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Abstract

We shall generalize the Black-Scholes option pricing formula by incorporating stochastic interest rates. Although the existing literatures have obtained some formulae for stock options under stochastic interest rates, the closed-form solutions have been known only under the Gaussian (Merton type) interest rate processes. We shall show that an explicit solution, which is an extended Black-Scholes formula under stochastic interest rates in certain asymptotic sense, can be obtained by extending the asymptotic expansion approach when the interest rate volatility is small. This method called the small-disturbance asymptotics for Itô processes has been recently developed by Kunitomo and Takahashi (1995, 1998), and Takahashi (1997). We found that the extended Black-Scholes formula is decomposed into the original Black-Scholes formula under the deterministic interest rates and the adjustment term driven by the volatility of interest rates. We illustrate the numerical accuracy of our new formula by using the Cox-Ingersoll-Ross model for the interest rates.

Key Words

Asymptotic Expansion Approach, Black-Scholes Economy, Stochastic Interest Rates, Cox-Ingersoll-Ross Model

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1 Introduction

Over a few years a considerable number of studies have been devoted to various generalizations of the Black-Scholes formula. (See Black and Scholes (1973).) Stochastic interest rates, dividends, and stochastic volatilities for the underlying assets processes are important examples. The valuation of options on risky assets in the stochastic interest economy, which is the main forcus of this paper, has been studied recently by two approaches.

The first approach is the general equilibrium-based option pricing. Turnbull and Milne (1991) and Amin and Ng (1993) proposed the closed form solutions for discrete time version of an extended Black-Scholes formula by this approach. The second approach is the arbitrage-based option pricing, which has been rapidly developed after the work by Harrison and Kreps (1979). At the earliest, Merton (1973) considered the Black-Scholes economy under stochastic interest rates and derived some option pricing formula under the assumption of Gaussian interest rate process. Duffie (1988) also described an stochastic interest rate ecomony, but in his case the interest rate process is assumed to be a function of underlying asset and time, and the closed-form option pricing formula in his economy has not been obtained. Cheng (1989) considered the class of bond price process which is consistent with the arbitrage-free valuation. Amin and Jarrow (1992) generalized the approach utilized by Heath, Jarrow and Morton (1992) by imbedding their stochastic interest rate economy into the one containing an arbitrary number of risky assets and obtained some option pricing formula for various options on interest rate sensitive risky assets. However, these studies have derived the closed-form solutions for the option pricing formulae under the assumption that the underlying interest rates are Gaussian processes.

In this paper, we shall derive an explicit extension of the Black-Scholes formula under stochastic interest rate by using the asymptotic expansion of the solution when the interest rate volatility is small. Here, we mean 'explicit' by that our option pricing formula can be analytically expressed only as the function of parameters in the assumed system in certain asymptotic sense. The asymptotic expansion method for the valuation problem of contingent claims under continuous stochastic processes has been recently developed by Kunitomo and Takahashi (1995, 1998), and Takahashi (1997).

The plan of this paper is as follows. In Section 2 we explain the extended Black-Scholes economy when the spot interest rate is stochastic and present our main theoretical result from the asymptotic expansion approach. Section 3 investigates the hedging issue when the interest rate is stochastic. Then in Section

4, we shall give an example in which the spot interest rate is the same process of diffusion type proposed by Cox, Ingersoll, and Ross (1985). The related tables of our numerical analyses in this case are summarized in Appendices. Finally, some concluding remarks are given in Section 5.

2 The Black-Scholes Economy under Stochastic Interest Rates

We consider an economy in which there are two primitive securities. Fix a probability space $(\Omega, \mathcal{F}, P)$ with the Brownian filtration $\{\mathcal{F}_t : t \in [0, T]\}$ generated by the two-dimensional Brownian motion $\{\tilde{W} = (\tilde{W}_1, W_2) : t \in [0, T]; T < \infty\}$ with the zero initial value. The first security, called stock, has the price process of diffusion type given by

$$(2.1) \qquad S_t = S_0 + \int_0^t \mu(S_s, s)\, ds + \int_0^t \sigma(S_s, s)\, d\tilde{W}_{1s}$$

where $S_0 > 0$ and $\mu : R \times [0, T] \to R$ and $\sigma : R \times [0, T] \to R$ are $\mathcal{F}_t$-adapted and satisfy technical conditions, which guarantee the existence of the non-negative solution to (2.1) [1].

The second security, called bond, is implicitly defined by the spot interest rate process, $r_t^{(\epsilon)}$, which is given by

$$(2.2) \qquad r_t^{(\epsilon)} = r_0 + \int_0^t \zeta(r_s^{(\epsilon)}, s)\, ds + \epsilon \int_0^t \nu(r_s^{(\epsilon)}, s)\, dW_{2s},$$

where ϵ is a parameter $(0 < \epsilon \leq 1)$, and $\zeta : R \times [0, T] \to R$ and $\nu : R \times [0, T] \to R$ are adapted with respect to $\mathcal{F}_t$ and satisfy technical conditions, which guarantee the existence of the non-negative solution to (2.2). We shall set the above two asset prices to have possibly non-zero covariation.

As the numeraire, let us introduce an accumulation factor corresponding to a continuously rolled over money market account (or short-term deposits, or cash bond) by

$$\beta_t^{(\epsilon)} = \beta_0 \exp\left(\int_0^t r_s^{(\epsilon)}\, ds\right),$$

where $\beta_0 > 0$ and $r_t^{(\epsilon)}$ is defined by (2.2).

By the satndard argument of the arbitrge-free valuation, we assume that the stock price process is determined by

$$(2.3) \qquad S_t^{(\epsilon)} = S_0 + \int_0^t r_s^{(\epsilon)} S_s^{(\epsilon)}\, ds + \int_0^t \sigma(S_s^{(\epsilon)}, s)\, dW_{1s}$$

[1]See Chapter IV of Ikeda and Watanabe (1989) for the sufficient conditions on drift and volatility terms for the existence and uniqueness of the strong solution, for instance.

and $\{W_{1s}\}$ is the Brownian motion under the probability measure Q such that the discounted price process $S_t^{(\epsilon)}/\beta_t^{(\epsilon)}$ is a martingale. Here we use the notation for the price process $S_t^{(\epsilon)}$ with a parameter ϵ under Q. We note that the normalized bond price processes are also martingales under the equivalent martingale measure Q and the drift $\zeta(r_t^{(\epsilon)}, t)$ in (2.2) is now interpreted as the function of the market price for risk under the equivalent martingale measure Q.

If we set $Z_T^{(\epsilon)} \equiv \exp\left(-\int_0^T r_s^{(\epsilon)}\, ds\right)\left[S_T^{(\epsilon)} - K\right]$, the value of derivative security called European stock call option at the initial date, $V(0)$, under the complete market is determined by

$$V(0) = E^Q\left[\left[Z_T^{(\epsilon)}\right]^+\right], \tag{2.4}$$

where $[\cdot]^+ = \max[0, \cdot]$.

