CIRJE-F-8

Seasonal Cycles and Endogenous Business Cycles in a Monetary Economy

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June 1998

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Abstract

The purpose of this paper is to investigate how seasonal fluctuations in preference may change the dynamic stability and make multiple equilibria more likely outcome in a standard monetary economy. In the analysis, we investigate a model of money-in-the-utility function where real money balances held at the beginning of the period induce utility. Unless the utility function has seasonal patterns, the steady state equilibrium is a unique equilibrium unless the degree of risk aversion is incredibly large. However, when the utility function has some seasonal patterns, the dynamic system may have multiple dynamic paths or limit cycles even if the degree of risk aversion takes a reasonable value in most of the seasons. In particular, when the number of seasons is large, there may exist multiple dynamic paths even if the observed money demand is decreasing in interest rates.

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* I would like to thank seminar participants at the Japan Development Bank for their helpful comments. Financial support from the Japan Ministry of Education Scientific Research Grant is gratefully acknowledged.

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1. Introduction

In almost all industrial economies, it is widely observed that seasonal fluctuations account for a quantitatively important part of the total variations in most GDP components. This is because both preference and technology have seasonal patterns. For example, consumers usually have higher propensity to consume in holiday seasons than in other seasons. Productivity may also vary over seasons, especially in the agricultural sector. Given these seasonal patterns in preference and technology, it is not surprising that not a few macro variables have large seasonal fluctuations. However, given these seasonal patterns, it is far from obvious whether these seasonal patterns can be a source of business cycles, that is, macroeconomic fluctuations of seasonally adjusted data. The purpose of this paper is to investigate whether seasonal fluctuations in preference can be a source of endogenous equilibrium business cycles in a monetary economy. Specifically, we explore how seasonal fluctuations in the utility functions may change the dynamic stability and make multiple equilibria and limit cycles more likely outcome in a standard monetary economy.

In the previous literature, there are several studies that have investigated the existence of multiple dynamic paths (i.e., multiple convergence equilibria, sunspots, limit cycles, and chaos) in monetary economies. For example, Benhabib and Day (1982), Grandmont (1985, 1986), and Azariadis and Guesnerie, (1986) showed the existence of perfect foresight deterministic cycles and sunspot equilibria in overlapping generations models. In a cash-in-advance model of infinitely lived agents, Woodford (1994) similarly found the existence of sunspot equilibria.[1] Throughout these studies on monetary economies, it was shown that there exist multiple dynamic paths when the degree of relative risk aversion is very large. This is because the relative importance of the income and substitution effects plays an important role in producing multiple dynamic paths. However, such a large degree of risk aversion was not supported by previous empirical studies because it means that saving rates are decreasing in real interest rates.[2] In particular, in the models of money, the large degree of risk aversion is not consistent with previous empirical studies because it implies that the money demand function is increasing in nominal interest rate.

In the following analysis, we investigate a model of money-in-the-utility function where real money balances held at the beginning of the period induce utility.[3] Assuming that the utility function is separable in consumption and real money balances, the first-order conditions

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[1] Michener and Ravikumar (1994) showed the existence of deterministic cycles and chaos in a similar cash-in-advance model.

[2] For example, Kydland and Prescott (1982) found that the degree of relative risk aversion needs a value between one and two to mimic the observed relative variability of consumption and investment. Hansen and Singleton (1982) also showed that the degree of relative risk aversion is close to zero in the estimates from stochastic Euler equations.

[3] This type of model was used in McCallum (1984), Danthine et al. (1986, 1987), Den Haan (1990), and Woodford (1990).

generally lead to the well-defined backward perfect foresight dynamics that the future value of real money supply determines its current value. When the utility function has no seasonal patterns, the steady state equilibrium is a unique equilibrium path unless the degree of risk aversion is incredibly large.[4] Even if the utility function has seasonal patterns, the degree of risk aversion needs to be large at least in one season for the existence of multiple dynamic paths. However, if the degree of risk aversion is large in one season, the monetary economy can have multiple dynamic paths even if the degree of risk aversion takes a reasonable value in all of the other seasons. In particular, when the number of seasons is large, there exist multiple dynamic paths even if the observed money demand function is decreasing in nominal interest rate.

What is crucial in the following analysis is the assumption that individuals who are not so risk averse becomes very risk averse in some special season. Some of the readers may think that this assumption is strange and unrealistic. However, recalling our activities in a year, we can easily find economic behavior for which the assumption is satisfied. For example, individuals who are not so risk averse can be very risk averse in Christmas Eve. This is because Christmas Eve is a special day for all Christians and they hate to have miserable Christmas Eve. The following analysis will show that this type of seasonal patterns in preference may cause not only seasonal fluctuations but also business cycles without exogenous fundamental shocks.

