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Global cross section and its associated decomposable distributions

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Abstract

We study properties of global cross sections and characterize the class of all global cross sections. Then generalizing the cross-sectionally contoured distributions of Takemura and Kuriki (1996), we define the decomposable distributions with respect to a global cross section. The distributional results about the invariant and equivariant parts remain to hold for the decomposable distributions. We also investigate the relation between the actions of a group and one of its subgroups.

Key words: invariance, equivariance, isotropy subgroup, normalizer, cross-sectionally contoured distribution, invariant measure.

1 Introduction

Generalizing the elliptically contoured distributions, Takemura and Kuriki (1996) defined the star-shaped distributions, in which the contours of the density functions are arbitrary star-shaped sets. Furthermore they defined the cross-sectionally contoured distributions in the general framework of group invariance and applied their theory to random matrices. However their theoretical discussion as well as their examples was restricted to the case of free actions.

In the statistical literature non-free actions are more common. For example the action of the orthogonal group on the set of positive definite matrices is not free. The present paper started out as an effort to extend the theory in Takemura and Kuriki (1996) to actions which are not necessarily free.

For general actions, cross sections have to be taken in such a way that the isotropy subgroups are the same at all points of them. We call the cross sections possessing this property the global cross sections. By taking a global cross section, we still have the orbital decomposition; the global cross section works as the invariant part, whereas the coset space modulo the common isotropy subgroup plays the role of the equivariant part. As in Takemura and Kuriki (1996) we are concerned with construction and characterization of non-standard global cross sections. We will characterize the class of all global cross sections in terms of the normalizer of the common isotropy subgroup.

Generalizing the cross-sectionally contoured distributions of Takemura and Kuriki (1996) we define the decomposable distributions with respect to a given global cross section. We show that the distributional results about the invariant and equivariant parts in the case of free actions remain to hold for decomposable distributions.

Sometimes a global cross section does not exist for the whole sample space. In this case we can partition the sample space into equivalence classes under equivalence relation based on the conjugacy of the isotropy subgroups. These equivalence classes are called orbit types. On each orbit type a global cross section exists and our theory remains to hold on each orbit type.

Another problem we consider is the relation between the actions of a group and one of its subgroups. Let $\mathcal{G}$ and $\mathcal{H}$ act on the sample space $\mathcal{X}$ with $\mathcal{H}$ being a subgroup of $\mathcal{G}$. For example consider the group $\mathcal{H} = LT(p)$ of lower triangular matrices with positive diagonal elements and the real general linear group $\mathcal{G} = GL(p)$ acting on the set $\mathcal{X}$ of pairs of positive definite matrices. We will show that under appropriate conditions $\mathcal{H}$ leads to a further hierarchical decomposition of the orbital decomposition with respect to $\mathcal{G}$.

The organization of the paper is as follows.

In Section 2.1 we summarize properties of global cross sections and discuss orbit types. In Section 2.2 we characterize the class of all global cross sections and discuss construction of arbitrary global cross sections from a given global cross section. In Section 3 we define the decomposable distributions. In Section 3.1 we show that the results about the distributions of the invariant and equivariant parts in Takemura and Kuriki (1996) remain to hold for general actions and in Section 3.2 we give some examples of the decomposable distributions. In Section 4 we consider actions of a group and one of its subgroups. In Section 4.1 we derive a further hierarchical decomposition of the orbital decomposition by means of a subgroup action. In Section 4.2 we define distributions with the corresponding further decomposability properties.

2 The orbital decomposition and the global cross sections

In this section we investigate properties of global cross sections. Our discussion is purely group-theoretic and we make no topological or measure theoretic assumptions in this section. Existence of a measurable global cross section is discussed in Kamiya (1996) and references therein.

2.1 The orbital decomposition

Let a group $\mathcal{G}$ act on a space $\mathcal{X}$ from the left:

$$(g, x) \mapsto gx : \mathcal{G} \times \mathcal{X} \to \mathcal{X}.$$

We list some common symbols and well known results about the orbits of the action of $\mathcal{G}$ on $\mathcal{X}$. Let

$$\mathcal{G}x = \{gx : g \in \mathcal{G}\}$$

denote the *orbit* containing $x \in \mathcal{X}$ and let $\mathcal{X}/\mathcal{G} = \{\mathcal{G}x : x \in \mathcal{X}\}$ denote the *orbit space*, i.e., the set of orbits. Let

$$\mathcal{G}_x = \{g \in \mathcal{G} : gx = x\}$$

denote the *isotropy subgroup* at $x \in \mathcal{X}$. The *left coset space* of $\mathcal{G}$ modulo $\mathcal{G}_x$ is denoted by

$$\mathcal{G}/\mathcal{G}_x = \{g\mathcal{G}_x : g \in \mathcal{G}\}$$

with the *canonical map*

$$\pi(g) = g\mathcal{G}_x.$$

The group $\mathcal{G}$ acts on $\mathcal{G}/\mathcal{G}_x$ by

$$(g, h\mathcal{G}_x) \mapsto (gh)\mathcal{G}_x.$$

The action of $\mathcal{G}$ on an orbit $\mathcal{G}x$, $x \in \mathcal{X}$, is isomorphic to the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{G}_x$

$$\begin{array}{ccc} \mathcal{G}x & \leftrightarrow & \mathcal{G}/\mathcal{G}_x, \\ gx & \leftrightarrow & g\mathcal{G}_x. \end{array} \tag{1}$$

The isotropy subgroups at two points on a common orbit are conjugate to each other:

$$\mathcal{G}_{gx} = g\mathcal{G}_x g^{-1}, \quad g \in \mathcal{G},\ x \in \mathcal{X}.$$

We now discuss properties of cross sections. Compared with orbits, the properties of cross sections are often not fully discussed in standard treatments of group invariance. A *cross section* is defined to be a set $\mathcal{Z} \subset \mathcal{X}$ which intersects each orbit $\mathcal{G}x$, $x \in \mathcal{X}$, exactly once. Therefore $\mathcal{Z}$ is in one-to-one correspondence with the orbit space. We denote this correspondence by $\iota_Z : \mathcal{X}/\mathcal{G} \to Z$

$$\iota_{\mathcal{Z}}(Gz) = z \quad \text{for } z \in \mathcal{Z}\ .$$

A cross section $\mathcal{Z}$ is called a *global cross section* if the isotropy subgroups are common at all the points of $\mathcal{Z}$:

$$\mathcal{G}_z = \{g \in \mathcal{G} : gz = z\} = \mathcal{G}_0, \qquad \forall z \in \mathcal{Z}.$$

For an arbitrary cross section $\mathcal{Z}$, $g\mathcal{Z} = \{gz : z \in \mathcal{Z}\}$ is again a cross section for each $g \in \mathcal{G}$. We call $g\mathcal{Z}$ a cross section *proportional* to $\mathcal{Z}$. Note that $\mathcal{Z}$ and $g\mathcal{Z}$ are in one-to-one correspondence

$$g\mathcal{Z} \leftrightarrow \mathcal{Z} \tag{2}$$

by $gz \leftrightarrow z$. Since $\mathcal{Z}$ meets each orbit we have

$$\mathcal{X} = \bigcup_{g \in \mathcal{G}} g\mathcal{Z}. \tag{3}$$

We are interested in the case where (3) gives a partition of $\mathcal{X}$, that is,

$$g_1\mathcal{Z} \cap g_2\mathcal{Z} \neq \emptyset \quad \Rightarrow \quad g_1\mathcal{Z} = g_2\mathcal{Z}$$

for $g_1, g_2 \in \mathcal{G}$, where $\emptyset$ is the empty set. If (3) gives a partition of $\mathcal{X}$, then in view of (2) $\mathcal{X}$ is in one-to-one correspondence with $\{g\mathcal{Z} : g \in \mathcal{G}\} \times \mathcal{Z}$:

$$\begin{aligned} \mathcal{X} \quad &\leftrightarrow \quad \{g\mathcal{Z} : g \in \mathcal{G}\} \times \mathcal{Z}, \\ x \quad &\leftrightarrow \quad (g\mathcal{Z},\ z), \quad x = gz. \end{aligned} \tag{4}$$

Moreover, for an arbitrarily fixed $z_0 \in \mathcal{Z}$, we have a one-to-one correspondence between $\{g\mathcal{Z} : g \in \mathcal{G}\}$ and $\mathcal{G}z_0$:

$$\begin{aligned} \{g\mathcal{Z} : g \in \mathcal{G}\} \quad &\leftrightarrow \quad \mathcal{G}z_0, \\ g\mathcal{Z} \quad &\leftrightarrow \quad gz_0. \end{aligned} \tag{5}$$

The following lemma shows that a necessary and sufficient condition for $\{g\mathcal{Z} : g \in \mathcal{G}\}$ to be a partition of $\mathcal{X}$ is that $\mathcal{Z}$ is a global cross section.

Lemma 2.1 *A cross section $\mathcal{Z}$ is global if and only if the family $\{g\mathcal{Z} : g \in \mathcal{G}\}$ of proportional cross sections gives a partition of $\mathcal{X}$.*

Proof. Suppose that $\{g\mathcal{Z} : g \in \mathcal{G}\}$ gives a partition of $\mathcal{X}$. Let z_1 and z_2 be two arbitrary points of $\mathcal{Z}$. Let $g \in \mathcal{G}_{z_1}$. Then $gz_1 = z_1$ and $z_1 \in g\mathcal{Z} \cap \mathcal{Z} \neq \emptyset$, and hence $g\mathcal{Z} = \mathcal{Z}$. Thus there exists a $z \in \mathcal{Z}$ such that $gz_2 = z$. Since $\mathcal{Z}$ is a cross section, we have $z_2 = z$ and hence $gz_2 = z_2$. This observation shows that $g \in \mathcal{G}_{z_1}$ implies $g \in \mathcal{G}_{z_2}$. By interchanging the roles of z_1 and z_2, we see that the converse is true as well and thus $\mathcal{G}_{z_1} = \mathcal{G}_{z_2}$. Hence $\mathcal{Z}$ is global.

Conversely, suppose that $\mathcal{Z}$ is global and let $\mathcal{G}_0$ be the common isotropy subgroup. Suppose $g_1\mathcal{Z} \cap g_2\mathcal{Z} \neq \emptyset$ for $g_1, g_2 \in \mathcal{G}$. Then there exists a pair $z_1, z_2 \in \mathcal{Z}$ such that $g_1z_1 = g_2z_2$. Since $\mathcal{Z}$ is a cross section, we have $z_1 = z_2$ and thus $g_1z_1 = g_2z_1$. Therefore, $g_1^{-1}g_2 \in \mathcal{G}_0$ and hence $g_1z = g_2z$ for all $z \in \mathcal{Z}$. Thus we obtain $g_1\mathcal{Z} = g_2\mathcal{Z}$.
∎

For a global cross section $\mathcal{Z}$ we call the partition $\{g\mathcal{Z} : g \in \mathcal{G}\}$ the *family of proportional global cross sections.*

Let $\mathcal{Z}$ be a global cross section with the common isotropy subgroup $\mathcal{G}_0$ and fix $z_0 \in \mathcal{Z}$. Then (4) and (5) together with $\mathcal{G}z_0 \leftrightarrow \mathcal{G}/\mathcal{G}_{z_0}$ in (1) yield the following one-to-one correspondence:

$$\begin{aligned} \mathcal{X} \quad &\leftrightarrow \quad \mathcal{Y} \times \mathcal{Z}, \\ x \quad &\leftrightarrow \quad (y, z), \quad x = gz, \quad y = \pi(g), \end{aligned} \tag{6}$$

where $\mathcal{Y} = \mathcal{G}/\mathcal{G}_0$ is the left coset space modulo $\mathcal{G}_0$ and $\pi : \mathcal{G} \to \mathcal{Y}$ is the canonical map $\pi(g) = g\mathcal{G}_0$. The bijection $x \leftrightarrow (y, z)$ is called the *orbital decomposition* of $x \in \mathcal{X}$. In the orbital decomposition we can think of y and z as functions $y = y(x)$ and $z = z(x)$

of x. If $x \leftrightarrow (y, z)$ then $gx \leftrightarrow (gy, z)$, $g \in \mathcal{G}$. Therefore $y(x)$ is equivariant and $z(x)$ is invariant:

$$y(gx) = gy(x), \qquad z(gx) = z(x), \qquad g \in \mathcal{G}.$$

Thus $y \in \mathcal{Y}$ is called the *equivariant part* and $z \in \mathcal{Z}$ the *invariant part*.

Note that the choice of point $z_0 \in \mathcal{Z}$ is arbitrary so that we can identify all the orbits in a natural way:

$$\begin{aligned} \mathcal{G}z &\leftrightarrow \mathcal{G}/\mathcal{G}_0, \\ gz &\leftrightarrow g\mathcal{G}_0 \end{aligned}$$

for all $z \in \mathcal{Z}$.

In the discussion above, a global cross section $\mathcal{Z}$ was given first and the equivariant function y was induced by the orbital decomposition with respect to $\mathcal{Z}$; conversely, we can construct a global cross section from a given equivariant function in the following way.

Theorem 2.1 *Let a group $\mathcal{G}$ act on a space $\mathcal{Y}$ (not, a priori, a coset space) as well as on $\mathcal{X}$, and let $\tilde{y}$ be an equivariant function from $\mathcal{X}$ to $\mathcal{Y}$. Suppose that the action of $\mathcal{G}$ on $\mathcal{Y}$ is transitive and that the function $\tilde{y}$ satisfies the following condition:*

$$\tilde{y}(x) = \tilde{y}(gx) \;\Leftrightarrow\; x = gx$$

for $g \in \mathcal{G}$ and $x \in \mathcal{X}$. Then we have the following:

1. *For any $y_0 \in \mathcal{Y}$, the inverse image $\tilde{y}^{-1}(y_0) = \{x \in \mathcal{X} : \tilde{y}(x) = y_0\}$ is a global cross section.*
2. *$\{\tilde{y}^{-1}(y) : y \in \mathcal{Y}\}$ is a family of proportional global cross sections:*

$$\{\tilde{y}^{-1}(y) : y \in \mathcal{Y}\} = \{g\tilde{y}^{-1}(y_0) : g \in \mathcal{G}\} \text{ for } y_0 \in \mathcal{Y}.$$

Proof. First we prove part 1. Fix an arbitrary $y_0 \in \mathcal{Y}$ and put $\mathcal{Z} = \tilde{y}^{-1}(y_0)$. Suppose $z \in \mathcal{Z}$ and $gz \in \mathcal{Z}$ for some $g \in \mathcal{G}$. Then,

$$\tilde{y}(z) = y_0 = \tilde{y}(gz),$$

and thus $z = gz$ by the assumed condition on $\tilde{y}$. Therefore $\mathcal{Z}$ intersects each orbit at most once.

Now we show that $\mathcal{Z}$ intersects each orbit at least once. Take an arbitrary $x \in \mathcal{X}$. Then by the transitivity of the action of $\mathcal{G}$ on $\mathcal{Y}$, there exists a $g \in \mathcal{G}$ such that $g\tilde{y}(x) = y_0$. By the equivariance of $\tilde{y}$ we have $\tilde{y}(gx) = y_0$ and thus $gx \in \tilde{y}^{-1}(y_0) = \mathcal{Z}$. Therefore $\mathcal{Z}$ intersects each orbit at least once.

We have shown that $\mathcal{Z}$ is a cross section. It remains to be shown that it is global. Fix two arbitrary points $z_1, z_2 \in \mathcal{Z}$. Then, for $g \in \mathcal{G}$, we have

$$\begin{aligned} gz_1 = z_1 \quad &\Leftrightarrow \quad \tilde{y}(gz_1) = \tilde{y}(z_1) \\ &\Leftrightarrow \quad gy_0 = y_0 \\ &\Leftrightarrow \quad \tilde{y}(gz_2) = \tilde{y}(z_2) \\ &\Leftrightarrow \quad gz_2 = z_2. \end{aligned}$$

This observation shows that the isotropy subgroups are the same at all the point of $\mathcal{Z}$.

Next we verify part 2. Fix an arbitrary $y_0 \in \mathcal{Y}$. Note that $\tilde{y}^{-1}(gy_0) = g\tilde{y}^{-1}(y_0)$ for all $g \in \mathcal{G}$:

$$\begin{aligned} x \in \tilde{y}^{-1}(gy_0) \quad &\Leftrightarrow \quad \tilde{y}(x) = gy_0 \\ &\Leftrightarrow \quad \tilde{y}(g^{-1}x) = y_0 \\ &\Leftrightarrow \quad g^{-1}x \in \tilde{y}^{-1}(y_0) \\ &\Leftrightarrow \quad x \in g\tilde{y}^{-1}(y_0). \end{aligned}$$

Now by the transitivity of the action of $\mathcal{G}$ on $\mathcal{Y}$ we have

$$\begin{aligned} \{\tilde{y}^{-1}(y) : y \in \mathcal{Y}\} &= \{\tilde{y}^{-1}(gy_0) : g \in \mathcal{G}\} \\ &= \{g\tilde{y}^{-1}(y_0) : g \in \mathcal{G}\}. \end{aligned}$$

Here $\tilde{y}^{-1}(y_0)$ is a global cross section by part 1. ∎

Let $h : \mathcal{X} \to R$ be a real-valued function on $\mathcal{X}$. Given a global gross section $\mathcal{Z}$ we sometimes want to know if h depends only on y, i.e., if there exists some $s : \mathcal{Y} = \mathcal{G}/\mathcal{G}_0 \to R$ such that

$$h(x) = s(y(x)).$$

For a given $s : \mathcal{Y} \to R$ define $t : \mathcal{G} \to R$ by $t(g) = s(g\mathcal{G}_0)$. Then

$$t(gg') = t(g), \quad \forall g' \in \mathcal{G}_0, \tag{7}$$

namely t is invariant with respect to the action of $\mathcal{G}_0$ on $\mathcal{G}$ from the right. Conversely if $t : \mathcal{G} \to R$ satisfies (7) then we can define $s : \mathcal{G}/\mathcal{G}_0 = \mathcal{Y} \to R$ by $s(g\mathcal{G}_0) = t(g)$. Therefore $s : \mathcal{Y} \to R$ and $t : \mathcal{G} \to R$ satisfying (7) can be identified by the correspondence $t(g) = s(g\mathcal{G}_0)$.

