95-F-27

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ABSTRACT

In this paper, we investigate nonparametric restrictions of dynamic optimization behavior under risk for the case of finite horizon and discrete time period with one good per period. In order to find the restrictions of an optimization behavior on observed data, we follow the revealed preference tradition: we ask what conditions must the data satisfy in order for the data to be compatible with an optimization behavior? The classes of utility functions investigated in this paper are (A) time-additive expected utility functions and (B) time-additive expected utility functions with a constant discount factor.

[1]This research started in the summer of 1992 when Ichimura was visiting the Department of Economics, Osaka University. The research continued at the Institute of Monetary and Economic Studies, the Bank of Japan between September of 1992 and April of 1993, during which he was a visiting scholar at the institute. He is grateful for the hospitalities he received at the institutes.

1 Introduction

We investigate nonparametric restrictions of dynamic optimization behavior under risk for the case of finite horizon and discrete time period with one good per period. When a utility function is specified parametrically, Hall (1978), Hansen and Singleton (1982), and Epstein and Zin (1989)(1991) obtained some observable implications of dynamic optimization behavior under a similar environment we are concerned. We wish to characterize the observable implications that do not rely on a parametric specification of a utility function.

The work suggests testable implications of the behavioral hypothesis and also gives a method to verify whether particular demand data are compatible with the optimization behavior. In addition the work is a first step toward understanding what aspects of preference relations are identifiable from observed data.

In order to find the restrictions of an optimization behavior on observed data, we follow the revealed preference tradition: we ask what conditions must the data satisfy in order for the data to be compatible with an optimization behavior? An immediate question is, "which optimization behavior?"

A typical approach to modeling an individual choice in this environment is to specify an additively separable utility function over time with a constant discount factor and further hypothesize expected utility maximization, but this is not the only approach. Clearly, there is no intrinsic reason why utility functions should be additive over time and also there are behavioral hypotheses other than expected utility maximization. In fact Epstein and Zin (1989) and others specify a recursive intertemporal utility function and consider non-expected utility maximization behavior, for example. Thus there is no unique way to define a "rational consumer" whose preference we attempt to reveal from given data. Therefore, instead, we define varieties of classes of utility functions and, for each class, seek necessary and sufficient conditions for data to be compatible with a utility function in the class,i.e., the nonparametric restrictions on the data derived from the utility maximization.

In this paper we consider state independent utility functions. The classes of utility functions are (A) time-additive expected utility functions and (B) time-additive expected utility functions with a constant discount factor.[2] In Ichimura and Kamiya (1995), we deal with nonexpected utility functions.

[2]Precise definitions of these classes are given below.

Using these classes we define rationality; we say that a data set is time-additive expected utility rational if we can find a time-additive expected utility function which is compatible with the data, for example.

In this paper, we present three theorems. Theorems 3.1 and 3.2 deal with the case of time-additive expected utility and Theorem 4.1 deals with the case of time-additive expected utility with a constant discount factor. Theorems 3.1 and 3.2 can be considered as generalizations of Border's results (Theorem 2.4, 1992) on atemporal models. He showed that the agent's choices are compatible with the expected utility maximization if and only if there does not exist a prior for which the choices are *ex ante* stochastically dominated. Note that we employ a strong notion of rationality as well as the weak notion of rationality of Border (1992). That is he just requires that the choice maximizes expected utility while we may further require that the choice is the unique expected utility maximizer. (For the discussion, see Richter (1971)(1987).) For revealed preference approaches to decision making under uncertainty in different frameworks, see Green and Osband (1991) and Kim (1991).

He and Huang (1994) investigated an integrability problem in a continuous time, finite horizon asset market model, where the asset price process follows a general diffusion process. That is, for a given consumption-portfolio *policy*, they provided a necessary and sufficient condition for it to be compatible with the maximization of some increasing, strictly concave, time additive, and state independent utility function. Recall that we employ revealed preference approach in a finite horizon, discrete time model and we do not impose any restrictions on the underlying stochastic process. It is worthwhile noting that our approach can be applicable to the case of an infinite number of data as well as the case of a finite number of data. (See Section 5.) Moreover, we deal with the case of a constant discount factor, which was not investigated in He and Huang (1994).

The next section specifies the economic environment in which our consumers make choices and the data we assume to observe. We then proceed to provide necessary and sufficient conditions for a data set to be rationalizable in Sections 3 and 4. Finally, in Section 5, we discuss extensions of the results in Sections 3 and 4.

2 The Model

We consider a discrete time finite horizon dynamic optimization model (T period model) with risk. We allow our data to be generated from general stochastic processes and ask if they are compatible with time-additive expected utility maximization.

Let $M_1^+(X)$ denote the set of all Borel probability measure over a compact set $X \subset R$, where R is the set of real numbers. The domain of the agent's choice is the cartesian product of $M_1^+(X)$, i.e., $M_1^+(X)^T = \underbrace{M_1^+(X) \times \cdots \times M_1^+(X)}_{T}$. The data consist of a finite number, N, of choices in the first period, $y^1 = (m_1^1, \ldots, m_T^1), \ldots, y^N = (m_1^N, \ldots, m_T^N)$, and corresponding feasible sets, $B^1, ..., B^N$, from which these choices are made, i.e., $y^n \in B^n, n = 1, \ldots, N$. We assume that $B^1, ..., B^N$ are all subsets of $M_1^+(X)^T$. Let $D = \{(y^n, B^n)\}_{n=1}^N$.

The following example illustrates a typical case we have in mind. Typically, a feasible set B^n is spanned by a finite number of assets.

Example 1

An agent wishes to choose consumption schedule, x_t $(t = 1, 2)$, over two periods $(T = 2)$ in period 0, knowing that he/she receives income of I_1 and I_2 in the two periods, respectively. No consumption takes place in period 0. The agent can transfer a consumption good in period 1 to the consumption good in period 2 only by holding a security, in the amount of s_1. The prices of the consumption good and the security are p_t and q_t $(t = 1, 2)$, respectively, and we assume that income I_t, the price of the consumption good, and the price of the security q_t $(t = 1, 2)$ are all random at $t-1$. More specifically, we define two sigma-algebras $\mathbf{A_1}$ and $\mathbf{A_2}$, such that $\mathbf{A_1} \subset \mathbf{A_2}$, and assume that I_1, p_1, q_1, x_1, and s_1 are measurable with respect to $\mathbf{A_1}$ and that I_2, p_2, q_2, and x_2 are measurable with respect to $\mathbf{A_2}$. The first period budget constraint is

$$p_1 x_1 + q_1 s_1 = I_1,$$

and the second period budget constraint is

$$p_2 x_2 = I_2 + q_2 s_1.$$

Given random variables q_1 and q_2, measurable with respect to $\mathbf{A_1}$ and $\mathbf{A_2}$, respectively, we assume that the agent chooses $\mathbf{A_1}$-measurable functions x_1

and s_1 and an $\mathbf{A}_2$-measurable function x_2 from all possible such random variables that satisfy the budget constraints. The set B^1 denotes the subset of $M_1^+(X) \times M_1^+(X)$ that corresponds to the pair of random variables (x_1, x_2). (Note that the number of data is one in this example.) The agent chooses $y^1 = (m_1^1, m_2^1) \in B^1$.

3 Time-Additive Expected Utility Rationalization

In this section, we consider the following time-additive expected utility function:

$$\sum_{t=1}^{T} \int u_t dm_t,$$

where m_t, $t = 1, \ldots, T$, is a probability distribution of the t period consumption.