In the following discussion, we shall suppress the superscript of the probability measure Q in expectation operations for the notational ease without making any ambiguity. We shall evaluate the equation (2.4) explicitly and compare with the original Black-Scholes option pricing formula. For this purpose, we start to expand the interest rate process, $r_t^{(\epsilon)}$, as $\epsilon \downarrow 0$ by using the asymptotic expansion approach developed [2] by Kunitomo and Takahashi (1995, 1998), and Takahashi (1997).

By expanding $r_t^{(\epsilon)}$ with respect to ϵ formally, we write

$$r_t^{(\epsilon)} = r_t + \epsilon\, A(t) + \epsilon^2\, B^*(t) + \cdots, \tag{2.5}$$

where $r_t = r_t^{(0)}$,

$$r_t = r_0 + \int_0^t \zeta(r_s, s)\, ds, \tag{2.6}$$

$$A(t) = \left.\frac{\partial r_t^{(\epsilon)}}{\partial \epsilon}\right|_{\epsilon=0}, \tag{2.7}$$

and

$$B^*(t) = \left.\frac{1}{2}\frac{\partial^2 r_t^{(\epsilon)}}{\partial \epsilon^2}\right|_{\epsilon=0}. \tag{2.8}$$

By comparing both terms of (2.2) and (2.5), the stochastic process (2.7) can be represented by

$$A(t) = \int_0^t \partial\zeta(r_s^{(0)}, s)\, A(s)\, ds + \int_0^t \nu(r_s, s)\, dW_{2s}\,, \tag{2.9}$$

[2] We do not discuss the rigorous mathematical validity of the asymptotic expansion approach in this paper. It has been discussed in Kunitomo and Takahashi (1998) to certain extent, which is based on the Watanabe-Yoshida theory on Malliavin Calculus recently developed in stochastic analysis.

where

$$\partial\zeta(r_s^{(0)}, s) \equiv \frac{\partial\zeta(r_s^{(\epsilon)}, s)}{\partial r^{(\epsilon)}}\Big|_{r^{(\epsilon)}=r^{(0)}} . \tag{2.10}$$

In order to have concise representations of processes, let Y_t be the solution of the differential equation

$$dY_t = \partial\zeta(r_t, t)\, Y_t\, dt$$

with $Y_0 = 1$. Then the stochastic differential equation (2.9) can be solved as

$$A(t) = \int_0^t Y_t Y_s^{-1} \nu(r_s, s)\, dW_{2s} . \tag{2.11}$$

By using the same procedure, we can also express $B^*(t)$ as

$$B^*(t) = \frac{1}{2}\int_0^t Y_t Y_s^{-1} [\partial^2\zeta(r_s^{(0)}, s)\, A^2(s)\, ds + 2\,\partial\nu(r_s^{(0)}, s)\, A(s)\, dW_{2s}] , \tag{2.12}$$

where

$$\partial\nu(r_s^{(0)}, s) \equiv \frac{\partial\nu(r^{(\epsilon)}, s)}{\partial r^{(\epsilon)}}\Big|_{r^{(\epsilon)}=r^{(0)}} . \tag{2.13}$$

From the equation (2.11), we can notice that $A(t)$ follows the Gaussian process with $A(t) \sim N\left[0, \Sigma_{A_t}\right]$, where

$$\Sigma_{A_t} = \int_0^t Y_t^2\, Y_s^{-2}\, \nu(r_s, s)^2\, ds. \tag{2.14}$$

Then the interest-rate-sensitive stock price process (2.3) can be expanded with respect to ϵ formally as

$$S_t^{(\epsilon)} = S_0 + \int_0^t \left[r_s + \epsilon A(s) + \epsilon^2 B^*(s) + \cdots\right] S_s^{(\epsilon)} ds + \int_0^t \sigma\left(S_s^{(\epsilon)}, s\right) dW_{1s}, \tag{2.15}$$

where

$$E\left[\begin{pmatrix} dW_{1t} \\ dW_{2t} \end{pmatrix} \begin{pmatrix} dW_{1t} & dW_{2t} \end{pmatrix}\right] = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt .$$

In the above notation, ρ is the instantaneous correlation coefficient between two standard Brownian motions. We notice that when ϵ goes to zero, the equation (2.15) becomes

$$S_t^{(0)} = S_0 + \int_0^t r_s\, S_s^{(0)}\, ds + \int_0^t \sigma(S_s^{(0)}, s)\, dW_{1s} , \tag{2.16}$$

where r_s is a deterministic function satisfying the integral equation (2.6). Further, if we set $\zeta(r_s, s) = 0$, $r_s = r_0$, and $\sigma(S_s^{(0)}, s) = \sigma S_s^{(0)}$, then (2.15) becomes the geometric Brownian motion with a constant risk free rate, which corresponds to

the original Black-Scholes economy.
In the rest of this paper, we shall make the following assumption to derive some useful results.

Assumption I : The volatility function of the risky asset $S_t^{(\epsilon)}$ is given by $\sigma(S_s^{(\epsilon)}, s) = \sigma_s S_s^{(\epsilon)}$, where σ_s is a deterministic function of time.

Let $X_t^{(\epsilon)}$ be $\log S_t^{(\epsilon)}$ with $X_0 = \log S_0$. Then by using the Itô's lemma, the equation (2.15) leads to

$$(2.17) X_t^{(\epsilon)} = X_0 + \int_0^t \left(r_s - \frac{1}{2}\sigma_s^2 + \epsilon A(s) + \epsilon^2 B^*(s) + o_p(\epsilon^2)\right) ds + \int_0^t \sigma_s dW_{1s} .$$

Next, we need to evaluate the present value of money being worth 1 Yen at time t when ϵ is small. From the equation (2.5), we expand the present value formally with respect to ϵ as

$$\begin{aligned} &\exp\left(-\int_0^t r_s^{(\epsilon)} ds\right) \\ &= \exp\left(-\int_0^t r_s ds - \epsilon\int_0^t A(s)ds - \epsilon^2\int_0^t B^*(s)ds + \cdots\right) \\ &= \exp\left(-\int_0^t r_s ds\right)\left[1 - \epsilon\int_0^t A(s)ds - \epsilon^2\int_0^t B^*(s)ds + \frac{\epsilon^2}{2}\left(\int_0^t A(s)ds\right)^2 + \cdots\right] \\ &= E_0 + \epsilon\, E_1 + \epsilon^2\, E_2 + \cdots , \end{aligned}$$

where we denote

$$\begin{aligned} E_0 &= \exp\left(-\int_0^t r_s ds\right) \\ E_1 &= -\exp\left(-\int_0^t r_s ds\right)\int_0^t A(s)ds \\ E_2 &= \exp\left(-\int_0^t r_s ds\right)\left[-\int_0^t B^*(s)ds + \frac{1}{2}\left(\int_0^t A(s)ds\right)^2\right] . \end{aligned}$$

In (2.17) we have obtained an asymptotic expansion of the stochastic process $X_t^{(\epsilon)}$ with respect to ϵ. Then by combining (2.17) with the above expression, we can obtain an asymptotic expansion for the random variable $Z_T^{(\epsilon)}$ as

$$\begin{aligned} Z_T^{(\epsilon)} &= S_0 \exp\left[X_{1T} - \frac{1}{2}\int_0^T \sigma_s^2 ds\right] - \left[E_0 + \epsilon E_1 + \epsilon^2 E_2 + \cdots\right] \times K \\ &= Z_0 + \epsilon Z_1 + \epsilon^2 Z_2 + \cdots , \end{aligned}$$

where we have implicitly defined

$$(2.18) \qquad X_{1T} = \int_0^T \sigma_s dW_{1s} ,$$

$$Z_0 = \exp\left(-\int_0^T r_s ds\right)\left[S_0 \exp\left(X_{1T} + \int_0^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds\right) - K\right] ,$$

and $Z_i = -E_i \times K$ $(i = 1, 2)$.