Now let $g(y) : \mathcal{Y} \to \mathcal{G}$ be an arbitrary *selection*

$$g(y) \in y \in \mathcal{Y} = \mathcal{G}/\mathcal{G}_0, \tag{8}$$

that is,

$$\pi(g(y)) = y, \quad y \in \mathcal{Y}.$$

Then by (6) $x = gz = g(y(x))z(x)$. If t satisfies (7) then $s(y) = t(g(y)) : \mathcal{Y} \to R$ does not depend on the choice of the selection $g(y)$. Conversely any $s : \mathcal{Y} \to R$ can be written as $s(y) = t(g(y))$, where t is defined by $t(g) = s(g\mathcal{G}_0)$ and $g(y)$ is an arbitrary selection. The above considerations can be summarized in the following lemma.

Lemma 2.2 *Let $x \leftrightarrow (y, z)$ be the orbital decomposition for a given global cross section $\mathcal{Z}$. A function $h : \mathcal{X} \to R$ depends only on y if and only if there exists $t : \mathcal{G} \to R$ satisfying (7) such that*

$$h(x) = t(g(y(x))),$$

where $g(y)$ is an arbitrary selection.

We illustrate this lemma with the following example.

Example 2.1 *Spectral decomposition*
Consider the action of the orthogonal group $\mathcal{O}(p)$ on

$$\mathcal{X} = \{W \in PD(p) \ : \ \text{the } p \text{ roots of } W \text{ are all distinct }\},$$

where $PD(p)$ is the set of $p \times p$ positive definite matrices. The action of $\mathcal{O}(p)$ is

$$(G, W) \mapsto GWG', \quad G \in \mathcal{O}(p).$$

As a standard cross section take

$$\mathcal{Z} = \{\Lambda \ : \ \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_p), \ \lambda_1 > \cdots > \lambda_p > 0 \ \}$$

with the isotropy subgroup

$$\mathcal{G}_0 = \{\mathrm{diag}(\epsilon_1, \dots, \epsilon_p) \ : \ \epsilon_1 = \pm 1, \dots, \epsilon_p = \pm 1\}.$$

The orbital decomposition amounts to the usual spectral decomposition of W :

$$W = G\Lambda G' = G(W)\Lambda(W)G(W)', \qquad G(W) \in \mathcal{O}(p). \tag{9}$$

Note that G is unique up to the sign of each column of G corresponding to the left coset $G\mathcal{G}_0$. In order to deal with the indeterminacy of G , one usually takes G such that the elements of the first row of G are all positive. But this is just an example of selection and any other selection works equally well. Let $G(W)$ in (9) be an arbitrary selection. Then $h(W)$ is a function of $G(W)\mathcal{G}_0 \in \mathcal{Y} = \mathcal{G}/\mathcal{G}_0$ if and only if $h(W)$ can be written as

$$h(W) = t(G(W))$$

where t satisfies $t(G) = t(G\,\mathrm{diag}(\epsilon_1, \dots, \epsilon_p))$, $\epsilon_1 = \pm 1, \dots, \epsilon_p = \pm 1$.

So far we have discussed properties of a global cross section, assuming that it exists. However a global cross section does not always exist. For example, for the action of the multiplicative group R_+^* of positive real numbers on the p-dimensional Euclidean space R^p, a global cross section does not exist, and that is why in the discussion of the star-shaped distributions, the origin is omitted from the sample space R^p (Takemura and Kuriki (1996), Section 3). As a second example, take the action of the group $LT(p)$ of lower triangular matrices with positive diagonal elements on a set $\mathcal{X}$ of $n \times p$ matrices. In this case, $\mathcal{X}$ is usually taken to be the Stiefel manifold $V'_{p,n}$ of p-frames in n-space. That is, matrices of rank less than p are excluded (Takemura and Kuriki (1996), Section 4.1).

A global cross section exists if and only if all the isotropy subgroups $\mathcal{G}_x$, $x \in \mathcal{X}$, are conjugate to each other. One can confirm this easily by noting $\mathcal{G}_{gx} = g\mathcal{G}_x g^{-1}$ for $g \in \mathcal{G}$ and $x \in \mathcal{X}$. Now define the equivalence relation $\sim_{\mathcal{X}}$ in $\mathcal{X}$ by the conjugacy of the isotropy subgroups:

$$x \sim_{\mathcal{X}} x' \ \Leftrightarrow \ \mathcal{G}_x = g\mathcal{G}_{x'}g^{-1} \ \text{ for some } \ g \in \mathcal{G}. \tag{10}$$

Then even when a global cross section does not exist for the action of $\mathcal{G}$ on $\mathcal{X}$, there does exist a global cross section if we restrict our attention to the action of $\mathcal{G}$ on each equivalence class under $\sim_{\mathcal{X}}$. These equivalence classes are called *orbit types.* See Section 1.8 of Kawakubo (1991) or Section 1.4 of Bredon (1972).

Example 2.2 *Action of $LT(p)$ on $n \times p$ matrices*
Consider the action of $\mathcal{G} = LT(p)$ on the set $\mathcal{X}$ of all $n \times p$ matrices: $(T, X) \mapsto XT'$, where the prime denotes the transpose. For this action, a global cross section does not exist, and the orbit types can be explicitly given as follows.

For a given $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p) \in \mathcal{X}$, let $r = \operatorname{rank} X$ and define $(i_1, \ldots, i_r)$, $1 \le i_1 < \cdots < i_r \le p$, by

$$i_k = \min\{i : r(i) = k,\ 1 \le i \le p\}, \quad k = 1, \ldots, r,$$

where $r(i) = \operatorname{rank}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i)$. Then, $X \in \mathcal{X}$ and $\tilde{X} \in \mathcal{X}$ are equivalent $X \sim_{\mathcal{X}} \tilde{X}$ if and only if they have the same rank r and $(i_1, \ldots, i_r)$; in other words

$$\{\ \mathcal{X}(i_1, \ldots, i_r) : 0 \le r \le p,\ 1 \le i_1 < \cdots < i_r \le p\ \}$$

gives the partition of $\mathcal{X}$ into the equivalence classes under $\sim_{\mathcal{X}}$, where $\mathcal{X}(i_1, \ldots, i_r)$ is the set of $n \times p$ matrices which correspond to $(i_1, \ldots, i_r)$.

The proof of this fact is given in Appendix A.1. For other examples, see Bredon (1972), pp.42–44 and Barndorff-Nielsen, Blæsild and Eriksen (1989), pp.115–116.

Now, let us consider Theorem 2.1 when a global cross section does not exist for the whole $\mathcal{X}$. The existence of a global cross section is equivalent to the existence of $\mathcal{Y}$ and $\tilde{y}$ in Theorem 2.1. Thus, for each equivalence class under $\sim_{\mathcal{X}}$ in $\mathcal{X}$, we can find $\mathcal{Y}$ and $\tilde{y}$. We illustrate this fact with the following simple examples.

Consider the action of R_+^* on $\mathcal{X} = R^p$, which was considered in the case of the star-shaped distributions, but here the origin is not removed from R^p. In this case the equivalence classes are $\mathcal{X}_1 = R^p - \{\mathbf{0}\}$ and $\mathcal{X}_2 = \{\mathbf{0}\}$. Correspondingly let $\mathcal{Y}_1 = R_+$ be the set of positive real numbers and $\mathcal{Y}_2 = \{0\}$ and let $\tilde{y}_1(\cdot) = \tilde{y}_2(\cdot) = \|\cdot\|$ be the Euclidean norm. Then we can apply Theorem 2.1 to each triple $(\mathcal{X}_i, \mathcal{Y}_i, \tilde{y}_i)$ $(i = 1, 2)$.

As another example, consider the action of the group $\mathcal{O}(p)$ of $p \times p$ real orthogonal matrices on $\mathcal{X} = R^p$ under matrix multiplication. The equivalence classes are again $\mathcal{X}_1 = R^p - \{\mathbf{0}\}$ and $\mathcal{X}_2 = \{\mathbf{0}\}$. We can take $\mathcal{Y}_1 = S^{p-1}$ (the unit sphere in R^p), $\mathcal{Y}_2 = \{\mathbf{0}\}$, $\tilde{y}_1(\boldsymbol{x}) = (1/\|\boldsymbol{x}\|)\boldsymbol{x}$ and $\tilde{y}_2(\mathbf{0}) = \mathbf{0}$.

2.2 The class of global cross sections

One of the motivations of Takemura and Kuriki (1996) was to consider non-standard cross sections and they discussed construction of a general cross section from a standard cross section $\mathcal{Z}$ by arbitrarily moving the points of $\mathcal{Z}$ within the orbits (Takemura and Kuriki (1996), Section 2.3). However this construction does not work when the action is not free. The difficulty is that even if $\mathcal{Z}$ is a global cross section, cross section $\mathcal{Z}'$ obtained by arbitrarily moving the points of $\mathcal{Z}$ within their orbits is not necessarily global.

In this subsection we discuss construction of general global cross sections from a given global cross section and we characterize the class of global cross sections in terms of the normalizer of the common isotropy subgroup. We derive our results in a series of lemmas and summarize our results in Theorem 2.2 below.

The following lemma states how we can construct a general global cross section from a given global cross section.

Lemma 2.3 *Let $\mathcal{Z}$ be a global cross section with the common isotropy subgroup $\mathcal{G}_0$. Let $\mathcal{N}$ denote the normalizer of $\mathcal{G}_0 : \mathcal{N} = \{g \in \mathcal{G} : g\mathcal{G}_0 g^{-1} = \mathcal{G}_0\}$. Then for any coset $g\mathcal{N} \in \mathcal{G}/\mathcal{N} = \{g\mathcal{N} : g \in \mathcal{G}\}$ and any mapping $f : \mathcal{Z} \to g\mathcal{N}$, the set $\mathcal{Z}' \subset \mathcal{X}$ defined by*

$$\mathcal{Z}' = \{f(z)z : z \in Z\}$$

is a global cross section.

Proof. The set $\mathcal{Z}' = \{f(z)z : z \in \mathcal{Z}\}$ is obviously a cross section. We shall show that $\mathcal{Z}'$ is global. Note that we can write $f(z) \in g\mathcal{N}$, $z \in \mathcal{Z}$, as

$$f(z) = gn(z), \quad n(z) \in \mathcal{N}.$$

Write an arbitrary $z' \in \mathcal{Z}'$ as $z' = f(z)z = gn(z)z$, $z \in \mathcal{Z}$. Then the isotropy subgroup at z' is

$$\begin{aligned} \mathcal{G}_{z'} &= \mathcal{G}_{gn(z)z} \\ &= (gn(z))\mathcal{G}_z(gn(z))^{-1} \\ &= gn(z)\mathcal{G}_0 n(z)^{-1}g^{-1} \\ &= g\mathcal{G}_0 g^{-1}, \end{aligned}$$

which does not depend on z'. This observation shows that $\mathcal{Z}'$ is global. ∎

Converse of Lemma 2.3 can be given as follows.

Lemma 2.4 *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two global cross sections with the respective isotropy subgroups $\mathcal{G}_0$ and $\mathcal{G}_0'$. Denote by $\mathcal{N}$ and $\mathcal{N}'$ the normalizers of $\mathcal{G}_0$ and $\mathcal{G}_0'$, respectively. Then there exist cosets $g\mathcal{N} \in \mathcal{G}/\mathcal{N}$, $g'\mathcal{N}' \in \mathcal{G}/\mathcal{N}'$ and mappings $f : \mathcal{Z} \to g\mathcal{N}$, $f' : \mathcal{Z}' \to g'\mathcal{N}'$ such that*

$$\mathcal{Z}' = \{f(z)z : z \in \mathcal{Z}\}, \quad \mathcal{Z} = \{f'(z')z' : z' \in \mathcal{Z}'\}.$$

Here $g\mathcal{N}$ and $g'\mathcal{N}'$ are unique and they satisfy

$$gg' \in \mathcal{N}', \quad g'g \in \mathcal{N}.$$

Furthermore

$$\begin{aligned} f(\cdot)\mathcal{G}_0 : \mathcal{Z} &\to g(\mathcal{N}/\mathcal{G}_0) = \{gn\mathcal{G}_0 : n \in \mathcal{N}\} \\ f'(\cdot)\mathcal{G}_0' : \mathcal{Z}' &\to g'(\mathcal{N}'/\mathcal{G}_0') = \{g'n'\mathcal{G}_0' : n' \in \mathcal{N}'\} \end{aligned}$$

are uniquely determined and they are related by

$$f'(z')\mathcal{G}_0' = (f(z)\mathcal{G}_0)^{-1}$$

for z, z' on a common orbit.

Proof. Since $\mathcal{Z}'$ and $\mathcal{Z}$ are cross sections, we can write $\mathcal{Z}' = \{f(z)z : z \in \mathcal{Z}\}$ for some $f : \mathcal{Z} \to \mathcal{G}$. Fix an arbitrary point $z_0 \in \mathcal{Z}$. Since the isotropy subgroups are common at all the points of $\mathcal{Z}'$, we have $f(z)\mathcal{G}_0 f(z)^{-1} = f(z_0)\mathcal{G}_0 f(z_0)^{-1} = \mathcal{G}_0'$ for all $z \in \mathcal{Z}$. Thus $f(z_0)^{-1}f(z) \in \mathcal{N}$ for all $z \in \mathcal{Z}$. If we put $f(z_0) = g$, we have

$$f(z) \in g\mathcal{N} \text{ for all } z \in \mathcal{Z}.$$

Now we show that $g\mathcal{N}$ and $f(\cdot)\mathcal{G}_0$ are unique. Suppose there exist another $\tilde{g}\mathcal{N} \in \mathcal{G}/\mathcal{N}$ and $\tilde{f} : \mathcal{Z} \to \tilde{g}\mathcal{N}$ such that

$$\mathcal{Z}' = \{\tilde{f}(z)z : z \in \mathcal{Z}\}.$$

We want to show $f(\cdot)\mathcal{G}_0 = \tilde{f}(\cdot)\mathcal{G}_0$ and $g\mathcal{N} = \tilde{g}\mathcal{N}$. Take an arbitrary $z \in \mathcal{Z}$. Then there exists a $\tilde{z} \in \mathcal{Z}$ such that $f(z)z = \tilde{f}(\tilde{z})\tilde{z}$. But since $\mathcal{Z}$ is a cross section, we have $z = \tilde{z}$ and thus $f(z)z = \tilde{f}(z)z$. This observation implies $(\tilde{f}(z))^{-1}f(z) \in \mathcal{G}_0$ or $f(z)\mathcal{G}_0 = \tilde{f}(z)\mathcal{G}_0$ for all $z \in \mathcal{Z}$. This proves $f(\cdot)\mathcal{G}_0 = \tilde{f}(\cdot)\mathcal{G}_0$.

Next we show that $g\mathcal{N} = \tilde{g}\mathcal{N}$. From the proof above we have $f(z) \in \tilde{f}(z)\mathcal{G}_0 \subset \tilde{g}\mathcal{N}\mathcal{G}_0 = \tilde{g}\mathcal{N}$. On the other hand, $f(z) \in g\mathcal{N}$. Therefore $g\mathcal{N}$ and $\tilde{g}\mathcal{N}$ are not disjoint and hence $g\mathcal{N} = \tilde{g}\mathcal{N}$.

The same is true with the roles of $\mathcal{Z}$ and $\mathcal{Z}'$ interchanged.

Now we show $gg' \in \mathcal{N}'$ and $g'g \in \mathcal{N}$. Since $f(z)$ is in $g\mathcal{N}$, we can write $f(z) = gn(z)$ with $n(z) \in \mathcal{N}$, and as in the proof of the previous lemma we have $\mathcal{G}_0' = g\mathcal{G}_0 g^{-1}$. Similarly $\mathcal{G}_0 = g'\mathcal{G}_0' g'^{-1}$. Therefore $\mathcal{G}_0' = (gg')\mathcal{G}_0'(gg')^{-1}$ and thus $gg' \in \mathcal{N}'$. The relation $g'g \in \mathcal{N}$ can be proved similarly.

