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Evolutionary Game Theory in Economics

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1 INTRODUCTION

The success of game theory in the 1980's has revolutionized economics. In addition to optimization and competitive market equilibrium, the concept of Nash equilibrium became a basic analytical tool and a common language of economists in almost all fields. In his text book, Paul Samuelson once quoted an epigram, "You can make even a parrot into a learned economist; all it must learn are the two words, 'supply' and 'demand'.", but now the parrot needs two more words, 'Nash equilibrium', to be academically correct.

As the game theoretic approach penetrated into many fields, however, some basic problems became apparent. First, it is not clear *how* players come to play a Nash equilibrium. Although Nash equilibrium was once perceived as the outcome of perfectly rational reasoning, active research in the past decade revealed that common knowledge of rationality only implies rationalizability, which is much weaker than Nash equilibrium. Second, game theoretic models quite often possess multiple equilibria *which have markedly different properties*. This is in contrast to the general equilibrium model where all equilibria are efficient. Hence in applications of game theory it is vital to pin down *which* equilibrium is selected. A host of refinements literature tried to solve this problem by defining a stronger notion of rationality than Nash equilibrium assumes, but it was not entirely successful. We are left with a number of new solution concepts, and there seems to be no clear consensus among economists as to which one is right.

Partly due to these difficulties in the rationality approach, the study of boundedly rational adjustment process towards Nash equilibrium became an active field of research in the past few years. The literature is (vaguely) classified into two categories, *learning* and *evolution*. The learning literature typically assumes that players can calculate the best response and examines how players update their beliefs about their opponents' strategies in a *fixed match*. In contrast, the evolutionary approach does not

necessarily assume the ability to optimize and analyzes the evolution of sophisticated behavior through trial and error and natural selection in a *population* of players.

The basic idea of the evolutionary approach is in line with the oral tradition in economics about the justification of rationality. This basic premise of neoclassical economics has often been justified in the following way: Economic agents may not consciously optimize but behave *as if* they were rational, because economic competition selects optimizing agents. This was articulated in particular by Alchian (1950) and Friedman (1953)[1].

A big impetus for the formal study of such a process came from biology. Following the pioneering work on sex ratio by Fisher (1930) and Hamilton (1967), Maynard Smith and Price (1973) introduced the notion of *evolutionary stable strategy (ESS)*, and asserted that the observed traits of animals and plants can be explained by Nash equilibria of suitably defined games. The idea is that the combination of natural selection and mutation lead the population to a stable Nash equilibrium (ESS) in the long run. Since then, this view has been verified in a number of field studies. Here, the 'as if' explanation is not just a plausible parable but rather a quite accurate description of reality. Animals and plants have little or no ability for rational reasoning, yet their behavior can be explained by Nash equilibrium!

Encouraged by the success in biology, a number of economic theorists recently became involved in active research in evolutionary game theory. For economists, this has a feeling of *déjá vu*, because the study of adjustment dynamics has a long history before evolutionary game theory. The *tâtonnement* stability of general equilibrium was extensively studied in the late 1950's, and the adjustment dynamics in oligopoly models goes back to the original work of Cournot in 1838. After extensive research,

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[1] Nelson and Winter (1982) also elaborate on natural selection in market competition.

however, those theories have been criticized as being *ad hoc* and dismissed by some economists. Given this experience, why should we be interested in such an approach again? What differentiates evolutionary game theory from the previous literature on adjustment dynamics?

The answer may be found in the aforementioned two basic problems of game theory. Firstly, unlike previous literature, evolutionary game theory in economics clearly acknowledges that something *ad hoc* is *necessary*. Only after the extensive study of game theory, which clarifies what rationality means in strategic situations and what it implies, the necessity became apparent. Rationality alone fails to justify Nash equilibrium, and the theory must search for some elements other than rationality (i.e. something which is necessarily *ad hoc*) to explain equilibrium behavior.

Secondly, the necessity of equilibrium selection, which became a pressing issue in the course of extensive applications of game theory to a variety of concrete economic problems, is something which the previous literature on adjustment dynamics did not encounter. In the Arrow-Debreu model which equilibrium is selected is rarely discussed, and the traditional oligopoly theory did not enjoy a wide range of applications because it was plagued by a host of alternative 'solution' concepts and deemed as unreliable.

The study of evolutionary game theory in economics has just started and it is rather early to judge how successful it is to achieve its goals. The present paper tries to review the present state of research[2], and is organized as follows. Section 2 presents

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[2] A survey of evolutionary game theory can also be found in Banerjee and Weibull (1992), Bomze and Pötschner (1988), Fudenberg and Levine (1995), Hammerstein and Selten (1994), Hines (1987), Hofbauer and Sigmund (1988), Mailath (1992,93,95), Matsui (1995), Maynard Smith (1982), Samuelson (1993), Selten (1991), van Damme (1987, 93), Vega-Redondo (1995), Weibull (1995) and Young (1995).

the basic concepts developed in biology, and its extensions are discussed in the next section. Section 4 reviews what evolutionary game theory has to say about the 'as if' justification of rationality, and the emergence of efficiency is addressed in Section 5. The next section examines the long run implications of repeated random mutations, and concluding remarks are given in Section 6. For convenience, major results are somewhat roughly summarized in the form of *Claims*.

2 BIOLOGICAL GAMES – ESS AND REPLICATOR DYNAMICS

In this section we review the basic concepts of evolutionary game theory developed in biology, namely, evolutionarily stable strategy and replicator dynamics. *Evolutionarily stable strategy (ESS)* was introduced by Maynard Smith and Price in their seminal paper (1973). This solution concept is meant to capture the stable outcome of evolutionary process, which is driven by natural selection and mutation. Consider a large population of a single species. Individuals are randomly matched in pairs and play a two-person symmetric game $g:A{\times}A{\to}\mathbb{R}$. $A{=}\{1,\ldots,K\}$ is a finite set of pure strategies, and $g(a,a')$ is a payoff of a player when he plays a and his opponent chooses a'. Let S be the set of mixed strategies, and the expected payoff from a mixed strategy profile (s,s') is denoted $g(s,s')$ with an abuse of notation[3]. A strategy may represent a mode of behavior (how aggressive one fights, for example), or a characteristic (such as sex or body size), and each individual is genetically programmed to choose a particular strategy. The payoff g is interpreted as the number of offspring, and called *fitness*. Reproduction is assumed to be asexual and offspring inherit the parent's strategy, unless mutations occur. The strategy distribution in the population can be represented by $s{\in}S$, and this

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[3] Also we do not distinguish pure strategy k and the mixed strategy which plays it with probability one.

admits alternative interpretations. When each individual is programmed to play a pure strategy, s represents the population frequencies of pure strategies. Alternatively, all players may be choosing exactly the same mixed strategy s. In general, when different (possibly mixed) strategies coexist, we say that the population is *polymorphic*: otherwise, we have a *monomorphic* population.

Definition 2.1 A strategy s of a symmetric two-person game is an ***evolutionarily stable strategy (ESS)*** if there is ϵ^0 such that for any $s' \neq s$ and any $\epsilon \in (0, \epsilon^0]$

$$g(s, (1-\epsilon)s+\epsilon s') > g(s', (1-\epsilon)s+\epsilon s') \qquad (2.1)$$

An ESS represents a stable state of population which is resistant to mutant invasion. When the incumbent strategy s is invaded by a small fraction ($\epsilon\%$) of mutants playing s', equation (2.1) says that the incumbents do strictly better than the mutants. As the payoffs represent the number of offspring, this means that the population share of the mutants eventually vanishes.

Given $g(x, (1-\epsilon)y+\epsilon z) = (1-\epsilon)g(x,y)+\epsilon g(x,z)$, the above condition can be simplified: strategy s is an ESS if and only if

(E1) s constitutes a Nash equilibrium ($g(s,s) \geq g(s',s)$ $\forall s'$), and

(E2) if $s' \neq s$ satisfies $g(s',s) = g(s,s)$, then $g(s,s') > g(s',s')$.

Condition (E2) highlights how an ESS refines Nash equilibrium. The unique feature of an ESS is to consider the possibility that a mutant encounters its copy (another mutant).