Although we have obtained an asymptotic expansion for $Z_t^{(\epsilon)}$, we need to cope with one technical issue in our problem. Because the payoff function of call option value is a non-linear and nonnegative function, the valuation of option should be done by taking into account of the condition that $S_T^{(\epsilon)} - K \geq 0$. In order to accomplish this, we define the random vector of $\mathbf{X} = (X_{1T}, X_{2T})$, where $X_{2T} = \int_0^T A(s)ds$.

We note that

$$\begin{aligned} \int_0^T A(s)\, ds &= \int_0^T \left(\int_0^s Y_s Y_v^{-1} \nu(r_v, v)\, dW_{2v}\right) ds \\ &= \int_0^T \left(\int_v^T Y_s\, ds\right) Y_v^{-1} \nu(r_v, v)\, dW_{2v} . \end{aligned} \tag{2.19}$$

Hence we have shown that $\mathbf{X}$ is a two dimensional Gaussian random vector. By using direct calculations, the variances and covariance of two random variables are given by

$$\Sigma_{22} \equiv \mathbf{Var}\left[X_{2T}\right] = \int_0^T \left(\int_v^T Y_s\, ds\right)^2 Y_v^{-2} \nu(r_v, v)^2 dv , \tag{2.20}$$

$$\Sigma_{11} \equiv \mathbf{Var}\left[X_{1T}\right] = \int_0^T \sigma_s^2 ds , \tag{2.21}$$

and

$$\Sigma_{12} \equiv \mathbf{Cov}\left[X_{1T}, X_{2T}\right] = \int_0^T \left(\int_v^T Y_s\, ds\right) Y_v^{-1} \nu(r_v, v) \sigma_v \rho dv . \tag{2.22}$$

By using the property of the Gaussian distribution, the conditional ditribution is given by

$$X_{2T} | X_{1T} = x \sim N\left[\frac{\Sigma_{12}}{\Sigma_{11}} x,\ \Sigma_{22.1}\right] ,$$

where $\Sigma_{22.1} = \Sigma_{22} - {\Sigma_{12}}^2 / \Sigma_{11}$. Then we can write

$$\int_0^T A(s)ds = \frac{\Sigma_{12}}{\Sigma_{11}} x + \sqrt{\Sigma_{22.1}}\, z ,$$

where $z \sim N[0, 1]$, which is independently distributed from X_{1T} . By using this representation, we rewrite the random variables Z_0 and Z_1 in terms of X_{1T} and z as

$$Z_0 = S_0 \exp\left(X_{1T} - \frac{1}{2}\int_0^T \sigma_s^2 ds\right) - K \exp\left(-\int_0^T r_s ds\right) , \tag{2.23}$$

and

$$(2.24) \qquad Z_1 = K\exp\left(-\int_0^T r_s ds\right)\left[\frac{\Sigma_{12}}{\Sigma_{11}}X_{1T} + z\sqrt{\Sigma_{22.1}}\right] .$$

We notice that the condition $S_T^{(\epsilon)} - K \geq 0$ is equivalent to

$$(2.25) X_{1T} \geq \log\frac{K}{S_0} - \int_0^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds - \epsilon\left[\frac{\Sigma_{12}}{\Sigma_{11}}X_{1T} + z\sqrt{\Sigma_{22.1}}\right] + O_p(\epsilon^2) .$$

By ignoring higher order terms in the expansion, a little algebra shows that this condition is formally equivalent to

$$\begin{aligned} X_{1T} &\geq \left\{1 - \epsilon\frac{\Sigma_{12}}{\Sigma_{11}} + O(\epsilon^2)\right\}\left\{\left[\log\frac{K}{S_0} - \int_0^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds\right] + \epsilon\left[-z\sqrt{\Sigma_{22.1}}\right] + \cdots\right\} \\ &= C_0 + \epsilon\left(zC_{11} + C_{12}\right) + \cdots , \end{aligned}$$

where we denote

$$(2.26) \qquad C_0 = \log\frac{K}{S_0} - \int_0^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds ,$$

$$(2.27) \qquad C_{11} = -\sqrt{\Sigma_{22.1}} ,$$

$$(2.28) \qquad C_{12} = \left[-\frac{\Sigma_{12}}{\Sigma_{11}}\right]\left[\log\frac{K}{S_0} - \int_0^T \left(r_s - \frac{1}{2}\sigma_s^2\right) ds\right] .$$

Our next task is to evaluate

$$E\left[\exp\left(-\int_0^T r_s^{(\epsilon)} ds\right)[S_T^{(\epsilon)} - K]^+\right]$$

as $\epsilon \downarrow 0$. For this purpose, first we rewrite this as

$$\begin{aligned} & E\left[Z_0 I(S_T^{(\epsilon)} - K \geq 0) + \epsilon Z_1 I(S_T^{(\epsilon)} - K \geq 0)\right] + o(\epsilon) \\ = \; & \Lambda_1 + \epsilon\Lambda_2 + o(\epsilon) , \end{aligned}$$

where $I(\omega)$ is the indicator function (it is 1 if ω being true and 0 otherwise), $\Lambda_1 = E\left[Z_0 I(S_T^{(\epsilon)} - K \geq 0)\right]$, and $\Lambda_2 = E\left[Z_1 I(S_T^{(\epsilon)} - K \geq 0)\right]$.
We now have to evaluate Λ_i $(i = 1, 2)$. For the notational convenience, we set $\Sigma_{11} = \sigma^2(T)$ and $\phi_{\sigma^2(T)}(x)$ being the density function of the normal random variable x with mean 0 and variance $\sigma^2(T)$.
Then by using the repeated expectation operation given $X_{1T} = x$, we have

$$\begin{aligned} \Lambda_1 &= E\left(\left[S_0\exp\left(x - \frac{1}{2}\int_0^T \sigma_s^2 ds\right) - K\exp\left(-\int_0^T r_s ds\right)\right] I(S_T^{(\epsilon)} - K \geq 0)\right) \\ &= E\left[\int_{x \geq C_0 + \epsilon(zC_{11} + C_{12}) + \cdots} S_0\exp\left(x - \frac{1}{2}\int_0^T \sigma_s^2 ds\right)\phi_{\sigma^2(T)}(x)dx\right] \\ &\quad - K\exp\left(-\int_0^T r_s ds\right) E\left[\int_{x \geq C_0 + \epsilon(zC_{11} + C_{12}) + \cdots} \phi_{\sigma^2(T)}(x)dx\right] . \end{aligned}$$