In the previous literature, there are a few theoretical studies that analyzed seasonal fluctuations in equilibrium models. Chatterjee and Ravikumar (1992) and Braun and Evans (1994) are one of these exceptional studies. They modified the standard real business cycle model by allowing seasonal shifts in taste and technology and compare the seasonal implications of their models to the observed seasonal movements in the data. However, contrary to our study, they derived no welfare implications because seasonal variations are the efficient responses of the economy to changes in preferences or technological opportunities in their models.

In previous empirical studies, several authors have stressed the importance of seasonal fluctuations in analyzing business cycles.[5] In particular, there exist empirical studies which have demonstrated that business cycles may not be independent of seasonal fluctuations. For example, Barsky and Miron (1989) and Beaulieu and Miron (1992) showed empirical evidence that seasonal fluctuations have several common characteristics with business cycles. Beaulieu,

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4 This observation follows Fukuda (1993, 1997). In the previous literature, Matsuyama (1990, 1991) have also investigated the existence of non-convergent dynamic paths (i.e, sunspots, limit cycles, and topological chaos) in a money-in-utility model of infinitely lived. However, his model is different from our model in that he did not impose liquidity advance constraint.

5 For example, Miron (1986), Miron and Zeldes (1988), and Birchenhall et al. (1989).

MacKie-Mason, and Miron (1992) showed that countries and industries with large seasonal cycles also have large business cycles. Our theoretical results might be consistent with these empirical studies because the economy with large seasonal patterns in preference can have endogenous business cycles for reasonable parameters in our model.

The paper proceeds as follows. Section 2 presents a basic framework in our model. Section 3 reviews previous results on the dynamic stability of no seasonal fluctuation. Section 4 discusses the steady state equilibrium with seasonal fluctuations. Section 5 investigates the dynamic stability for the case where seasonal patterns exist in preference and section 6 reconfirms its main result by specific utility functions. Section 7 explores the existence of periodic cycles and section 8 presents some simulation results. Section 9 summarizes our main results and refers to their possible extensions.

2. A Basic Framework

We consider an economy inhabited by identical agents, each maximizing their utility over an infinite lifetime. There is a single, perishable, consumption good in the economy. Each representative agent has the following utility function :

$$(1) \quad \sum_{i=0}^{\infty} \beta^i \left[u_{t+i}(c_{t+i}) + v_{t+i}\left(M_{t+i} / p_{t+i} \right) \right],$$

where c_t is consumption at period t, p_t is the price level at period t, and M_t is the amount of currency at the beginning of period t. β is a discount factor satisfying $0 < \beta < 1$.

A noteworthy feature in the above utility function is that the functional forms of $u_t(\cdot)$ and $v_t(\cdot)$ might be different over time.[6] In the following analysis, we assume that the economy has n seasons and suppose that :

$$(2) \quad u_t(\cdot) = u_j(\cdot) \quad \text{and} \quad v_t(\cdot) = v_j(\cdot) \quad \text{when } t = ns + j,$$

where n, s, and j are positive integers and $1 \leqq j \leqq n$. The well-defined utility functions $u_j(\cdot)$ and $v_j(\cdot)$ are strictly concave (i.e., $u_j''(\cdot) < 0$ and $v_j''(\cdot) < 0$) and increasing (i.e., $u_j'(\cdot) > 0$ and $v_j'(\cdot) > 0$) for all j. We also assume that $v_j(\cdot)$ satisfies the Inada conditions such that $\lim_{M/p \to +\infty} v_j'(M/p) = 0$ and $\lim_{M/p \to 0} v_j'(M/p) = +\infty$ for j.

The budget constraint of the representative agent is :

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[6] The utility function is a special case of state-dependent utility functions where states are deterministically defined as seasons.

(3) $\quad c_t + T_t + M_{t+1}/p_t \leqq M_t/p_t + y,$

where y is constant exogenous income and T_t represents lump-sum taxes (or transfers, if negative) at time t. This budget constraint indicates that only real money balances held at the beginning of the period induce utility.

We assume that there is no government consumption nor the growth of the nominal money supply. Then, since the balanced government's budget constraint implies that $T_t = 0$, it holds that $c_t = y$ in the goods market equilibrium for all t. Thus, under perfect foresight, the first-order conditions lead to:

(4) $\quad m_t = \beta\, m_{t+1}\, [\, A_j + v_{j+1}'(m_{t+1})/u_j'(y)\,] \quad$ when $t = ns + j$,

where $m_t \equiv M_t/p_t$ and $A_j \equiv u_{j+1}'(y)/u_j'(y)$. [7]

The above equation determines the dynamic system of real money supply m_t in our model. The dynamic system is the well defined backward perfect foresight dynamics that the future value of real money supply determines its current value. For notational simplicity, the following analysis rewrites the dynamic equation (4) as

(5) $\quad m_t = f_j\,(m_{t+1}) \quad$ when $t = ns + j$,

where $f_j\,(m) \equiv \beta\, m_{t+1}\, [\, A_j + v_{j+1}'(m_{t+1})/u_j'(y)\,]$ and $f_{j+n}\,(m) \equiv f_j\,(m)$. Then, the self-contained dynamics of season j ($1 \leqq j \leqq n$) is written as the following one-dimensional mappings:

(6) $\quad m_t = h_j\,(m_{t+n}) \equiv f_j\,(f_{j-1}\,(\ldots(f_{j-n+2}\,(f_{j-n+1}\,(m_{t+n})))\ldots))\quad$ when $t = ns + j$.