As an example, consider the infinitely repeated prisoner's dilemma game with no discounting, whose stage game is given in Figure 2.1.

Figure 2.1 here

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Axelrod and Hamilton (1981) showed that the population of "always defect(D)" can be invaded by "tit-for-tat(TFT)"[4]. When TFT meets D, it is cheated in the first stage but then reverts to defection. Since the loss in the first stage is negligible under no discounting, TFT fares as good as D against the incumbent strategy D (both yielding an average payoff of 2). However, when matched with TFT, TFT achieves cooperation (yielding a payoff of 4) and hence does strictly better than D, which yields a payoff of 2 against TFT. Hence, when the population of D is invaded by a small number of TFT, the TFT population grows faster than that of D and eventually takes over the whole population. The upshot is that D is not an ESS.

ESS has been quite successfully applied in biology, especially for a mixed strategy equilibrium. In typical applications, the actual population frequencies of strategies, such as body size, sex, or timing of emergence, are measured and compared with the evolutionarily stable mixed strategy equilibrium. In this regard, it should be stressed that one of the merits of ESS is its ability to differentiate stable versus unstable mixed strategy equilibria. Figure 2.2 may be helpful to understand this point. One can see, by checking (E2), that $s=(1/2,1/2)$ is an ESS in Figure 2.2 (a) but not in (b).

Figure 2.2 here

Note that the traditional refinements fail to capture the stability of completely mixed equilibria: the non-ESS $s=(1/2,1/2)$ in Figure 2.2 (b) passes all the tests of the traditional refinements in economics that are based on trembles (such as perfection, properness or stability). The reason is that the mathematical definition of trembles fails to perturb completely mixed strategies.

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[4] TFT starts with c and then mimics the action taken by the opponent in the last period.

While the definition of ESS does not formalize an explicit dynamic, it is meant to capture a locally stable point of a dynamic process of evolution. In biology, the following process, called the *replicator dynamic*, is widely used. First, we consider the case where each player plays a pure strategy. In this case, the replicator dynamic is given by

$$ds_k/dt = s_k[g(k,s)-g(s,s)]. \qquad (2.2)$$

This is derived as follows. Consider a large (continuum)[5] polymorphic population with pure strategies and suppose that $\alpha\Delta\%$ of players with each strategy breed according to the fitness function g within a small time interval Δ. Let M and κ be the total population and the population of strategy k at time t, and let M' and κ' denote those variables at $t+\Delta$. Similarly, we denote $s=s(t)$ $s'=s(t+\Delta)$ and let $g_k=g(k,s)$, $g=g(s,s)$. Then we have

$$\frac{s_k'-s_k}{\Delta} = \frac{1}{\Delta}\left(\frac{\kappa'}{\kappa}-\frac{M'}{M}\right)\frac{\kappa}{M'} = \alpha\,(g_k-g)\,\frac{\kappa}{M'}. \qquad (2.3)$$

As $\Delta\to 0$, we get $ds_k/dt=\alpha s_k(g_k-g)$, and aside from the speed of adjustment α this is the replicator equation (2.2).

According to the replicator dynamic, the population share of a strategy is increased (decreased) if it does better (worse) than the population average. If this is a reasonable description of an actual population dynamic, it is desirable that its asymptotically stable[6] rest points exactly correspond to the ESS's. This issue was studied by Taylor and Jonker (1978) under a certain restriction,

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[5] Boylan (1992, 95) examines the relationship between the continuum, countable and finite population cases.

[6] We say s^* is *asymptotically stable* if for any $\epsilon>0$ there is $\delta>0$ such that $\|s(0)-s^*\|<\delta$ implies (i) $\|s(t)-s^*\|<\epsilon$ for all $t>0$ and (ii) $\lim_{t\to\infty}s(t)=s^*$.

followed by generalizations by Hofbauer, Schuster and Sigmund (1979) and Zeeman (1981):

Claim 2.1 *ESS is always an asymptotically stable point of the pure strategy replicator dynamic (2.2), but the converse is not true.*

An example of an asymptotically stable point of (2.2) which fails to be an ESS is a mixed strategy equilibrium which works as follows (see van Damme (1987) for the details). Since it is not an ESS, an invading group of mutants can do better than the *average* incumbents. However, as the incumbent population is polymorphic, there can be a particular incumbent pure strategy which does strictly better than the mutants. Then it is possible that the mutants will die out.

The above explanation shows that the definition of ESS (2.2) is motivated by a *dimorphic* situation where a monomorphic population *s* is invaded by a small monomorphic group of mutants *s'*. In such a situation, selection pressure operates only among the two existing strategies, *s* and *s'*, and clearly the system returns to *s* if and only if (2.1) holds. In particular, contrary to the polymorphic situation, *s* cannot be stable unless it is an ESS. Hence, we can expect that the converse in Claim 2.1 is true when players can play mixed strategies. The replicator dynamics where each player chooses a mixed strategy are examined by Hines (1980), Zeeman (1981), Cressman (1990), Weissing (1990) and Bomze and van Damme (1992) and basically their works show:

Claim 2.2 *Under the mixed strategy replicator dynamics asymptotic stability and ESS exactly coincide[7].*

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[7] There can potentially be infinitely many mixed strategies and there is a question of how to reformulate (2.2) if a continuum of mixed strategies coexists (see Hines (1980) and Zeeman (1981)). Also, there are a variety of individual strategy distributions whose population average equals *s*, so the 'stability of *s*' must be

3 EXTENSIONS OF THE BASIC CONCEPTS

Some modifications are necessary when one tries to apply the basic biological concepts to various economic problems. First, the original definition of an ESS applies only to *symmetric two-person* games. Clearly, extension to more general classes is necessary both in economics and in biology. Such modifications are possible but involve some subtle issues, as we will see. Secondly, the basic premises in biology, that strategy is genetically programmed and a payoff represents the number of offspring, do not literally apply in economics. Reinterpretations and alternative formulations are in order in economic applications.

Let us examine the second issue first. In economics, successful strategies may proliferate by means of imitation rather than reproduction. Consider a continuum of players each of whom is randomly matched with another player in each period. Suppose that in each period with probability α each player randomly and independently samples one player from the population and observes the sampled player's payoff with observation error ϵ. The observation error for each player is i.i.d. with uniform distribution on $[-c, c]$, where c is sufficiently large[8]. Assume that a player switches to the sampled strategy if and only if the observed payoff is better than her own payoff. When the current strategy distribution is s, the following fraction will use strategy k in the next round (within the α% of the population who can potentially adjust):

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defined carefully. The cited works basically show that s is an ESS if and only if it is an asymptotically stable population average for *all* underlying individual strategy distributions.

[8] More precisely, $c > \max_{i,j,k,h}[g(i,j) - g(k,h)]$.

$$z_k \equiv s_k \sum_{i,j,h} [Pr(g(k,h)+\varepsilon > g(i,j)) + Pr(g(i,j)+\varepsilon < g(k,h))] s_i s_j s_h.$$

The first term is the fraction of players who sample k and find the observed payoffs better, and the second term represents the original k-users whose observed payoffs are worse than their own. A simple calculation shows $z_k = s_k[g(k,s)-g(s,s)]/c+s_k$, and we have the replicator equation

$$s_k(t+1)-s_k(t) = (\alpha/c)s_k(t)[g(k,s(t))-g(s(t),s(t))].$$

With a similar calculation one can derive the replicator dynamic if (1) there is no observation error, (2) a player switches to the sampled strategy only when the latter is better, and (3) the switching probability is proportional to the payoff difference (Cabrales (1994) and Schlag (1994)). Point (3) can be justified by a positive switching cost which is uniformly distributed. Schlag provides an axiomatic foundation for imitation behavior (2) and (3). Thus we conclude that the replicator dynamic can be justified by imitation, provided that imitation is done in plausible but rather specific ways[9].