If we take $y_1 = (x - \sigma^2(T))/\sigma(T)$ and $y_2 = x/\sigma(T)$, the above equation can be further rewritten as

$$
\begin{aligned}
\Lambda_1 &= E\left[\int_{y_1 \geq (C_0 - \sigma^2(T) + \epsilon(zC_{11}+C_{12})+\cdots)/\sigma(T)} S_0\phi(y_1)\,dy_1\right] \\
&\quad -K\exp\left(-\int_0^T r_s ds\right) E\left[\int_{y_2 \geq (C_0 + \epsilon(zC_{11}+C_{12})+\cdots)/\sigma(T)} \phi(y_2)dy_2\right] \\
&= S_0 E\left\{\Phi\left[\frac{\sigma^2(T) - C_0}{\sigma(T)} + \epsilon\left(\frac{-z\,C_{11} - C_{12}}{\sigma(T)}\right)\right]\right\} \\
&\quad -K\exp\left(-\int_0^T r_s ds\right) E\left\{\Phi\left[\frac{-C_0}{\sigma(T)} + \epsilon\left(\frac{-zC_{11} - C_{12}}{\sigma(T)}\right)\right]\right\},
\end{aligned}
$$

where $\Phi(\cdot)$ is the distribution function of the standard normal random variable. By expanding each term with respect to ϵ in the above expression, we have an asymptotic expansion given by

$$
\begin{aligned}
\Lambda_1 &= S_0 E\left\{\Phi\left[\frac{\sigma^2(T) - C_0}{\sigma(T)}\right] + \phi\left(\frac{\sigma^2(T) - C_0}{\sigma(T)}\right)\epsilon\left(\frac{-zC_{11} - C_{12}}{\sigma(T)}\right)\right\} \\
&\quad - K\exp\left(-\int_0^T r_s ds\right) E\left\{\Phi\left[\frac{-C_0}{\sigma(T)}\right] + \phi\left(\frac{-C_0}{\sigma(T)}\right)\epsilon\left(\frac{-zC_{11} - C_{12}}{\sigma(T)}\right)\right\} + \cdots \\
&= \left[S_0\Phi(d_1) - K\exp\left(-\int_0^T r_s ds\right)\Phi(d_2)\right] \\
&\quad + \epsilon\left(\frac{-C_{12}}{\sigma(T)}\right)\left\{S_0\phi(d_1) - K\exp\left(-\int_0^T r_s ds\right)\phi(d_2)\right\} + \cdots,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
d_1 &= \sigma(T) - \frac{C_0}{\sigma(T)} \\
&= \frac{1}{\sigma(T)}\left\{\log\frac{S_0}{K} + \int_0^T \left(r_s + \frac{1}{2}\sigma_s^2\right) ds\right\}
\end{aligned}
$$

and $d_2 = -C_0/\sigma(T) = d_1 - \sigma(T)$.
Furthermore, if we rewrite (2.28) as

$$
C_{12} = \frac{\Sigma_{12}}{\Sigma_{11}} \times \sigma(T) \times d_2\,,
$$

we can express the above asymptotic expansion as

$$
\begin{aligned}
\Lambda_1 &= \left[S_0\,\Phi(d_1) - K\exp\left(-\int_0^T r_s ds\right)\Phi(d_2)\right] \\
&\quad - \epsilon\left(d_2\frac{\Sigma_{12}}{\Sigma_{11}}\right)\left\{S_0\phi(d_1) - K\exp\left(-\int_0^T r_s ds\right)\phi(d_2)\right\} + \cdots.
\end{aligned}
$$

By using a similar argument as to Λ_1, we can also evaluate $\epsilon\Lambda_2$ as

$$
\begin{aligned}
\epsilon\Lambda_2 &= \epsilon E\left[K\exp\left(-\int_0^T r_s ds\right)\int_0^T A(s)ds\; I(S_T^{(\epsilon)}-K\geq 0)\right] \\
&= \epsilon K\exp\left(-\int_0^T r_s ds\right)E\left[\int_{x\geq C_0+\epsilon(zC_{11}+C_{12})}\left(\frac{\Sigma_{12}}{\Sigma_{11}}x+z\sqrt{\Sigma_{22.1}}\right)\phi_{\sigma^2(T)}(x)dx\right] \\
&\cong \epsilon K\exp\left(-\int_0^T r_s ds\right)\frac{\Sigma_{12}}{\Sigma_{11}}E\left[\int_{x\geq C_0+\epsilon(zC_{11}+C_{12})} x\phi_{\sigma^2(T)}(x)dx\right] \\
&= \epsilon K\exp\left(-\int_0^T r_s ds\right)\frac{\Sigma_{12}}{\Sigma_{11}}\sigma(T)\phi\left(d_2\right).
\end{aligned}
$$

In the above derivations, the expansion in the third equation deserves a separate technical consideration, which is given in Appendix A. By collecting all terms in Λ_i $(i=1,2)$, we are now in a position to state our theoretical result on the European call option.

Theorem 2.1 *Under the Assumption I, an asymptotic expansion of the value of European call option with stochastic interest rate, $V(0)$, is given by*

$$
\begin{aligned}
(2.29)\quad V(0) &= \left[S_0\Phi(d_1)-K\exp\left(-\int_0^T r_s ds\right)\Phi(d_2)\right] \\
&\quad + \epsilon C_1\left\{d_2 S_0\phi(d_1)-d_1 K\exp\left(-\int_0^T r_s ds\right)\phi(d_2)\right\}+o(\epsilon)
\end{aligned}
$$

as $\epsilon\downarrow 0$, where $\Phi(\cdot)$ is the distribution function of the standard normal variable and $\phi(\cdot)$ is its density function, $C_1=-\Sigma_{12}/\sigma^2(T)$,

$$
\begin{aligned}
d_1 &= \frac{1}{\sigma(T)}\left[\log\frac{S_0}{K}+\int_0^T\left(r_s+\frac{1}{2}\sigma_s^2\right)ds\right], \\
d_2 &= d_1-\sigma(T), \\
\sigma^2(T) &= \int_0^T\sigma_s^2 ds,
\end{aligned}
$$

and Σ_{12} is defined by (2.22).