Definition 3.1 *A set of functions $u_t : X \to R, t = 1, \ldots, T$, is said to be a weak time-additive expected utility rationalization (WTAR) for D if*

1. for $t = 1, \ldots, T$, u_t is strictly increasing and continuous,
2. for $n = 1, ..., N$,

$$\sum_{t=1}^{T} \int u_t dm_t^n \geq \sum_{t=1}^{T} \int u_t dm_t \quad \text{for} \quad (m_1, \ldots, m_T) \in B^n.$$

Definition 3.2 *A set of functions $u_t : X \to R, t = 1, \ldots, T$, is said to be a strong time-additive expected utility rationalization (STAR) for D if*

1. for $t = 1, \ldots, T$, u_t is strictly increasing and continuous,
2. for $n = 1, ..., N$,

$$\sum_{t=1}^{T} \int u_t dm_t^n > \sum_{t=1}^{T} \int u_t dm_t \quad \text{for} \quad (m_1, \ldots, m_T) \in B^n \backslash \{(m_1^n, \ldots, m_T^n)\}.$$

Remark 1. In Definition 3.1, the feasible set B^n may contain other maximal elements that are not chosen, while, in Definition 3.2, the choice is the unique maximizer.

Let $\overline{co}\ B^n$ be the weak* closed, convex hull of $B^n, n = 1, ..., N$. (For the definition of weak* topology, see the appendix.) Note that $\overline{co}\ B^n$ is equal to the weak* closure of the convex hull of B^n. (See, for example, Dunford and Schwartz (1958)).

We use the following notations.

$C(X)$: the set of continuous real valued functions on X with the sup norm.

$M(X)$: the set of finite countably additive Borel signed measures on X.

U : the set of strictly increasing real valued functions on X.

$\overline{U}$: the set of nondecreasing functions on X.

U_c: the set of continuous members of U.

$\overline{U}_c$: the set of continuous members of $\overline{U}$.

For $m_i \in M(X), i = 1, \ldots, k$, and $\alpha_i \in R, i = 1, \ldots, k$, $\sum_{i=1}^k \alpha_i m_i \in M(X)$ denotes the measure such that, for all Borel set $E \subset X$, the measure of E is $\sum_{i=1}^k \alpha_i m_i(E)$. Let

$$A = \left\{ m \in M(X) \mid \forall u \in \overline{U}_c, \int u dm \geq 0 \right\},$$

and denote the T-fold Cartesian products of A and $M(X)$ by A^T and $M(X)^T$, respectively. Also let

$$G = \left\{ \left(\sum_{n=1}^N \alpha^n (\overline{m}_1^n - m_1^n), \ldots, \sum_{n=1}^N \alpha^n (\overline{m}_T^n - m_T^n) \right) \mid (\overline{m}_1^n, \ldots, \overline{m}_T^n) \in \overline{co} B^n \right., \left. n = 1, \ldots, N, \alpha = (\alpha^1, \ldots, \alpha^N) \in S^N \right\},$$

where S^N is the $(N-1)$-dimensional unit simplex.

Note that G is clearly weak* compact and convex. Let

$$\begin{aligned} cn\ G &= \{(m_1, \ldots, m_T) \in M(X) \times \cdots \times M(X) \mid \exists (\overline{m}_1, \ldots, \overline{m}_T) \in G, \exists \lambda \geq 0, \\ &\quad (m_1, \ldots, m_T) = (\lambda \overline{m}_1, \ldots, \lambda \overline{m}_T)\}, \end{aligned}$$

the cone generated by G.[3] Note that clearly $cn\, G$ is convex since G is convex. Let $cl\ (cn\ G)$ denote the weak* closure of $cn\ G$.

Example 2 Let $T = 1$, $N = 3$, and $X = \{1, 2, 3\} \subset R$. A probability distribution on X is represented by a vector $(x_1, x_2, x_3) \in S^3$, where S^3 is the two dimensional unit simplex. Suppose we have the following data:

$$B^1 = \{(1/2, 0, 1/2), (0, 1, 0)\}, \qquad m^1 = (0, 1, 0),$$

[3] In general, for a set K in a linear space E, $cn\ K$ denotes the cone generated by K, i.e,. $cn\ K = \{x \in E | \exists x' \in K, \exists \alpha \geq 0, x = \alpha x'\}$.

$$B^2 = \{(1/3, 0, 2/3), (0, 3/4, 1/4)\}, \qquad m^2 = (0, 3/4, 1/4),$$
$$B^3 = \{(1/3, 1/3, 1/3), (0, 1, 0)\}, \qquad m^3 = (0, 1, 0).$$

Then G is the convex hull of vectors $(0,0,0)$, $(1/2,0,1/2)-(0,1,0)$, $(1/3,0,2/3)-(0,3/4,1/4)$, and $(1/3,1/3,1/3)-(0,1,0)$. (See Figure 1.) In this example, A is the cone spanned by $(-1,1,0)$ and $(0,-1,1)$. (See Figure 1.) In the example, $G \bigcap A = \{(0,0,0)\}$ holds. By the convexity of G and A, and the separating hyperplane theorem, there exists a vector (a_1, a_2, a_3) such that

$$a_1x_1 + a_2x_2 + a_3x_3 \leq 0 \quad \text{for} \quad (x_1, x_2, x_3) \in G$$

and

$$a_1x_1 + a_2x_2 + a_3x_3 > 0 \quad \text{for} \quad (x_1, x_2, x_3) \in A \setminus \{(0,0,0)\}.$$

Let $u(1) = a_1, u(2) = a_2, u(3) = a_3$. Then, for $(1/2, 0, 1/2) - (0, 1, 0) \in G$,

$$(1) \quad \frac{1}{2}u(1) + 0u(2) + \frac{1}{2}u(3) \leq 0u(1) + 1u(2) + 0u(3)$$

holds. Similarly, for $(1/3, 0, 2/3) - (0, 3/4, 1/4) \in G$ and $(1/3, 1/3, 1/3) - (0, 1, 0) \in G$,

$$(2) \quad \frac{1}{3}u(1) + 0u(2) + \frac{2}{3}u(3) \leq 0u(1) + \frac{3}{4}u(2) + \frac{1}{4}u(3)$$

$$(3) \quad \frac{1}{3}u(1) + \frac{1}{3}u(2) + \frac{1}{3}u(3) \leq 0u(1) + 1u(2) + 0u(3)$$

hold. On the other hand, for $(-1, 1, 0) \in A$ and $(0, -1, 1) \in A$,

$$-1u(1) + 1u(2) + 0u(3) > 0 \quad \text{and} \quad 0u(1) - 1u(2) + 1u(3) > 0$$

hold so that

$$(4) \quad u(1) < u(2) < u(3).$$

holds, i.e., u is strictly increasing. By (1)-(4), u is a WTAR for the data.

The above argument seems to indicate that $G \bigcap A = \{(0,0,0)\}$ is sufficient to imply the existence of a WTAR. The example in Figure 2 is a counter example to this conjecture. In the example, $G \bigcap A = \{(0,0,0)\}$ holds but there does not exist a WTAR. Since we require a strictly increasing utility function, $A \setminus \{(0,0,0)\}$ must be strictly separated, i.e.,

$$a_1x_1 + a_2x_2 + a_3x_3 > 0 \quad \text{for} \quad (x_1, x_2, x_3) \in A \setminus \{(0,0,0)\}.$$

We need more restrictive condition to guarantee this. When X consists of three elements, it is easy to see that $cl\ (cn\ G) \bigcap A = \{(0,0,0)\}$ is necessary and sufficient for the existence of a WTAR, where $cl\ (cn\ G)$ denotes the closure of $cn\ G$. In general, when $T > 1$ and X contains infinite number of elements, a similar condition is necessary and sufficient for the existence of a WTAR.(See Theorem 3.1.)

The following definition is a generalization of *ex ante* mixture undomination in the case of $T = 1$ of Border (1992).

DEFINITION **3.3** *G is said to be ex ante mixture undominated if*

$$cl\ (cn\ G) \bigcap A^T = \{(0,\ldots,0)\},$$

where 0 denotes the zero measure.

THEOREM **3.1** *The data set D has a WTAR if and only if G is ex ante mixture undominated.*

Proof. See the appendix.

In order to have a STAR, we need to strictly separate any points in G other than the origin from the separating hyperplane itself. The following condition, which says that the origin is an extreme point of G, guarantees this.