In economic application we may assume that players are more rational than are supposed in the above discussion. In particular, players may be aware of the payoff function and can optimize, while revising their expectations in an adaptive way. Gilboa and Matsui (1991) analyze such a case and propose *best response dynamics*:

$$d^+s(t)/dt = \alpha(h(t)-s(t)),$$

$$h(t) \in BR(s(t)). \tag{3.1}$$

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[9] Börgers and Sarin (1993) provides a learning model of *fixed match* (N players playing an N-person game) based on the satisfycing model of Bush and Mosteller (1955), and show that the players' mixed strategies follow the replicator dynamics.

Here, d^+s/dt denotes the right-derivative $\lim_{\Delta \downarrow 0}(s(t+\Delta)-s(t))/\Delta$ and $BR(s)=\Pi_i BR_i(s)$, where $BR_i(s)$ is the set of player i's mixed strategy best responses against s. The best response dynamics assume that $\alpha\Delta\%$ of players with each pure strategy switch to a best response against the current state within a small time interval Δ. Note that under the replicator dynamics a suboptimal strategy can grow if it does better than the average, while only the best response can grow under the best response dynamics. Gilboa and Matsui propose a **cyclically stable set(CSS)** as the stable outcome of their process. We say that s' is reachable from s, if s' is an accumulation point of a trajectory of (3.1) with $s(0)=s$. If s'' is reachable from s', which in turn is reachable from s, we also say that s'' is reachable from s. A CSS is a set such that (i) its elements are mutually reachable and (ii) no point outside is reachable from it[10]. Intuitively a CSS includes stable equilibria and limit cycles (and more complicated objects), and its relationship to evolutionary stability will be discussed later.

Now let us turn to the question of how to extend the notion of an ESS beyond symmetric two-person games. Let us introduce a general n-person game $g:A\rightarrow\mathbb{R}^n$, where $A=A_1\times\cdots\times A_n$ is the finite set of pure strategy profiles. As before, the payoff function g is extended to the set of mixed strategy profiles $S=S_1\times\cdots\times S_n$. First note that extension to *symmetric* n-person games is straightforward, if we reinterpret $g(x,(1-\epsilon)s+\epsilon s')$ in (2.1) as player 1's payoff when she plays x and each of her opponents plays pure strategy k with probability $(1-\epsilon)s(k)+\epsilon s'(k)$.

For a general n-person game, which is not necessarily symmetric, Selten (1980) considers the case where groups of n

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[10] This is the definition employed by Matsui (1991) and (1992). The original definition by Gilboa and Matsui has a weaker notion of reachability. They say that z' is reachable from z if z' is reachable from all neighborhoods of z. This is the stability condition against small perturbation.

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players are randomly chosen from a single population, and then each player in a group is randomly assigned to one of the n players' roles in the stage game. Note that a strategy in this setting is a contingent action plan, specifying one action for each role. Then we can apply condition (2.1) for this symmetrized version to define an ESS in any game. However, this approach does not provide a new concept of equilibrium refinement, as Selten (*op.cit.*) showed:

Claim 3.1 *ESS is equivalent to strict equilibrium under random role assignment*[11].

To see this, take a stage game played by a buyer and a seller, and consider a non-strict equilibrium where the buyer has an alternative best reply. If some players' actions for the buyer's role mutate to the alternative best reply, while their actions in the seller's position remain unchanged, the mutants clearly fare equally well as the incumbents, and the condition (2.1) is violated (it is satisfied with equality). The essential feature underlying the definition of an ESS in a symmetric contest is that new actions are matched together, but under random role assignment this is not necessarily true.

A similar problem arises in extensive form games, where there is no selection pressure upon the actions off the path of play. For example, consider the repeated prisoner's dilemma discussed in Section 2. While D can be invaded by TFT, TFT itself is not an ESS because it can be invaded by strategy C, which always cooperates (both sides of condition (2.1) are equal). In general, *an ESS in an extensive form game fails to exist if there are off-the-equilibrium actions*, because mutants whose actions differ only off the path of play fare equally well as the incumbents.

Let us say that mutants are *neutral* if they fare equally well as the incumbents do, and let us call invasion by neutral mutants

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[11] A strict equilibrium is a Nash equilibrium with the unique best reply for each player.

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evolutionary drift. As we have seen, when we go beyond symmetric two-person normal form games, quite often the possibility of evolutionary drifts causes the non-existence of an ESS. Hence how to treat evolutionary drifts is a crucial point in the extension of ESS to general games.

An easy way to deal with this problem is to simply allow for neutral mutations. Strategy *s* in a symmetric game is called a **weak ESS** or **neutrally stable strategy (NSS)** if (2.1) is satisfied with weak inequality[12]. The population share of neutral mutants remains unchanged until additional mutations happen, so NSS captures the stability in the short or medium run. However, this may not capture the stability in the long run, where the system can drift away from a NSS with enough accumulation of neutral mutations, leading to an unstable point as Figure 3.1 (a) shows.

Figure 3.1 here

This observation motivates the following set-valued concept, introduced by Thomas (1985). First, when we have $g(x,x)=g(y,x)$ and $g(x,y)=g(y,y)$, we write $x \, D \, y$. This is similar to condition (E2), and it shows that y is a neutral mutant which can remain in the incumbent population of x. Note that the system can drift from x to y in the long run by the accumulation of such neutral mutations. Then we define the following.

Definition 3.1 A closed set $X \subset S$ is **evolutionary stable (ES)** in a symmetric two-person game if (i) each element of X is NSS and (ii) $x \in X$ and $x \, D \, y$ implies $y \in X$.

The state can drift within an ES set but cannot go outside; if a small fraction of mutants move the state outside the set, the mutants do strictly worse than the incumbents and the state moves

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[12] Accordingly, the strict inequality in condition (E2) should be replaced with **weak** inequality to define NSS.

back to the set, as Figure 3.1 (b) shows. In fact, Thomas (1985) proves:

Claim 3.2 *An ES set is asymptotically stable under the replicator dynamic (2.2).*

We can see that tit-for-tat (TFT) in the repeated prisoner's dilemma is a NSS but it does not belong to an ES set, as TFT D C but C, which always cooperates, is not Nash (hence not a NSS).

The last point suggests that ES may be modified in economic applications to preclude such fragile mutations as C. Although the drift to C is possible in biological evolution, rational economic agents may hesitate to invade the population of TFT with such a fragile plan. Swinkels (1992a) proposed the following concept:

Definition 3.2 A set $X \subset S$ is *equilibrium evolutionarily stable* *(EES)* in a n-person game if it is a minimal closed non-empty set such that (i) each element of X is a Nash equilibrium and (ii) $\exists \epsilon' > 0$ such that $\forall \epsilon \in (0, \epsilon')$, $\forall x \in X$ and $\forall y \in S$, $y \in BR((1-\epsilon)x+\epsilon y)$ implies $(1-\epsilon)x+\epsilon y \in X$.

A motivation for this concept is to endow a certain amount of rationality for the mutants. If we have $y \in BR((1-\epsilon)x+\epsilon y)$, $\epsilon\%$ of the incumbent population x can be invaded by y, because such a plan is self-enforcing among the mutants. An EES set is robust to such *equilibrium entrants*; the population can never leave an EES set by a series of small equilibrium entries. Swinkels goes on to show that for any extensive form all points in an ESS set generically differ only in off-the-equilibrium behavior and possess the same equilibrium path[13].

Matsui (1992) provided another motivation for EES based on the

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[13]   This is based on the Kohlberg and Mertens' result (1986) that a connected set of Nash equilibria has a single realized outcome for generic extensive form games.

best response dynamics (3.1). First let us modify Definition 3.2(ii) to allow ϵ' to depend on x and y.[14] Also let us allow only piecewise linear trajectories of the best response dynamics in the definition of CSS. Under these modified definitions, Matsui showed:

Claim 3.3 *A set of Nash equilibria is EES if and only if it is a CSS.*

In other words, an EES is a set of Nash equilibria which cannot be upset by the best response dynamics. The basic reason for this result is the following. It can be shown that for any given $x \in X$ and y, if $y \in BR((1-\epsilon)x+\epsilon y)$ holds for a small enough $\epsilon = \epsilon'$, it also holds *for all* $\forall \epsilon \in (0, \epsilon')$. Then, clearly there is a best response path from x to $(1-\epsilon')x+\epsilon'y$.