Finally we show $f'(z')\mathcal{G}_0' = (f(z)\mathcal{G}_0)^{-1}$ for z, z' on a common orbit. Since

$$z' = f(z)z = f(z)f'(z')z',$$

we have $f(z)f'(z') \in \mathcal{G}_0'$. Similarly $f'(z')f(z) \in \mathcal{G}_0$. Therefore we have

$$\begin{aligned} f'(z')\mathcal{G}_0' &= f(z)^{-1}\mathcal{G}_0' \\ &= f(z)^{-1}g\mathcal{G}_0 g^{-1} \\ &= \mathcal{G}_0 f(z)^{-1}gg^{-1} \\ &= \mathcal{G}_0 f(z)^{-1} \\ &= (f(z)\mathcal{G}_0)^{-1}, \end{aligned}$$

in which the third equality follows from $f(z)^{-1}g \in \mathcal{N}$. ∎

It is useful to explicitly write down how the equivariant part transforms by the construction of Lemma 2.3. Let $x \leftrightarrow (y, z)$ be the orbital decomposition with respect to $\mathcal{Z}$ and let $x \leftrightarrow (y', z')$ be the orbital decomposition with respect to

$$\mathcal{Z}' = \{f(z)z : z \in \mathcal{Z}\} = \{g_0 n(z)z : z \in \mathcal{Z}\},$$

where $n(z) \in \mathcal{N}$. For simplicity we only consider the case $g_0 = e$, the identity element of $\mathcal{G}$. Actually there is no essential loss of generality in assuming $g_0 = e$. Then $\mathcal{Z}'$ has the same isotropy subgroup $\mathcal{G}_0$ as $\mathcal{Z}$. Now we have the following lemma.

Lemma 2.5 *Let* $\mathcal{Z}' = \{n(z)z : z \in \mathcal{Z}\}$ *and let* $x \leftrightarrow (y', z')$ *be the orbital decomposition with respect to* $\mathcal{Z}'$ *. Write* $x \in \mathcal{X}$ *as* $x = gz = g'z'$, $z \in \mathcal{Z}, z' \in \mathcal{Z}', \pi(g) = g\mathcal{G}_0 = y, \pi(g') = g'\mathcal{G}_0 = y'$ *. Then*

$$y' = g'\mathcal{G}_0 = g\mathcal{G}_0 n(z)^{-1} = yn(z)^{-1}. \tag{11}$$

Proof. Substituting $z' = n(z)z$ into $x = gz = g'z'$, we have $gz = g'n(z)z$. Hence $g\mathcal{G}_0 = g'n(z)\mathcal{G}_0 = g'\mathcal{G}_0 n(z)$ since $n(z) \in \mathcal{N}$. Multiplying by $n(z)^{-1}$ from the right we obtain the lemma. ∎

Concerning the arbitrary selection discussed in Lemma 2.2, Lemma 2.5 implies the following. Let $g(y) \in y = g\mathcal{G}_0$ be an arbitrary selection in the orbital decomposition $x = g(y)z \leftrightarrow (y, z)$. Define

$$g'(x) = g(y)n(z)^{-1}, \quad y = y(x),\ z = z(x). \tag{12}$$

Then $g'(x) = g(y)n(z)^{-1} \in g\mathcal{G}_0 n(z)^{-1} = y'$ and $g'(x)$ is again a "selection" for $x \leftrightarrow (y', z')$. We make use of this fact in Example 3.3.

The following lemma captures the interchangeability of the roles of $\mathcal{Z}$ and $\mathcal{Z}'$ in the form of the conjugacy of their isotropy subgroups and normalizers.

Lemma 2.6 *Under the assumptions of Lemma 2.4,* $\mathcal{G}_0$ *and* $\mathcal{G}'_0$ *are conjugate and so are* $\mathcal{N}$ *and* $\mathcal{N}'$:

$$\mathcal{G}'_0 = g\mathcal{G}_0 g^{-1}, \quad \mathcal{N}' = g\mathcal{N}g^{-1},$$

where g *is the* $g \in g\mathcal{N}$ *in Lemma 2.4. Moreover the factor groups* $\mathcal{N}/\mathcal{G}_0$ *and* $\mathcal{N}'/\mathcal{G}'_0$ *are isomorphic.*

Proof. We have already noted $\mathcal{G}'_0 = g\mathcal{G}_0 g^{-1}$ in the proof of Lemma 2.4. Using this relation, we can show $\mathcal{N}' = g\mathcal{N}g^{-1}$ as follows: For any $n \in \mathcal{N}$, we have

$$\begin{aligned}(gng^{-1})\mathcal{G}'_0(gng^{-1})^{-1} &= gn\mathcal{G}_0 n^{-1}g^{-1}\\ &= g\mathcal{G}_0 g^{-1}\\ &= \mathcal{G}'_0\end{aligned}$$

and thus $gng^{-1} \in \mathcal{N}'$. This observation shows $g\mathcal{N}g^{-1} \subset \mathcal{N}'$. The reverse inclusion is proved similarly. Thus the proof of $\mathcal{N}' = g\mathcal{N}g^{-1}$ is completed.

Now we show that factor groups $\mathcal{N}/\mathcal{G}_0$ and $\mathcal{N}'/\mathcal{G}'_0$ are isomorphic. By $\mathcal{N}' = g\mathcal{N}g^{-1}$ and $\mathcal{G}'_0 = g\mathcal{G}_0 g^{-1}$, we have

$$\begin{aligned}\mathcal{N}'/\mathcal{G}'_0 &= g\mathcal{N}g^{-1}/g\mathcal{G}_0 g^{-1}\\ &= \{(gng^{-1})g\mathcal{G}_0 g^{-1} : n \in \mathcal{N}\}\\ &= \{g(n\mathcal{G}_0)g^{-1} : n \in N\}.\end{aligned}$$

Now the mapping

$$n\mathcal{G}_0 \mapsto g(n\mathcal{G}_0)g^{-1}$$

gives an isomorphism from $\mathcal{N}/\mathcal{G}_0$ to $\mathcal{N}'/\mathcal{G}'_0$. ∎

Let $\mathcal{Z}$ be a global cross section with the common isotropy subgroup $\mathcal{G}_0$, and denote by $\mathcal{N}$ the normalizer of $\mathcal{G}_0$. Then we have by Lemmas 2.3 and 2.4 that $\mathcal{Z}'$ is a global cross section if and only if $\mathcal{Z}'$ is of the form

$$\mathcal{Z}' = \{g_0 n(z) z : z \in \mathcal{Z}\} \tag{13}$$

for some $g_0 \in \mathcal{G}$ and $n : \mathcal{Z} \to \mathcal{N}$. We give two examples of this construction.

Example 2.3 *Orthogonal Group*

Consider the action of $\mathcal{G} = \mathcal{O}(p)$ on $\mathcal{X} = R^p - \{0\}$, $p \geq 2$, by the usual multiplications of matrices and vectors: $(C, x) \mapsto Cx$. This action is not free but there exists a global cross section. A natural global cross section is $\mathcal{Z} = \{(x_1, 0, \ldots, 0)' : x_1 > 0\}$ with the common isotropy subgroup

$$\mathcal{G}_0 = \{\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \tilde{C} \end{pmatrix} : \tilde{C} \in \mathcal{O}(p-1)\}, \quad \mathbf{0} = (0, \ldots, 0)' \in R^{p-1}. \tag{14}$$

Using $\mathcal{Z}$ as a building block, we obtain general global cross sections $\mathcal{Z}'$ in the following way: Since the normalizer $\mathcal{N}$ of $\mathcal{G}_0$ is

$$\mathcal{N} = \{\begin{pmatrix} \epsilon & \mathbf{0}' \\ \mathbf{0} & \tilde{C} \end{pmatrix} : \epsilon = \pm 1, \ \tilde{C} \in \mathcal{O}(p-1)\},$$

we have by (13) that $\mathcal{Z}' \subset \mathcal{X}$ is a global cross section iff it is of the form

$$\begin{aligned} \mathcal{Z}' &= \{C_0 \begin{pmatrix} \epsilon(x_1) & \mathbf{0}' \\ \mathbf{0} & \tilde{C}(x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ \mathbf{0} \end{pmatrix} : x_1 > 0\} \\ &= \{C_0 \begin{pmatrix} \epsilon(x_1) x_1 \\ \mathbf{0} \end{pmatrix} : x_1 > 0\}, \end{aligned} \tag{15}$$

with $C_0 \in \mathcal{O}(p)$ and $\epsilon(x_1) = \pm 1$, $\tilde{C}(x_1) \in \mathcal{O}(p-1)$ $(x_1 > 0)$. In other words, a cross section is a global one iff it is contained in a line through the origin, which is to be expected.

The above example is somewhat trivial. The next example is more substantial from statistical viewpoint.

Example 2.4 *Two-sample Wishart problem*

Consider the action related to the two-sample Wishart problem. Let $\mathcal{G} = GL(p)$ be the real general linear group and let

$$\begin{aligned} \mathcal{X} = \{ & (W_1, W_2) \in PD(p) \times PD(p) : \\ & \text{the } p \text{ roots of } \det(W_1 - \lambda(W_1 + W_2)) = 0 \text{ are all distinct } \}. \end{aligned} \tag{16}$$

The action is

$$(B, (W_1, W_2)) \mapsto (BW_1B', BW_2B').$$

If we take

$$\mathcal{Z} = \{(\Lambda, I - \Lambda) \ : \ \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p), \ 1 > \lambda_1 > \cdots > \lambda_p > 0\}, \tag{17}$$

we have

$$\mathcal{G}_0 = \{\mathrm{diag}(\epsilon_1, \ldots, \epsilon_p) \ : \ \epsilon_1 = \pm 1, \ldots, \epsilon_p = \pm 1\}$$

and

$$\mathcal{N} = \{P \in GL(p) \quad : \quad P \text{ has exactly one nonzero element in each row and in each column}\}.$$

The normalizer $\mathcal{N}$ is the group generated by permutation matrices and nonsingular diagonal matrices. A subset $\mathcal{Z}' \subset \mathcal{X}$ is a global cross section iff it is of the form

$$\{ \ (\ BP(\Lambda)\Lambda P(\Lambda)'B', \ BP(\Lambda)(I - \Lambda)P(\Lambda)'B' \ ) : \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p), \ 1 > \lambda_1 > \cdots > \lambda_p > 0 \ \}, \tag{18}$$

with $B \in GL(p)$, $P(\Lambda) \in \mathcal{N}$, where I is the identity matrix.

Although Lemmas 2.3–2.6 characterize the class of global sections, their statements are somewhat lengthy. By introducing appropriate equivalence relations, we can state the results of these lemmas more succinctly.

Note that the g_0 in (13) is not essential since $\mathcal{Z}'$ and $g_0^{-1}\mathcal{Z}'$ are proportional and thus induce the same family of proportional global cross sections:

$$\{g\mathcal{Z}' : g \in \mathcal{G}\} = \{g(g_0^{-1}\mathcal{Z}') : g \in \mathcal{G}\}.$$

Moreover n in (13) is essentially unique. These two points are clearly stated if we introduce the following equivalence relations.

We define the equivalence relation $\sim$ among the global cross sections by proportionality:

$$\mathcal{Z} \sim \mathcal{Z}' \ \Leftrightarrow \ \exists g \in \mathcal{G} : \mathcal{Z} = g\mathcal{Z}'.$$

Now for any global cross section $\mathcal{Z}$ with the isotropy subgroup $\mathcal{G}_0$ and its normalizer $\mathcal{N}$, denote the factor group $\mathcal{N}/\mathcal{G}_0$ by $\mathcal{M}_\mathcal{Z}$. $\mathcal{M}_\mathcal{Z}$ for different global cross sections $\mathcal{Z}$ are isomorphic to each other in view of Lemma 2.6. Therefore we can identify $\mathcal{M}_\mathcal{Z}$ with a particular factor group $\mathcal{M}$. Let $\tilde{n} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ be a mapping from the orbit space to $\mathcal{M}$. We introduce an equivalence relation $\sim_\mathcal{M}$ among the mappings $\tilde{n} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ by

$$\tilde{n} \sim_\mathcal{M} \tilde{n}' \ \Leftrightarrow \ \exists m \in \mathcal{M} \quad \forall \mathcal{G}x : \tilde{n}(\mathcal{G}x) = m\tilde{n}'(\mathcal{G}x). \tag{19}$$

We shall denote the equivalence classes under $\sim$ and $\sim_\mathcal{M}$ by $[\ \cdot\ ]$ and $[\ \cdot\ ]_\mathcal{M}$, respectively.

Fix an arbitrary global cross section $\mathcal{Z}$. Abusing the notation, we simply write $\tilde{n}(z)$ instead of $\tilde{n}(\iota_\mathcal{Z}^{-1}(z))$, where $\iota_\mathcal{Z} : \mathcal{X}/\mathcal{G} \to \mathcal{Z}$ is the one-to-one correspondence between the orbit space $\mathcal{X}/\mathcal{G}$ and $\mathcal{Z}$. Then we can write $\tilde{n}(z) = n(z)\mathcal{G}_0$, $z \in \mathcal{Z}$, for some $n : \mathcal{Z} \to \mathcal{N}$. Thus

$$\tilde{n}(z)z = n(z)\mathcal{G}_0 z = n(z)z, \quad z \in \mathcal{Z}.$$

If we denote $\{\tilde{n}(z)z : z \in \mathcal{Z}\}$ by $\tilde{n}\mathcal{Z}$, we have

$$\tilde{n}\mathcal{Z} = \{n(z)z : z \in \mathcal{Z}\}.$$

We are now in a position to summarize the results of this subsection in the following theorem.

Theorem 2.2 *Let $\mathcal{Z}, \mathcal{Z}'$ be global cross sections with the associated isotropy subgroups $\mathcal{G}_0, \mathcal{G}_0'$ and their normalizers $\mathcal{N}, \mathcal{N}'$, respectively. Then there exists a mapping $\tilde{n} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ such that $[\mathcal{Z}'] = [\tilde{n}\mathcal{Z}]$. Here $[\tilde{n}]_{\mathcal{M}}$ is unique, that is, if $[\mathcal{Z}'] = [\tilde{n}_1\mathcal{Z}] = [\tilde{n}_2\mathcal{Z}]$ then $[\tilde{n}_1]_{\mathcal{M}} = [\tilde{n}_2]_{\mathcal{M}}$. Furthermore if $\mathcal{Z}' = \tilde{n}\mathcal{Z}$ and $\mathcal{Z} = \tilde{m}\mathcal{Z}'$, then $\tilde{m}\tilde{n} = \tilde{e}$, where the mapping $\tilde{m}\tilde{n} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ is $\mathcal{G}z \mapsto \tilde{m}(\mathcal{G}z)\tilde{n}(\mathcal{G}z)$ (multiplication in $\mathcal{M}$) and $\tilde{e} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ is $\tilde{e}(\mathcal{G}z) \equiv \tilde{e} = \mathcal{G}_0$ (the identity element of $\mathcal{M}$).*

Proof. As was mentioned earlier, there exist $g_0 \in \mathcal{G}$ and $n : \mathcal{Z} \to \mathcal{N}$ such that (13) holds. For this n, we define $\tilde{n} : \mathcal{X}/\mathcal{G} \to \mathcal{M}$ by $\tilde{n}(z) = n(z)\mathcal{G}_0$, $z \in \mathcal{Z}$. Then we have

$$\tilde{n}\mathcal{Z} = \{n(z)z : z \in \mathcal{Z}\} = g_0^{-1}\mathcal{Z}',$$

and hence $[\tilde{n}\mathcal{Z}] = [\mathcal{Z}']$.

Next we show the uniqueness of $[\tilde{n}]_{\mathcal{M}}$. Suppose $[\mathcal{Z}'] = [\tilde{n}_1\mathcal{Z}] = [\tilde{n}_2\mathcal{Z}]$ for $\tilde{n}_1, \tilde{n}_2 : \mathcal{X}/\mathcal{G} \to \mathcal{M}$. Then we have

$$\tilde{n}_1\mathcal{Z} = g(\tilde{n}_2\mathcal{Z}) \tag{20}$$

for some $g \in \mathcal{G}$. If we write $\tilde{n}_1$ and $\tilde{n}_2$ as $\tilde{n}_1(\cdot) = n_1(\cdot)\mathcal{G}_0$ and $\tilde{n}_2(\cdot) = n_2(\cdot)\mathcal{G}_0$, respectively, with $n_1 : \mathcal{Z} \to \mathcal{N}$, $n_2 : \mathcal{Z} \to \mathcal{N}$, we have by (20) that

$$\{n_1(z)z : z \in \mathcal{Z}\} = \{gn_2(z)z : z \in \mathcal{Z}\}. \tag{21}$$

Since $\mathcal{Z}$ is a cross section, $n_1(z)z = gn_2(z)z$, $z \in \mathcal{Z}$. therefore $n_1(z)\mathcal{G}_0 = gn_2(z)\mathcal{G}_0$ or

$$\tilde{n}_1(z) = g\tilde{n}_2(z), \qquad z \in \mathcal{Z}. \tag{22}$$

Now $n_1(z)^{-1}gn_2(z) \in \mathcal{G}_0$ and $g \in n_1(z)\mathcal{G}_0 n_2(z)^{-1} \subset \mathcal{N}$. Therefore $m = g\mathcal{G}_0$ belongs to $\mathcal{M} = \mathcal{N}/\mathcal{G}_0$. Furthermore $g\mathcal{G}_0 n_2(z)z = gn_2(z)\mathcal{G}_0 z = gn_2(z)z$. This implies that (22) can be written as $\tilde{n}_1(z) = m\tilde{n}_2(z)$ and we have $[\tilde{n}_1]_{\mathcal{M}} = [\tilde{n}_2]_{\mathcal{M}}$.