We use the notations BS_0 for the first term and BS_1 for the coefficient of ϵ of the second term on the right hand side of (2.29). Then our proposition states that

$$
\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\left[V(0)-BS_0-\epsilon\, BS_1\right]=0.
$$

We notice that BS_0 corresponds to the Black-Scholes formula when the interest rate is a deterministic function of time. Then $\epsilon\, BS_1$ corresponds to the adjustment term induced by the volatility of interest rate and the instantaneous correlation between the stock price process and interest rate process. Therefore, the option value can be decomposed into

$$V(0) = BS + [BS_0 - BS] + \epsilon\, BS_1 + o(\epsilon),$$

where BS is the original Black-Scholes formula. The second term in this expression reflects the adjustment by the non-constant, but deterministic interest rate, and the third term represents the adjustment by the volatility of stochastic interest rate. Also we note that C_1 in Theorem 2.1 can be expressed by

$$-\frac{\rho}{\sigma^2(T)}\int_0^T \left(\int_v^T Y_s ds\right) Y_v^{-1}\nu(r_v, v)\sigma_v dv\ ,$$

we immediately obtain the following result.

Corollary 2.1 *When* $\rho = 0$, $BS_1 = 0$ *and* $\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\left[V(0) - BS_0\right] = 0$.

This corollary confirms the standard argument that the option value equals to BS_0 when the stock price moves irrespectively with the interest rate. Since the adjustment term, $\epsilon\, BS_1$, is linear in ρ in the general case, however, it has the symmetric effect as ρ goes from -1 to 1. This theoretical result can be confirmed numerically if we cast a glance at tables in Section 4. The sensitivity of the option value with respect to ρ is summarized in the next corollary.

Corollary 2.2 *Under Assumption I, an asymptotic expansion of the response of option value to the correlation coefficient between stock price and interest rate is given by*

$$\begin{aligned}\frac{\partial V(0)}{\partial \rho} &= -\frac{\epsilon}{\sigma^2(T)}\int_0^T \left(\int_v^T Y_s ds\right) Y_v^{-1}\nu(r_v, v)\sigma_v dv \\ &\quad \times \left\{d_2\, S_0\, \phi(d_1) - d_1\, K\, \exp\left(-\int_0^T r_s\, ds\right)\phi(d_2)\right\} + o(\epsilon)\end{aligned}$$

as $\epsilon \downarrow 0$.

By the same method for the valuation of call option, we can derive the theoretical value for the put option whose payoff function is given by $[K - S_T]^+$ at the maturity.

Theorem 2.2 *Under the Assumption I, an asymptotic expansion of the value of European put option with stochastic interest rate,* $V^*(0)$, *is given by*

$$\begin{aligned}(2.30)\quad V^*(0) &= \left[K\exp\left(-\int_0^T r_s ds\right)\Phi(-d_2) - S_0\Phi(-d_1)\right] \\ &\quad + \epsilon C_1\left\{d_2 S_0\phi(d_1) - d_1 K\exp\left(-\int_0^T r_s ds\right)\phi(d_2)\right\} + o(\epsilon)\end{aligned}$$

as $\epsilon \downarrow 0$, *where* $\Phi(\cdot)$ *is the distribution function of the standard normal variable and* $\phi(\cdot)$ *is its density function, and* d_i $(i = 1, 2)$, C_1, *and* $\sigma^2(T)$ *are the same as in Theorem 2.1.*

In this expression, we notice that the correction term to the Black-Scholes formula with the deterministic interest rate for the put option is the same as the corresponding term to the call option. This can be also derived from the Put-Call parity in the standard option pricing theory.

3 Hedging Problem

This section investigates the hedging problem for the exposure generated by the options. Replicating the option value under the original Black-Scholes economy can be straightforwardly extended to the Black-Scholes economy under the stochastic interest rate. Hence we can execute a self-financing hedging strategy by using (Δ_t, Ψ_t) in the stock and short-term deposit to replicate the payoff of target options. In particular, we are concerned with Δ_t, which is often called option delta. An asymptotic expansion of the option delta under stochastic interest rates can be obtained by utilizing the theoretical results in Section 2.

By making use of the fact that $\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{1}{S_0 \sigma(T)}$, we can directly calculate the option delta as

$$
\begin{aligned}
(3.31) \qquad \Delta &= \frac{\partial V(0)}{\partial S_0} \\
&= \Phi(d_1) + \epsilon C_1 \Bigg\{ d_2 \phi(d_1) + [\frac{\partial d_2}{\partial S_0} - d_1 d_2 \frac{\partial d_1}{\partial S_0}] \\
&\quad \times \left[S_0 \phi(d_1) - K \exp\left(-\int_0^T r_s ds\right) \phi(d_2) \right] \Bigg\} + o(\epsilon) .
\end{aligned}
$$

Then by using the identity that

$$
\phi(d_1) = \phi(d_2) \frac{K}{S_0} \exp\left(-\int_0^T r_s ds\right) ,
$$

the last parenthesis in (3.31) is zero and hence we immediately obtain the next result.

Theorem 3.1 *Under Assumption I in Section 2, an asymptotic expansion of the option delta at time* 0, Δ, *under the stochastic interest rates is given by*

$$
(3.32) \qquad \Delta = \Phi(d_1) + \epsilon\, C_1\, d_2\, \phi(d_1) + o(\epsilon)
$$

as $\epsilon \downarrow 0$, *where* d_i $(i = 1, 2)$ *and* C_1 *are given in Theorem 2.1.*

As we have discussed on the fair option value under stochastic interest rates, the option delta under stochastic interest rates can be decomposed into

$$\Delta = \Delta^{const} + \left[\Delta^{deter} - \Delta^{const}\right] + \Delta^{stoch} + o(\epsilon), \tag{3.33}$$

where Δ^{const} is the option delta in the original Black-Scholes economy when the interest rate is constant. $\Delta^{deter} \equiv \Phi(d_1)$ is the options delta under deterministic interest rates and $\Delta^{stoch} = \epsilon C_1 \, d_2 \, \phi(d_1)$. Then, the second term on right hand side of (3.33) corresponds to the option delta bias due to the drift term of interest rate process while the third term corresponds to the effects of interest rate volatility. Therefore, these terms represent the adjustment sizes generated by incorporating stochastic interest rates into the standard Black-Scholes economy.

4 The Black-Scholes Economy with CIR Interest Rate Process : An Example

In order to clarify our theoretical results in Sections 2 and 3, we shall illustrate one example of the Black-Scholes economy with stochastic interest rate. Many stochastic processes have been proposed to describe the short term interest rates in this regard. Among them, a considerable attention has been paid on the one proposed by Cox, Ingersoll, and Ross (1985) (hereafter, the CIR model) partly because it is a typical diffusion process and has some attractive features in theory and practice. Thus we shall investigate the Black-Scholes economy with the CIR interest rate model when the risky asset and the spot interest rate are correlated. Also we shall give some numerical analyses in details for the CIR case, but the analysis in this section can be extended to other interest rate models. In this section we assume that the volatility function of the risky asset S_t is constant for the resulting simplicity of our analysis. This implies that $\sigma_s = \sigma$ and $\sigma(T) = \sigma\sqrt{T}$.