DEFINITION **3.4** *G is said to be irreversible if there do not exist $(m_1,\ldots,m_T)$, $(m'_1,\ldots,m'_T) \in G \setminus \{(0,\ldots,0)\}$, $\alpha > 0$, and $\beta > 0$ such that $\alpha m_t + \beta m'_t = 0$ for all $t = 1,\ldots,T$.*

THEOREM **3.2** *Suppose that $B^n \subset M(X)^T, n = 1,\ldots,N$, is a weak* closed, convex set and that $cn\ G$ is a weak* closed set. Then the data set D has a STAR if and only if G is ex ante mixture undominated and irreversible.*

Proof. See the appendix.

Both of the theorems show that we may interpret the separating hyperplanes as the time-additive expected utility function with appropriate properties. Below, we sketch the reasons why this may be so.

Note first that for each weak* continuous linear functional ξ on $M(X)^T$, there exists $(u_1, \ldots, u_T) \in C(X)^T$ such that

$$\xi(m_1, \ldots, m_T) = \sum_{t=1}^{T} \int u_t dm_t \quad \text{for all} \quad (m_1, \ldots, m_T) \in M(X)^T.$$

(See the appendix.) In the proof of Theorem 3.1, by using $cl\ (cn\ G) \bigcap A^T = \{(0, \ldots, 0)\}$, we show the existence of a weak* continuous linear functional ξ on $M(X)^T$ which separates G and A^T. That is there exists $(u_1, \ldots, u_T) \in C(X)^T$ satisfying

(1) $\sum_{t=1}^{T} \int u_t dm_t \leq 0$ for $(m_1, \ldots, m_T) \in cl(cn\ G)$ and
(2) $\sum_{t=1}^{T} \int u_t dm_t > 0$ for $(m_1, \ldots, m_T) \in A^T \setminus \{(0, \ldots, 0)\}$.

Thus, for $(m_1, \ldots, m_T) \in B^n, (m_1, \ldots, m_T) - (m_1^n, \ldots, m_T^n) \in G$ holds so that, by (1),

$$\sum_{t=1}^{T} \int u_t dm_t^n \geq \sum_{t=1}^{T} \int u_t dm_t \quad \text{for all} \quad (m_1, \ldots, m_T) \in B^n.$$

Note that (2) guarantees that $u_t \in U_c$; for the details, see the appendix.

4 Time-Additive Expected Utility Rationalization with a Constant Discount Factor

As in the previous section, we consider a set of data $D = \{(y^n, B^n)\}_{n=1}^{N}$ $\left(B^n \subset M_1^+(X)^T, y^n = (m_1^n, \ldots, m_T^n) \in B^n\right), n = 1, \ldots, N$. In this section, we only employ the weak notion of rationality.

DEFINITION 4.1 *A pair of a function $u : X \to R$ and a real number $\gamma > 0$ is said to be a weak time-additive expected utility rationalization with a constant discount factor (WTARC) for D if*
1. u is a strictly increasing continuous function,
2. for all $n = 1, \ldots N$,

$$\sum_{t=1}^{T} \gamma^{t-1} \int u dm_t^n \geq \sum_{t=1}^{T} \gamma^{t-1} \int u dm_t \quad \textit{for all} \quad (m_1, \ldots, m_T) \in B^n.$$

The following example shows that some data sets are WTAR but not WTARC.

Example 3

For simplicity we consider the implication of the WTARC in the certainty case. Let $T = 2$. Suppose such a representation is possible for (x_1, x_2) and if we find (x_1, x_2), $x_1 > x_2$ and (x'_1, x'_2), $x'_1 < x'_2$ such that

$$u(x_1) + \gamma u(x_2) > u(x_2) + \gamma u(x_1)$$

and

$$u(x'_1) + \gamma u(x'_2) > u(x'_2) + \gamma u(x'_1).$$

Then we have a contradiction. Thus the following data is not WTARC. The data consist of two observations. In the first observation a consumer faces price $(p_1, p_2) = (1, 2)$ with income 4 and chooses $(x_1, x_2) = (1, 1.5)$. In the second observation the consumer faces price $(p_1, p_2) = (1, 1/8)$ with income 4 and chooses $(x_1, x_2) = (35/9, 8/9)$. Note that $(1.5, 1)$ is in the first feasible set and $(8/9, 35/9)$ is in the second feasible set inducing a contradiction as observed earlier. Clearly this is WTAR.

In the proof of Theorem 3.1, we constructed the set $cl(cn\ G)$ and proved the existence of a linear functional $(u_1, \ldots, u_T)$ which separates $cl(cn\ G)$ and A^T. In this section, we need to separate two sets by a linear functional which has the form $(u, \gamma u, \ldots, \gamma^{T-1} u)$. In order to prove the existence of such a linear functional, we first construct the following set.

Let $G_\alpha \subset M(X)^T$ be a set satisfying

(i) $G \subset G_\alpha$,
(ii) G_α is weak* closed and convex,
(iii) $(m_1, \ldots, m_{T-1}, 0) \in G_\alpha$ implies $(0, m_1, \ldots, m_{T-1}) \in G_\alpha$, and
(iv) $(0, \ldots, 0, \underbrace{m}_{t}, 0, \ldots, 0) \in G_\alpha$ for some $t \in \{1, \ldots, T\}$ implies $(0, \ldots, 0, \underbrace{m}_{s}, 0, \ldots, 0) \in G_\alpha$ for all $s \in \{1, \ldots, T\}$.

Let $\tilde{G}$ be the intersection of all of such G_α's, i.e., $\tilde{G} = \bigcap_\alpha G_\alpha$. Note that $\tilde{G}$ is not empty. Obviously, (i) $G \subset \tilde{G}$, (ii) $\tilde{G}$ is weak* closed and convex, (iii) $(m_1, \ldots, m_{T-1}, 0) \in \tilde{G}$ implies $(0, m_1, \ldots, m_{T-1}) \in \tilde{G}$, and (iv) $(0, \ldots, 0, \underbrace{m}_{t}, 0, \ldots, 0) \in \tilde{G}$ for some $t \in \{1, \ldots, T\}$ implies $(0, \ldots, 0, \underbrace{m}_{s}, 0, \ldots, 0) \in \tilde{G}$ for all $s \in \{1, \ldots, T\}$.

Then if $\tilde{G}$ satisfies certain conditions, there exists a linear functional which separates $\tilde{G}$ and A^T. It will be shown that among such functionals, there exists a linear functional which has the form $(u, \gamma u, \ldots, \gamma^{T-1} u)$. Clearly, (iv) is necessary for the existence of a functional which has the above form. By the following discussion, the readers may understand the reason why (iii) is necessary. For simplicity, we consider the case $T = 3$. Suppose there exists a linear functional $(u, \gamma u, \gamma^2 u)$ separating G and A^3. Then, for $(m_1, m_2, 0) \in G$,

$$\int u dm_1 + \gamma \int u dm_2 + \gamma^2 \int u d0 \leq 0 \text{ implies } \int u d0 + \gamma \int u dm_1 + \gamma^2 \int u dm_2 \leq 0.$$

Thus if $(u, \gamma u, \gamma^2 u)$ separates G and A^3 then it also separates $\tilde{G}$ and A^3.

For $m, m' \in M_1^+(X)$, m strictly first order dominates m', denoted mFm', if $\int u dm > \int u dm'$ holds for all $u \in U_c$.

DEFINITION **4.2** *$\tilde{G}$ is said to be ex ante mixture undominated if* $cl\left(cn\ \tilde{G}\right) \bigcap A^T = \{(0, \ldots, 0)\}$.

THEOREM **4.1** *Suppose*

(a) for some n, there exist $(m_1, \ldots, m_T)$, $(m_1', \ldots, m_T') \in B^n$ such that $m_T F m_T^n$, $m_t^n F m_t$ for $t = 1, \ldots, T-1$, $m_T^n F m_T'$, and $m_t' F m_t^n$ for $t = 1, \ldots, T-1$, hold,

(b) for some n, there exists a measure $m_t \in M_1^+(X)$ such that $m_t^n F m_t$ and $(m_1^n, \ldots, m_{t-1}^n, m_t, m_{t+1}^n, \ldots, m_T^n) \in B^n$ for all $t = 1, \ldots, T$,

(c) for all $(m_1, \ldots, m_T) \in \tilde{G} \backslash \{(0, \ldots, 0)\}$, there exists t such that $m_t \neq 0$ and $(0, \ldots, 0, m_t, 0, \ldots, 0) \in \tilde{G}$.

