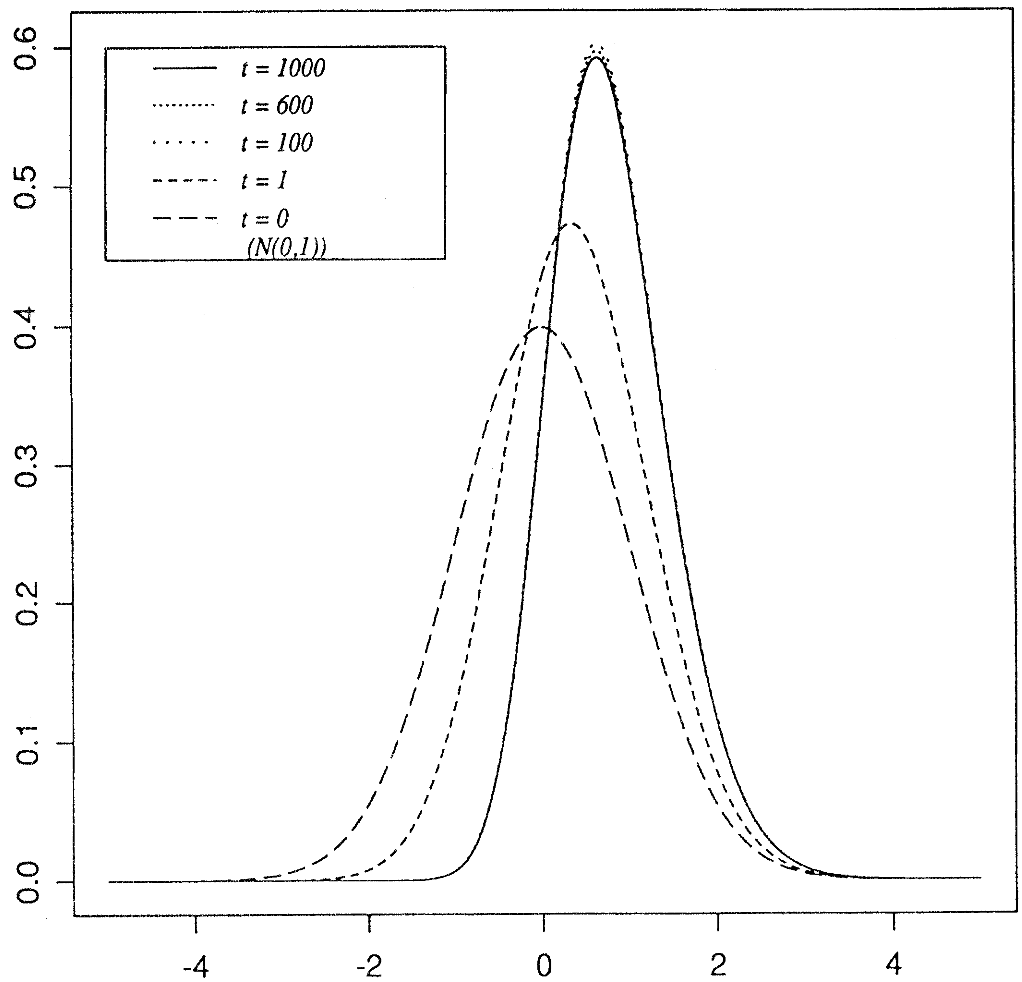


Figure 5.3: The stationary distribution when $A = 0.2$, $B = 0.8$, $\sigma_1 = 1$, $\sigma_2 = 0.25$

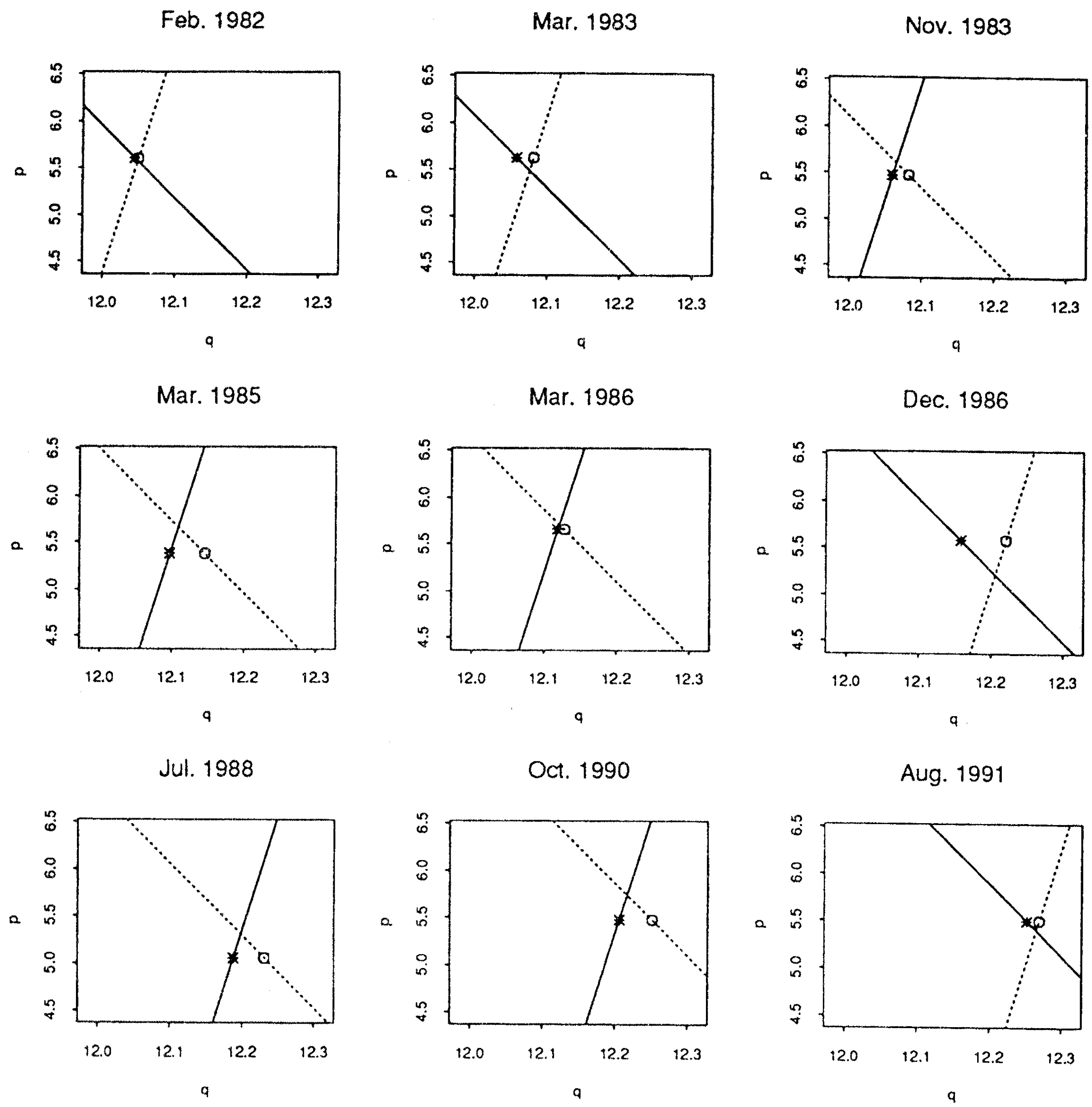


Figure 6.1: The estimated demand function and supply function of egg market

- "$*$" shows the observed price and quantity at each time.
- "o" shows the unrealized demand or supply.
- The solid line shows the demand or supply function on which the price and quantity of the market is eatmated to be determined.