The CIR model is a special case of (2.2) when we take $\zeta(r_s^{(\epsilon)}, s) = \kappa(\bar{r} - r_s^{(\epsilon)})$ and $\nu(r_s^{(\epsilon)}, s) = \sqrt{r_s^{(\epsilon)}}$. The solution of (2.6) is given by

$$r_t = r_0 \exp(-\kappa t) + \bar{r}(1 - \exp(-\kappa t))$$

and $Y_t = \exp(-\kappa t)$. Then using (2.22) in Section 2 and the integration operation we can derive C_1 in (2.29) explicitly as

$$C_1 = -\frac{\rho}{\sigma T \kappa} \int_0^T \Big(1 - \exp(-\kappa(T - v))\Big)\Big(\exp(-\kappa v)(r_0 - \bar{r}) + \bar{r}\Big)^{\frac{1}{2}} dv$$

$$= -\frac{\rho}{\sigma T}\Big[\frac{2\sqrt{\bar{r}}((1+2\exp(\kappa T))\sqrt{r_0}-3\gamma)+(r_0-\bar{r}(1+2\exp(\kappa T))\,\lambda}{2\exp(\kappa T)\kappa^2\sqrt{\bar{r}}}\Big],$$

where $\gamma \equiv \exp(\frac{\kappa T}{2})\sqrt{r_0-\bar{r}(1-\exp(\kappa T))}$ and

$$\lambda \equiv \log(\frac{(\sqrt{r_0}+\sqrt{\bar{r}})^2}{r_0-\bar{r}(1-2\exp(\kappa T))+2\gamma\sqrt{\bar{r}}}).$$

Also we can obtain

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\log\frac{S_0}{K}+\bar{r}T+\frac{1}{\kappa}(r_0-\bar{r})(1-\exp(-\kappa T))+\frac{\sigma^2}{2}T\right)$$

and $d_2 = d_1 - \sigma\sqrt{T}$.

From Table 1 to Table 6 in Appendix B, we provide the European stock call option values and option deltas based on our approach for at-the-money case with the CIR spot rate processes. To incorporate the covariation between the stock price process and interest rate process, we give a set of numerical values for the cases of $\rho = -1.0, -0.5, 0.0, 0.5, 1.0$. When we consider that the initial interest rate is in the transitory phases (the downward phase or the upward phase), it starts at $\bar{r} + 0.04$ (or $\bar{r} - 0.04$), $\bar{r} = 0.07$, and $\kappa = 2.0$. In all cases [3] we set $S_0 = K = 100$, $T = 1$, (a year), $\sigma = 0.2$, (stock price volatility) and $\epsilon = 0.1, 0.3$ (interest rate volatility). When $\kappa = 0$ and $\epsilon = 0$, the case corresponds to the original Black-Scholes economy with a constant risk free rate. Our approach in each table corresponds to the numerical value of the approximation up to $O(\epsilon)$ based on the asymptotic expansion in Theorem 2.1 by ignoring higer order terms. The call option value and option delta under the original Black-Scholes model have been also given for the comparative purpose. As the benchmark, we also provide the Monte Carlo simulation results in the first row of each table. The number of simulated sample paths is 16,384 and the time interval is 128. As the discretization method of sample paths, we have adopted the Euler-Maruyama approximation [4]. All results are the mean of 200 simulation trials and we choose 0.0001 as the stock price increment in the re-simulation. We have found that it gives sufficient accuracy to get the option delta, for instance.

The figures in Table 1 and Table 2 in Appendix B tell us that for the downward interest rate process our asymptotic option value and option delta are very close

[3]We have investigated many other cases as well. Since the results are basically similar, however, we have omitted their details in this paper.

[4]See Duffie and Glynn (1995) for the details of the optimal choice of grid size, the order of the dicrete approximation, and the number of replications for a given fixed computaion budget, for instance.

to those of true values. For example, in Table 1 we find that for $\rho = -0.5$, $\sigma = 0.2$ and $\epsilon = 0.1$, the option value based on the simulation is 12.3811 and the option value based on our approach is 12.3773, hence the difference between them is only 0.0038. Besides, for the same set of parameters, the true option delta and our estimate are 0.707 and 0.7067, respectively ; the difference between them is very small. Although the deterministic parts of interest rate play a crucial role in determining option value, the magnitude of the adjustment term due to the volatility of interest rate can not be ignored, that is, for the same parameters the latter is -0.1476. In Table 2, as it is stated in Theorem 2.1, we find that the size of adjustment by interest rate volatility increases as the interest rate volatility increases. For the same parameters with $\epsilon = 0.3$, it increases from -0.1476 to -0.4429. Furtherfmore, we numerically observe that the effect of interest rate volatility on the adjustment term is about linear in ρ. Therefore, as the instantaneous correlation coefficient between the stock price process and the interest rate process increases, the adjustment term increases.

For the upward phase of the interest rate process, we can easily check the figures in Table 3 and Table 4 in Appendix B, which indicate that the direction of various effects are opposite. From Table 7 to Table 10 we can notice that although the option values under stochastic interest rate are not far from the original Balck-Scholes values with constant interest rate process, the size of the adjustment by the interest rate volatility can not be disregarded. For in-the-money and out of money case, similar numerical results have been obtained. The detailed numerical results in these cases are given in Appendix C.

The above numerical results confirm our argument that the trend of interest rate plays a crucial role in determining the stock option values. The effects of the volatility of interest rate process as well as the correlation beteen the stock price and interest rate are of secondary importance.

5 Concluding Remarks

In this paper we have developed a new valuation method based on the asymptotic expansion approach for the option prices when the interest rate process is of the diffusion type and it is correlated with the risky asset prices. We have illustrated our extension of the Black-Scholes formula by using the CIR interest rate process. Our approach has some advantage in the sense that we have explicit formula for the option prices, which is based on the asymptotic expansion of interest rate process when its volatility is small. Although we did not give the rigorous mathematical validity of the asymptotic expansion approach adopted in this paper, they have been developed by Kunitomo and Takahashi (1998).

Our theoretical results also give accurate numerical values in most cases when we have a set of reasonable parameter values in practice. As we have shown in numerical examples, a set of simulations have strongly suggested the numerical accuracy of our formula, which is the truncation of the corresponding asymptotic expansion of the stochastic process for the interest rate with respect to its volatility.

Finally, we should mention to the fact that it is further possible to extend our asymptotic expansion approach to several interesting cases. For instance, the valuation of options when the underlying asset pays a stochastic dividend and the valuation of currency options under stochastic interest rates may be immediate examples. We shall report other applications of the asymptotic expansion approach in another occasion.