Then there exists a WTARC for D if and only if $\tilde{G}$ is ex ante mixture undominated.

Proof. See the appendix.

(a) and (b) are obviously satisfied in standard security market models. Using the concept of equilibrium price measure, (c) is satisfied in standard security market models when the security price processes do not allow for arbitrage. (For the equilibrium price measure, see, for example, Dothan [1990].) The reason is as follows. The nonarbitrage condition guarantees that the feasible set can be expressed by a price functional generated by the equilibrium price measure. Suppose the choice $(m_1^n, \ldots, m_T^n)$ satisfies the feasible constraint with equality for $n = 1, \ldots, N$. Using the representation of the feasible set by a price functional, it is easy to show that $(\bar{m}_1, \ldots, \bar{m}_T) \in B^n$ implies that $\exists t, (m_1^n, \ldots, m_{t-1}^n, \bar{m}_t, m_{t+1}^n, \ldots, m_T^n) \in B^n$. Hence $(m_1, \ldots, m_T) \in G \setminus \{(0, \ldots, 0)\}$ implies that $\exists t$, $m_t \neq 0$ and $(0, \ldots, m_t, \ldots, 0) \in G$. Then let

$$\Xi = \{(m_1, \ldots, m_T) \in M(X)^T \setminus \{(0, \ldots, 0)\} \mid \exists t, (0, \ldots, m_t, \ldots, 0) \in \tilde{G}\}.$$

Since (i) $G \subset (\Xi \bigcup \{(0, \ldots, 0)\})$, (ii) $\Xi \bigcup \{(0, \ldots, 0)\}$ is weak* closed and convex, (iii) $(m_1, \ldots, m_{T-1}, 0) \in \Xi \bigcup \{(0, \ldots, 0)\}$ implies $(0, m_1, \ldots, m_{T-1}) \in \Xi \bigcup \{(0, \ldots, 0)\}$, and (iv) $(0, \ldots, 0, \underbrace{m}_{t}, 0, \ldots, 0) \in \Xi \bigcup \{(0, \ldots, 0)\}$ for some t implies $(0, \ldots, 0, \underbrace{m}_{s}, 0, \ldots, 0) \in \Xi \bigcup \{(0, \ldots, 0)\}$ for all s. Thus $\tilde{G} \bigcap (\Xi \bigcup \{(0, \ldots, 0)\}) = \tilde{G}$ holds. Hence (c) holds.

5 Concluding Remarks

In this section, we discuss extensions of our results in Sections 3 and 4.

First of all, Theorems 3.1 and 3.2 can be easily extended to the case of an infinite number of data. Suppose that the set of observation, Ω, is a compact Hausdorff space. Let $B : \Omega \to M_1^+(X)^T$ be a feasible correspondence, i.e., $B(\omega) \subset M_1^+(X)^T$ is the feasible set at observation ω. Let $y : \Omega \to M_1^+(X)^T$ be a choice function, i.e., $y(\omega) = (m_1(\omega), \ldots, m_T(\omega))$ is the choice at observation ω. Then the set of data is $D = \{(y(\omega), B(\omega)) \mid \omega \in \Omega\}$. The set G can be defined as follows:

$$\begin{aligned} G \;=\; & \bigcup \{(\textstyle\int (\bar{m}_1(\omega) - m_1(\omega)) d\lambda, \ldots, \int (\bar{m}_T(\omega) - m_T(\omega)) d\lambda) \mid \lambda \in M(\Omega), \\ & (\bar{m}_1(\omega) \ldots, \bar{m}_T(\omega)) \in \overline{co}\ B(\omega)\}, \end{aligned}$$

where the integral is the *Gel'fand integral* and $M(\Omega)$ denotes the set of all Borel probability measure over Ω. As in Section 3, the concepts of mixture undominatedness and irreversibility can be defined. The following theorem can be easily proved using similar arguments as in the appendix and in Border (1992). The proof is left for the readers.

THEOREM 5.1 *Suppose B is a continuous correspondence. Then the data set D has a WTAR if and only if G is ex ante mixture undominated.*

THEOREM **5.2** *Suppose (a) B is a continuous correspondence, (b)* $B(\omega) \subset M(X)^T, \omega \in \Omega$, *is a weak* closed, convex set, and (c) cn G is a weak* closed set. Then the data set D has a STAR if and only if G is ex ante mixture undominated and irreversible.*

Similarly, Theorem 4.1 can be easily extended to the case of infinite number of data. Let $G_\alpha \subset M(X)^T$ be a set such that (i) $G \subset G_\alpha$, (ii) G_α is weak* closed and convex, (iii) $(m_1, \ldots, m_{T-1}, 0) \in G_\alpha$ implies $(0, m_1, \ldots, m_{T-1}) \in G_\alpha$, and (iv) $(0, \ldots, 0, \underbrace{m}_{t}, 0, \ldots, 0) \in G_\alpha$ for some $t \in \{1, \ldots, T\}$ implies $(0, \ldots, 0, \underbrace{m}_{s}, 0, \ldots, 0) \in G_\alpha$ for all $s \in \{1, \ldots, T\}$.

Let $\tilde{G}$ be the intersection of all of such G_α's, i.e., $\tilde{G} = \bigcap_\alpha G_\alpha$.

The following theorem can be easily proved using the arguments in the appendix and in Border (1992). The proof is also left for the readers.

THEOREM **5.3** *Suppose cn* $\tilde{G}$ *is weak* closed. Moreover, suppose*

(a) for some ω, *there exist* $(m_1, \ldots, m_T), (m'_1, \ldots, m'_T) \in B(\omega)$ *such that* $m_T F m_T(\omega)$, $m_t(\omega) F m_t$ *for* $t = 1, \ldots, T-1$, $m_T(\omega) F m'_T$ *and* $m'_t F m_t(\omega)$ *for* $t = 1, \ldots, T-1$,

(b) for some ω, *there exists a measure* $m_t \in M_1^+(X)$ *such that* $m_t^n F m_t$ *and* $(m_1^n, \ldots, m_{t-1}^n, m_t, m_{t+1}^n, \ldots, m_T^n) \in B^n$ *for all* $t = 1, \ldots, T$,

(c) for all $(m_1, \ldots, m_T) \in \tilde{G}$, *there exists t such that* $m_t \neq 0$ *and* $(0, \ldots, 0, m_t, 0, \ldots, 0) \in \tilde{G}$.

Then there exists a WTARC for D if and only if $\tilde{G}$ *is ex ante mixture undominated.*

In economics, the utility functions are often assumed to be concave. By replacing A^T by a suitable cone in the definition of mixture undominatedness, we can prove the existence of a WTAR, a STAR, and a WTARC with concave utility functions. (For the fefinition of such a cone, see Border (1991).)

Clearly, there is no intrinsic reason why utility functions should be additive over time and also there are behavioral hypotheses other than expected utility maximization. In fact Epstein and Zin (1989) and others specify a recursive intertemporal utility function and consider non-expected utility maximization behavior, for example. In Ichimura and Kamiya (1995), we give a necessary and sufficient conditions for the existence of a nonexpected utility rationalization.

6 Appendix

The weak* Topology on $M(X)^T$

First, we explain the topological dual of $C(X)^T = \underbrace{C(X) \times \cdots \times C(X)}_{T}$. (For the details, see Schatten (Chapter 1,1950).) It is well-known that the topological dual of $C(X)$ is $M(X)$. We introduce a norm on $C(X)^T$ in such a way that a sequence $\{(f_1^q, \ldots, f_T^q)\}_{q=1}^{\infty}$ in $C(X)^T$ converges to $(f_1^*, \ldots, f_T^*) \in C(X)^T$ if and only if $\lim_{q\to\infty} f_t^q = f_t^*$ for all $t = 1, \ldots, T$. For example, $\|(f_1, \ldots, f_T)\| = \sum_{t=1}^{T} \|f_t\|$, where $\|f_t\|$ is the sup norm of f_t, satisfies the condition.