Finally, the last assertion is just a particular case of Lemma 2.4. ∎

3 The decomposable distributions

Takemura and Kuriki (1996) defined the cross-sectionally contoured distributions for free actions and obtained distributional results about the invariant and equivariant parts. In this section we generalize their results to the case where the group does not act freely. We also generalize their discussion in another direction: we define the decomposable distributions, which include the cross-sectionally contoured distributions as a special case.

We deal with the situation at the beginning of Section 2.1:

$$\begin{array}{ccl} \mathcal{X} & \leftrightarrow & \mathcal{Y} \times \mathcal{Z}, \\ x & \leftrightarrow & (y,z), \quad x = gz, \quad y = \pi(g) \in \mathcal{Y} = \mathcal{G}/\mathcal{G}_0. \end{array}$$

In order to discuss distributional results, we need to assume some regularity conditions. In this paper integrals and measures are interpreted in the sense of Section 6.3 of Wijsman (1990). Here we make regularity conditions on the topologies of the relevant spaces.

Assumption 3.1

1. *$\mathcal{X}$ is a locally compact Hausdorff space.*
2. *$\mathcal{G}$ is a second countable, locally compact Hausdorff topological group acting continuously on $\mathcal{X}$.*
3. *$\mathcal{G}_0$ is compact.*

We agree that a quotient space receive the quotient topology when regarded as a topological space. This applies to $\mathcal{Y} = \mathcal{G}/\mathcal{G}_0$ as well as to the orbit space $\mathcal{X}/\mathcal{G}$. Because of 2 of Assumption 3.1 there exists a left invariant measure $\mu_{\mathcal{G}}$ on $\mathcal{G}$, which is unique up to a multiplicative constant. For most of our theoretical discussion we do not need to specify the multiplicative constant. As in Takemura and Kuriki (1996) we consider densities with respect to a dominating measure λ which is relatively invariant with multiplier χ:

$$\lambda(d(gx)) = \chi(g)\lambda(dx), \quad g \in \mathcal{G}.$$

Concerning cross sections, we make the following assumption.

Assumption 3.2 *There exists a global cross section $\mathcal{Z}$ such that the bijection $x \leftrightarrow (y,z)$ is a homeomorphism, where the topology on $\mathcal{Z}$ is the relative topology of $\mathcal{Z}$ as a subset of $\mathcal{X}$.*

Note that under Assumptions 3.1 and 3.2, $\mathcal{Z}$ is a closed subset of $\mathcal{X}$ and is thus locally compact.

Actually we want to deal with an arbitrary measurable global cross section for which $x \leftrightarrow (y,z)$ is not necessarily a homeomorphism. A global cross section $\mathcal{Z}$ is said to be measurable if it is a measurable subset of $\mathcal{X}$. In Appendix A.2 we argue that Assumption 3.2 concerning a standard global cross section is sufficient to guarantee the factorization of the relatively invariant measure in (23) below for an arbitrary measurable global cross section as well.

We are now in a position to define the decomposable distributions.

Definition 3.1 *A distribution on $\mathcal{X}$ is said to be decomposable with respect to a global cross section $\mathcal{Z}$ iff it is of the form*

$$f(x)\lambda(dx) = f_{\mathcal{Y}}(y(x)) f_{\mathcal{Z}}(z(x))\lambda(dx).$$

In particular it is said to be cross-sectionally (resp. orbitally) contoured if $f_{\mathcal{Z}}(z) \equiv 1$ (resp. $f_{\mathcal{Y}}(y) \equiv 1$).

By $\mathcal{F}_{\mathcal{G}}$ we denote the family of decomposable distributions with respect to all possible global cross sections $\mathcal{Z}$ and relatively invariant measures λ.

Obviously a distribution $f(x)\lambda(dx)$ is cross-sectionally contoured iff $f(x)$ is constant on each proportional global cross section $g\mathcal{Z}$, $g \in \mathcal{G}$. Similarly, $f(x)\lambda(dx)$ is orbitally contoured iff $f(x)$ is constant on each orbit $\mathcal{G}x$, $x \in \mathcal{X}$.

Before we examine the distributions of the invariant and equivariant parts in Section 3.1, we observe the following two points.

First, a decomposable distribution $f_{\mathcal{Y}}(y(x))f_{\mathcal{Z}}(z(x))\lambda(dx)$ could always be thought of as a cross-sectionally contoured distribution:

$$f_{\mathcal{Y}}(y(x))f_{\mathcal{Z}}(z(x))\lambda(dx) = f_{\mathcal{Y}}(y(x))\tilde{\lambda}(dx)$$

with $\tilde{\lambda}(dx) = f_{\mathcal{Z}}(z(x))\lambda(dx)$.

Next, we can take various global cross sections, including non-standard ones (Section 2.2), and thus we can consider the cross-sectionally contoured distributions associated with different kinds of global cross sections. On the other hand, once an action is given, there is no room for choosing the orbits; the orbits are determined by the action in question and usually those orbits are standard subsets of $\mathcal{X}$. Hence we cannot produce the orbitally contoured distributions based on the orbits which are unfamiliar subsets of $\mathcal{X}$. For those reasons, we shall be concerned primarily with the cross-sectionally contoured distributions.

3.1 Distributions of the invariant and equivariant parts

For the reason previously stated, without loss of generality, we deal with the cross-sectionally contoured distributions in this subsection. We shall see that the distributional results corresponding to Theorems 2.1, 2.2 and 2.3 of Takemura and Kuriki (1996) remain to hold for non-free actions.

First we confirm the independence of the invariant and equivariant parts.

Thanks to the assumption that $\mathcal{G}_0$ is compact, we can induce a measure $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$ by

$$\mu_{\mathcal{Y}}(B) = \mu_{\mathcal{G}}(\pi^{-1}(B)), \quad B \subset \mathcal{Y}$$

(Wijsman (1990), 2.3.5. Proposition, 7.4.4. Corollary). Also, by the same assumption we can define $\bar{\chi}(y)$, $y \in \mathcal{Y}$, by

$$\bar{\chi}(y) = \chi(g) \quad \text{for } y = \pi(g),$$

where χ is the multiplier of λ. With some abuse of notation, we shall write $\chi(y)$ for $\bar{\chi}(y)$.

Now since $\mathcal{G}/\mathcal{G}_0$ and $\mathcal{Z}$ in Assumption 3.2 are locally compact Hausdorff and the action of $\mathcal{G}$ on $\mathcal{G}/\mathcal{G}_0$ is proper (Wijsman (1990), 2.3.11. Proposition), by Theorem 7.5.1 of Wijsman (1990) $\lambda(dx)$ is factored as

$$\lambda(dx) = \chi(y)\mu_{\mathcal{Y}}(dy)\nu_{\mathcal{Z}}(dz), \tag{23}$$

where $\nu_{\mathcal{Z}}(dz)$ is a probability measure on $\mathcal{Z}$. Although Theorem 7.5.1 of Wijsman (1990) only covers the case of a standard $\mathcal{Z}$ in Assumption 3.2, the factorization actually holds also for arbitrary measurable global cross sections as discussed in Appendix A.2.

Theorem 3.1 *Suppose that x is distributed according to a cross-sectionally contoured distribution $f_{\mathcal{Y}}(y(x))\lambda(dx)$. Then we have:*

1. *$y = y(x)$ and $z = z(x)$ are independently distributed.*
2. *The distribution of z does not depend on $f_{\mathcal{Y}}$.*
3. *The distribution of y is $f_{\mathcal{Y}}(y)\chi(y)\mu_{\mathcal{Y}}(dy)$.*

Given the factorization of $\lambda(dx)$ in (23), the proof is straightforward.

Now consider the situation where there are more than one orbit types. As was mentioned in Section 2.1, there does exist a global cross section for each orbit type; furthermore, given a particular orbit type, the invariant and equivariant parts are conditionally independent in the following sense.

We assume that the number of the orbit types are at most countable. It is known that if $\mathcal{G}$ is compact then the number of the orbit types is actually finite (see Bredon (1972) Sec.4.1). Let $\{\mathcal{X}_i : i \geq 1\}$ be the partition of $\mathcal{X}$ into the orbit types. By restricting the action $(\mathcal{G}, \mathcal{X})$ of $\mathcal{G}$ on $\mathcal{X}$, we obtain the action $(\mathcal{G}, \mathcal{X}_i)$ of $\mathcal{G}$ on each $\mathcal{X}_i$, $i \geq 1$. For each $i \geq 1$, let $\mathcal{Z}_i$ be a global cross section for $(\mathcal{G}, \mathcal{X}_i)$, and denote by $\mathcal{G}_i$ the common isotropy subgroup at the points of $\mathcal{Z}_i$. Then for each $i \geq 1$, we have the orbital decomposition of $\mathcal{X}_i$:

$$
\begin{aligned}
&\mathcal{X}_i \leftrightarrow \mathcal{Y}_i \times \mathcal{Z}_i, \\
&x_i \leftrightarrow (y_i, z_i), \qquad x_i = g_i z_i, \quad y_i = g_i \mathcal{G}_i \in \mathcal{Y}_i = \mathcal{G}/\mathcal{G}_i.
\end{aligned}
$$

Write $\mathcal{Y} = \bigcup_i \mathcal{Y}_i$ and $\mathcal{Z} = \bigcup_i \mathcal{Z}_i$, and define the functions $y : \mathcal{X} \to \mathcal{Y}$ and $z : \mathcal{X} \to \mathcal{Z}$ by

$$
y(x) = y_i(x), \quad z(x) = z_i(x) \ \text{ if } \ x \in \mathcal{X}_i, \quad i \geq 1. \tag{24}
$$

Note that $\mathcal{Z}$ is a cross section for $(\mathcal{G}, \mathcal{X})$.

Concerning topological questions, we assume 1, 3 of Assumption 3.1 and Assumption 3.2 with $\mathcal{X}$, $\mathcal{G}_0$, $\mathcal{Z}$, $x \leftrightarrow (y, z)$ replaced by $\mathcal{X}_i$, $\mathcal{G}_i$, $\mathcal{Z}_i$, $x_i \leftrightarrow (y_i, z_i)$, respectively. On $\mathcal{X}_i$ we consider a dominating measure λ_i which is relatively invariant with multiplier χ_i. We note that by 2 of Assumption 3.1, $\mathcal{G}$ is metrizable (Ash (1972), A5.16 Theorem). We regard the elements of $\mathcal{Y}$ as subsets of $\mathcal{G}$. By endowing $\mathcal{Y}$ with the Hausdorff distance, we make $\mathcal{Y}$ a metric space. For details see Appendix A.3.

Let $\lambda(dx) = \sum_i I_{\mathcal{X}_i}(x)\lambda_i(dx)$, where $I_{\mathcal{X}_i}$ is the indicator function of $\mathcal{X}_i$. Note that λ is not necessarily a relatively invariant measure. Now suppose that x is distributed according to

$$
f_{\mathcal{Y}}(y(x))\lambda(dx) \tag{25}
$$

for some $f_{\mathcal{Y}} : \mathcal{Y} \to R$. Here we assume $\int_{\mathcal{X}_i} f_{\mathcal{Y}}(y(x))\lambda(dx) > 0$ for each $i \geq 1$. Under these conditions it is easy to show that for each $i \geq 1$,

$$
P(y(x) \in A, \ z(x) \in B \mid x \in \mathcal{X}_i) = P(y(x) \in A \mid x \in \mathcal{X}_i) P(z(x) \in B \mid x \in \mathcal{X}_i) \tag{26}
$$

for each measurable $A \subset \mathcal{Y}_i$ and $B \subset \mathcal{Z}_i$. Therefore $y(x)$ and $z(x)$ are conditionally independent given $\mathcal{X}_i$. An example of the conditional independence will be given in Example 3.4.

Now let us go back to the situation where a global cross section $\mathcal{Z}$ exists. We shall confirm the results corresponding to Theorems 2.2 and 2.3 of Takemura and Kuriki (1996).

Let $\mathcal{Z}'$ be another measurable global cross section. Suppose without essential loss of generality that $\mathcal{Z}'$ is taken in such a way that the common isotropy subgroup at the points of $\mathcal{Z}'$ is the same as that at the points of $\mathcal{Z}$:

$$\mathcal{G}_{z'} = \mathcal{G}_z = \mathcal{G}_0, \quad z' \in \mathcal{Z}',\ z \in \mathcal{Z}.$$

Denote the invariant and equivariant parts with respect to $\mathcal{Z}'$ by $z' = z'(x)$ and $y' = y'(x)$, respectively:

$$\begin{array}{ccl} \mathcal{X} & \leftrightarrow & \mathcal{Y} \times \mathcal{Z}', \\ x & \leftrightarrow & (y', z'), \quad x = g'z', \quad y' = \pi(g') = g'\mathcal{G}_0. \end{array}$$

Now denote by $g(y)$ an arbitrary selection in (8). From now on, we shall write $g(x)$ for $g(y(x))$:

$$x = g(x)z(x), \quad x \in \mathcal{X}.$$

Define $g'(x)$ in a similar way:

$$x = g'(x)z'(x), \quad x \in \mathcal{X}.$$

Now consider the map $w : \mathcal{X} \to \mathcal{X}$ defined by

$$w = w(x) = g(x)z'(x), \quad x \in \mathcal{X}, \tag{27}$$

Note that since $\mathcal{G}_{z'} = \mathcal{G}_0$, $z' \in \mathcal{Z}'$, w does not depend on the choice of the selection $g(y)$. Furthermore noting that x and $w(x)$ are on the same orbit, we call w *within-orbit bijection.* The within-orbit bijection is a basic tool for deriving a new cross-sectionally contoured distribution from a given cross-sectionally contoured distribution.

Theorem 3.2 *Suppose that x is distributed according to a cross-sectionally contoured distribution $f_{\mathcal{Y}}(y(x))\lambda(dx)$. Then the distribution of $w = w(x)$ is*

$$f_{\mathcal{Y}}(y'(w))\chi(g(w)^{-1}g'(w))\Delta^{\mathcal{G}}(g(w)^{-1}g'(w))\lambda(dw),$$

where $\Delta^{\mathcal{G}}$ is the right-hand modulus of $\mathcal{G}$:

$$\mu_{\mathcal{G}}(d(gg_1)) = \Delta^{\mathcal{G}}(g_1)\mu_{\mathcal{G}}(dg), \quad g_1 \in \mathcal{G}.$$

Proof. First note that

$$\begin{aligned} w &= g(x)z'(x) \\ &= g(x)g'(x)^{-1}x \\ &= g(x)g'(z(x))^{-1}z(x). \end{aligned}$$

Now we regard $w = w(x)$ as a function of $y = y(x)$ and $z = z(x)$: $w = w(x) = w(y, z)$. Noting that the integration over $\mathcal{Y}$ can be carried out by the integration over $\mathcal{G}$, we have for an arbitrary measurable subset $B \subset \mathcal{X}$ that

$$\begin{aligned} P(w \in B) &= \int_{\mathcal{X}} I_B(w(x)) f_{\mathcal{Y}}(y(x)) \lambda(dx) \\ &= \int_{\mathcal{Z}} \int_{\mathcal{Y}} I_B(w(y,z)) f_{\mathcal{Y}}(y) \chi(y) \mu_{\mathcal{Y}}(dy) \nu_{\mathcal{Z}}(dz) \\ &= \int_{\mathcal{Z}} \int_{\mathcal{G}} I_B(g g'(z)^{-1} z) \hat{f}_{\mathcal{Y}}(g) \chi(g) \mu_{\mathcal{G}}(dg) \nu_{\mathcal{Z}}(dz), \end{aligned}$$

where $\hat{f}_{\mathcal{Y}} = f_{\mathcal{Y}} \circ \pi$. The rest can be shown by the same calculation as in the proof of Theorem 2.2 of Takemura and Kuriki (1996). ∎

For notational simplicity write

$$\Delta(g) = \chi(g)\Delta^{\mathcal{G}}(g), \quad g \in \mathcal{G},$$

which is a continuous homomorphism from $\mathcal{G}$ to R_+^* . Because of 3 of Assumption 3.1 $\Delta(g) = 1$ for all $g \in \mathcal{G}_0$ and $\Delta(g(w)^{-1}g'(w))$ does not depend on the choice of the selections $g(w)$ and $g'(w)$. Also note as in Section 2.3 of Takemura and Kuriki (1996) that

$$\tilde{\lambda}(dw) = \Delta(g(w)^{-1}g'(w))\lambda(dw) \tag{28}$$

is relatively invariant with multiplier χ and that w is distributed with cross-sectionally contoured density $f_{\mathcal{Y}}(y'(w))$ with respect to $\tilde{\lambda}(dw)$.

Now we turn to the distribution of z' . Corresponding to the orbital decomposition with respect to $\mathcal{Z}'$, the measure $\lambda(dx)$ is factored as

$$\lambda(dx) = c\chi(y')\mu_{\mathcal{Y}}(dy')\nu_{\mathcal{Z}'}(dz'),$$

where $\nu_{\mathcal{Z}'}$ is a (not necessarily probability) measure on $\mathcal{Z}'$, and $c = 1/\int_{\mathcal{Y}} f_{\mathcal{Y}}(y')\chi(y')\mu_{\mathcal{Y}}(dy')$. In terms of $\nu_{\mathcal{Z}'}$, the distribution of z' is written as follows.