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6 Appendices

6.1 Appendix A

In this appendix we give the validity of our approximation in the evaluation of $\epsilon\,\Lambda_2$, which has been used in the derivation of Theorem 2.1. We need a careful argument on this. We note that $\epsilon\,\Lambda_2$ can be rewritten as

$$
(6.34)\ \epsilon\Lambda_2 = \epsilon K \exp\left(-\int_0^T r_s ds\right)\left\{E\left[\frac{\Sigma_{12}}{\Sigma_{11}}\int_{x\geq C_0+\epsilon(zC_{11}+C_{12})} x\phi_{\sigma^2(T)}(x)dx\right]\right.
$$
$$
\left. +E\left[\sqrt{\Sigma_{22.1}}\int_{x\geq C_0+\epsilon(z\,C_{11}+C_{12})} z\,\phi_{\sigma^2(T)}(x)dx\right]\right\}.
$$

Then we need to evalulate the expectation terms in the parentheses. By setting $y = x/\sigma(T)$ and using the property of the Gaussian density such that $\frac{\partial\phi(y)}{\partial y} = -y\phi(y)$, the extected value of the first parenthesis can be rewritten as

$$
\begin{aligned}
& \frac{\Sigma_{12}}{\Sigma_{11}}\sigma(T)E\left[\int_{y\geq\frac{C_0+\epsilon(z\,C_{11}+C_{12})}{\sigma(T)}} y\phi(y)dy\right] \\
= \;& \frac{\Sigma_{12}}{\Sigma_{11}}\sigma(T)E\left[\phi\left(\frac{C_0+\epsilon(z\,C_{11}+C_{12})}{\sigma(T)}\right)\right] \\
= \;& \frac{\Sigma_{12}}{\Sigma_{11}}\sigma(T)E\left[\phi\left(\frac{C_0}{\sigma(T)}\right)+\phi'\left(\frac{C_0}{\sigma(T)}\right)[\epsilon(z\,C_{11}+C_{12})]+\cdots\right] \\
= \;& \frac{\Sigma_{12}}{\Sigma_{11}}\sigma(T)\left[\phi\left(\frac{C_0}{\sigma(T)}\right)-\left(\frac{C_0}{\sigma(T)}\right)\phi\left(\frac{C_0}{\sigma(T)}\right)\epsilon\,C_{12}\right]+\cdots .
\end{aligned}
$$

Next, we need to evaluate the second expectation term in Λ_2. By construction two random variables z and x are independent. By setting $y = x/\sigma(T)$, we have

$$
\begin{aligned}
& \sqrt{\Sigma_{22.1}}E\left[z\int_{y\geq\frac{C_0+\epsilon(z\,C_{11}+C_{12})}{\sigma(T)}} \phi(y)dy\right] \\
= \;& \sqrt{\Sigma_{22.1}}E\left[z\,\Phi\left(-\frac{C_0+\epsilon(zC_{11}+C_{12})}{\sigma(T)}\right)\right] \\
= \;& \sqrt{\Sigma_{22.1}}E\left\{z\left[\Phi\left(-\frac{C_0}{\sigma(T)}\right)+\phi\left(-\frac{C_0}{\sigma(T)}\right)\frac{\epsilon(z\,C_{11}+C_{12})}{\sigma(T)}\right]+\cdots\right\} \\
= \;& \sqrt{\Sigma_{22.1}}\phi\left(-\frac{C_0}{\sigma(T)}\right)\frac{1}{\sigma(T)}\epsilon\,C_{11}+\cdots ,
\end{aligned}
$$

which is in the order of $o(\epsilon)$. Because we can ignore all terms except the leading term in the asymptotic expansion of $\epsilon\,\Lambda_2$, which is in the order of $o(\epsilon)$, we have the desired result.

6.2 Appendix B

We give some numerical results for stock option values for the at-the-money case from Table 1 to Table 6.

Table 1: European Call Option Value on Equity and Option Delta under Downward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.11 > \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 13.868.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	12.2271	12.3811	12.5288	12.6731	12.8127
(2) Our Approach	12.2297	12.3773	12.525	12.6726	12.8203
(2-1) B-S value under deter.	12.525	12.525	12.525	12.525	12.525
(2-2) Adjustment by stoch.	-0.2953	-0.1476	0	0.1476	0.2953
(1)-(2)	-0.0026	0.0038	0.0038	0.0005	-0.0076
(2-1)-Original B-S Value	-1.3430	-1.3430	-1.3430	-1.3430	-1.3430
(2) - Original B-S Value	-1.6383	-1.4907	-1.3430	-1.1954	-1.0477
Option Delta by Simul.	0.709	0.706	0.705	0.702	0.699
Our Option Delta	0.7092	0.7067	0.7042	0.7017	0.6992
B-S Option Delta	0.7422	0.7422	0.7422	0.7422	0.7422

Table 2: European Call Option Value on Equity and Option Delta under Downward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.3$

We set $r_0 = 0.11 > \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.3$,$S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 13.868.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	11.6306	12.1267	12.5671	12.9701	13.3439
(2) Our Approach	11.6391	12.082	12.525	12.9679	13.4108
(2-1) B-S value under deter.	12.525	12.525	12.525	12.525	12.525
(2-2) Adjustment by stoch.	-0.8859	-0.4429	0	0.4429	0.8859
(1)-(2)	-0.0085	0.0447	0.0421	0.0022	-0.0669
(2-1)-Original B-S Value	-1.3430	-1.3430	-1.3430	-1.3430	-1.3430
(2) - Original B-S Value	-2.2289	-1.7860	-1.3430	-0.9001	-0.4572
Option Delta by Simul.	0.715	0.709	0.703	0.695	0.687
Our Option Delta	0.7191	0.7116	0.7042	0.6967	0.6893
B-S Option Delta	0.7422	0.7422	0.7422	0.7422	0.7422

Table 3: European Call Option Value on Equity and Option Delta under Upward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.03 < \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 9.4134.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	10.3592	10.4794	10.5953	10.7088	10.8212
(2) Our Approach	10.3615	10.4783	10.5952	10.7120	10.8288
(2-1) B-S value under deter.	10.5952	10.5952	10.5952	10.5952	10.5952
(2-2) Adjustment by stoch.	-0.2337	-0.1168	0	0.1168	0.2337
(1)-(2)	-0.0023	0.0011	0.0001	-0.0032	-0.0076
(2-1)-Original B-S Value	1.1818	1.1818	1.1818	1.1818	1.1818
(2) - Original B-S Value	0.9481	1.0649	1.1818	1.2986	1.4154
Option Delta by Simul.	0.643	0.642	0.642	0.641	0.639
Our Option Delta	0.6438	0.6429	0.6419	0.6409	0.6400
B-S Option Delta	0.5987	0.5987	0.5987	0.5987	0.5987

Table 4: European Call Option Value on Equity and Option Delta under Upward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.3$

We set $r_0 = 0.03 < \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.3$, $S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 9.4134.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	9.9045	10.2750	10.6179	10.9383	11.2403
(2) Our Approach	9.8942	10.2447	10.5952	10.9456	11.2961
(2-1) B-S value under deter.	10.5952	10.5952	10.5952	10.5952	10.5952
(2-2) Adjustment by stoch.	-0.7010	-0.3505	0	0.3505	0.7010
(1)-(2)	0.0103	0.0303	0.0227	-0.0073	-0.0558
(2-1)-Original B-S Value	1.1818	1.1818	1.1818	1.1818	1.1818
(2) - Original B-S Value	0.4808	0.8313	1.1818	1.5322	1.8827
Option Delta by Simul.	0.6426	0.643	0.641	0.637	0.632
Our Option Delta	0.6476	0.6448	0.6419	0.6390	0.6362
B-S Option Delta	0.5987	0.5987	0.5987	0.5987	0.5987