An element in the topological dual of $C(X)^T$ naturally corresponds to an element of $M(X)^T$. Indeed, for a continuous linear functional φ on $C(X)^T$,

$$\varphi(f_1, \ldots, f_T) = \varphi(f_1, 0, \ldots, 0) + \cdots + \varphi(0, \ldots, 0, f_T), \quad \text{for } (f_1, \ldots, f_T) \in C(X)^T$$

holds so that, by the definition of the norm on the space, there exists $m_t \in M(X), t = 1, \ldots, T$, such that $\varphi(0, \ldots, 0, f_t, 0, \ldots, 0)$ can be written as $\int f_t dm_t$ for all $f_t \in C(X)$. Thus

$$\varphi(f_1, \ldots, f_T) = \sum_{t=1}^{T} \int f_t dm_t \quad \text{for} \quad (f_1, \ldots, f_T) \in C(X)^T$$

holds.

Conversely, an element of $M(X)^T$ corresponds to an element of the topological dual of $C(X)^T$. Indeed, obviously

$$\sum_{t=1}^{T} \int f_t dm_t$$

is a continuous linear functional on $C(X)^T$.

Then the dual space of $C(X)^T$ can be identified with $M(X)^T$. Thus we can introduce the weak* topology on $M(X)^T$ using $C(X)^T$. A weak* continuous linear functional ξ on $M(X)^T$ can be identified with some function $(f_1, \ldots, f_T) \in C(X)^T$, since

$$\xi(m_1, \ldots, m_T) = \xi(m_1, 0, \ldots, 0) + \cdots + \xi(0, \ldots, 0, m_T)$$

holds and, by the definition of weak* topology, each $\xi(0, \ldots, 0, m_t, 0, \ldots, 0)$ can be written as $\int f_t dm_t$ for some $f_t \in C(X)$.

Proof of Theorem 3.1

Obviously, the conditions are necessary for the existence of a WTAR. Below, we prove the sufficiency of the conditions.

If each B^n is a singleton, then obviously there exists a WTAR. Thus, in what follows, we assume that at least one B^n has more than one element.

We use the following lemmas.

LEMMA **6.1** *(Border (1992)) Fix $u_0 \in U_c$. Then $\mathbf{B} = \{m \in A \mid \int u_0 dm = 1\}$ is a weak* closed convex base for A. That is $0 \notin \mathbf{B}$ and A is equal to $\{\beta m \mid \beta \geq 0, m \in \mathbf{B}\}$.*

LEMMA **6.2** *For $b_t \in M(X)$, let $\boldsymbol{b}_t$ be the element of $M(X)^T$ such that the t-th coordinate is b_t and the other coordinates are 0. Then*

$$\mathbf{B}^T = \left\{ \sum_{t=1}^{T} \beta_t \boldsymbol{b}_t | \beta = (\beta_1, \ldots, \beta_T) \in S^T, b_t \in \mathbf{B} \right\}$$

is a weak closed convex base for A^T, where S^T is the $(T-1)$-dimensional unit simplex.*

Proof. Let

$$F = \overline{co} \left(\bigcup_{t=1}^{T} \{\boldsymbol{b}_t \mid b_t \in \mathbf{B}\} \right).$$

By Lemma 6.1 and Border (Lemma 5.10,1992), $\mathbf{B}^T = F$ holds so that $\mathbf{B}^T$ is weak* closed and convex. Obviously, $(0, \ldots, 0) \notin \mathbf{B}^T$. Finally, if $(m_1, \ldots, m_T) \in A^T$, then by Lemma 6.1,there exist $\gamma_t > 0$ and $b_t \in \mathbf{B}, t = 1, \ldots, T$, such that $m_t = \gamma_t b_t, t = 1, \ldots, T$. Thus $(m_1, \ldots, m_T) = \left(\sum_{t=1}^{T} \gamma_t\right) \left(\sum_{t=1}^{T} \frac{\gamma_t}{\sum_{t=1}^{T} \gamma_t} \boldsymbol{b}_t\right)$.

Q.E.D.

By $cl(cn\ G) \bigcap A = \{(0, \ldots, 0)\}$ and the separating hyperplane theorem (see, for example, Dunford and Schwartz (v.2,7,10,1958)), there exists a linear functional $(u_1, \ldots, u_T) \in C(X)^T$ strongly separating $cl(cn\ G)$ and $\mathbf{B}^T$, *i.e.*, for some $c \in R$,

$$\sum_{t=1}^{T} \int u_t dm_t < c \qquad \text{for} \qquad (m_1, \ldots, m_T) \in cl(cn\ G)$$

and

$$\sum_{t=1}^{T} \int u_t dm_t > c \qquad \text{for} \qquad (m_1, \ldots, m_T) \in \mathbf{B}^T.$$

Since $\mathbf{B}^T$ is the base of A^T and $cl(cn\ G)$ is a cone,

(1) $\sum_{t=1}^{T} \int u_t dm_t \leq 0 \quad$ for $\quad (m_1, \ldots, m_T) \in cl(cn\ G)$

and

(2) $\sum_{t=1}^{T} \int u_t dm_t > 0 \quad$ for $\quad (m_1, \ldots, m_T) \in A^T \setminus \{(0, \ldots, 0)\}$

hold.

Since, by Lemma 6.2, $m_1 \in \mathbf{B}$ implies $(m_1, 0, \ldots, 0) \in \mathbf{B}^T$, then, for all $m_1 \in \mathbf{B}, \int u_1 dm_1 > 0$ holds. Thus, by Border (Corollary 5.4,1992), u_1 is strictly increasing. Similarly, for all $t = 2, \ldots, T$, u_t is strictly increasing.

Finally, for $(m_1, \ldots, m_T) \in B^n, (m_1, \ldots, m_T) - (m_1^n, \ldots, m_T^n) \in G$ holds so that, by (1),

$$\sum_{t=1}^{T} \int u_t dm_t^n \geq \sum_{t=1}^{T} \int u_t dm_t.$$

Proof of Theorem 3.2

First, we prove the necessity of the conditions. Obviously, if there exists a STAR, then G is *ex ante* mixture undomonated. Suppose that there exists a STAR, $(u_1, \ldots, u_T)$, and that G is not irreversible. Then there exist $m = (m_1, \ldots, m_T) \in G \setminus \{(0, \ldots, 0)\}, m' = (m'_1, \ldots, m'_T) \in G \setminus \{(0, \ldots, 0)\}, \alpha > 0, \beta > 0$ such that $\alpha m + \beta m' = \{(0, \ldots, 0)\}$. By the weak* closedness and convexity of B^n and the definition of G, there exist $a = (a^1, \ldots, a^N) \in S^N, b = (b^1, \ldots, b^N) \in S^N, \bar{m}^n = (\bar{m}_1^n, \ldots, \bar{m}_T^n) \in B^n$, and $\bar{m}'^n = (\bar{m}_1'^n, \ldots, \bar{m}_T'^n) \in B^n, n = 1, \ldots, N$, such that $m = \sum_{n=1}^{N} a^n(\bar{m}^n - m^n)$ and $m' = \sum_{n=1}^{N} b^n(\bar{m}'^n - m^n)$. Since $m \neq 0, m' \neq 0$, and $(u_1, \ldots, u_T)$ is a STAR,

$$a^n \sum_{t=1}^{T} \int u_t dm_t^n \geq a^n \sum_{t=1}^{T} \int u_t d\bar{m}_t^n \quad n = 1, \ldots, N$$

with at least one strict inequality, and

$$b^n \sum_{t=1}^{T} \int u_t dm_t^n \geq b^n \sum_{t=1}^{T} \int u_t d\bar{m}_t'^n \qquad n = 1, \ldots, N$$

with at least one strict inequality. Thus

$$\alpha \sum_{t=1}^{T} \int u_t dm_t + \beta \sum_{t=1}^{T} \int u_t dm_t' < 0$$

holds. This is a contradiction. Thus G is irreversible.