We investigate the stability of this dynamic equation in the following sections.

3. The Stability without Seasonal Fluctuations

Assuming that $u_t(\cdot\,) = u(\cdot\,)$ and $v_t(\cdot\,) = v(\cdot\,)$ for all t, we first review the dynamic stability of our model for the case where there exist no seasonal patterns in the utility functions. Without seasonal fluctuations, our dynamic system (4) can be simply written as

7 For expositional simplicity, we define that $A_n \equiv v_1'(m_{t+1})/u_n'(y)$ and $u_{n+1}'(y) \equiv u_1'(y)$.

(7) $\quad m_t = \beta\, m_{t+1}\,[\,1 + v'(m_{t+1})/u'(y)]\quad$ for all t.

The non-zero steady state equilibrium m^0 is thus defined so as to satisfy:

(8) $\quad 1 = \beta\,[\,1 + v'(m^0)/u'(y)]$.

Because of the Inada conditions, this steady state equilibrium always exists uniquely.

On the stability of dynamic equation (7), the following four facts are well known in previous studies (see Fukuda (1993, 1997) for details).

Fact 1: The steady state equilibrium m^0 is a unique non-zero equilibrium if and only if

(9) $\quad - m^0\, v''(m^0) \,/\, v'(m^0) \;<\; 2/(1-\beta\,)$.

Fact 2: There exist multiple convergence equilibria to the steady state if and only if :

(10) $\quad - m^0\, v''(m^0) \,/\, v'(m^0) \;>\; 2/(1-\beta\,)$.

Fact 3: If the condition (10) is satisfied, there exist period-two cycle equilibria and stationary sunspot equilibria around the steady state. In particular, when $- m^0\, v''(m^0) \,/\, v'(m^0) = 2/(1-\beta\,)$, period doubling bifurcation occurs.

Fact 4: Suppose that $v(m_t) \equiv \frac{m_t^{\;1-R}}{1-R}$ for some positive parameter R. Then, if R is large enough, there exist period-three cycle equilibria. Thus, the result of Li and Yorke (1975) implies that if R is large enough, there exists topological chaos in our model.

Define $F(m) \equiv \beta\, m[1 + v'(m)/u'(y)]$. Then, noting that m^0 is a unique non-zero steady state, facts 2 and 3 can be derived from the well-known condition that $F'(m) < -1$ at the steady state m^0 (see, for example, Grandmont (1985), Woodford (1986), and Chiappori, et al. (1992)).

However, the above four facts imply that the existence of multiple dynamic paths requires a very large degree of relative risk aversion for $v(m)$. This is because the relative importance of the income and substitution effects plays an important role in producing endogenous cycles and sunspots. For example, suppose that $v(m_t) \equiv \frac{m_t^{\;1-R}}{1-R}$. Then, since $R = - m^0\, v''(m^0) \,/\, v'(m^0)$, facts 2 and 3 imply that there exist multiple convergence equilibria, stationary sunspot equilibria, and period-two cycle equilibria if $R > 2/(1-\beta)$. This indicates that when $\beta = 0.95$, the existence of multiple dynamic paths requires R to be greater than 40! Needless to say, such a large degree of risk aversion is not supported by previous empirical studies. Therefore, without seasonal patterns in preference, the monetary model may not have multiple dynamic paths for empirically plausible parameters.

4. The Steady State Equilibrium

When the utility functions have seasonal fluctuations, the steady state equilibrium needs to be defined for each season because it shows seasonal fluctuations. When $t = ns + j$ $(1 \leqq j \leqq n)$, the non-zero steady state equilibrium of m_t is defined as $m_t \equiv m_j^* \neq 0$ which satisifes

$$(11) \quad m_j^* = h_j(m_j^*) \equiv f_j(f_{j+1}(\ldots(f_{j+n-2}(f_{j+n-1}(m_j^*)))\ldots)).$$

To the extent that the utility functions have seasonal fluctuations, the steady state equilibrium in season j is not equal to the steady state equilibrium in season i if $j \neq i$.

Under some mild conditions, it is not difficult to show that the non-zero steady state equilibrium m_j^* always exists for all j. In addition, when $1 > f_i'(m) > 0$ for all $m > 0$ and $i = 1, 2, \ldots, n$, the non-zero steady state equilibrium is unique because $1 > dh_j(m)/dm = f_j' f_{j+1}' \cdots f_{j+n-2}' f_{j+n-1}' > 0$ for all $m > 0$.