Theorem 3.3 *Suppose that x is distributed according to a cross-sectionally contoured distribution $f_{\mathcal{Y}}(y(x))\lambda(dx)$. Then the distribution of $z' = z'(x)$ is $\Delta(g(z'))^{-1}\nu_{\mathcal{Z}'}(dz')$.*

Proof. We have $\Delta(g(w)^{-1}g'(w)) = \Delta(g(z'(w)))^{-1}$. Writing $y' = y'(w)$ and $z' = z'(w)$, we have by Theorem 3.2 that the distribution of w is

$$\begin{aligned} f_{\mathcal{Y}}(y'(w))\Delta(g(z'(w)))^{-1}\lambda(dw) &= f_{\mathcal{Y}}(y')\Delta(g(z'))^{-1}c\chi(y')\mu_{\mathcal{Y}}(dy')\nu_{\mathcal{Z}'}(dz') \\ &= c f_{\mathcal{Y}}(y')\chi(y')\mu_{\mathcal{Y}}(dy')\Delta(g(z'))^{-1}\nu_{\mathcal{Z}'}(dz'). \end{aligned}$$

Accordingly, the distribution of $z' = z'(w)$ is

$$\Delta(g(z'))^{-1}\nu_{\mathcal{Z}'}(dz').$$

Since x and $w = g(x)g'(x)^{-1}x$ are on the same orbit, we have $z'(x) = z'(w)$ so that the distribution of $z'(x)$ is the same as that of $z'(w)$. ∎

3.2 Examples of the decomposable distributions

In this subsection we give some examples of the decomposable distributions.

Example 3.1 *Rotations in* R^2

Let $\mathcal{G} = \mathcal{O}^+(2)$ be the group of 2×2 proper orthogonal matrices acting on $\mathcal{X} = R^2 - \{\mathbf{0}\}$ by matrix multiplication. Note that unlike the actions of $\mathcal{O}^+(p)$, $p \geq 3$, the action of $\mathcal{O}^+(2)$ is free and this fact leads to the following class of decomposable distributions. Also, if we include reflections and consider the action of $\mathcal{O}(2)$, then by Example 2.3 the action is not free and the following construction does not work.

A general cross section $\mathcal{Z}$ is of the form

$$\mathcal{Z} = \{\ z\ :\ z = z(t) = (\, t\cos\eta(t), t\sin\eta(t)\,),\ t > 0\ \},\quad \eta : R_+ = (0,\infty) \to [0, 2\pi),$$

For a given η, cross section $\mathcal{Z}$ is parameterized by $t \in R_+$ and we identify $\mathcal{Z}$ with R_+ by $z(t) \leftrightarrow t$. Let

$$C(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\quad 0 \leq \theta < 2\pi.$$

Then $\mathcal{G} = \{C(\theta) : 0 \leq \theta < 2\pi\}$ is parameterized by the angle θ and we identify $\mathcal{G}$ with $[0, 2\pi)$. Now write

$$x = (\|x\|\cos(\arg x),\ \|x\|\sin(\arg x)\,)',\qquad \arg x \in [0, 2\pi).$$

Then $x = Cz$ reads

$$\begin{pmatrix} \|x\|\cos(\arg x) \\ \|x\|\sin(\arg x) \end{pmatrix} = \begin{pmatrix} t\cos(\theta + \eta(t)) \\ t\sin(\theta + \eta(t)) \end{pmatrix}$$

and thus $t = \|x\|$ and $\theta = \theta(x) = \arg x - \eta(\|x\|) \bmod 2\pi$. Therefore

$$x \leftrightarrow (\arg x - \eta(\|x\|) \bmod 2\pi\ ,\|x\|)$$

is the orbital decomposition with respect to $\mathcal{Z}$. Let $\lambda(dx) = \exp(-x'x/2)dx$. Then $t = t(x) = \|x\|$ and $\theta = \theta(x) = \arg x - \eta(\|x\|) \bmod 2\pi$ are independently distributed under the distribution

$$f_t(\|x\|)f_\theta(\theta(x))\lambda(dx). \tag{29}$$

If $\eta(t) = ct$ for some real c, then the cross section $\mathcal{Z}$ is "helical" and the surface of the density of (29) looks like a twisted circular cone.

Example 3.2 *Orthogonal group (Example 2.3 continued)*

By considering the action of R_+^* on $R^p - \{\mathbf{0}\}$, Takemura and Kuriki (1996) obtained the following distribution on $R^p - \{\mathbf{0}\}$:

$$\frac{1}{(2\pi)^{p/2}(\det\Sigma)^{1/2}}\exp(-\frac{1}{2}w'w)(\frac{w'w}{w'\Sigma^{-1}w})^{p/2}dw, \tag{30}$$

where Σ is a $p \times p$ positive definite matrix. They obtained (30) as the distribution of $w = (x'\Sigma^{-1}x/x'x)^{1/2}x$ when x is distributed according to the p-dimensional normal distribution $N_p(\mathbf{0}, \Sigma)$. Now we show that (30) can also be obtained as a decomposable distribution under the action of $\mathcal{G} = \mathcal{O}(p)$ on $\mathcal{X} = R^p - \{\mathbf{0}\}$, i.e., we show that (30) belongs to $\mathcal{F}_{\mathcal{O}(p)}$ as well as $\mathcal{F}_{R_+^*}$.

As in Example 2.3 take $\mathcal{Z} = \{(x_1, 0, \ldots, 0)' : x_1 > 0\}$ with the associated isotropy subgroup $\mathcal{G}_0$ in (14). Then

$$\begin{aligned} y = y(x) &= (x/\|x\|, B_x)\,\mathcal{G}_0 = \{\,(x/\|x\|, B_x\tilde{C}) \;:\; \tilde{C} \in \mathcal{O}(p-1)\},\\ z = z(x) &= (\|x\|, 0, \ldots, 0)', \end{aligned}$$

where B_x is any $p \times (p-1)$ matrix satisfying $(x/\|x\|, B_x) \in \mathcal{O}(p)$. Let $\lambda(dx) = dx$. Then $\lambda(dx)$ is invariant and we can write (30) as $f_{\mathcal{Y}}(y(x)) f_{\mathcal{Z}}(z(x))\lambda(dx)$ with

$$f_{\mathcal{Y}}(y) = \frac{1}{(2\pi)^{p/2}(\det\Sigma)^{1/2}}(y_1'\Sigma^{-1}y_1)^{-p/2}, \quad f_{\mathcal{Z}}(z) = \exp(-\frac{1}{2}z'z),$$

where $y_1 = y_1(y)$ is the first column of any $g \in y \subset \mathcal{G}$.

We now show that there is no inclusion relation between $\mathcal{F}_{R_+^*}$ and $\mathcal{F}_{\mathcal{O}(p)}$. For example the multivariate normal distribution $N_p(0, \Sigma)$, $\Sigma \neq I$, is in $\mathcal{F}_{R_+^*}$ but not in $\mathcal{F}_{\mathcal{O}(p)}$. An example in the opposite direction is harder to find. We need to take some non-standard global cross section in (15). In $R^2 - \{\mathbf{0}\}$ take the following global cross section with respect to the action of $\mathcal{O}(2)$:

$$\mathcal{Z} = \{(x_1, 0) : 0 < x_1 < 1\} \cup \{(-x_1, 0) : x_1 \geq 1\}. \tag{31}$$

Now consider the following density with respect to $dx_1 dx_2$:

$$f(x_1, x_2) = \begin{cases} 1/2\pi, & \text{if } \sqrt{x_1^2 + x_2^2} < 1 \text{ and } x_1 > 0,\\ 1/2\pi, & \text{if } 1 \leq \sqrt{x_1^2 + x_2^2} < 2 \text{ and } x_1 < 0,\\ 0, & \text{otherwise.} \end{cases}$$

It is easily verified that this density does not belong to $\mathcal{F}_{R_+^*}$ but is decomposable with respect to the global cross section (31) for the action of $\mathcal{O}(2)$.

Example 3.3 *Two-sample Wishart problem (Example 2.4 continued)*

By considering the action of $\mathcal{G} = LT(p)$ on $\mathcal{X} = \{(W_1, W_2) : W_1, W_2 \in PD(p)\}$, Takemura and Kuriki (1996) generalized the distribution of the pair (W_1, W_2) of independent Wishart matrices to the cross-sectionally contoured distributions associated with arbitrary cross sections for this action. Now we consider the same problem with the action of $GL(p)$ on $\mathcal{X}$ in (16) instead. As in Section 4.2 of Takemura and Kuriki (1996) we can take the dominating measure of the form

$$\lambda(d(W_1, W_2)) = (\det W_1)^{a-(p+1)/2}(\det W_2)^{b-(p+1)/2} dW_1 dW_2, \tag{32}$$

where $a, b > (p-1)/2, W_1 = (w_{1,ij}), W_2 = (w_{2,ij}), dW_1 = \prod_{i \geq j} dw_{1,ij}, dW_2 = \prod_{i \geq j} dw_{2,ij}$, with multiplier

$$\chi(B) = (\det B)^{2(a+b)}, \quad B \in GL(p).$$

$W = (W_1, W_2)$ can be written as

$$\begin{aligned} (W_1, W_2) &= (B\Lambda B', B(I-\Lambda)B') \\ &= (\ B(W)\Lambda(W)B(W)', B(W)(I-\Lambda(W))B(W)'\ ), \end{aligned}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$, $1 > \lambda_1 > \cdots > \lambda_p > 0$, and $B \in GL(p)$. Here $\Lambda(W)$ is uniquely determined by W but $B(W)$ is only unique up to the sign of each column of $B(W)$ as in Example 2.1. Here we use some appropriate selection $B(W)$ in the sense of Lemma 2.2. A general global cross section $\mathcal{Z}'$ for this problem is given in (18). Without loss of generality, we take $B = I$ in (18). By (12) a selection of the equivariant part for $\mathcal{Z}'$ is given as

$$B(W)P(\Lambda(W))^{-1}.$$

Let $t : GL(p) \to R$ be a real-valued function which does not depend on the sign of each column of G in $t(G)$. Then a density of the form

$$f(W) \ \propto \ t(B(W)P(\Lambda(W))^{-1}) \tag{33}$$

with respect to λ gives a cross-sectionally contoured distribution with respect to $\mathcal{Z}'$. Under this density $\Lambda(W)$ and $B(W)P(\Lambda(W))^{-1}$ are independently distributed.

Example 3.4 *Projection to the cone of non-negative definite matrices*

Here we present an example with more than one orbit types. In most examples of statistical invariance, there exists a particular orbit type $\mathcal{O}_0$ such that all other orbit types $\mathcal{O}_1, \mathcal{O}_2, \ldots,$ are null sets with respect the dominating measure λ, i.e., $\lambda(\mathcal{O}_i) = 0, i \geq 1$. In these cases, although we have different orbit types we can remove $\mathcal{O}_i, i \geq 1$, from the sample space without changing the distribution. For example, under the Lebesgue measure on R^p we can remove the origin. Also, from the set of $n \times p$ matrices we can remove the set of matrices X with $\mathrm{rank}\, X < p$ under the Lebesgue measure. The remaining main orbit type $\mathcal{O}_0$ is called the *principal orbit type.* See Section 4.3 of Bredon (1972). Therefore we need an example where non-principal orbit types receive positive probability. Although somewhat artificial, this can be accomplished by considering projection to lower dimensional spaces.

An example we consider here is the distribution considered in Kuriki (1993) and Section 3 of Takemura and Kuriki (1995). Let $p \times p$ symmetric random matrix $U = (u_{ij})$ be distributed according to the symmetric normal distribution, i.e., $u_{ij}, i \geq j$, are mutually independent normal variables with means 0 and variances

$$\mathrm{Var}(u_{ii}) = 1, \qquad \mathrm{Var}(u_{ij}) = 1/2, \quad i > j.$$

With probability 1, U is nonsingular and the characteristic roots of U are all distinct. Denote the spectral decomposition of U as

$$U = HLH' = (\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p)\, \mathrm{diag}(l_1, \ldots, l_p)(\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p)',$$

where $l_1 > \cdots > l_p$ and $H \in \mathcal{O}(p)$. Let $0 \le r = r(U) \le p$ be defined by $l_r > 0 > l_{r+1}$. Now let

$$X = H\,\mathrm{diag}(l_1,\ldots,l_r,0,\ldots,0)H' = \sum_{i=1}^{r} l_i \boldsymbol{h}_i \boldsymbol{h}_i'.$$

Note that X is obtained from U by replacing negative roots of U by 0. Actually X is an orthogonal projection of U onto the cone of non-negative definite matrices. Now define

$$\mathcal{X} = \{\ X\ :\ X \text{ is non-negative definite and positive roots of } X \text{ are all distinct }\}.$$

and let P denote the probability distribution of X on $\mathcal{X}$. $\mathcal{X}$ is partitioned as $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_p$, where

$$\mathcal{X}_r = \{X \in \mathcal{X}\ :\ \mathrm{rank}\, X = r\}.$$

Now consider the action of the orthogonal group $\mathcal{O}(p)$ on $\mathcal{X}$ acting as $(G,X) \mapsto GXG'$, where $G \in \mathcal{O}(p)$. Under this action $\mathcal{X}_r$ are different orbit types and $\mathcal{X} = \cup_{r=0}^{p} \mathcal{X}_r$ coincides with the partition into orbit types. Furthermore $P(\mathcal{X}_r) > 0$ for $r = 0,\ldots,p$. The probabilities $P(\mathcal{X}_r), r = 0,\ldots,p,$ are investigated in Kuriki (1993) and Takemura and Kuriki (1995) in detail. On $\mathcal{X}_r$ we can take the standard global cross section

$$\mathcal{Z}' = \{\Lambda_r : \Lambda_r = \mathrm{diag}(l_1,\ldots,l_r,0,\ldots,0),\ l_1 > \cdots > l_r > 0\}$$

with the isotropy subgroup

$$\mathcal{G}_{0,r} = \{\begin{pmatrix} E_r & 0 \\ 0 & C \end{pmatrix} :\quad E_r = \mathrm{diag}(\epsilon_1,\ldots,\epsilon_r),\ \epsilon_i = \pm 1, 1 \le i \le r,\quad C \in \mathcal{O}(p-r)\}.$$

The set $(\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r)$ of the first r columns of an element of $y \in \mathcal{Y}_r = \mathcal{O}(p)/\mathcal{G}_{0,r}$ consists of r orthonormal characteristic vectors $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r$ of X, where the signs of these r vectors are ignored. P is orthogonally invariant on each $\mathcal{X}_r$. Therefore given $X \in \mathcal{X}_r$, Λ_r and $\boldsymbol{h}_1,\ldots,\boldsymbol{h}_r$ are conditionally independently distributed.

4 Further hierarchical decomposition by a subgroup action

In this section we consider the relations between the actions of a group and one of its subgroups. Let a group $\mathcal{G}$ act on a space $\mathcal{X}$, and let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. By restricting the action $(\mathcal{G},\mathcal{X})$ to $\mathcal{H}$, we have the action $(\mathcal{H},\mathcal{X})$:

$$(h,x) \mapsto hx, \quad h \in \mathcal{H},\ x \in \mathcal{X}.$$

Obviously each $\mathcal{H}$-orbit is contained in a $\mathcal{G}$-orbit and $\mathcal{H}$ acts on each $\mathcal{G}$-orbit by

$$(h,gx) \mapsto (hg)x, \quad h \in \mathcal{H},\ g \in \mathcal{G}.$$

Therefore we actually need to consider the relations of three actions: $(\mathcal{G},\mathcal{X})$, $(\mathcal{H},\mathcal{X})$ and $(\mathcal{H},\mathcal{G}x)$.

The relations between these actions are not obvious. For example consider the following conditions:

(i) There exists a global cross section $\mathcal{Z}_{(\mathcal{G},\mathcal{X})}$ for action $(\mathcal{G},\mathcal{X})$.

(ii) There exists a global cross section $\mathcal{Z}_{(\mathcal{H},\mathcal{X})}$ for action $(\mathcal{H},\mathcal{X})$.

(iii) There exists a global cross section $\mathcal{Z}_{(\mathcal{H},\mathcal{G}x)}$ for action $(\mathcal{H},\mathcal{G}x)$, $x \in \mathcal{X}$.

We see that (ii) implies (iii) for each $\mathcal{G}x, x \in \mathcal{X}$: Simply take $\mathcal{Z}_{(\mathcal{H},\mathcal{G}x)} = \mathcal{G}x \cap \mathcal{Z}_{(\mathcal{H},\mathcal{X})}$. However, there are no other implication relations among (i), (ii) and (iii). Furthermore there are no inclusion relations between the classes of cross-sectionally contoured distributions for actions $(\mathcal{G},\mathcal{X})$ and $(\mathcal{H},\mathcal{X})$. These points are discussed in Appendix A.4.

However under certain conditions the orbital decomposition with respect to $\mathcal{G}$ and $\mathcal{H}$ are nested and this leads to a further hierarchical decomposition of the $\mathcal{G}$-orbital decomposition by the action of $\mathcal{H}$.