Below, we prove the sufficiency of the conditions.

We use the following lemma.

LEMMA **6.3** *(Phelps (Theorem 11.6,1966)) Suppose K is a closed, locally compact, convex cone in a locally convex space E such that $K \bigcap (-K) = \{0\}$. Then K has a compact convex base H.*[4]

By the irreversibility of G, $K = cn\ G$ and $E = M(X)^T$ satisfy the conditions of the above lemma. Indeed, since we assumed the closedness of $cn\ G$, $cn\ G \bigcap -(cn\ G) = \{(0, \ldots, 0)\}$ immediately follows from the irreversibility of G. Thus there exists a weak* compact convex base H of $cn\ G$.

We have so far shown that A^T and $cn\ G$ have weak* closed convex bases $\mathbf{B}^T$ and H, respectively, and H is weak* compact. Then, by $(cn\ G) \bigcap A^T = cl(cn\ G) \bigcap A^T = \{(0, \ldots, 0)\}$ and the separating hyperplane theorem (see, for example, Dunford and Schwartz (v.2,7,10,1958)), there exists a linear functional $(u_1, \ldots, u_T) \in C(X)^T$ strongly separating H and $\mathbf{B}^T$, *i.e.*, for some $c \in R$,

$$\sum_{t=1}^{T} \int u_t dm_t < c \qquad \text{for} \qquad (m_1, \ldots, m_T) \in H$$

and

$$\sum_{t=1}^{T} \int u_t dm_t > c \qquad \text{for} \qquad (m_1, \ldots, m_T) \in \mathbf{B}^T.$$

Since H and $\mathbf{B}^T$ are the bases of $cn\ G$ and A^T, respectively,

(1) $\sum_{t=1}^{T} \int u_t dm_t < 0 \quad$ for $\quad (m_1, \ldots, m_T) \in cn\ G \backslash \{(0, \ldots, 0)\}$

[4] Note that, in Phelps (1966), the definition of a base includes its convexity.

and

(2) $\sum_{t=1}^{T}\int u_t dm_t \geq 0$ for $(m_1,\ldots,m_T)\in A^T$

holds.

Since, by Lemma 6.2, $m_1 \in \mathbf{B}$ implies $(m_1,0,\ldots,0)\in \mathbf{B}^T$, then, for all $m_1 \in \mathbf{B}, \int u_1 dm_1 > 0$ holds. Thus, by Border (Corollary 5.4,1992), u_1 is strictly increasing. Similarly, for all $t = 2,\ldots,T$, u_t is strictly increasing.

Finally, for $(m_1,\ldots,m_T)\in B^n\setminus\{(m_1^n,\ldots,m_T^n)\}, (m_1,\ldots,m_T)-(m_1^n,\ldots,m_T^n)\in G$ holds so that, by (1),

$$\sum_{t=1}^{T}\int u_t dm_t^n > \sum_{t=1}^{T}\int u_t dm_t$$

holds for all $(m_1,\ldots,m_T)\in B^n\setminus\{(m_1^n,\ldots,m_T^n)\}$.

Proof of Theorem 4.1

First, we prove the sufficiency of the conditions.

Let

$$G_1 = \{m \in M(X) \mid (m,0,\ldots,0)\in \tilde{G}\}.$$

By the *ex ante* mixture undominated condition, $cl\,(cn\ G_1)\bigcap A = \{0\}$. Thus, by the same argument as in the proof of Theorem 3.1, there exists a linear functional $u \in C(X)$ such that

(1) $\int udm \leq 0$ for $m \in cl(cn\ G_1)$

and

(2) $\int udm > 0$ for $m \in A\setminus\{0\}$.

As in the proof of Theorem 3.1, $u \in U_c$.

Let

$$\Gamma = \left\{\left(\int udm_1,\ldots,\int udm_T\right) \mid (m_1,\ldots,m_T)\in \tilde{G}\right\}.$$

LEMMA **6.4** *Γ is a closed convex set such that $\Gamma\bigcap R_{++}^T = \emptyset$ and $(0,\ldots,0)\in \partial\Gamma$.*[5]

[5]For a set $K \subset R^T$, *int* K and ∂K denote the interior of K and the boundary of K, respectively.

Proof. The convexity and closedness of Γ immediately follow from the convexity and weak* closedness of $\tilde{G}$. Note that, by $(0,\dots 0) \in \tilde{G} \subset M(X)^T$, $(0,\dots,0) \in \Gamma \subset R^T$ holds.

First, we show that $int\ \Gamma \neq \emptyset$. Since, by the assumption (b) and the property (ii) of $\tilde{G}$, there exists a data (y^n, B^n) such that, for some $(m_1^*,\dots,m_T^*) \in \tilde{G}$, $m_t^* = m_t^n$, $t = 1,\dots,T-1$, and $m_T^n F m_T^*$ hold. Thus there exists a real number $b_T > 0$ such that $(0,\dots,0,-b_T) \in \Gamma$. Similarly, there exist positive real numbers $b_1,\dots,b_{T-1}$ such that $(0,\dots,0,-b_t,0,\dots,0) \in \Gamma$ for $t = 1,\dots,T-1$. Thus, by the convexity of Γ and $(0,\dots,0) \in \Gamma$, $int\ \Gamma \neq \phi$ holds.

$(0,\dots,0) \in \partial\Gamma$ follows from (c). Indeed, if $(0,\dots,0) \notin \partial\Gamma$, then there exist $(a_1,\dots,a_T) \in R^T$ and $m = (m_1,\dots,m_T) \in \tilde{G} \setminus \{(0,\dots,0)\}$ such that $a_t > 0$ and $\int u dm_t = a_t$, $t = 1,\dots,T$. By (c), there exists t such that $\int u dm_t \leq 0$ holds. This is a contradiction.

By the assumption (b) and the property (ii) of $\tilde{G}$, there exists a measure $\bar{m}_t \in M(X)$ such that $(0,\dots,0,\bar{m}_t,0,\dots,0) \in \tilde{G}$ and $\int u d\bar{m}_t < 0$ for $t = 1,\dots,T$. Let $\int u d\bar{m}_t = -a_t$. Then $(0,\dots,0,-a_t,0,\dots,0) \in \Gamma$ holds for $t = 1,\dots,T$. Thus, by the convexity of Γ, $(0,\dots,0) \notin int\ \Gamma$ implies $\Gamma \bigcap R^T_{++} = \phi$.

Q.E.D.

Let

$$Q = \{\alpha \in R^T_+ \mid \sum_{t=1}^{T} \alpha_t = 1, \sum_{t=1}^{T} \alpha_t c_t \leq 0 \quad \text{for all} \quad (c_1,\dots,c_T) \in \Gamma\}.$$

By the above lemma and the separating hyperplane theorem, Q is a nonempty closed convex set in R^T. Then let

$$G^*(\alpha) = \{(\sum_{t=1}^{T-1} \alpha_{t+1} c_t, c_T) \in R^2 \mid (c_1,\dots,c_T) \in \Gamma\}.$$

We use the following Lemma.

LEMMA **6.5** *For all $\alpha \in Q$, (i) $G^*(\alpha)$ is a closed convex set, (ii) $(0,0) \in G^*(\alpha)$, (iii) $R^2_- \subset cn\ G^*(\alpha)$, and (iv) $(0,0) \in \partial G^*(\alpha)$.*

Proof. (i) follows from the closedness and convexity of Γ. (ii) follows from $(0,\dots,0) \in \Gamma$.

Next, we prove (iii). By the assumption (a), there exist $(c_1, \ldots, c_T) \in \Gamma$ and $(c'_1, \ldots, c'_T) \in \Gamma$ such that $c_t < 0, t = 1, \ldots, T-1$, $c_T > 0$, $c'_t > 0, t = 1, \ldots, T-1$, and $c'_T < 0$. Thus $\alpha \in Q$ cannot have the form $(0, \ldots, 0, \alpha_T)$ or $(\alpha_1, \ldots, \alpha_{T-1}, 0)$. Thus, by the assumption (b), there exist vectors $(-a, 0) \in G^*(\alpha)$ and $(0, -b) \in G^*(\alpha)$, where a and b are positive real numbers. Together with the convexity of $G^*(\alpha)$, this leads to (iii).