However, when $f_i'(m) < 0$ for some $m > 0$ and $i = 1, 2, \ldots, n$, the non-zero steady state equilibrium is not necessarily unique. A special case where the non-zero steady state equilibrium is not unique arises when $u_t(\cdot) = u(\cdot)$ and $v_t(\cdot) = \frac{m^{1-R}}{1-R}$ for all t and that $R > 2/(1-\beta)$. In this case, because of no seasonal patterns in the utility functions, $m_j^* = m^0$ for all j is an equilibrium. However, because of fact 4 in the previous subsection, the dynamic equation (6) has a period two cycle equilibrium such that $m_t = m^1 \neq m^0$ when t is odd (even) and $m_t = m^2 \neq m^0$ when t is even (odd) for some $m^1 \neq m^2$. Thus, when n is even, both m^1 and m^2 are another equibribria of m_j^*.[8]

The above special case implies that multiple steady state equilibria arise because period-two cycles of seasonally unadjusted data is observationally equivalent to seasonal cycles when the number of seasons is even. In order to avoid this observationally equivalence problem, the following analysis implicitly assumes that the non-zero steady state equilibrium is unique and

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[8] Because of continuity, we can also verify that m_j^* is not unique when $u_1(c)$ and $u_2(c)$ are different but very similar when $v_t(\cdot) = \frac{m^{1-R}}{1-R}$ and $R > 2/(1-\beta)$.

investigates how seasonal patterns in preference may cause the existence of multiple equilibria around the unique non-zero steady state equilibrium.

5. The Dynamic Stability

In this section, we investigate how seasonal patterns in the utility functions can change the dynamic stability of our model. Recall that the self-contained dynamics of season j ($1 \leqq j \leqq n$) is written as equation (6). Then, if we define

$$(12) \qquad g_j(m) \equiv f_{j-1}(\ldots(f_{j-n+2}(f_{j-n+1}(m)) \ldots),$$

the dynamic equation (6) is written as

$$(13) \qquad m_t = h_j(m_{t+n}) \equiv f_j(g_j(m_{t+n})) = \beta\, g_j(m_{t+n})\,[\, A_j + v_{j+1}'(g_j(m_{t+n}))/u_j'(y)\,]$$

when $t = ns + j$ ($1 \leqq j \leqq n$). The non-zero steady state equilibrium of m_t at season j satisfies

$$(14) \qquad m_j^* = h_j(m_j^*) \equiv \beta\, g_j(m_j^*)\,[\, A_j + v_{j+1}'(g_j(m_j^*))/u_j'(y)\,]$$

Thus, it holds that

$$(15) \qquad dh_j(m_j^*)/d\, m_j^*$$
$$= \beta\, g_j'(m_j^*)\,[\, A_j + v_{j+1}'(g_j(m_j^*))/u_j'(y)\,] + \beta\, g_j(m_j^*)\, g_j'(m_j^*)\, v_{j+1}''(g_j(m_j^*))/u_j'(y),$$
$$= \beta\, g_j'(m_j^*)\, v_{j+1}'(m_{j+1}^*)/u_j'(y)\,[1 + A_j\,\{u_j'(y)/\, v_{j+1}'(m_{j+1}^*)\} + m_{j+1}^* v_{j+1}''(m_{j+1}^*)/\, v_{j+1}'(m_{j+1}^*)].$$

where $m_{j+1}^* = g_j(m_j^*)$.

Equation (15) implies that when $g_j'(m_j^*) > 0$, it holds that $dh_j(m_j^*)/d\, m_j^* < -1$ if and only if [9]

$$(16) \qquad -\, m_{j+1}^*\, v_{j+1}''(m_{j+1}^*)/\, v_{j+1}'(m_{j+1}^*) > 1 + (A_j + 1/\{\beta\, g_j'(m_j^*)\})\,\{u_j'(y)/\, v_{j+1}'(m_{j+1}^*)\}.$$

Since there exist multiple convergence equilibria if $dh_j(m_j^*)/d\, m_j^* < -1$, this implies that the utility function $v_{j+1}(m)$ needs to have a large degree of relative risk aversion for the existence of multiple convergence equilibria when $g_j'(m_j^*) > 0$. In fact, because

[9] Without seasonal patterns in preference and technology, $A_j = 1$, $g_j(m_j^*) = m_j^*$ and $g_j'(m_j^*) = 1$. Thus, the condition (16) reduces to the condition (10).

(17) $1 + (A_j + 1/\{\beta g_j'(m_j^*)\})\{u_j'(y)/v_{j+1}'(m_{j+1}^*)\}$

$$= \frac{1+g_j(m_j^*)/(m_j^* g_j'(m_j^*))}{1-A_j\beta g_j(m_j^*)/m_j^*} > 2,$$

(16) implies that the degree of risk aversion of $v_{j+1}(m)$ needs to be greater than two when $g_j'(m_j^*) > 0$.