Throughout Sections 4.1 and 4.2 we assume that a measurable global cross section $\mathcal{Z}$ exists for action $(\mathcal{G},\mathcal{X})$, and denote by $\mathcal{G}_0$ the common isotropy subgroup at the points of $\mathcal{Z}$.

4.1 Decomposition of the equivariant part by a subgroup action

In this subsection, we give a further factorization of the orbital decomposition for $(\mathcal{G},\mathcal{X})$ by decomposing the equivariant part by means of $(\mathcal{H},\mathcal{G}x)$.

As in Section 2.1 we have the decomposition

$$\mathcal{X} \leftrightarrow \mathcal{G}/\mathcal{G}_0 \times \mathcal{Z}. \tag{34}$$

Now $\mathcal{H}$ acts on $\mathcal{G}/\mathcal{G}_0$ by

$$(h, g\mathcal{G}_0) \mapsto (hg)\mathcal{G}_0, \quad h \in \mathcal{H},\ g \in \mathcal{G}.$$

Note that we may equivalently consider the action of $\mathcal{H}$ on $\mathcal{G}z_0 : (h, gz_0) \mapsto (hg)z_0,\ z_0 \in \mathcal{Z}$.

Now suppose furthermore that a global cross section $\mathcal{V}$ exists for action $(\mathcal{H},\mathcal{G}/\mathcal{G}_0)$. Existence of a global cross section $\mathcal{V}$ leads to a further hierarchical decomposition of (34) as follows.

Denote the common isotropy subgroup at the points of $\mathcal{V}$ by $\mathcal{H}_0$. Then $\mathcal{G}/\mathcal{G}_0$ is decomposed as

$$\mathcal{G}/\mathcal{G}_0 \leftrightarrow \mathcal{H}/\mathcal{H}_0 \times \mathcal{V}. \tag{35}$$

We can take $\mathcal{V}$ in such a way that $\mathcal{G}_0 \in \mathcal{V}$; in that case, we can write $\mathcal{H}_0$ as

$$\mathcal{H}_0 = \mathcal{H}_{\mathcal{G}_0} = \{h \in \mathcal{H} : h\mathcal{G}_0 = \mathcal{G}_0\} = \mathcal{H} \cap \mathcal{G}_0.$$

From now on we always take $\mathcal{V}$ in this way. (We do not make a notational distinction between the actions of $\mathcal{H}$ on $\mathcal{G}z_0$, $z_0 \in \mathcal{Z}$, and on $\mathcal{G}/\mathcal{G}_0$, and by abuse of notation, we write the isotropy subgroups as $\mathcal{H}_{gz_0}$ and $\mathcal{H}_{g\mathcal{G}_0}$, $g \in \mathcal{G}$.)

Combining (34) and (35), we have the decomposition

$$\mathcal{X} \leftrightarrow \mathcal{H}/\mathcal{H}_0 \times \mathcal{V} \times \mathcal{Z}.$$

Therefore our question is:

(i) specifying the condition for $\mathcal{V}$ to exist, and

(ii) expressing $\mathcal{V}$ in a concrete form.

Note that $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$-orbits in $\mathcal{G}/\mathcal{G}_0$ are of the form

$$\mathcal{H}g\mathcal{G}_0, \quad g \in \mathcal{G}.$$

$\mathcal{H}g\mathcal{G}_0$ is a double coset of $\mathcal{H}$ and $\mathcal{G}_0$ in $\mathcal{G}$. This suggests that the above questions are closely related to the properties of the double cosets $Hg\mathcal{G}_0$, $g \in \mathcal{G}$, in $\mathcal{G}$.

The following lemma indicates this fact.

Lemma 4.1 *Let $\mathcal{G}' \subset \mathcal{G}$. Then $\{g'\mathcal{G}_0 : g' \in \mathcal{G}'\}$ is a cross section for the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$ if and only if $\mathcal{G}'$ is a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0$ in $\mathcal{G}$.*

The proof is straightforward and omitted.

Concerning the existence of $\mathcal{V}$, we now state the following theorem.

Theorem 4.1 *Suppose that there exists a global cross section $\mathcal{Z}$ for the action of $\mathcal{G}$ on $\mathcal{X}$, with the common isotropy subgroup denoted by $\mathcal{G}_0$. Then a global cross section $\mathcal{V}$ exists for the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$ if and only if*

$$\mathcal{H} \cap g_i\mathcal{G}_0 g_i^{-1}, \quad i \in I,$$

are all conjugate in $\mathcal{H}$, where $\{g_i : i \in I\}$ is a complete set of representatives of double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in G$.

Proof. First note that a global cross section $\mathcal{V}$ exists for action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$ if and only if the isotropy subgroups $\mathcal{H}_{g\mathcal{G}_0}$, $g \in \mathcal{G}$, are all conjugate in $\mathcal{H}$.

Let $\{g_i : i \in I\}$ be a complete set of representatives of double cosets $\mathcal{H}g\mathcal{G}_0$. Then, we can write every $g \in \mathcal{G}$ in the form

$$g = hg_ig_0, \quad h \in \mathcal{H},\ i \in I,\ g_0 \in \mathcal{G}_0$$

and thus we have

$$\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{hg_ig_0\mathcal{G}_0} = \mathcal{H}_{hg_i\mathcal{G}_0} = h\mathcal{H}_{g_i\mathcal{G}_0}h^{-1}.$$

Therefore $\mathcal{V}$ exists if and only if $\mathcal{H}_{g_i\mathcal{G}_0}$, $i \in I$, are all conjugate in $\mathcal{H}$.

Here we can write $\mathcal{H}_{g_i\mathcal{G}_0}$ as

$$\begin{aligned}
\mathcal{H}_{g_i\mathcal{G}_0} &= \{h \in \mathcal{H} : hg_i\mathcal{G}_0 = g_i\mathcal{G}_0\} \\
&= \{h \in \mathcal{H} : g_i^{-1}hg_i \in \mathcal{G}_0\} \\
&= \mathcal{H} \cap g_i\mathcal{G}_0 g_i^{-1}.
\end{aligned}$$

∎

Remark 4.1 *The condition that all $\mathcal{H} \cap g_i\mathcal{G}_0 g_i^{-1}$, $i \in I$, be conjugate does not depend on the choice of $\{g_i : i \in I\}$.*

Corollary 4.1 *If the action of $\mathcal{H}$ on $\mathcal{X}$ is free, then so is the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$.*

Proof. The isotropy subgroup at $g\mathcal{G}_0 \in \mathcal{G}/\mathcal{G}_0$ is

$$\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{gz_0} = \{e\}, \quad z_0 \in \mathcal{Z},$$

where e is the identity element of $\mathcal{H}$. ∎

Let us now consider the second problem—expressing $\mathcal{V}$ in a more concrete form.

Lemma 4.2 *Suppose that $\mathcal{G}' \subset \mathcal{G}$ is a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in \mathcal{G}$, such that the isotropy subgroups $\mathcal{H}_{g'\mathcal{G}_0}$, $g' \in \mathcal{G}'$, do not depend on $g' \in \mathcal{G}'$. Then*

$$\mathcal{V} = \{g'\mathcal{G}_0 : g' \in \mathcal{G}'\}$$

is a global cross section for the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$.

The proof is obvious.

Under the assumption of Lemma 4.2, $\mathcal{X}$ is decomposed as

$$\mathcal{X} \leftrightarrow \mathcal{H}/\mathcal{H}_0 \times \mathcal{V} \times \mathcal{Z},$$

where $\mathcal{H}_0$ is the common isotropy subgroup at the points of $\mathcal{V}$ for action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$.

We now prove that $\mathcal{V} \times \mathcal{Z}$ is a global cross section for action $(\mathcal{H}, \mathcal{X})$.

Theorem 4.2 *Suppose that there exists a global cross section $\mathcal{Z}$ for the action of $\mathcal{G}$ on $\mathcal{X}$, with the common isotropy subgroup $\mathcal{G}_0$. Suppose that $\mathcal{G}'$ is a complete set of representatives of double cosets $\mathcal{H}g\mathcal{G}_0$ of $\mathcal{G}$ satisfying the condition of Lemma 4.2, and let $\mathcal{V} = \{g'\mathcal{G}_0 : g' \in \mathcal{G}'\}$. Then $\mathcal{V} \times \mathcal{Z}$ is in one-to-one correspondence with*

$$\mathcal{G}'\mathcal{Z} = \{g'z : g' \in \mathcal{G}', z \in \mathcal{Z}\}, \tag{36}$$

and the latter is a global cross section for the action of $\mathcal{H}$ on $\mathcal{X}$.

Proof. The correspondence $(g'\mathcal{G}_0, z) \leftrightarrow g'z$ between $\mathcal{V} \times \mathcal{Z}$ and $\mathcal{G}'\mathcal{Z}$ is bijective. Now we show that $\mathcal{G}'\mathcal{Z}$ is a global cross section for action $(\mathcal{H}, \mathcal{X})$.

Every $x \in \mathcal{X}$ can be written as $x = gz$, $g \in \mathcal{G}$, $z \in \mathcal{Z}$. Furthermore g can be written as $g = hg'g_0$, where $h \in \mathcal{H}$, $g' \in \mathcal{G}'$ and $g_0 \in \mathcal{G}_0$. Hence $x = hg'g_0z = hg'z$. This implies that $\mathcal{G}'\mathcal{Z}$ intersects each $(\mathcal{H}, \mathcal{X})$-orbit $\mathcal{H}x$ at least once.

Next we show that $\mathcal{G}'\mathcal{Z}$ intersects each $\mathcal{H}x$, $x \in \mathcal{X}$, at most once. Suppose that

$$hg'z \in \mathcal{G}'\mathcal{Z}$$

for $h \in \mathcal{H}$, $g' \in \mathcal{G}'$ and $z \in \mathcal{Z}$. Then there exist $g'' \in \mathcal{G}'$ and $z' \in \mathcal{Z}$ such that $hg'z = g''z'$. Since $\mathcal{Z}$ is a cross section for action $(\mathcal{G}, \mathcal{X})$, we have $z = z'$ and $hg'\mathcal{G}_0 = g''\mathcal{G}_0$. Because $hg'\mathcal{G}_0 \in \mathcal{H}g'\mathcal{G}_0$ and $g''\mathcal{G}_0 \in \mathcal{H}g''\mathcal{G}_0$ we have $g' = g''$. Thus we obtain

$$hg'z = g'z.$$

This observation shows that $\mathcal{G}'\mathcal{Z}$ intersects each $\mathcal{H}x$, $x \in \mathcal{X}$, at most once.

It remains to be proved that the isotropy subgroups $\mathcal{H}_{g'z}$ at the points $g'z \in \mathcal{G}'\mathcal{Z}$ are all common. This is shown as follows. For any $g'z \in \mathcal{G}'\mathcal{Z}$, we have $\mathcal{H}_{g'z} = \mathcal{H}_{g'\mathcal{G}_0}$. Here $\mathcal{H}_{g'\mathcal{G}_0}$ does not depend on g' by the assumption of Lemma 4.2. Thus $\mathcal{H}_{g'z}$ does not depend on $g'z \in \mathcal{G}'\mathcal{Z}$. ∎

We call a global cross section of the form (36) a *decomposable global cross section* for action $(\mathcal{H}, \mathcal{X})$. Of course a general global cross section for $(\mathcal{H}, \mathcal{X})$ is not necessarily decomposable. See examples in Appendix A.5.

Let us consider special cases.

First, consider the case where the action of $\mathcal{G}$ on $\mathcal{X}$ is free. In that case, we want to decompose $\mathcal{G}$ by considering the action of $\mathcal{H}$ on $\mathcal{G} : (h, g) \mapsto hg$.

Corollary 4.2 *The action of $\mathcal{H}$ on $\mathcal{G}$ is free, and any complete set $\{g_i : i \in I\}$ of representatives of the right cosets $\mathcal{H}g$, $g \in \mathcal{G}$, of $\mathcal{H}$ is a cross section for this action.*

Proof. It is trivial to see that action $(\mathcal{H}, \mathcal{G})$ is free. In order to show that $\{g_i : i \in I\}$ is a cross section for $(\mathcal{H}, \mathcal{G})$, we apply Lemma 4.2 with $\mathcal{G}_0 = \{e\}$: The set $\{g_i : i \in I\}$ serves as $\mathcal{G}'$ in Lemma 4.2. ∎

In the case of Corollary 4.2, $\mathcal{G}$ is decomposed as

$$\begin{aligned} \mathcal{G} &\leftrightarrow \mathcal{H} \times \{g_i : i \in I\} \\ &\leftrightarrow \mathcal{H} \times \mathcal{H}\backslash\mathcal{G}, \end{aligned}$$

where $\mathcal{H}\backslash\mathcal{G}$ is the right coset space

$$\mathcal{H}\backslash\mathcal{G} = \{\mathcal{H}g : g \in \mathcal{G}\}.$$

Corollary 4.3 *Suppose the action of $\mathcal{G}$ on $\mathcal{X}$ is free, and let $\mathcal{Z}$ be a cross section for this action. Then the action of $\mathcal{H}$ on $\mathcal{X}$ is free, and for any complete set $\mathcal{G}' = \{g_i : i \in I\}$ of representatives of the right cosets $\mathcal{H}g$, $g \in \mathcal{G}$, of $\mathcal{H}$, the set $\mathcal{G}'\mathcal{Z} = \{g_i z : i \in I,\ z \in \mathcal{Z}\}$ is a cross section for this action.*

Proof. It is trivial to see that the action of $\mathcal{H}$ on $\mathcal{X}$ is free. As in the proof of Corollary 4.2, $\mathcal{G}' = \{g_i : i \in I\}$ satisfies the conditions of Lemma 4.2. Thus we have by Theorem 4.2 that $\mathcal{G}'\mathcal{Z}$ is a cross section for action $(\mathcal{H}, \mathcal{X})$. ∎

As an example, consider the actions related to the star-shaped distributions—the actions of $\mathcal{G} = R^*_\times$ (the multiplicative group of nonzero real numbers) and $\mathcal{H} = R^*_+$ on $\mathcal{X} = R^p - \{0\}$ by scalar multiplication. In that case, $\mathcal{G}$ acts on $\mathcal{X}$ freely. Moreover, $\{\pm 1\}$ is a complete set of representatives of $\mathcal{H}g$, $g \in G$, and is thus a cross section for action $(\mathcal{H}, \mathcal{G})$. Accordingly, we have one-to-one correspondence

$$R^*_\times \leftrightarrow R^*_+ \times \{\pm 1\}.$$

Furthermore, we have by Corollary 4.3 that $\mathcal{G}'\mathcal{Z}$ with $\mathcal{G}' = \{\pm 1\}$ is a cross section for the action of R_+^* on $R^p - \{\mathbf{0}\}$. Let us take $\mathcal{Z}$ as

$$\mathcal{Z} = \{(x_1, \ldots, x_p)' \in S^{p-1} : x_p > 0\} \cup \{(x_1, \ldots, x_{p-1}, 0)' \in S^{p-1} : (x_1, \ldots, x_{p-1})' \in \tilde{\mathcal{Z}}\},$$

where $\tilde{\mathcal{Z}}$ is a cross section for the action of $R_\times^*$ on $R^{p-1} - \{\mathbf{0}\}$, $\mathbf{0} \in R^{p-1}$. Since it is clear that $\mathcal{G}'\mathcal{Z} = S^{p-1}$ is true for $p = 2$, we see by induction on p that for all p,

$$\begin{aligned} \mathcal{G}'\mathcal{Z} &= \{(x_1, \ldots, x_p)' \in S^{p-1} : x_p \neq 0\} \cup \{(x_1, \ldots, x_{p-1}, 0)' \in S^{p-1}\} \\ &= S^{p-1}, \end{aligned}$$

which is clearly a cross section for the action of R_+^* on $R^p - \{\mathbf{0}\}$.

Next we treat the case which covers the two-sample Wishart problem.

Corollary 4.4 *Let $\mathcal{G}_0$ be a subgroup of $\mathcal{G}$. Suppose there exists a subgroup $\mathcal{K}$ of $\mathcal{G}$ satisfying the following conditions:*

(i) Every $g \in \mathcal{G}$ can be written uniquely in the form $g = hk$, $h \in \mathcal{H}$, $k \in \mathcal{K}$.

(ii) $\mathcal{G}_0 \subset \mathcal{K}$.

Then, the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$ is free, and

$$\mathcal{V} = \mathcal{K}/\mathcal{G}_0 = \{k\mathcal{G}_0 : k \in \mathcal{K}\}$$

is a cross section for this action.

Proof. Noting that $\mathcal{H} \cap \mathcal{K} = \{e\}$ by assumption (i) and that $k\mathcal{G}_0 k^{-1} \subset \mathcal{K}$, $k \in \mathcal{K}$, by assumption (ii), we have $\mathcal{H} \cap k\mathcal{G}_0 k^{-1} = \{e\}$ for any $k \in \mathcal{K}$. Therefore $\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H} \cap g\mathcal{G}_0 g^{-1}$ is trivial for any g in $\mathcal{K}$ and thus in $\mathcal{G}$:

$$\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{hk\mathcal{G}_0} = h\mathcal{H}_{k\mathcal{G}_0} h^{-1} = \{e\}, \quad g = hk, \ h \in \mathcal{H}, \ k \in \mathcal{K}.$$

Hence action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$ is free.