Selten (1983) proposes a different line of approach to deal with the evolutionary drifts in extensive form games. We may expect that the information sets off the path of play are reached with small but strictly positive probabilities in reality by various reasons which are ignored in the formal model. If there are such 'trembles', selection pressure is at work off the equilibrium path, and such effects provide a sharper prediction. Selten defines the notion of **limit ESS** to formulate this idea for a symmetric two-person extensive form game[15]. Let Γ be a symmetric two person extensive

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[14] Formally, $\forall x \in X$ and $\forall y \in S$, $\exists \epsilon' > 0$ such that for any $\epsilon \in (0, \epsilon')$, $y \in BR((1-\epsilon)x+\epsilon y)$ implies $(1-\epsilon)x+\epsilon y \in X$. Swinkels (1992a) conjectures that this is equivalent to the original definition 3.2(ii).

[15] As we have seen, an asymmetric extensive form game can always be symmetrized by means of random role assignment. On the other hand, a given extensive form sometimes admits different ways to define symmetry. For the latter problem, see Selten (1983) or van Damme (1987) for details. The argument below presumes that a

form game, and let b, h, and $C(h)$ denote a behavioral strategy for player 1, an information set for either player and the set of available actions at h. We will consider symmetric 'trembles', which they can be represented by a function δ from $\cup_h C(h)$ to $\mathbb{R}$ that satisfies (i) $\delta(c) \geq 0$ $\forall c$ and (ii) $\Sigma_{c \in C(h)} \delta(c) \leq 1$ $\forall h$, and (iii) $\delta(c) = \delta(c')$ whenever player 1's action c corresponds to player 2's action c' under the given symmetry. A perturbed game (Γ, δ) is a game where each action c must be taken with a probability greater than or equal to $\delta(c)$. Suppose b_k and its symmetric counterpart for player 2 constitute an ESS (defined with respect to behavioral strategies) of (Γ, δ_k) for each k, where $\delta_k \to 0$ and $b_k \to b^*$ as $k \to \infty$. Then we say that b^* is a limit ESS. Note that an ESS is always a limit ESS because $\delta_k \equiv 0$ $\forall k$ is allowed.

Finally, we comment on the existence of various solution concepts:

Claim 3.4 *ESS, NSS, ES set, EES set, and limit ESS do not always exist, while CSS always exists.*

The reason is that those concepts try to capture the stable outcome of the evolutionary process but there is no guarantee that the process converges to a Nash equilibrium. In some cases, a limit cycle can be a global attractor. Therefore, all those concepts which require the solution to be Nash equilibrium fail to have the existence property. To get existence, we must allow for other behavior patterns such as limit cycle, as CSS does. In this respect it is interesting to note that limit cycles can provide a certain justification for Nash equilibria. Schuster, Sigmund, Hofbauer and Wolff (1981) show that if all strategies appear on the limit cycle of the replicator dynamic (2.2), the time average on the limit cycle corresponds to a completely mixed strategy equilibrium.

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symmetry has already been defined for the game under consideration.

4 EVOLUTION AND RATIONALITY

In this section we examine if the evolutionary models developed above actually support Alchian-Friedman's thesis of 'as if' rationality. In a strategic situation, various degrees of rationality can be defined, ranging from the use of undominated strategy, rationalizability, Nash equilibrium, sequential rationality and forward induction. We examine how they are justified by evolutionary processes, and also show that evolution can explain certain types of irrational behavior.

4.1 Optimization

Absent strategic interaction (i.e. for a single person decision problem), combination of selection pressure and mutation should lead to the optimal choice. This rather straightforward intuition can fail when we have uncertainty. Suppose that a choice s induces a probability distribution $q(s)=(q_1(s),\ldots,q_I(s))$ over the set of realized payoffs $\{R_1,\ldots,R_I\}$. The underlying shocks are *global*, such as weather, so that *in each period the players with the same strategy receive the same realization of R*, but the shocks are independent over time. After T periods, the number of offspring of a player is $W=R(1)\times\cdots\times R(T)$, where $R(t)$ is the realized payoff at time t. Since $E[W]=\Pi_t E[R(t)]$, the strategy that maximizes the *number* of offspring in the long run is equal to the expected payoff maximizer. However, it does not maximize the long-run *population share*. Suppose a player and its offspring adopt a strategy which induces distribution q. By the law of large numbers, if T is sufficiently long, R_i realizes approximately in $q_i T$ periods. Then, with a large probability the number of offspring after T is approximately

$$(R_1^{q_1}\cdots R_I^{q_I})^T \equiv L^T.$$

Hence if a strategy (uniquely) maximizes L, its expected population

share is almost 1 for a large enough T. As maximizing L is equivalent to maximizing $\log L = \Sigma_i q_i \log R_i = E[\log R]$, we have:

Claim 4.1 *If payoff R is subject to common random shocks across players, the strategy which maximizes $E[\log R]$ dominates the population share with probability one.*

As $R = e^{\log R}$, the log of payoff (fitness) is interpreted as the growth rate. Hence, biologists use expected growth rate, rather than expected fitness under the kind of uncertainty described above. In economics, a strategy may be interpreted as an investment opportunity and R as the return for an asset. The above analysis shows that the market selects the optimal choice for Neumann-Morgenstern utility $u(R) = \log R$. Hence, when the investors are risk neutral, economic natural selection leads to a suboptimal, rather cautious behavior in the long run, and Alchian-Friedman's thesis fails for this important class of situations. See Blume and Easley (1992, 93, 95) for a systematic treatment.

4.2 Domination and Rationalizability

If a game is played by rational players, and if this fact is common knowledge, the outcome should be rationalizable as defined by Bernheim (1984) and Pearce (1984). Let us say that a player is first order rational if she can calculate the best responses. A player is n^{th} order rational if she takes a best response given that the other players are $(n-1)^{\text{th}}$ order rational. A rationalizable strategy is defined to be an ∞-order rational choice[16]. Suppose

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[16] In a two-player game, iterative elimination of strictly dominated strategies, when performed in the space of mixed strategies, results in the set of all rationalizable strategies. In general, a best response is undominated but not vice versa, so that rationalizable strategies are included in the set of iteratively strictly undominated strategies.

players are first order rational and update their beliefs in an *adaptive* fashion, in the sense that a strategy which is never played after a finite period will asymptotically receive zero probability. Fictitious play in the learning literature and the best response dynamics satisfy this property. Then, a player's rationality evolves towards a higher order over time, and in the long run players will end up choosing rationalizable strategies. See Milgrom and Roberts (1991) for a comprehensive treatment of this issue:

Claim 4.2 *If players take best response against expectations which are revised adaptively, only iteratively strictly undominated strategies survive.*

The above argument needs to be modified when the players are not even first order rational. Under natural selection or imitation where players switch to *better* strategies, a strategy doing sufficiently well may survive even if it is never a best response. The following example, due to Dekel and Scotchmer (1992) serves as such an example. Figure 4.1 is the column player's payoff matrix of a symmetric 2-person game, which is basically a Rock-Paper-Scissors game with an additional strategy D. Assume that D is strictly dominated by the mixture of the first three strategies (i.e. $(a+b+c)/3 > a+\epsilon$). Absent D, the population share under a variety of better response dynamics, including the discrete time replicator dynamic (2.3) with simultaneous reproduction ($\alpha=\Delta=1$), can approach the cycle of Rock→Paper→Scissors→···, as depicted in Figure 4.2.

Figure 4.1 and Figure 4.2 here

Because $\epsilon>0$, D can proliferate when a majority of the population play a single strategy, and therefore it can survive near this cycle. Hence we conclude that the strictly dominated strategy can survive under some better response dynamics.

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This example has the property that D is strictly dominated by a *mixed* strategy. In contrast, Samuelson and Zhang (1991) show:

Claim 4.3 *If a pure strategy is strictly dominated by another pure strategy (or, iteratively strictly dominated in pure strategies), it cannot survive under any monotonic dynamic, provided that all strategies exist initially.*

Monotone dynamic[17] is a class of selection process over pure strategies

$$ds_i/dt = s_i F_i(g(1,s),\ldots,g(K,s),s) \tag{4.1}$$

whose 'growth rate function' F_i satisfies

$$g(i,s)>g(j,s) \Rightarrow F_i > F_j. \tag{4.2}$$

Claim 4.3. holds because the relative share s_j/s_i under such a dynamic tends to zero, if j is strictly dominated by i.