However, when $g_j'(m_j^*) < 0$, (15) implies that $dh_j(m_j^*)/dm_j^* < -1$ if and only if

(18) $-m_{j+1}^* v_{j+1}''(m_{j+1}^*)/v_{j+1}'(m_{j+1}^*) < 1 + (A_j + 1/\{\beta g_j'(m_j^*)\})\{u_j'(y)/v_{j+1}'(m_{j+1}^*)\}$.

Therefore, when $g_j'(m_j^*) < 0$, there exist multiple convergence equilibria even if the degree of risk aversion of $v_{j+1}(m)$ is arbitrarily small.

Because $dh_j(m)/dm = f_j' f_{j+1}' \ldots f_{j+n-2}' f_{j+n-1}'$, it holds that $f_k' < 0$ for some k when $dh_j(m)/dm < -1$. Since $g_j'(m_j^*) = f_{j+1}' \ldots f_{j+n-2}' f_{j+n-1}'$, this indicates that there always exists $g_k'(m_k^*) > 0$ for some k when $dh_j(m)/dm < -1$. However, when $f_k' < 0$ for some k, it is possible that $f_i' > 0$ for all $i \neq k$ even if $dh_j(m)/dm < -1$ for all j. Therefore, when $g_k'(m_k^*) < 0$ for some k, it is possible that $g_i'(m_i^*) > 0$ for all $i \neq k$ even if $dh_j(m)/dm < -1$ for all j. This leads to the following proposition.

Proposition 1: If there exist multiple convergence equilibria to the steady state, then it must hold that $-m_k^* v_k''(m_k^*)/v_k'(m_k^*) > 2$ for some k. However, if $-m_k^* v_k''(m_k^*)/v_k'(m_k^*)$ is large enough for some k, there exist multiple convergence equilibria to the steady sate even if $-m_i^* v_i''(m_i^*)/v_i'(m_i^*)$ is arbitrarily small for all $i \neq k$.

The above proposition implies that in order for our monetary economy to have multiple convergence equilibria to the steady state, the degree of risk aversion of $v_j(m)$ needs to be large at least in one season among n seasons. However, if the degree of risk aversion is large in one season, the monetary economy can have multiple convergence equilibria even if the degree of risk aversion is small in the other n-1 seasons. Therefore, if the number of seasons n is large enough, there exist multiple convergence equilibria and stationary sunspot equilibria even if the degree of risk aversion is small for almost all of the periods.

6. Two Examples

The purpose of this section is to reconfirm Proposition 1 by specifying utility functions in two alternative ways. The first is the case where the utility functions are written as follows:

(19) $u_i(y) \equiv u(y)$ for all i, $v_2(m) \equiv \frac{Bm^{1-R}}{1-R}$, and $v_j(m) \equiv Dm$ when $j \neq 2$.

When R is large, these utility functions indicate that each agent is very risk averse in season 2 but is risk neutral in the other seasons.[10] For these utility functions, equation (4) is written as

(20a) $$m_t = \beta\, m_{t+1}\left[1 + \frac{B}{u'(y)}\left(\frac{1}{m_{t+1}}\right)^R\right] \qquad \text{when } t = ns + 1,$$

(20b) $$m_t = \beta\left[1 + \frac{D}{u'(y)}\right] m_{t+1} \qquad \text{when } t \neq ns + 1.$$

Thus, equations (20a) and (20b) lead to the self-contained dynamic equations of season 1 as follows

(21) $$m_t = \beta^n\left[1 + D/u'(y)\right]^{n-1} m_{t+n}\left\{1 + \frac{B}{u'(y)}\left(\frac{1}{\beta\left[1 + D/u'(y)\right]}\right)^{R(n-1)}\left(\frac{1}{m_{t+n}}\right)^R\right\}$$

when $t = ns + 1$. It is easy to see that this dynamic equation has multiple convergence equilibria to the steady state if

(22) $$R > \frac{2}{1 - \beta^n\left[1 + D/u'(y)\right]^{n-1}}.$$

In other words, as long as $v_2(m)$ satisfies the condition (22), there exist multiple convergence equilibria and stationary sunspot equilibria even if the utility function is linear in all of the other seasons.

Recall that when $v_j(m) \equiv Bm^{1-R}/(1-R)$ for all j, that is, when there exist no seasonal patterns in the utility functions, there exist multiple convergence equilibria if and only if $R > 2/(1-\beta)$. Because

[10] Although the risk neutral utility function violates the Inada conditions, the violation is not essential in the following argument.

$$\frac{2}{1-\beta^n\left[1+D/u'(y)\right]^{n-1}} < \frac{2}{1-\beta} \quad \text{when} \quad \beta\,[1+\frac{D}{u'(y)}] < 1,$$

this implies that multiple convergence equilibria and stationary sunspot equilibria may exist even if the risk averse utility function satisfying (22) is not so risk averse as that satisfying (10). In other words, although the existence of multiple dynamic paths requires some large degree of risk aversion in one season, the required degree of risk aversion may not be so large as what was derived in previous studies.