Finally, we prove (iv). By $(0,0) \in G^*(\alpha)$, it is suffices to prove $(0,0) \notin int\, G^*(\alpha)$. Suppose the contrary. Then there exist a real number $a > 0$ and a vector $(c_1, \ldots, c_{T-1}, 0) \in \Gamma$ such that $(a, 0) \in int\ G^*(\alpha)$ and $\sum_{t=1}^{T-1} \alpha_{t+1} c_t = a > 0$. By $(c_1, \ldots, c_{T-1}, 0) \in \Gamma$ and the property (iii) of $\tilde{G}$, $\sum_{t=1}^{T-1} \alpha_{t+1} c_t \leq 0$ holds. This is a contradiction.

Q.E.D.

By the above lemma and the separating hyperplane theorem, there exists a vector $(\beta_1, \beta_2) \in R^2_+ \setminus \{(0,0)\}$ such that

$$\beta_1 d_1 + \beta_2 d_2 \leq 0 \quad \text{for all} \quad (d_1, d_2) \in G^*(\alpha).$$

Let

$$Q^2(\alpha) = \{(\beta_1, \beta_2) \in R^2_+ \mid \beta_1 + \beta_2 = 1, \text{ and } \forall (d_1, d_2) \in G^*(\alpha),\ \beta_1 d_1 + \beta_2 d_2 \leq 0\}.$$

Lemma 6.6 *(i) $Q^2(\alpha)$ is a nonempty, closed, convex set and (ii) $Q^2(\alpha) \subset int\ R^2_+$.*

Proof. (i) obviously holds. By the assumption (a), there exist vectors $(-a, b) \in G^*(\alpha)$ and $(c, -d) \in G^*(\alpha)$, where a, b, c, and d are positive real numbers. Thus $Q^2(\alpha) \subset int\ R^2_+$.

Q.E.D.

Lemma 6.7 *For $\alpha \in Q$ and $(\beta_1, \beta_2) \in Q^2(\alpha)$, let $\xi = \sum_{t=1}^{T-1} \beta_1 \alpha_{t+1} + \beta_2$. Then*

$$\alpha' = \xi^{-1}(\beta_1 \alpha_2, \ldots, \beta_1 \alpha_T, \beta_2) \in Q.$$

Note that, by the above lemma, $\xi > 0$ holds.

Proof. Obviously, $\sum_{t=1}^{T} \alpha'_t = 1$. For all $(c_1, \ldots, c_T) \in \Gamma$, obviously $\beta_1 \sum_{t=1}^{T-1} \alpha_{t+1} c_t + \beta_2 c_T \leq 0$ holds. Thus $\alpha' \in Q$.

Q.E.D.

Let $\Theta : Q \times S^2 \to Q$ be

$$\Theta(\alpha, \beta) = \xi^{-1}(\beta_1 \alpha_2, \ldots, \beta_1 \alpha_T, \beta_2),$$

where $S^2 = \{\beta \in R^2_+ | \beta_1 + \beta_2 = 1\}$. Then let $\varphi : Q \to Q$ be

$$\varphi(\alpha) = \{\Theta(\alpha, \beta) \mid \beta \in Q^2(\alpha)\}.$$

LEMMA **6.8** *φ is an upper hemicontinuous correspondence with compact convex values.*

Proof. Since $Q^2(\alpha)$ is closed, then, by $\varphi(\alpha) \subset S^1$, $\varphi(\alpha)$ is compact. For $\Theta(\alpha, \beta), \Theta(\alpha, \overline{\beta}) \in \varphi(\alpha)$, and $t \in (0, 1)$, there exists a real number $s \in (0, 1)$ such that

$$t\Theta(\alpha, \beta) + (1 - t)\Theta(\alpha, \overline{\beta}) = \Theta(\alpha, s\beta + (1 - s)\overline{\beta}),$$

because Θ is a continuous function of β and the vectors of the first $T - 1$ components of $\Theta(\alpha, \beta)$, $\Theta(\alpha, \overline{\beta})$, and $\Theta(\alpha, s\beta + (1 - s)\overline{\beta})$ have the common direction. Thus $\varphi(\alpha)$ is convex.

Finally, we prove the upper hemicontinuity of φ. By the continuity of Θ, if Q^2 is upper hemicontinuous then so is φ. Below, we prove the upper hemicontinuity Q^2.

We consider sequences $\{\alpha^q\}(\alpha^q \in Q, \lim_{q \to \infty} \alpha^q = \alpha^*)$ and $\{\beta^q\}(\beta^q \in Q^2(\alpha^q), \lim_{q \to \infty} \beta^q = \beta^*)$. Suppose $\beta^* \notin Q^2(\alpha^*)$. Then there exists a vector $(c_1, \ldots, c_T) \in \Gamma$ such that

$$\beta_1^* \sum_{t=1}^{T-1} \alpha_{t+1}^* c_t + \beta_2^* c_T > 0.$$

Thus there exists $\overline{q}$ such that, for all $q \geq \overline{q}$,

$$\beta_1^q \sum_{t=1}^{T-1} \alpha_{t+1}^q c_t + \beta_2^q c_T > 0.$$

This contradicts $\beta^q \in Q^2(\alpha^q)$. Thus Q^2 is upper hemicontinuous.

Q.E.D.

By the above lemma and Kakutani's fixed point theorem, there exist vectors $\alpha^* \in \varphi(\alpha^*)$ and $\beta^* \in Q^2(\alpha^*)$ such that

$$(\alpha_1^*, \ldots, \alpha_{T-1}^*, \alpha_T^*) = (\xi^*)^{-1}(\beta_1^*\alpha_2^*, \ldots, \beta_1^*\alpha_T^*, \beta_2^*).$$

Thus, by $\alpha_1^* = (\xi^*)^{-1}\beta_1^*\alpha_2^*$, $\alpha_2^* = \frac{\xi^*}{\beta_1^*}\alpha_1^*$ holds. Similarly, $\alpha_t^* = \left(\frac{\xi^*}{\beta_1^*}\right)^{t-1}\alpha_1^*, t = 3, \ldots, T$, holds. By $\left(1, \frac{\alpha_2^*}{\alpha_1^*}, \ldots, \frac{\alpha_T^*}{\alpha_1^*}\right) \in cn\ Q$, $\left(1, \frac{\xi^*}{\beta_1^*}, \left(\frac{\xi^*}{\beta_1^*}\right)^2, \ldots, \left(\frac{\xi^*}{\beta_1^*}\right)^{T-1}\right) \in cn\ Q$ holds. Let $\gamma = \frac{\xi^*}{\beta_1^*}$. Then $(1, \gamma, \gamma^2, \ldots, \gamma^{T-1}) \in cn\ Q$ so that the utility function

$$\sum_{t=1}^{T} \gamma^{t-1} \int u dm_t$$

rationalizes the data. Indeed, for $(m_1, \ldots, m_T) \in B^n, (m_1, \ldots, m_T) - (m_1^n, \ldots, m_T^n) \in \tilde{G}$ holds so that, by the definition of Γ,

$$\left(\int ud(m_1 - m_1^n), \ldots \int ud(m_T - m_T^n)\right) \in \Gamma$$

holds. Thus, by $(1, \gamma, \gamma^2, \ldots, \gamma^{T-1}) \in cn\ Q$,

$$\sum_{t=1}^{T} \gamma^{t-1} \int udm_t^n \geq \sum_{t=1}^{T} \gamma^{t-1} \int udm_t$$

holds.

Finally, *ex ante* mixture undominated condition of $\tilde{G}$ is obviously necessary for the existence of a $WTARC$.