Let $\mathcal{G}' \subset \mathcal{K}$ be a complete set of representatives of left cosets $k\mathcal{G}_0$ in $\mathcal{K}$. By Lemma 4.2 it suffices to show that $\mathcal{G}'$ is a complete set of representatives of double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in \mathcal{G}$. Since

$$\bigcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0 = \mathcal{H}\mathcal{G}'\mathcal{G}_0 = \mathcal{H}\mathcal{K} = \mathcal{G}$$

it remains to show that

$$\mathcal{H}g'\mathcal{G}_0 = \mathcal{H}g''\mathcal{G}_0, \quad g', g'' \in \mathcal{G}', \tag{37}$$

implies $g' = g''$. If (37) holds, then there exist $h \in \mathcal{H}$ and $g_0 \in \mathcal{G}_0$ such that $g' = hg''g_0$. By assumptions (i) and (ii) we have $g' = g''g_0$. Thus

$$g'\mathcal{G}_0 = g''g_0\mathcal{G}_0 = g''\mathcal{G}_0,$$

and we have $g' = g''$. ∎

In the case of Corollary 4.4, $\mathcal{G}/\mathcal{G}_0$ is decomposed as

$$\mathcal{G}/\mathcal{G}_0 \leftrightarrow \mathcal{H} \times \mathcal{K}/\mathcal{G}_0.$$

Corollary 4.5 *Suppose there exists a global cross section* $\mathcal{Z}$ *for the action of* $\mathcal{G}$ *on* $\mathcal{X}$*, with the common isotropy subgroup denoted by* $\mathcal{G}_0$ *. Suppose moreover that there exists a subgroup* $\mathcal{K}$ *satisfying conditions (i) and (ii) in Corollary 4.4. Then, the action of* $\mathcal{H}$ *on* $\mathcal{X}$ *is free, and* $\mathcal{K}\mathcal{Z}$ *is a cross section for this action.*

Proof. $\mathcal{K}\mathcal{Z} = \mathcal{G}'\mathcal{Z}$ for any complete set $\mathcal{G}' \subset \mathcal{K}$ of representatives of left cosets $k\mathcal{G}_0$, $k \in \mathcal{K}$. Therefore $\mathcal{K}\mathcal{Z}$ is a (global) cross section for action $(\mathcal{H}, \mathcal{X})$ by Theorem 4.2.

Now we show that action $(\mathcal{H}, \mathcal{X})$ is free. As was shown in the proof of Corollary 4.4, $\mathcal{H}_{k\mathcal{G}_0}$ is trivial for each $k \in \mathcal{K}$. Accordingly, by noting that $\mathcal{H}_{kz} = \mathcal{H}_{k\mathcal{G}_0}$, $k \in \mathcal{K}$, $z \in \mathcal{Z}$, we see that $\mathcal{H}_{kz}$ is trivial for each $k \in \mathcal{K}$ and $z \in \mathcal{Z}$. But since $\mathcal{K}\mathcal{Z}$ is a cross section for action $(\mathcal{H}, \mathcal{X})$, we have that $(\mathcal{H}, \mathcal{X})$ is free. ∎

As an example, consider the two-sample Wishart problem in Example 2.4 with $\mathcal{H} = LT(p)$ and $\mathcal{Z}$ in (17). Then $\mathcal{O}(p)$ can serve as $\mathcal{K}$ in Corollary 4.4. Corollary 4.5 implies that the action of $\mathcal{H} = LT(p)$ on $\mathcal{X}$ is free and that $\mathcal{K}\mathcal{Z}$ with $\mathcal{K} = \mathcal{O}(p)$ is a cross section for $(\mathcal{H}, \mathcal{X})$. We can write $\mathcal{K}\mathcal{Z}$ as

$$\begin{aligned}\mathcal{K}\mathcal{Z} &= \{(C\Lambda C', I - C\Lambda C') : C \in \mathcal{O}(p), \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p), 1 > \lambda_1 > \cdots > \lambda_p > 0\} \\ &= \{(U, I - U) : O < U < I, \text{ and the eigenvalues of } U \text{ are all distinct}\},\end{aligned}$$

where O denotes the null matrix and $<$ means the Löwner order.

4.2 Distributions further decomposable with respect to a subgroup action

In this subsection, we discuss distributional aspects of the further hierarchical decompositions in Section 4.1.

Suppose there exists a subgroup $\mathcal{L}$ of $\mathcal{G}$ of the form

$$\mathcal{L} = \mathcal{G}'\mathcal{G}_0, \tag{38}$$

where $\mathcal{G}'$ is a complete set of representatives of double cosets $\mathcal{H}g\mathcal{G}_0$ in $\mathcal{G}$ such that $\mathcal{H}_{g'\mathcal{G}_0}$ does not depend on $g' \in \mathcal{G}'$. Then we have $\mathcal{H}\mathcal{L} = \mathcal{H}\mathcal{G}'\mathcal{G}_0 = \mathcal{G}$. Therefore every $g \in \mathcal{G}$ can be written in the form $g = hl$, $h \in \mathcal{H}, l \in \mathcal{L}$. Moreover, by considering the action of the product group $\mathcal{H} \times \mathcal{L}$ on $\mathcal{G}$:

$$((h, l), g) \mapsto hgl^{-1},$$

we have the bijection $\mathcal{G} \leftrightarrow (\mathcal{H} \times \mathcal{L})/\mathcal{F}^*$, where $\mathcal{F}^* = \{(g, g) : g \in \mathcal{F}\}$, $\mathcal{F} = \mathcal{H} \cap \mathcal{L}$, is the isotropy subgroup at the identity element of $\mathcal{G}$.

Here we note the following.

Consider the situation in Corollaries 4.2 and 4.3. Suppose the action of $\mathcal{G}$ on $\mathcal{X}$ is free and that a complete set $\{g_i : i \in I\}$ of representatives of the right cosets $\mathcal{H}g$, $g \in \mathcal{G}$, of $\mathcal{H}$ forms a subgroup of $\mathcal{G}$. Then we can take $\mathcal{L} = \mathcal{G}' = \{g_i : i \in I\}$. For instance, consider the example immediately after Corollary 4.3—the actions related to the star-shaped distributions. Then $\{\pm 1\}$ forms a subgroup of $\mathcal{G} = R^*_\times$ and thus can serve as $\mathcal{L}$.

On the other hand, consider the situation in Corollaries 4.4 and 4.5. Suppose a subgroup $\mathcal{K}$ of $\mathcal{G}$ satisfies conditions (i) and (ii) of Corollary 4.4. Then we can take $\mathcal{L} = \mathcal{K}$. For instance, consider the example immediately after Corollary 4.5—the actions related to the two-sample Wishart problem. Then $\mathcal{O}(p)$ can serve as $\mathcal{L}$.

Now we have by Lemma 4.2 that $\mathcal{V} = \mathcal{L}/\mathcal{G}_0 = \{l\mathcal{G}_0 : l \in \mathcal{L}\}$ is a global cross section for action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$, and thus we obtain the decomposition

$$\begin{array}{lll} \mathcal{X} & \leftrightarrow & \mathcal{U} \times \mathcal{V} \times \mathcal{Z} \qquad (\mathcal{U} = \mathcal{H}/\mathcal{H}_0), \\ x & \leftrightarrow & (u, v, z), \quad x = hlz, \quad u = h\mathcal{H}_0, \ v = l\mathcal{G}_0, \end{array} \tag{39}$$

where $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$ since $\mathcal{L}$ is a group and thus contains the identity element of $\mathcal{G}$.

Concerning topological questions, we assume Assumptions 3.1 and 3.2 and further make the following assumptions:

Assumption 4.1

1. *$\mathcal{H}$ and $\mathcal{L}$ are closed subgroups of $\mathcal{G}$.*
2. *$\mathcal{F}$ is compact.*

Note that under our assumptions, $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$ is compact since $\mathcal{G}_0$ is compact and $\mathcal{H} \cap \mathcal{G}_0$ is closed in the relative topology of $\mathcal{G}_0$. Note also that the one-to-one correspondence $\mathcal{G} \leftrightarrow (\mathcal{H} \times \mathcal{L})/\mathcal{F}^*$ is a homeomorphism since $\mathcal{G}$ is second countable (p.92 of Wijsman (1990)).

Now let λ be a measure on $\mathcal{X}$ relatively invariant under the action of $\mathcal{G}$ with multiplier χ. We define the extended decomposable distributions as follows.

Definition 4.1 *A distribution on $\mathcal{X}$ is said to be an extended decomposable distribution with respect to $(\mathcal{U}, \mathcal{V}, \mathcal{Z})$ iff it is of the form*

$$f(x)\lambda(dx) = f_{\mathcal{U}}(u(x))f_{\mathcal{V}}(v(x))f_{\mathcal{Z}}(z(x))\lambda(dx).$$

The next theorem gives the distributions of u, v and z when x is distributed according to an extended decomposable distribution.

Theorem 4.3 *Suppose that x is distributed according to an extended decomposable distribution $f_{\mathcal{U}}(u(x))f_{\mathcal{V}}(v(x))f_{\mathcal{Z}}(z(x))\lambda(dx)$. Then $u = u(x) = h\mathcal{H}_0$, $v = v(x) = l\mathcal{G}_0$ and $z = z(x)$ $(x = hlz)$ are independently distributed with the joint distribution*

$$\begin{aligned} &f_{\mathcal{U}}(u)\chi(u)\mu_{\mathcal{U}}(du) \\ &\quad \times f_{\mathcal{V}}(v)\chi(v)\Delta^{\mathcal{G}}(v)\Delta^{\mathcal{L}}(v)^{-1}\mu_{\mathcal{V}}(dv) \\ &\quad \times f_{\mathcal{Z}}(z)\nu_{\mathcal{Z}}(dz), \end{aligned}$$

where $\Delta^{\mathcal{L}}$ is the right-hand modulus of $\mathcal{L}$, measure $\mu_{\mathcal{U}}$ (resp. $\mu_{\mathcal{V}}$) is a version of the invariant measures on $\mathcal{U} = \mathcal{H}/\mathcal{H}_0$ (resp. $\mathcal{V} = \mathcal{L}/\mathcal{G}_0$), and $\nu_{\mathcal{Z}}$ is the probability measure in (23).

This theorem can be proved by 7.6.1 Proposition and (7.6.5) of Wijsman (1990).

Let us apply Theorem 4.3 to the above-mentioned two examples.

Example 4.1 *Symmetric star-shaped distribution*

Consider the actions related to the star-shaped distributions—the actions of $\mathcal{G} = R^*_{\times}$ and $\mathcal{H} = R^*_{+}$ on $\mathcal{X} = R^p - \{\mathbf{0}\}$. Then we have $\mathcal{G}_0 = \{e\}$ and $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0 = \{e\}$. If we take $\mathcal{L} = \{\pm 1\}$, we obtain the bijection

$$\begin{array}{ccl} \mathcal{X} & \leftrightarrow & \mathcal{H} \times \mathcal{L} \times \mathcal{Z}, \\ \boldsymbol{x} & \leftrightarrow & (h, \epsilon, \boldsymbol{z}), \quad \boldsymbol{x} = \epsilon h \boldsymbol{z}, \end{array}$$

where $\mathcal{Z}$ is a cross section for action $(\mathcal{G}, \mathcal{X})$. Furthermore, we have that $\mathcal{G}$ and $\mathcal{L}$ are unimodular: $\Delta^{\mathcal{G}} = 1$, $\Delta^{\mathcal{L}} = 1$.

Now suppose that $\boldsymbol{x}$ is distributed according to a star-shaped distribution—that is, a cross-sectionally contoured distribution for action $(\mathcal{H}, \mathcal{X})$:

$$f(h(\boldsymbol{x}))d\boldsymbol{x}.$$

This distribution can also be regarded as the extended decomposable distribution with

$$\begin{array}{rcl} f_{\mathcal{U}}(h) & = & f_{\mathcal{H}}(h) = f(h), \\ f_{\mathcal{V}}(\epsilon) & = & f_{\mathcal{L}}(\epsilon) \equiv 1, \\ f_{\mathcal{Z}}(\boldsymbol{z}) & \equiv & 1, \end{array}$$

and

$$\lambda(d\boldsymbol{x}) = d\boldsymbol{x}.$$

Measure λ is relatively invariant under the action of $\mathcal{G}$ with multiplier

$$\chi(g) = |g|^p, \quad g \in \mathcal{G},$$

where $|\cdot|$ denotes the absolute value. Therefore, we have by Theorem 4.3 that h, ϵ and $\boldsymbol{z}$ are independently distributed according to

$$c_0^{-1} f(h) h^p h^{-1} dh = c_0^{-1} f(h) h^{p-1} dh, \quad \mu_{\mathcal{V}} = \mu_{\mathcal{L}}, \quad \nu_{\mathcal{Z}},$$

respectively, where $c_0 = \int_0^\infty f(h) h^{p-1} dh$ and $\mu_{\mathcal{L}}(\{1\}) = \mu_{\mathcal{L}}(\{-1\}) = 1/2$. Under the additional assumption that $h(\boldsymbol{x})$ is piecewise of class C^1, we have

$$\nu_{\mathcal{Z}}(d\boldsymbol{z}) = c_0 \langle \boldsymbol{z}, \boldsymbol{n}_{\boldsymbol{z}} \rangle d\boldsymbol{z},$$

where $\boldsymbol{n}_{\boldsymbol{z}}$ is the outward unit normal vector of $\mathcal{Z}$, $d\boldsymbol{z}$ on the right hand side is the volume element of $\mathcal{Z}$, and $\langle \cdot\,, \cdot \rangle$ denotes the inner product (Takemura and Kuriki (1996) Section 3).

Example 4.2 *Two-sample Wishart problem (Example 2.4, 3.3 continued)*

Take $\mathcal{L} = \mathcal{K} = \mathcal{O}(p)$. Then we obtain the bijection

$$\begin{aligned} \mathcal{X} &\leftrightarrow \mathcal{H} \times \mathcal{L}/\mathcal{G}_0 \times \mathcal{Z}, \\ (W_1, W_2) &\leftrightarrow (T, C\mathcal{G}_0, (\Lambda, I - \Lambda)), \quad (W_1, W_2) = (TC\Lambda C'T', TC(I - \Lambda)C'T'). \end{aligned}$$

Furthermore, we have that $\mathcal{G}$ and $\mathcal{L}$ are unimodular: $\Delta^{\mathcal{G}} = 1$, $\Delta^{\mathcal{L}} = 1$.

Suppose that the random matrices W_1 and W_2 are independently distributed according to $W_p(n_1, \Sigma)$ and $W_p(n_2, \Sigma)$, respectively. The dominating measure is as in (32) with $a = n_1/2, b = n_2/2$. Then the distribution of (W_1, W_2) is the extended decomposable distribution with $f_{\mathcal{U}}(T) \propto \operatorname{etr}(-\frac{1}{2}\Sigma^{-1}TT')$, $f_{\mathcal{V}}(C\mathcal{G}_0) \equiv 1$ and $f_{\mathcal{Z}}((\Lambda, I - \Lambda)) \equiv 1$. Therefore by Theorem 4.3 T, $C\mathcal{G}_0$ and Λ are independently distributed. The distributions of these parts are given in standard textbooks of multivariate distribution theory (see Anderson (1984) or Muirhead (1982) for example). In particular $C\mathcal{G}_0$ is distributed according to the invariant probability measure on $\mathcal{V} = \mathcal{O}(p)/\mathcal{G}_0$ induced from the left invariant Haar measure on $\mathcal{O}(p)$.

A non-standard distribution with respect to the action of $\mathcal{G} = GL(p)$ is given in (33). Write (up to an arbitrary selection)

$$B(W)P(\Lambda(W))^{-1} = T(W)C(W), \qquad T(W) \in LT(p),\ C(W) \in \mathcal{O}(p).$$

If t can be written as

$$t(B(W)P(\Lambda(W))^{-1}) = f_{\mathcal{U}}(T(W))\ f_{\mathcal{V}}(C(W)\mathcal{G}_0)$$

then $T(W), C(W)\mathcal{G}_0$ and $\Lambda(W)$ are independently distributed. Suppose furthermore that t satisfies the following condition

$$t(B) = t(BC), \quad \forall C \in \mathcal{O}(p),\ B \in GL(p),$$

or equivalently suppose that $t(B)$ depends only on BB'. Then $f_{\mathcal{V}}$ is a constant function and $C(W)\mathcal{G}_0$ is again distributed according to the invariant probability measure on $\mathcal{V} = \mathcal{O}(p)/\mathcal{G}_0$ induced from the left invariant Haar measure on $\mathcal{O}(p)$.