The second special but more interesting example can be obtained when we specify the utility functions as follows:

(23) $u_i(y) \equiv u(y)$ for all i, $v_1(m) \equiv H\log(m - K)$, and $v_j(m) \equiv J\log(m)$ when $j \neq 1$.

Under these utility functions, each agent always has log utility functions. However, the agent has the minimum subsistence level of consumption K in season 1.[11] For these utility functions, equation (4) is written as

(24a) $m_t = \beta\,[m_{t+1} + H / (m_{t+1} - K)]$ when $t = ns$,

(24b) $m_t = \beta\,(m_{t+1} + J)$ when $t \neq ns$,

where $m_t > K$ when $t = ns+1$. Thus, equations (24a) and (24b) lead to the self-contained dynamic equations of season 1 as follows

(25) $$m_t = \beta^n m_{t+n} + \frac{\beta^n H}{m_{t+n} - K} + A, \quad \text{when } t = ns + 1,$$

where $A \equiv J\sum_{j=1}^{\infty}\beta^j$ and $m_{t+ns} > K$ for all s. This dynamic equation has multiple convergence equilibria to the steady state if

$$\beta^n - \frac{\beta^n H}{(m^* - K)^2} < -1, \quad \text{where} \quad \left(1-\beta^n\right)m^* = \frac{\beta^n H}{m^* - K} + A,$$

11 Fukuda (1998) showed the existence of multiple equilibria for this type of utility functions in an overlapping generations model.

or equivalently,

$$(26) \qquad K(1-\beta^n) > A + \sqrt{\frac{\beta^{3n}H}{1+\beta^n}}.$$

Therefore, even for the log utility functions, there exist multiple convergence equilibria and stationary sunspot equilibria if the minimum subsistence level of consumption is large in season 1.

7. The Existence of Periodic Cycles

In the dynamic equation (13), the function $h_j(m)$ is one-demensional mappings. Thus, if we assume that β is a bifurcation parameter and suppose that $m_j^* = h_j(m_j^*)$ when $\beta = \beta^*$, then Theorem 3.5.1 in Guckenheimer and Holmes (1983) leads to the following proposition.

Proposition 2: Suppose that :

$$(\partial h_j/\partial \beta)(\partial^2 h_j/\partial m) + 2(\partial^2 h_j/\partial m \partial \beta) \neq 0 \quad \text{at } (m_j^*, \ \beta^*),$$
$$(1/2)(\partial^2 h_j/\partial m^2)^2 + (1/3)(\partial^2 h_j/\partial m^2) \neq 0 \quad \text{at } (m_j^*, \ \beta^*).$$

Then, period-doubling bifurcation occurs when

$$(27) \qquad -m_k^* v_k''(m_k^*)/v_k'(m_k^*) = 1 + (A_{k-1} + 1/\{\beta g_{k-1}'(m_k^*)\})\{u_{k-1}'(y)/v_k'(m_k^*)\},$$

for some k. Thus, for the existence of period-two cycle equilibria, the degree of risk aversion of $v_k(m)$ needs to be large at least in one season among n seasons. However, there exist period-two cycle equilibria even if $-m_i^* v_i''(m_i^*)/v_i'(m_i^*) \leq 1$ for all $i \neq k$. Therefore, if the number of seasons n is large enough, period-two cycle equilibria exist even if the degree of risk aversion is small for almost all of the periods.

In no-linear dynamic models, the existence of period two cycles is particularly important because it is a necessary condition for the existence of any periodic cycles (Sarkovskii theorem). In fact, when we change the values of bifurcation parameters, we can see varieties of cycles in our model. For example, when the utility functions are written as (19), we can draw bifurcation maps for two alternative bifurcation parameters: β and R.

Figure 1a is a bifurcation map when we take β as a bifurcation parameter. In the figure, we set parameters as $B = 1$, $D = 0.1$, $u'(y) = 1$, $n = 12$, and $R = 10$. For these parameters, period-doubling bifurcation occurs when $\beta = 0.972664$. In addition, period cycles with higher frequencies also arise for smaller values of β. For example, period-four cycle arises when β is approximately equal to 0.967, period-eight cycle arises when β is approximately equal to 0.9655, and so on.

Figure 1b shows another bifurcation map when we take R as a bifurcation parameter. In this figure, we set parameters as $B = 1$, $\beta(1+D) = 0.95$, $u'(y) = 1$, $n = 12$, and $\beta = 0.9$. For these parameters, period-doubling bifurcation occurs when $R = 4.09769$, period-four cycle arises when R is approximately equal to 5.25, period-eight cycle arises when R is approximately equal to 5.6, and so on. In particular, when R is approximately equal to 6.7, we can see that period-three cycle arises, implying the existence of chaos in our model.