Table 5: European Call Option Value on Equity and Option Delta under Level Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 11.5415.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	11.2683	11.4085	11.5436	11.6757	11.8046
(2) Our Approach	11.2707	11.4061	11.5415	11.6768	11.8122
(2-1) B-S value under deter.	11.5415	11.5415	11.5415	11.5415	11.5415
(2-2) Adjustment by stoch.	-0.2707	-0.1354	0	0.1354	0.2707
(1)-(2)	-0.0024	0.0024	0.0021	-0.0011	-0.0076
(2-1)-Original B-S Value	-0.00003	-0.00003	-00003	-0.00003	-0.00003
(2) - Original B-S Value	-0.2708	-0.1354	-0.00003	0.1353	0.2707
Option Delta by Simul.	0.676	0.675	0.674	0.672	0.670
Our Option Delta	0.6770	0.6753	0.6736	0.6720	0.6703
B-S Option Delta	0.6736	0.6736	0.6736	0.6736	0.6736

Table 6: European Call Option Value on Equity and Option Delta under Level Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.3$

We set $r_0 = \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.3$, $S_0 = K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 11.5415.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	10.7290	11.1734	11.5751	11.9455	12.2913
(2) Our Approach	10.7293	11.1354	11.5415	11.9476	12.3537
(2-1) B-S value under deter.	11.5415	11.5415	11.5415	11.5415	11.5415
(2-2) Adjustment by stoch.	-0.8122	-0.4061	0	0.4061	0.8122
(1)-(2)	-0.0003	0.0380	0.0336	-0.0021	-0.0624
(2-1)-Original B-S Value	-0.00003	-0.00003	-0.00003	-0.00003	-0.00003
(2) - Original B-S Value	-0.8122	-0.4061	-0.00003	0.4061	0.8122
Option Delta by Simul.	0.679	0.677	0.672	0.667	0.660
Our Option Delta	0.6838	0.6787	0.6736	0.6686	0.6635
B-S Option Delta	0.6736	0.6736	0.6736	0.6736	0.6736

6.3 Appendix C

We give some numerical results for stock option values for the in-the-money case and the out-of-the-money case. The in-the-money cases are given in Table 7 and Table 8 while the out-of-the-money cases are given in Table 9 and Table 10.

Table 7: European Call Option Value on Equity and Option Delta under Downward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.11 > \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = 110 > K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 21.9837.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	20.0939	20.2126	20.3262	20.4356	20.5384
(2) Our Approach	20.0976	20.2099	20.3221	20.4344	20.5467
(2-1) B-S value under deter.	20.3221	20.3221	20.3221	20.3221	20.3221
(2-2) Adjustment by stoch.	-0.2245	-0.1123	0	0.1123	0.2245
(1)-(2)	-0.0037	0.0027	0.0042	0.0012	-0.0083
(2-1)-Original B-S Value	-1.6616	-1.6616	-1.6616	-1.6616	-1.6616
(2) - Original B-S Value	-1.8861	-1.7738	-1.6616	-1.5493	-1.4370
Option Delta by Simul.	0.852	0.848	0.845	0.841	0.836
Our Option Delta	0.8528	0.8486	0.8445	0.8403	0.8362
B-S Option Delta	0.8700	0.8700	0.8700	0.8700	0.8700

Table 8: European Call Option Value on Equity and Option Delta under Upward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.03 < \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = 110 > K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 16.2837.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	17.6541	17.7555	17.8527	17.9464	18.0348
(2) Our Approach	17.6594	17.7559	17.8524	17.9489	18.0453
(2-1) B-S value under deter.	17.8524	17.8524	17.8524	17.8524	17.8524
(2-2) Adjustment by stoch.	-0.1929	-0.0965	0	0.0965	0.1929
(1)-(2)	-0.0053	-0.0004	0.0003	-0.0025	-0.0105
(2-1)-Original B-S Value	1.5687	1.5687	1.5687	1.5687	1.5687
(2) - Original B-S Value	1.3758	1.4722	1.5687	1.6652	1.7616
Option Delta by Simul.	0.805	0.802	0.799	0.797	0.794
Our Option Delta	0.8052	0.8024	0.7996	0.7968	0.7940
B-S Option Delta	0.7662	0.7662	0.7662	0.7662	0.7662

Table 9: European Call Option Value on Equity and Option Delta under Downward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.11 > \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = 90 < K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 7.36263.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	6.1366	6.2945	6.4469	6.5970	6.7471
(2) Our Approach	6.1365	6.2899	6.4434	6.5968	6.7502
(2-1) B-S value under deter.	6.4434	6.4434	6.4434	6.4434	6.4434
(2-2) Adjustment by stoch.	-0.3069	-0.1534	0	0.1534	0.3069
(1)-(2)	0.0001	0.0046	0.0035	0.0003	-0.0031
(2-1)-Original B-S Value	-0.9193	-0.9193	-0.9193	-0.9193	-0.9193
(2) - Original B-S Value	-1.2262	-1.0727	-0.9193	-0.7658	-0.6124
Option Delta by Simul.	0.5003	0.5021	0.5038	0.5055	0.5065
Our Option Delta	0.5006	0.5022	0.5039	0.5055	0.5071
B-S Option Delta	0.5490	0.5490	0.5490	0.5490	0.5490

Table 10: European Call Option Value on Equity and Option Delta under Upward Stochastic Interest Rate in CIR type : $\sigma = 0.2$, $\epsilon = 0.1$

We set $r_0 = 0.03 < \bar{r} = 0.07$, $\kappa = 2.0$, $\epsilon = 0.1$, $S_0 = 90 < K = 100$, $\sigma = 0.2$, and $T = 1.0$. The call option value based on the original Black-Scholes formula is 4.44793.

	Corr. Coeff. bt. Stock Price and Interest Rate				
	$\rho = -1$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = 1$
(1) Simulation	4.9615	5.0738	5.1828	5.2912	5.4018
(2) Our Approach	4.9610	5.0718	5.1827	5.2935	5.4044
(2-1) B-S value under deter.	5.1827	5.1827	5.1827	5.1827	5.1827
(2-2) Adjustment by stoch.	-0.2217	-0.1108	0	0.1108	0.2217
(1)-(2)	0.0005	0.0020	0.0001	-0.0023	-0.0026
(2-1)-Original B-S Value	0.7347	0.7347	0.7347	0.7347	0.7347
(2) - Original B-S Value	0.5131	0.6239	0.7347	0.8456	0.9564
Option Delta by Simul.	0.4303	0.4329	0.4352	0.4373	0.4392
Our Option Delta	0.4307	0.4329	0.4352	0.4374	0.4396
B-S Option Delta	0.3910	0.3910	0.3910	0.3910	0.3910