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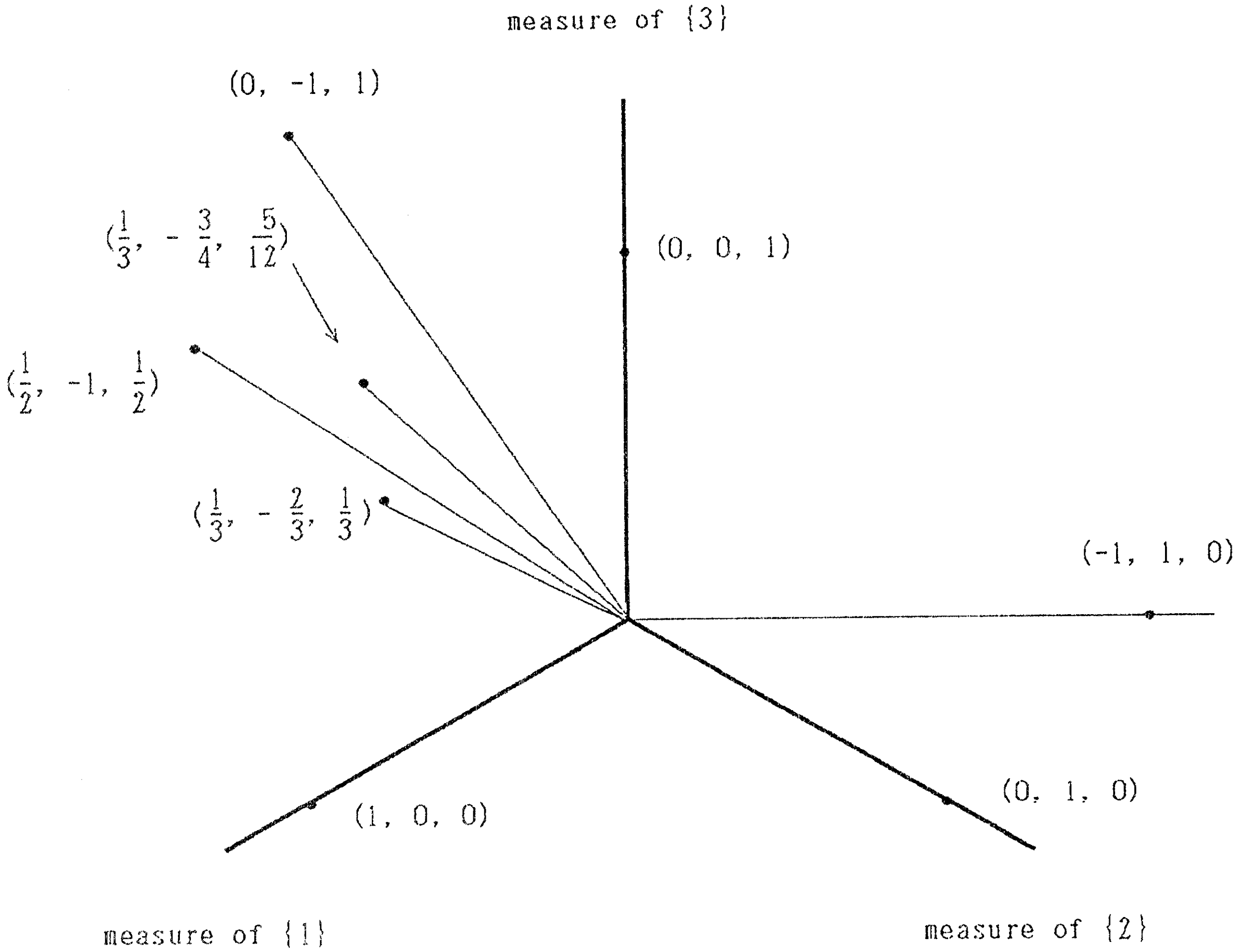
measure of {3}
(0, -1, 1)
(1/3, - 3/4, 5/12)
(0, 0, 1)
(1/2, -1, 1/2)
(1/3, - 2/3, 1/3)
(-1, 1, 0)
(1, 0, 0)
(0, 1, 0)
measure of {1}
measure of {2}

Figure 1

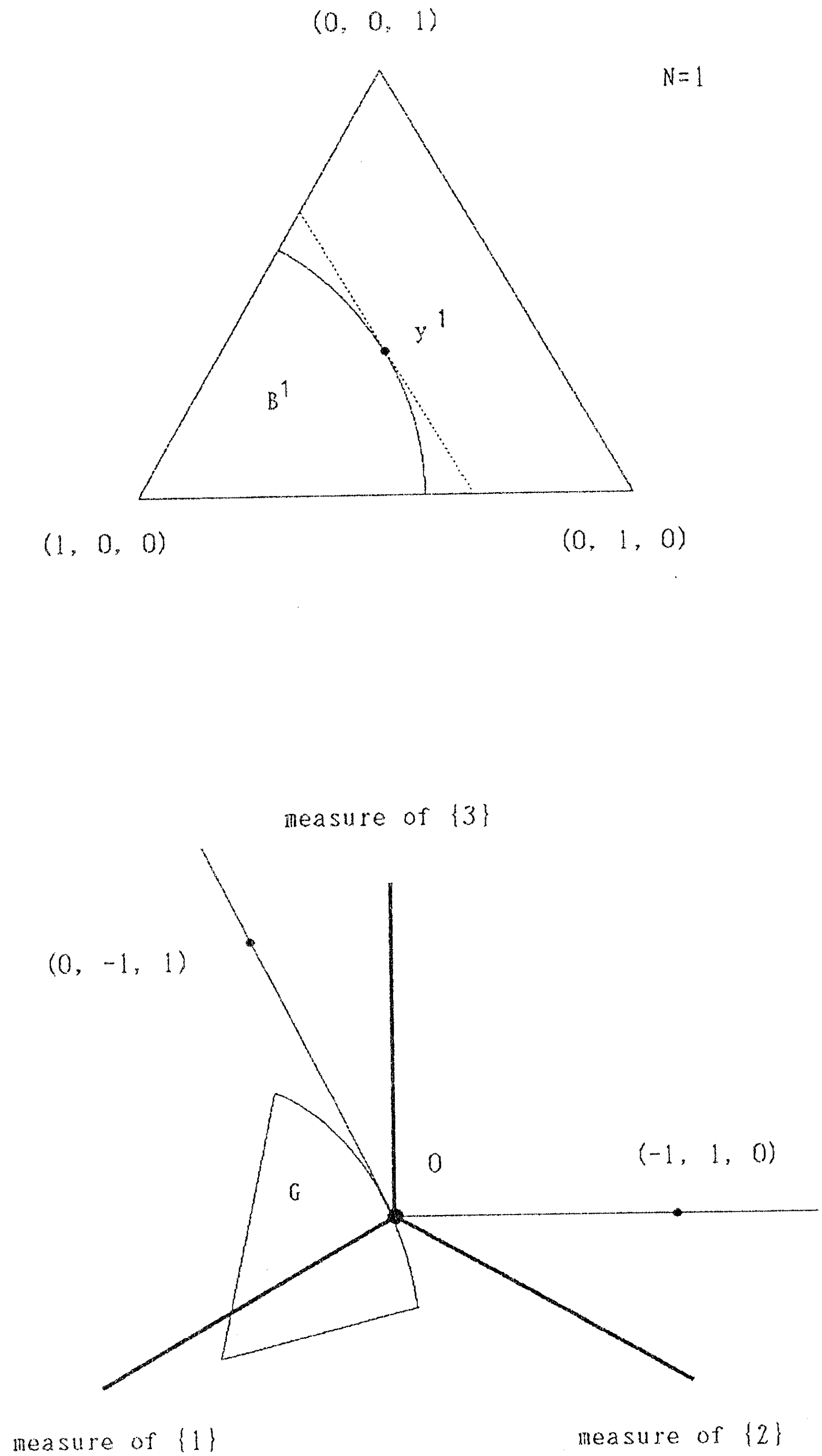
(0, 0, 1)
N=1
y 1
B 1
(1, 0, 0)
(0, 1, 0)
measure of {3}
(0, -1, 1)
0
(-1, 1, 0)
G
measure of {1}
measure of {2}

Figure 2