8. Some Simulation Results

Based on discussions in previous sections, this section investigates the time-series property of real money balances simulated by specific utility functions and parameter sets. In the following simulation, we specify the utility functions as follows:

(28a) $u_i(y) \equiv u(y)$ for all i,

(28b) $v_2(m) \equiv \dfrac{B * m^{1-R_1}}{1-R_1}$, and $v_j(m) \equiv \dfrac{B * m^{1-R_2}}{1-R_2}$ when $j \neq 2$.

For these utility functions, equation (4) is written as

(29a) $m_t = \beta\, m_{t+1} \left[1 + \dfrac{B}{u'(y)}\left(\dfrac{1}{m_{t+1}}\right)^{R_1}\right]$ when $t = ns + 1$,

(29b) $m_t = \beta\, m_{t+1} \left[1 + \dfrac{B}{u'(y)}\left(\dfrac{1}{m_{t+1}}\right)^{R_2}\right]$ when $t \neq ns + 1$.

Throughout the simulation, we set that $n = 12$, $\beta = 0.95$, $B = 0.01$, and $u'(y) = 1$ and assume that that $m_{2400} = 1.5$. Unless specified, we also set that $R_1 = 7$ and $R_2 = 0.1$. For this parameter set, Figure 2 depicts the dynamic paths of real money balances from $t = 1$ to 120. From the figure, we can see that movements of real money balances show some regularity for each 12 periods. Since the number of seasons is 12, this implies that there exist seasonal cycles in the simulated real money balances.

However, in Figure 2, we can also see that the movements of real money balances have some regularity whose cycles are longer than 12 periods. This indicates that there exist business cycles, that is, endogenous cycles of seasonal adjusted data in Figure 2. In fact, when we plot the dynamic paths of real money balances only for season 1, we can see a clear-cut business cycle regularity in the movements of real money balances. Figure 3 shows this by depicting the dynamic paths of m_t only when $t = ns + 1$ from $s = 1$ to 30. In the figure, we can see that the annual data of real money balances show quite regular movements that are close to period four cycles.

Needless to say, the existence of these endogenous business cycles crucially depends on the choice of parameter sets. In particular, the first part of Proposition 1 states that when both R_1 and R_2 are small, our model never has endogenous business cycles. However, the second part of Proposition 1 indicates that when R_1 is large, our model can have endogenous business cycles even if R_2 is small. In addition, our model tends to have larger endogenous cycles when R_1 is larger but when R_2 is smaller. Figures 4 and 5 show this property graphically.

Given that $R_2 = 0.1$, Figure 4 depicts the dynamic paths of real money balances in season 1 from s = 1 to 30 for four alternative values of R_1, that is, 5, 7, 10, and 15. When $R_1 = 5$, we can see no fluctuations of real money balances. However, for the other values of R_1, we can see significant fluctuations of real money balances. In particular, we can see larger endogenous fluctuations as R_1 becomes larger.

On the other hand, given that R_1 is large, endogenous fluctuations of real money balances are likely to arise when R_2 is smaller. Figure 5 shows this by depicting the dynamic paths of real money balances in season 1 for four alternative values of R_2, that is, 0.1, 1, 1.5, and 2. When $R_2 = 0.1$, 1, or 1.5, we can see significant fluctuations of real money balances. In particular, we can see larger fluctuations as R_2 becomes smaller. On the other hand, when $R_2 = 2$, we can see no fluctuations of real money balances.

In interpreting the above simulation results, it is important to recall that when $R_1 = R_2$, that is, when there is no seasonal patterns in preference, our parameter set never produces endogenous fluctuations of real money balances if $R_1 = R_2 < 40$. Because our simulations had endogenous cycles when $R_1 = 7$ and $R_2 = 0.1$, this indicates that the existence of seasonal patterns in preference make endogenous fluctuations more likely outcome not only for smaller value of R_2 but also for smaller value of R_1.

9. Concluding Remarks

This paper has investigated how seasonal fluctuations in preference can change the dynamic stability and may make multiple equilibria and limit cycles more likely outcome in a model of money-in-the-utility function. Without seasonal fluctuations, the steady state equilibrium is a unique equilibrium unless the degree of risk aversion is large enough. However, when the utility function has some seasonal patterns, we found that the dynamic system may have multiple dynamic paths or limit cycles around the steady state even if the degree of risk aversion takes a reasonable value in most of the seasons.

When the degree of risk aversion is always large in the models of money, the derived money demand function may not be consistent with previous empirical studies because it is increasing in nominal interest rate. However, when the degree of risk aversion is small in most of the periods, the observed money demand function can be decreasing in nominal interest rate. Therefore, when the number of seasons is large, our theoretical results on endogenous cycles can be consistent with previous empirical studies on money demand functions.