5 Appendix

A.1. Orbit types of Example 2.2

We give a proof of the orbit types described in Example 2.2. First we show that for each $(i_1, \ldots, i_r)$ $(1 \le i_1 < \cdots < i_r \le p,\ 0 \le r \le p)$, $X, \tilde{X} \in \mathcal{X}(i_1, \ldots, i_r)$ implies $X \sim_{\mathcal{X}} \tilde{X}$. It suffices to prove that there exists a global cross section for each $\mathcal{X}(i_1, \ldots i_r)$. Let

$$\mathcal{Z}(i_1, \ldots, i_r) = \{H = (\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p) \in \mathcal{X} : \boldsymbol{h}_{i_1}, \ldots, \boldsymbol{h}_{i_r} \text{ are orthonormal and } \boldsymbol{h}_j = \boldsymbol{0} \text{ for } j \notin \{i_1, \ldots, i_r\}\ \}.$$

Then $\mathcal{Z}(i_1, \ldots, i_r)$ is a global cross section for $\mathcal{X}(i_1, \ldots, i_r)$. This is shown as follows. It is easy to see that $\mathcal{Z}(i_1, \ldots, i_r)$ is a cross section for $\mathcal{X}(i_1, \ldots, i_r)$. On the other hand, for any $H \in \mathcal{Z}(i_1, \ldots, i_r)$, the isotropy subgroup $\mathcal{G}_H$ at H is the set of $T = (t_{ij}) = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_p) \in LT(p)$ with

$$\boldsymbol{t}_i = \begin{cases} \boldsymbol{e}_i \text{ for } i \in \{i_1, \ldots, i_r\}, \\ (0, \ldots, 0, t_{ii}, t_{i+1,i}, \ldots, t_{pi})' \text{ is arbitrary for } i \notin \{i_1, \ldots, i_r\}, \end{cases}$$

where $\boldsymbol{e}_i = (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0)'$ is the i-th coordinate vector. Thus $\mathcal{G}_H$ does not depend on $H \in \mathcal{Z}(i_1, \ldots, i_r)$.

Next we show that $X \sim_{\mathcal{X}} \tilde{X}$ implies $X, \tilde{X} \in \mathcal{X}(i_1, \ldots, i_r)$ for some $(i_1, \ldots, i_r)$ $(1 \leq i_1 < \cdots < i_r \leq p,\ 0 \leq r \leq p)$. Suppose that $X \sim_{\mathcal{X}} \tilde{X}$ and that $X \in \mathcal{X}(i_1, \ldots, i_r)$, $\tilde{X} \in \mathcal{X}(i'_1, \ldots, i'_{r'})$. We want to show $r = r'$ and $(i_1, \ldots, i_r) = (i'_1, \ldots, i'_{r'})$. Note that it is sufficient to consider the case $r, r' \geq 1$. Suppose on the contrary that $(i_1, \ldots, i_r) \neq (i'_1, \ldots, i'_{r'})$. Then there would exist either i such that $i \in \{i_1, \ldots, i_r\} \cap \{i'_1, \ldots, i'_{r'}\}^C$ or $i' \in \{i_1, \ldots, i_r\}^C \cap \{i'_1, \ldots, i'_{r'}\}$, where C denotes the complement. Without loss of generality let $i \in \{i_1, \ldots, i_r\} \cap \{i'_1, \ldots, i'_{r'}\}^C$. If we write X and $\tilde{X}$ as $X = HT'$ and $\tilde{X} = \tilde{H}\tilde{T}'$ $(T, \tilde{T} \in LT(p),\ H \in \mathcal{Z}(i_1, \ldots, i_r), \tilde{H} \in \mathcal{Z}(i'_1, \ldots, i'_{r'}))$, respectively, then we have $H \sim_{\mathcal{X}} \tilde{H}$:

$$T\mathcal{G}_H T^{-1} = \mathcal{G}_{\tilde{H}} \text{ for some } T = (t_{ij}) \in LT(p).$$

Now for any $c > 0$ there exists a matrix in $\mathcal{G}_{\tilde{H}}T$ which has c as the i-th diagonal element. On the other hand, all the matrices in $T\mathcal{G}_H$ have t_{ii} as the i-th diagonal element. This is a contradiction. Thus we have proved $(i_1, \ldots, i_r) = (i'_1, \ldots, i'_{r'})$. ∎

A.2. Factorization of measure for measurable $\mathcal{Z}$

Theorem 7.5.1 of Wijsman (1990) guarantees $\lambda(dx)$ can be factored as (23) for a standard global cross section $\mathcal{Z}$ in Assumption 3.2. For an arbitrary measurable global cross section $\mathcal{Z}'$, the factorization of the dominating measure λ with respect to the orbital decomposition can be obtained as follows. Consider a standard global cross section $\mathcal{Z}$ and factorize λ. Now by within-orbit bijection $g\mathcal{Z} \to g\mathcal{Z}'$ in (27), the dominating measure transforms as (28), and we see that factorizability of λ is equivalent to the factorizability of $\tilde{\lambda}$. We now see that Assumption 3.2 implies factorization of $\tilde{\lambda}$ for measurable global cross sections $\mathcal{Z}'$ as well.

Regarding factorizability we can also use the sufficiency approach in Section 2.2 of Takemura and Kuriki (1996).

A.3. Some topological questions about $\mathcal{Y}$ in Section 3.1

Here we discuss some topological questions about $\mathcal{Y} = \bigcup_i \mathcal{Y}_i$ with $\mathcal{Y}_i = \mathcal{G}/\mathcal{G}_i$ in Section 3.1.

Since $\mathcal{G}$ is a metrizable group, the topology of $\mathcal{G}$ can be defined by a right invariant distance d (Dieudonné (1976), (12.9.1)). For two nonempty subsets A, B of $\mathcal{G}$ write

$$d(A, B) = \inf_{g \in A,\ g' \in B} d(g, g').$$

For brevity we shall write $d(\{g\}, A)$ as $d(g, A)$ for $g \in \mathcal{G}$ and $A \subset \mathcal{G}$.

Now we regard each element $g\mathcal{G}_i$, $g \in \mathcal{G}$, $i \geq 1$, of $\mathcal{Y} = \bigcup_i \mathcal{Y}_i = \bigcup_i \mathcal{G}/\mathcal{G}_i$ as a subset of $\mathcal{G} : g\mathcal{G}_i \subset \mathcal{G}$. Noting that $g\mathcal{G}_i$ is compact for each $g \in \mathcal{G}$ and each $i \geq 1$, we endow $\mathcal{Y}$ with the Hausdorff distance h to make $\mathcal{Y}$ a metric space:

$$h(g\mathcal{G}_i, g'\mathcal{G}_{i'}) = \max\{\rho(g\mathcal{G}_i, g'\mathcal{G}_{i'}),\ \rho(g'\mathcal{G}_{i'}, g\mathcal{G}_i)\}, \quad g, g' \in \mathcal{G},\ i, i' \geq 1$$

with $\rho(g\mathcal{G}_i, g'\mathcal{G}_{i'}) = \sup_{g_i \in \mathcal{G}_i} d(gg_i, g'\mathcal{G}_{i'})$. By restricting $h : \mathcal{Y} \times \mathcal{Y} \to R$ to $\mathcal{Y}_i \times \mathcal{Y}_i$, we have the induced distance (also denoted by h) for each $\mathcal{Y}_i$, $i \geq 1$.

Here we show that for each $i \geq 1$, the topology on $\mathcal{Y}_i$ defined by h is the same as the quotient topology on $\mathcal{Y}_i = \mathcal{G}/\mathcal{G}_i$. Fix an arbitrary $i \geq 1$. Note that by the right invariance of d,

$$\forall g_i \in \mathcal{G}_i : d(gg_i, g'\mathcal{G}_i) = d(g, g'\mathcal{G}_i)$$

for $g, g' \in \mathcal{G}$. Using this relation, we have

$$\begin{aligned}
\rho(g\mathcal{G}_i, g'\mathcal{G}_i) &= \sup_{g_i \in \mathcal{G}_i} d(gg_i, g'\mathcal{G}_i) \\
&= d(g, g'\mathcal{G}_i) \\
&= \inf_{g_i \in \mathcal{G}_i} d(g, g'g_i) \\
&= \inf_{g_i \in \mathcal{G}_i} d(g'g_i, g) \\
&= \inf_{g_i \in \mathcal{G}_i} d(g', gg_i^{-1}) \\
&= d(g', g\mathcal{G}_i) \\
&= \rho(g'\mathcal{G}_i, g\mathcal{G}_i)
\end{aligned}$$

for $g, g' \in \mathcal{G}$. Accordingly, we obtain

$$\begin{aligned}
h(g\mathcal{G}_i, g'\mathcal{G}_i) &= \max\{\rho(g\mathcal{G}_i, g'\mathcal{G}_i),\ \rho(g'\mathcal{G}_i, g\mathcal{G}_i)\} \\
&= \rho(g\mathcal{G}_i, g'\mathcal{G}_i) \\
&= d(g, g'\mathcal{G}_i) \\
&= d(g\mathcal{G}_i, g'\mathcal{G}_i)
\end{aligned}$$

for $g, g' \in \mathcal{G}$.

Thus we have by (12.11.3) of Dieudonné (1976) that the topology on $\mathcal{Y}_i$ defined by h is the quotient topology on $\mathcal{Y}_i = \mathcal{G}/\mathcal{G}_i$.

A.4. Examples concerning existence of global cross sections for subgroup actions in Section 4

We show that there are no implication relations among the conditions (i), (ii) and (iii) at the beginning of Section 4 except for (ii) $\Rightarrow$ (iii).

First, we show that neither of (i) and (ii) implies the other.

In order to see that (i) does not imply (ii), consider the following example: $\mathcal{G} = GL(p)$, $\mathcal{H} = LT(p)$ and $\mathcal{X} = R^p - \{0\}$, and the actions of $\mathcal{G}$ and $\mathcal{H}$ on $\mathcal{X}$ are the usual multiplications of matrices and vectors. The action of $\mathcal{G}$ on $\mathcal{X}$ is transitive, i.e., there is only one orbit: $\mathcal{X} = \mathcal{G}x$, $x \in \mathcal{X}$. Thus there trivially exists a global cross section

$\mathcal{Z}_{(\mathcal{G},\mathcal{X})}$ for this action: $\mathcal{Z}_{(\mathcal{G},\mathcal{X})} = \{z\}$, $z \in \mathcal{X}$. On the other hand, a global cross section $\mathcal{Z}_{(\mathcal{H},\mathcal{X})}$ does not exist for the action of $\mathcal{H}$ on $\mathcal{X}$. One can see this by noting that the isotropy subgroups at $\mathbf{e}_1 = (1,0,\dots,0)'$ and $\mathbf{e}_p = (0,\dots,0,1)'$ are not conjugate in $\mathcal{H}$ since

$$\begin{aligned}\mathcal{H}_{\mathbf{e}_1} &= \{\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & T_{22}\end{pmatrix} : T_{22} \in LT(p-1)\},\\ \mathcal{H}_{\mathbf{e}_p} &= \{\begin{pmatrix} T_{11} & \mathbf{0} \\ \mathbf{t}_{21}' & 1\end{pmatrix} : T_{11} \in LT(p-1),\ \mathbf{t}_{21} \in R^{p-1}\},\end{aligned}$$

where $\mathbf{0} = (0,\dots,0)' \in R^{p-1}$. In addition, this example also indicates that (i) does not imply (iii).

In order to see that (ii) does not imply (i), we consider the actions related to the two-sample Wishart problem: $\mathcal{G} = GL(p)$, $\mathcal{H} = LT(p)$ and $\mathcal{X} = PD(p) \times PD(p) = \{(W_1, W_2) : W_1, W_2 \in PD(p)\}$. The actions are $(B,(W_1,W_2)) \mapsto (BW_1B', BW_2B')$, $B \in GL(p)$ and $(T,(W_1,W_2)) \mapsto (TW_1T', TW_2T')$, $T \in LT(p)$. Then, since the action of $\mathcal{H}$ on $\mathcal{X}$ is free, there exists a (global) cross section $\mathcal{Z}_{(\mathcal{H},\mathcal{X})}$ for this action. On the other hand there does not exist a global cross section $\mathcal{Z}_{(\mathcal{G},\mathcal{X})}$ for the action of $\mathcal{G}$ on $\mathcal{X}$. In order to ensure the existence of $\mathcal{Z}_{(\mathcal{G},\mathcal{X})}$, we have to exclude, for example, the pairs $(W_1, W_2) \in PD(p) \times PD(p)$ for which the characteristic equation $\det(W_1 - \lambda(W_1 + W_2)) = 0$ has multiple roots λ.

Next we see that (iii) does not imply (ii): Just take $\mathcal{H} = \mathcal{G}$ and consider any action of $\mathcal{G}$ on $\mathcal{X}$ which does not satisfy (i).

Finally we know from the implication relations above that (iii) does not imply (i). On the other hand, as was mentioned earlier, we have that (i) does not imply (iii).

A.5. The class of decomposable distributions for a subgroup action in Section 4

We compare the classes of decomposable distributions for actions $(\mathcal{G},\mathcal{X})$ and $(\mathcal{H},\mathcal{X})$ for a subgroup $\mathcal{H} \subset \mathcal{G}$.

Consider the situation where there exist global cross sections for both $(\mathcal{G},\mathcal{X})$ and $(\mathcal{H},\mathcal{X})$. Let $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$ be the families of decomposable distributions with respect to $(\mathcal{G},\mathcal{X})$ and $(\mathcal{H},\mathcal{X})$, respectively. One may wonder which, if any, of $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$ is wider. In general, however, neither of $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{H}}$ contains the other as a subclass.

For instance, consider the actions related to the star-shaped distributions—the actions of $\mathcal{G} = R^*_\times$ and $\mathcal{H} = R^*_+$ on $\mathcal{X} = R^p - \{\mathbf{0}\}$ under scalar multiplication. In that case, the actions of $\mathcal{G}$ and $\mathcal{H}$ are free, so that in the trivial sense, there exist global cross sections for both actions.

Let us take $p = 2$.

First we give an example of a distribution which is in $\mathcal{F}_{\mathcal{G}}$ but not in $\mathcal{F}_{\mathcal{H}}$. Consider the distribution $f(x_1,x_2)dx_1dx_2$ with

$$f(x_1,x_2) = \begin{cases} (1-e^{-2})/(4\pi) & \text{if } (x_1,x_2) = (t\cos\theta,\ t\sin\theta),\ 0 < t < 2,\ 0 \le \theta < \pi, \\ \phi(x_1,x_2) & \text{otherwise,}\end{cases}$$

where $\phi(x_1,x_2) = (2\pi)^{-1}\exp(-(x_1^2+x_2^2)/2)$. It is easy to see that this distribution can be realized as a cross-sectionally contoured distribution associated with $(\mathcal{G},\mathcal{X})$, but not with $(\mathcal{H},\mathcal{X})$.

Next we give an example of a distribution which is in $\mathcal{F}_{\mathcal{H}}$ but not in $\mathcal{F}_{\mathcal{G}}$. Let $\mathcal{Z} = \mathcal{Z}_{(\mathcal{H},\mathcal{X})}$ be (the boundary of) the triangle with vertices $(-1,2)$, $(2,-1)$ and $(-1,-1)$, (see Example 3.1 of Takemura and Kuriki (1996)) and let $y(x_1, x_2)$ be the equivariant part of the orbital decomposition with respective to $\mathcal{Z}$ for $(\mathcal{H}, \mathcal{X})$. Then the distribution of the form

$$f_{\mathcal{Y}}(y(x_1, x_2))dx_1dx_2 \quad \text{with } f \text{ injective}$$

is of course in $\mathcal{F}_{\mathcal{H}}$, but not in $\mathcal{F}_{\mathcal{G}}$. One can confirm the latter assertion by noting that we have only three pairs $\{x, \tilde{x}\}$, $x, \tilde{x} \in \mathcal{Z}$, such that $x = -\tilde{x}$, that is, $\{(0,1),(0,-1)\}$, $\{(1,0),(-1,0)\}$ and $\{(1,-1),(-1,1)\}$.

References

[1] Anderson, T.W. (1984). *An Introduction to Multivariate Statistical Analysis*, Second Edition. Wiley, New York.

[2] Ash, R.B. (1972). *Real Analysis and Probability.* Academic Press, New York.

[3] Bredon, G.E. (1972). *Introduction to Compact Transformation Groups.* Academic Press, New York.

[4] Barndorff-Nielsen, O.E., Blæsild, P. and Eriksen, P.S. (1989). *Decomposition and Invariance of Measures, and Statistical Transformation Models.* Lecture Notes in Statistics, 58, Springer-Verlag, New York.

[5] Dieudonné, J. (1976). *Treatise on Analysis, Volume II.* Enlarged and Corrected Printing. Academic Press, New York.

[6] Kamiya, H. (1996). Borel isomorphism between the sample space and a product space. *Mathematical Methods of Statistics*, **5**, 237–243.

[7] Kawakubo, K. (1991). *The Theory of Transformation Groups.* Oxford University Press. Oxford.

[8] Kuriki, S. (1993). One-sided test for the equality of two covariance matrices. *Ann. Statist.*, **21**, 1379–1384.

[9] Muirhead, R.J. (1982). *Aspects of Multivariate Statistical Theory*, Wiley, New York.

[10] Takemura, A. and Kuriki, S. (1995). Weights of $\bar{\chi}^2$ distribution for smooth or piecewise smooth cone alternatives. Discussion Paper Series 95-F-28, Faculty of Economics, University of Tokyo.

[11] Takemura, A. and Kuriki, S. (1996). Theory of cross sectionally contoured distributions and its applications. Discussion Paper Series 96-F-15, Faculty of Economics, University of Tokyo.

[12] Wijsman, R.A. (1990). *Invariant Measures on Groups and Their Use in Statistics.* Lecture Notes–Monograph Series, Vol. 14, Institute of Mathematical Statistics.