A strenghtnening of (4.2),

$$g(x,s)>g(j,s) \Rightarrow \Sigma_i F_i x_i > F_j \tag{4.3}$$

is called *convex monotonicity*. Hofbauer and Weibull (1995) showed a sweeping result:

Claim 4.4 *Convex monotonicity is necessary*[18] *and sufficient for a selection dynamic (4.1) to eliminate iteratively strictly dominated strategies, when all strategies are present initially.*

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[17] Monotone dynamics are also examined by Friedman (1991) and Nachbar (1990).

[18] When selection dynamic (4.1) is not convex monotone, it cannot eliminate the dominated strategy D, when applied to the game in Figure 4.1 for some parameter values.

Convex monotonicity admits a wide class of selection dynamics, including the replicator dynamic. Hence we conclude that players act as if rationality were common knowledge under a wide class of selection processes.

4.3 Refinements

Van Damme (1987) showed:

Claim 4.5 *An ESS is a proper equilibrium.*

Therefore ESS is also trembling-hand perfect. Van Damme also showed (1984) that a proper equilibrium in the normal form of an extensive form game induces a sequential equilibrium. Those two facts, taken together, have a remarkable implication that such a sophisticated behavior as sequentiality can evolve from a minimal amount of rationality. However, this observation is in a sense vacuous, because an extensive form most often possesses no ESS because of the evolutionary drifts off the path of play. Swinkels (1992b) handles this problem by his set theoretic notion of EES. With this concept, he provides an evolutionary justification not only for backwards induction (sequentiality), but also for forward induction (the 'never a weak best response' property of Kohlberg and Mertens (1986)). He first modifies the definition (ii) of EES as; (ii)" $\exists$ a neighborhood of X, denoted U, such that $\forall \epsilon \in (0,1)$, $\forall x \in X$ and $\forall y \in S$, $z = (1-\epsilon)x + \epsilon y \in U$ and $y \in BR(z)$ imply $z \in X$.[19] Then he shows the following.

Claim 4.6 *In a generic extensive form, if an EES set (defined with*

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[19] The difference between (ii) and (ii)" is that the latter measures perturbations by the distance the strategy distribution is moved, rather than the fraction of entrants. For generic two person extensive form games, the difference is inconsequential (Swinkels (1992a)).

(ii)") is convex[20], then the set possesses a single outcome, and it satisfies both backwards induction (sequentiality) and forward induction (the never a weak best response property).

This comes from the fact that, under the stated assumptions, the EES set contains a set that is both fully stable (implying sequentiality) and stable (implying forward induction)[21]. Swinkels demonstrates that in the beer-quiche example (Cho and Kreps (1987)) only the 'right' (beer drinking) equilibria correspond to an EES set.

4.4 Irrationality

Evolutionary game theory also has a potential to explain particular *irrational* behavior that we observe in reality. One prominent reason why irrational behavior may survive is that *commitment* to a suboptimal strategy pays in strategic situations. Rational people may give way to a crazy driver, and a stubborn player with a short temper may obtain better terms in negotiation than a sensible person does. Frank (1897, 88), Banerjee and Weibull (1995), Biasis and Shadur (1995), and Carmichael and MacLeod (1995) elaborate on this point. This idea is built on the assumption that there is a credible way to signal that a player is committed to a certain type of behavior. Frank argues that physical symptoms of emotional arousal, such as posture or facial expression, can serve as such signals. A weakness of this theory is to ignore the

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[20] Swinkels (1992b) employs a weaker topological condition, which admits the case where the EES set is homomorphic to a convex set. He goes on to show that an EES set is always convex in a generic two-person extensive form game.

[21] Also see Ritzerburger and Weibull (1994) and Swinkels (1993) for the relationship between dynamic stability and strategic stability.

possibility that mutants who are not committed but only mimic the signals can proliferate.

Casual observation suggests that people tend to put more emphasis on relative performance than is justified by rationality. ("peer pressure" or "jealousy"). In a small population where a player's action has a great impact on others, a suboptimal behavior which harms one's opponents more than oneself may prosper. Evolution of such *spiteful behavior* has been studied by Hamilton (1970), Schaffer (1988), Crawford (1991), Rhode and Stegeman (1995), and Vega-Redondo (1995). There is also a large body of literature on evolution of altruism in biology. See, for example, Hamilton (1972). Given those observations, one may take a fundamentalist view that evolution is the first principle and rationality is only one of its many consequences.

5 EVOLUTION AND EFFICIENCY – SECRET HANDSHAKE

One of the most notable implications of evolutionary game theory is that efficient outcomes emerge in some important classes of games in economics. This idea dates back to Axelrod and Hamilton's paper (1981) on the evolution of cooperation (see also Axelrod (1984)) in the repeated prisoners' dilemma, which has attracted much attention. As we have seen, "tit-for-tat" is evolutionarily stable (a NSS), while "always defect" is unstable. However, it turns out that some inefficient equilibria are also evolutionarily stable. For example, it is easy to check that the following strategy is a NSS: (i) it cooperates every other period and (ii) reverts to defection ever after the opponent deviates from the behavior pattern (i). Hence one cannot readily conclude that evolution always leads to cooperation. This problem was resolved by Fudenberg and Maskin (1990) and Binmore and Samuelson (1992), who found reasonable sets of assumptions under which the unique evolutionary outcome of the repeated prisoners' dilemma is efficient.

On the other hand, Matsui (1991), Wärneryd (1991), and Kim and Sobel (1995) discovered that evolutionary process derives meaningful pre-play communication and leads to an efficient outcome in a class of games. These two sets of results, evolution of cooperation in repeated games and evolution of meaningful pre-play communication, share the same basic logic, which Robson (1990) calls 'secret handshake'. As we already have an excellent survey of the former issue (Fudenberg (1992)), we will focus on the latter. Sobel (1993) provides a general survey of those issues.

Consider the game in Figure 5.1.

Figure 5.1 here

If players can discuss before playing this game, our intuition strongly suggests that they agree to play the efficient equilibrium (a,a). However, explaining such 'meaningful pre-play communication' had long been a notoriously difficult open problem before evolutionary game theory.

To see this, let us formally incorporate pre-play communication into the above game. In the first stage, each player simultaneously sends a costless message ('cheap talk') from a set M. After seeing the messages they choose actions in the above game. A strategy in this augmented game is a pair $s=(m,\sigma)$, where $m \in M$ and σ is a function from M^2 to $\{a,b\}$. Once this setup is made, it is easy to see that not all equilibria have meaningful pre-play communication. For example, strategy s^0 which sends a fixed message m^0 and always plays the inefficient action b constitutes a Nash equilibrium. Hence traditional equilibrium analysis fails to derive meaningful communication.

In contrast, evolution can explain how meaningful communication emerges. First we show that the population of s^0 can be invaded by s^1, which sends a different message m^1 and plays the efficient outcome a if and only if the exchanged message profile is (m^1,m^1). The mutant strategy takes the efficient action if and only if the opponent is another mutant, and otherwise it takes the same action

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as the incumbent does. Clearly, the mutant fares strictly better than the incumbent after invasion, and therefore s^0 is not an ESS (nor a NSS). In contrast, any strategy which plays the efficient action cannot be outperformed by mutants and therefore is a NSS[22].

Note that the following, which Robson (1990) calls '*secret handshake*', are the key to upset the inefficient equilibrium s^0.

(SH1) Mutants send a new message to identify themselves and
      achieve cooperation among themselves.
(SH2) Incumbents do not react to the new message.

To kill more complicated inefficient equilibria we need to apply the logic of the secret handshake in a more sophisticated manner. First, if the incumbent chooses all messages with positive probabilities and always plays *b*, mutants cannot identify themselves ((SH1) fails). Such a strategy is called the babbling equilibrium. Secondly, if the game is given by Figure 5.2, the incumbent can react to the mutant and (SH2) fails.

Figure 5.2 here

For example, consider strategy s^2, which sends m^0, and plays *b* if the opponent sends m^0, but otherwise plays *c*. Against this strategy, anyone sending a new message is punished, so mutants cannot directly invade.

One way to deal with these problems is to invoke evolutionary drifts. Since the choice of message is inconsequential in the babbling equilibrium, the probability distribution over messages can drift. When a particular message becomes sufficiently rare, mutants can invade by using that message. As for strategy s^2, the punishment is off the path of play in the incumbent population, so it can drift. When sufficiently many incumbents cease to choose *c*

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[22] We cannot have an ESS because the actions off the path of play can drift, as we have seen in Section 2

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against unsent messages, mutants can invade. In either case, for any inefficient equilibrium s' there is a non-NSS s'' such that s' D s''. Hence the inefficient strategies do not belong to ES sets[23]. This observation can be generalized. Consider a two-person game, which has a unique weakly Pareto efficient strategy profile. Such a game is called a *common interest game*.

Claim 5.1 *Assume that a common interest game is played with pre-play communication. Then there is an unique ES set, and each of its elements supports the efficient outcome.*

The above argument is based on substantial accumulation of evolutionary drifts and it may take a long time to kill inefficient equilibria. The availability of a new message may not be a fundamental problem because in reality there should always be some unsent messages. As for the drifts to derive (SH2), Bhaskar (1994) has shown that fast evolution is possible if there is a possibility of misperception of messages. If the receiver's perception is not known to the sender, it is not a Nash equilibrium to react to unsent messages, so without any drifts inefficient equilibria can be killed.

Finally, we briefly explain the evolution of efficiency in the repeated prisoners' dilemma. When we assume no discounting, actions in early stages of a repeated game, which do not affect the total payoff, can serve as costless messages to derive (SH1). For (SH2) to hold, players should not react harshly to such deviations, and Fudenberg and Maskin (1990) introduce trembling hands and Binmore and Samuelson (1992) employ cost of complexity to suppress harsh punishment.

6 STOCHASTIC EVOLUTION

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[23] The same conclusion holds if one replaces an ES set with an EES set or a CSS.

The theories reviewed above rely on mutation, and most models treat mutation as a small one-shot perturbation of the underlying dynamical system. There are also some results relying on a substantial accumulation of neutral mutations, or evolutionary drift. In either case, random mutation is not explicitly incorporated in the formal models. In this section we will see that explicitly modeled dynamics with random mutations can provide sharper predictions in a specific sense. Such an approach was initiated by the seminal paper by Foster and Young (1990).

6.1 A motivating story

Let us begin with a simple story (taken form Kandori, Mailath and Rob (1993), abbreviated KMR hereafter) to understand the basic logic and scope of potential applications. Imagine that there is a dormitory with ten graduate students. Each student is either using a Mac or IBM personal computer and the situation can be modelled as a random matching coordination game (Figure 6.1).

Figure 6.1 here

The stage game has two pure strategy equilibria, (Mac, Mac) and (IBM, IBM). The mixed strategy equilibrium assigns probability 1/3 to Mac, and this means that Mac is the best response if more than one third of the fellow students are Mac users. The students buy new computers every once in a while, and when they do, they switch to the current best response. We also assume that, with a small probability ϵ, the students 'mutate' to a suboptimal choice. This may be caused by temporal shocks to the payoffs or mistakes. Or, each student sometimes exits from the dormitory and is replaced by a new comer, who brings his/her own computer.

Although the two pure strategy equilibria are locally stable (ESS's), each equilibrium can be upset *in the long run* where a large number of repeated mutations can happen. As the mutation rate ϵ tends to zero, it becomes harder to upset either equilibrium, but

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upsetting the IBM equilibrium becomes much harder. This is because upsetting the IBM equilibrium requires seven mutations, while upsetting the Mac equilibrium needs only four mutations. When the mutation rate is small, the former is much less likely to happen[24].

Hence, if we look at the relative proportion of time spent on the Mac equilibrium in an *infinite* time horizon, it converges to one as ϵ tends to zero. In this sense in the (very) long run the Mac equilibrium is much more likely to appear under a small mutation rate[25].

The above story is informal and will be made precise shortly. However, it illustrates both the strength and limitations of such an approach. First, note that stochastic evolution can select among strict equilibria, while this is impossible for virtually all existing equilibrium refinements, including ESS, based on the local stability of equilibria.

On the other hand, such a strong prediction is valid only under a specific class of situations. Whereas the crisp equilibrium selection requires a vanishing mutation rate, the same condition makes the waiting time to see the long run effects indefinitely long. For example, assuming fast (instantaneous) adjustment, KMR report that the expected time to upset each equilibrium is about 78 periods for IBM and 100,000 for Mac, when the mutation rate is .1. When the analyst's time horizon is less than 100, the initial condition (or the 'path dependence') is clearly more relevant than the long run prediction. On the other hand, if the relevant horizon is more than 1,000, for example, the long run effects cannot be

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[24] This assumes that a mutation always happens with the same probability ϵ. Bergin and Lipman (1994) point out that if the mutation rate is state dependent, almost anything can happen in the long run.

[25] A similar logic is utilized by a stochastic algorithm to find the global maximum (*'simulated annealing'* introduced by Kirkpatrick *et al.*(1983)).

ignored.

A similar caveat applies to the population size. If the dormitory in the above story had 1,000 students, upsetting the IBM equilibrium would require more than 333 mutations to Mac, and this is clearly very unlikely in any reasonable time horizon. Therefore stochastic evolution is most relevant for a small population.

6.2. The discrete model

Now we present formal models to capture the long run effects of repeated random mutation. As the present approach is most relevant for a small population, we first consider the discrete model, which has a finite population and discrete time. Such a model was first introduced by KMR (1993), and generalized by Young (1993a) and Kandori and Rob (1995). Consider a general n-person game $g:A \to \mathbb{R}^n$. We consider the situation where a finite number of players play this game by random matching[26]. State z represents the current strategy distribution. We assume that each player chooses a pure strategy, so the state space is a finite set and denoted $Z=\{1,\ldots,J\}$. In each period, a certain set of players are selected (possibly randomly) and they adjust to myopic best responses. In addition, we assume that at the end of each period, each player 'mutates' with probability ϵ. When a player mutates, she randomizes with a fixed probability distribution, which assigns a strictly positive probability for each action.

This defines a time-homogeneous Markov chain. The transition probability is denoted by $p_{ij}=\Pr(z(t+1)=j \mid z(t)=i)$, and let P be the matrix whose ij element is p_{ij}. With this notation the law of motion

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[26] The present formulation incorporates various matching schemes, such as the n-population case, random role assignment with a single population, random matching with a single population for symmetric games, or non-uniform random matching (local interaction).

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is given by $q(t+1)=q(t)P$, where $q(t)$ is the row vector representing the probability distribution over the state space Z at time t. The stationary point of this process μ ($\mu=\mu P$) is called a *stationary distribution*.

When the mutation rate ϵ is zero, the long run outcome generally depends on the initial condition. A possible long run outcome in the absence of mutation can be captured by a set $X{\subset}Z$ satisfying: (i) $\forall i, j{\in}X \exists t$ s.t. $\Pr(z(t)=j|z(0)=i)>0$, and (ii) $\Pr(z(1){\in}X|z(0){\in}X)=1$. Such a set is called a *recurrent communication class* or *limit set*, and it is similar to a CSS in the continuous framework. It may represent a Nash equilibrium or a cyclical behavior, and the system is eventually absorbed in one of such sets if mutations are absent.

In contrast, with a positive mutation rate the system fluctuates all over the state space, and the dependence on the initial condition vanishes in the long run[27]. More specifically, when $\epsilon>0$, there is a unique stationary distribution $\mu(\epsilon)$, and it satisfies the following two properties for any initial condition $q(0)$: (i) $\lim_{t\to\infty}q(t)=\mu(\epsilon)$, and (ii) $\mu(\epsilon)$ represents the relative proportion of time spent on each state (within infinite horizon). Such a system is called *ergodic*[28].

When ϵ is small, the long run behavior can be captured by the *limit distribution* $\mu^*=\lim_{\epsilon\to 0}\mu(\epsilon)$. It can be shown that the limit distribution always exists and its support is a collection of limit

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[27] Canning has a series of works emphasizing this point (see (1992)).

[28] The stated properties are known to hold for any irreducible and aperiodic finite Markov Chain. A Markov chain is irreducible when all states are mutually reachable within finite periods. When the greatest common divisor of $\{t=1,2,\ldots|\Pr(z(t)=i|z(0)=i)>0\}$ is 1 for any state i, the Markov chain is called aperiodic. It is easy to check that these two conditions are satisfied when $\epsilon>0$.

sets[29] A limit set in the support of μ^* is called a *long run equilibrium*, and its element is called a *stochastically stable state* or *long run state*. The Mac equilibrium in the above example is such a state, and it is most often observed in the long run when the mutation rate is small.

Long run equilibria can be found by the following graph theoretic technique. First, given a finite path $f=(z(0),\ldots,z(T))$, the associated cost $c(f)$ is defined to be the minimum number of mutations to realize f with a positive probability. For a pair of limit sets X and Y, the cost of transition from X to Y is defined by $C(X,Y)=\mathrm{Min}_f c(f)$, where the minimum is taken over the set of all paths from X to Y. Consider a tree whose set of nodes consists of all limit sets, and assume that it is directed into the root. Let us call such a graph a *transition tree*, and technically it is a collection of directed branches (X,Y) where X and Y are limit sets. We define the *cost of a transition tree h* by

$$\sum_{(X,Y)\in h} C(X,Y).$$

Then, we have the following 'mutation counting' technique, originally due to Freidlin and Wentzell (1984). Their method was modified by KMR (1993) to analyze small population (discrete) models, followed by Young (1993a)'s simplification to give the present form.

Claim 6.1 *A limit set is a long run equilibrium if and only if it is the root of a minimum cost transition tree.*

This procedure provides a selection among the limit sets, and some remarks are in order. First, in all existing results, the selection turns out to be unique for generic normal form games. Although the generic uniqueness of a long run equilibrium is still

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[29] The latter comes from the fact that the limit distribution is a stationary distribution for $\epsilon=0$.

a conjecture, it has been proved for a special class of games (see KMR (1993) for 2x2 games and Kandori and Rob (1992) for games with bandwagon effects). Second, for a class of games where the dynamic without mutation always converges to one of the Nash equilibria (Young (1993a) calls such a dynamic *acyclic*), stochastic evolution can provide a unique selection even among strict equilibria. Pure coordination games, supermodular games, and games with bandwagon effects fall in this category of games (Kandori and Rob (1992, 1995)).

6.2 Applications of the discrete model

The above technique has been applied to various games. KMR (1993) show:

Claim 6.2 *In a symmetric 2x2 game with two strict equilibria, the risk dominance equilibrium is the unique long run equilibrium.*

Figure 6.2 here

According to Harsanyi and Selten (1988), equilibrium $X=(x,x)$ in Figure 6.2 risk-dominates $Y=(y,y)$ if and only if $a-c > d-b$. Calculation shows that x is the best response if and only if the opponent plays x with a probability more than

$$p=(d-b)/(a-c+d-b). \tag{6.1}$$

Since $p<1/2$ if and only if $a-c > d-b$, the risk dominant equilibrium is the one with the larger basin of attraction. The game has two limit sets, corresponding to the two strict equilibria, and (ignoring the integer problem) the cost of transition is pM for the transition from Y to X and $(1-p)M$ for the converse. Therefore, the root of the minimum cost tree is the risk dominant equilibrium.

A $m \times m$ symmetric game satisfies the *marginal bandwagon property* if for any a and b, $g(a,c)-g(b,c)$ is maximized when $c=a$. Kandori

and Rob (1992) and Maruta (1995) show that if an equilibrium pairwise risk dominates other equilibria in a game with marginal bandwagon property, it becomes a unique long run equilibrium[30]. In a pure coordination game, which is a symmetric two-person game with $g(a,a)>0$ and $g(a,b)=0$ for $a \neq b$, the Pareto efficient equilibrium is the unique long run equilibrium (Kandori and Rob (1995)). Ellison (1994) synthesizes these results by showing that a 1/2-dominant equilibrium is the unique long run equilibrium. In a symmetric game, strategy x constitutes a 1/2-dominant equilibrium if x is the unique best reply when more than 1/2 of the population is playing x.

Young (1993b) analyzes the Nash demand game and shows a striking result:

Claim 6.3 *For the Nash demand game (simultaneous offer bargaining game), the unique long run equilibrium is the Nash bargaining solution.*

Consider two populations of players, such as buyers and sellers, who are going to bargain over a $1 surplus. A seller and a buyer simultaneously announce their shares, α and β. If $\alpha+\beta \leq 1$, they get the announced shares, and enjoy payoffs $u(\alpha)$ and $v(\beta)$. Otherwise they get zero payoffs. Note that any exhaustive division $\alpha+\beta=1$ is a Nash equilibrium, and moreover, it is a strict equilibrium if both α and β are strictly positive. Young discretizes the strategy space by $S=\{0,1/H,2/H,\ldots,1\}$, and shows that for a sufficiently large integer H, the long run equilibrium is the one that approximates the Nash bargaining solution $\text{Argmax}_{\alpha+\beta \leq 1} u(\alpha)v(\beta)$.

Nöldeke and Samuelson (1993) provide a useful characterization of the long run equilibria and analyze extensive form games. Their result formalizes the idea that the set of long run equilibria should be locally stable under dynamics without mutations. We say

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[30] Young (1993) provides an example, which shows that the assertion is not true without the marginal bandwagon condition.

that a limit set X' is weakly reachable from a limit set X if $C(X,X')=1$. If X'' is weakly reachable from X', which is in turn weakly reachable from X, we say that X'' is weakly reachable from X. A collection of limit sets constitutes a *locally stable component* if they are mutually weakly reachable and no other limit set is weakly reachable form them. This corresponds to a version of CSS (see footnote 10) and is similar in spirit to an ES set and an EES set. As we have seen, such a set-valued concept captures the effects of evolutionary drifts and is useful in analyzing extensive forms. Nöldeke and Samuelson show:

Claim 6.4 *The set of long run equilibria corresponds to a collection of locally stable components.*

Using this, they show that a subgame perfect equilibrium emerges in the long run as a unique outcome only under a strong set of assumptions (1993). In the following paper (1995), they examine signaling games and show:

Claim 6.5 *In Spence's job market signaling game with two types, if the long run equilibria possess a unique outcome, it must be an undefeated equilibrium[31].*

Ellison (1993) shows that fast convergence to the stationary distribution holds true even in a large population, when the players interact locally. Suppose each player interacts with a small number of neighbors. If the neighborhoods have meager overlaps, the model is essentially a collection of isolated small populations and trivially fast convergence occurs. Ellison's contribution lies in the opposite, non-trivial case, where the overlap is substantial. Take the Mac versus IBM game (Figure 6.1),

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[31] Undefeated equilibrium is introduced by Mailath, Okuno-Fujiwara and Postlewaite (1993), and gives somewhat different predictions than strategic stability.

and consider a large population covered by a number of small overlapping neighborhoods. Suppose more than 1/3 of each neighborhood overlaps with an adjacent one. (Recall that 1/3 is the critical mass of Mac users to make it a best response). Even if all players are initially using IBM, if one neighborhood mutates into Mac, the adjacent neighborhoods may switch to Mac, thanks to the substantial overlap. Then, this 'domino effect' continues until all players use Mac. Note that the opposite transition, from Mac to IBM, requires a substantial number of mutations in *each* neighborhood and therefore is very unlikely. Ellison also reports an example where different matching structures produce different long run equilibria. Other models of local interaction include Blume (1993), An and Kiefer (1992), Goyal and Janssen (1993), and Anderlini and Ianni (1994). Durlauf (1991) and Aoki (1994) apply related techniques to macroeconomic problems.

6.4 The Continuous Model

The effects of perpetual randomness can be examined in a large (continuum) population in continuous time, using Brownian motion, and such a formulation provides somewhat different results. Foster and Young (1990) presented a continuous model, followed by Fudenberg and Harris (1993) and Vaughan (1993). Here we present Vaughan's approach, which provides a simple closed form solution.

Consider the game in figure 6.2 and let $z \in [0,1]$ represent the population of players adopting strategy x. The law of motion is given by the stochastic differential equation

$$dz = G(z)\,dt + \sigma(z)\,dW, \tag{6.2}$$

where dW represents the standard Brownian motion. The drift term $G(z)$ reflects selection pressure, and its graph is depicted in Figure 6.3, where p corresponds to the mixed strategy equilibrium and is given by (6.1).

Figure 6.3 here

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Let $q(z,t)$ denote the probability density of state z at time t. It is known that the evolution of the system is described by the Fokker-Planck equation (or Kolmogorov's forward equation)

$$\frac{\partial}{\partial t} q(z,t) = -\frac{\partial}{\partial z}[G(Z)q(z,t)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[\sigma^2(z)q(z,t)]. \quad (6.3)$$

Clearly this is satisfied if $\partial \Pr(z(t)\leq z)/\partial t = \mathcal{F}q(z,t)$, where the operator $\mathcal{F}$ is defined by

$$\mathcal{F}q = -Gq + \frac{1}{2}\frac{\partial}{\partial z}\sigma^2 q.$$

We assume that (6.2) has reflecting boundaries, and this is formulated as the boundary conditions $\mathcal{F}q(0,t)=\mathcal{F}q(1,t)=0$. Once the system reaches the stationary distribution $\mu(z)$, for any state z the probability mass below z does not change ($\partial \Pr(z(t)\leq z)/\partial t=0$), so we must have

$$\mathcal{F}\mu(z)=0, \quad\quad\quad (6.4)$$

not only at the boundaries $z=0,1$ but everywhere. For simplicity, let us consider a special case where $\sigma^2(z)\equiv\sigma^2$. Solving the differential equation (6.4) with $\int_0^1\mu(z)=1$ yields

$$\mu(z) = \frac{e^{\psi(z)/\sigma^2}}{\int_0^1 e^{\psi(x)/\sigma^2}dx}, \quad\quad\quad (6.5)$$

where ψ/σ^2 is called the *potential* and given by

$$\psi(z) = 2\int_p^z G(x)\,dx.$$

As $\sigma^2\to 0$, both the denominator and numerator of (6.5) diverge, so the stationary distribution places probability 1 at the state which maximizes ψ. Hence we have:

Claim 6.6 *Under the continuous dynamic (6.2) for a symmetric 2x2 game, the long run equilibrium is the state with the maximum*

potential.

Hence equilibrium (x,x) (corresponding to state $z=1$) is the long run equilibrium if area B is larger than area A in Figure 6.3.

A couple of remarks are in order about the above result. First, the above model provides a simple closed form for all mutation rates (here measured by σ^2), whereas the discrete model in the previous section is intractable unless the mutation rate is vanishingly small. Second, the clean result depends on the assumption that the state space is one-dimensional (i.e. the underlying game is a 2×2 symmetric game). For a higher dimensional case, a closed form solution can only be obtained under some restrictions on the drift terms.

Third, the nature of equilibrium selection is somewhat different between the discrete and continuous models. In the discrete model, the relative size of the basin of attraction determines the long run equilibrium. In the continuous model, not only the size but also the strength of flow in each basin is important. For example, in Figure 6.2, equilibrium (x,x) is risk-dominated and has the smaller basin of attraction ($p<1/2$), but the flow of adjustment is much stronger there. As a result, area B is larger than area A, and the long run equilibrium is the risk dominted equilibrium (x,x).

Binmore, Samuelson and Vaughan (1993) clarify the reason for the difference between the discrete and continuous models. If we examine the finite population case in continuous time with Poisson adjustment, the stationary distribution has a clean closed form when the state space is one dimensional. Such a model is known as *the birth and death process*, and is employed by Amir and Berninghaus (1994) and Binmore and Samuelson (1993). Starting with the birth and death process, there are two ways to eliminate aggregate randomness. One method is to reduce the mutation rate for a fixed population size, which yields the prediction of the discrete model. The other way is to increase the population size with a fixed mutation rate. Binmore, Samuelson and Vaughan show that increasing

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the population size for a fixed mutation rate and then reducing the mutation rate yields the prediction of the continuous model.

7 CONCLUDING REMARKS

How should we evaluate evolutionary game theory in economics? Mainstream economics, which only admits rational behavior, may criticize such an approach on the following grounds. First, allowing various behavioral assumptions provides too much degree of freedom and loses the prediction power of the theory. Second, the particular behavioral assumptions utilized so far in evolutionary game theory are too naive and implausible in economic applications. Third, the literature thus far is strictly theoretical and lacks convincing applications to concrete economic problems.

Evolutionary game theorists may admit these shortcomings but point to the necessity of a good theory of bounded rationality. Rationality itself is unable to explain how players come to play a Nash equilibrium or which equilibrium is selected. The long history of the rationality approach, which culminated in extensive research in noncooperative game theory in the 1980's, finally revealed the necessity of another principle, and evolutionary game theory, although it is rather preliminary in its current form, hopefully provides a first step in this general direction.

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	c	d
c	4,4	1,5
d	5,1	2,2

Figure 2.1

0,0	1,1
1,1	0,0

(a)

1,1	0,0
0,0	1,1

(b)

Figure 2.2

evolutionary drifts

NSS

(a)

ES set

(b)

Figure 3.1

$$
\begin{array}{c c}
 & \begin{array}{cccc} R & P & S & D \end{array} \\
\begin{array}{c} R \\ P \\ S \\ D \end{array} &
\left(
\begin{array}{cccc}
a & c & b & \delta \\
b & a & c & \delta \\
c & b & a & \delta \\
a+\varepsilon & a+\varepsilon & a+\varepsilon & 0
\end{array}
\right)
\end{array}
$$
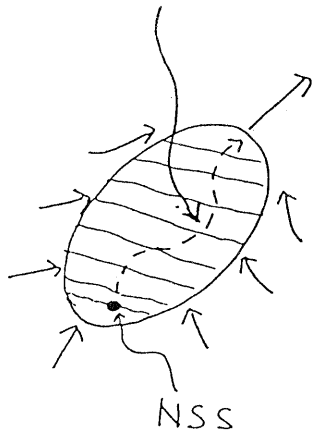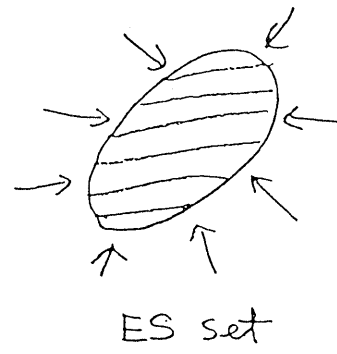
$$ c < a < b \;,\; 0 < \varepsilon < b - a \;,\; 0 < \delta $$
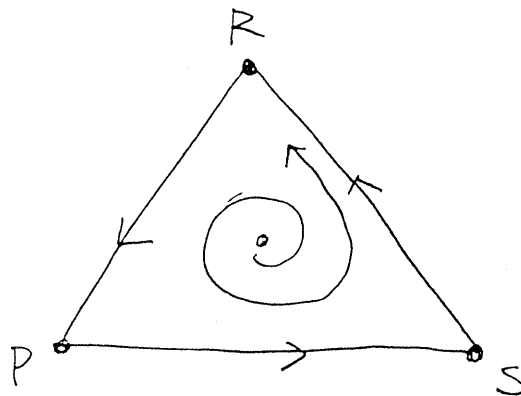
Figure 4.1

Figure 4.2

	a	b
a	3,3	0,0
b	0,0	2,2

Figure 5.1

	a	b	c
a	3,3	0,0	0,0
b	0,0	2,2	0,0
c	0,0	0,0	1,1

Figure 5.2

	Mac	IBM
Mac	2, 2	0, 0
IBM	0, 0	1, 1

Figure 6.1

	x	y
x	a, a	b, c
y	c, b	d, d

a > c , d > b

Figure 6.2

Figure 6.3