In the previous literature, there are some theoretical studies that analyzed seasonal fluctuations in equilibrium models. However, most of them derived no welfare implications because seasonal variations were the efficient responses of the economy in their models. Since derived multiple dynamic paths are Pareto-ranked in our model, our results have thus quite important welfare implications of seasonal fluctuations which were not discussed in previous studies.

Although we have analyzed a model of money-in-the-utility function in the text, similar results can hold true in other monetary models such as an overlapping generations model and a cash-in-advance model. This is because models of money-in-the-utility function are reduced forms of various types of monetary models (see Feenstra (1986)). In the Appendix, we show this for a cash-in-advance model by specifying utility functions.

One possible extension of our analysis is to investigate what effects seasonal patterns in technology will have on the dynamic stability. In particular, it would be interesting to see how seasonal patterns in preference and technology will affect the dynamic stability in optimal growth models. Without seasonal fluctuations, there exist a large number of studies that investigated the dynamic stability of optimal growth models, especially, two-sector models and models of externalities and increasing returns to scale. The results of this paper hint that allowing seasonal fluctuations may be also important in considering the issues of multiple dynamic paths and endogenous cycles in these dynamic models.

Appendix

The purpose of this Appendix is to see how the results derived in the text essentially carry through in a cash-in-advance model. The following model is based on a modified version of Lucas and Stokey (1987) and Woodford (1994). In the model, a representative individual has the utility function with seasonal patterns as follows:

(A1) $\sum_{i=0}^{\infty} \beta^i \; [\, \phi_{t+i}(c_{t+i}^{A}) + \phi_{t+i}(c_{t+i}^{B})],$

where c_{t+i}^{A} is cash-good consumption and c_{t+i}^{B} is credit-good consumption at period $t+i$. As in the text, we assume that the economy has n seasons and suppose that :

(A2) $\phi_t(\cdot) = \phi_j(\cdot)$ and $\phi_t(\cdot) = \phi_j(\cdot)$ when $t = ns + j$,

where n, s, and j are positive integers and $1 \leqq j \leqq n$. The utility functions $\phi_j(\cdot)$ and $\phi_j(\cdot)$ are well-defined and satisfy the conditions that $\phi_j'(\cdot) > 0$, $\phi_j''(\cdot) < 0$, $\phi_j'(\cdot) > 0$ and $\phi_j''(\cdot) < 0$ for all j.

The budget constraint of each individual is:

(A3) $c_{t+i}^{A} + c_{t+i}^{B} + T_t + M_{t+1}/P_t \leqq y + M_t/p_t,$ for all t.

In addition, each individual faces the cash-in-advance constraint that applies only to cash-good consumption purchases as follows

(A4) $c_t^{A} \leqq M_t/p_t.$

A representative individual's optimization problem is to maximize (A1) subject to the budget constraint (A3) and the cash-in-advance constraint (A4). The constraint optimization problem can be solved by using the following Lagrangean :

(A5) $L = \sum_{i=0}^{\infty} \beta^i \; [\, \phi_{t+i}(c_{t+i}^{A}) + \phi_{t+i}(c_{t+i}^{B})]$

$+ \lambda_{t+i} \{y + M_t/p_t - (c_{t+i}^{A} + c_{t+i}^{B} + T_t + M_{t+1}/p_t)\} + \gamma_{t+i} (M_t/p_t - c_t^{A})].$

Differentiating (A5) with respect to c_{t+i}^{A}, c_{t+i}^{B}, and M_{t+1}, we obtain

(A6a) $\phi_t'(c_t^{A}) = \lambda_t + \gamma_t,$

(A6b) $\phi_t'(c_t^{B}) = \lambda_t,$

(A6c) $\lambda_t / p_t = \beta\, (\lambda_{t+1} + \gamma_{t+1}) / p_{t+1}.$

As in the text, we assume that the nominal money supply M_t is constant and that there is no government consumption. We also assume that the cash-in-advance constraint is always binding. Then, because $c_t = y$ and $c_t = M_t/p_t$ in equilibrium, (A2) and (A6a) - (A6c) lead to

$$\text{(A7)} \qquad \phi_j'(x_t)\, x_t = \beta\, \phi_{j+1}'(x_{t+1})\, x_{t+1}, \qquad \text{when } t = ns + j,$$

where $x_t \equiv M_t/p_t$.

Equation (A7) determines the dynamic system of real money balances x_t in our cash-in-advance model. Since $\phi_j'(x_t)$ is increasing in x_t, it is the well-defined backward perfect foresight dynamics. In particular, defining that $\phi_j(c) \equiv u_j'(y)\mu(c)$, $\phi_j(c) \equiv \phi_j(c) + \pi_j(c)$, and $m_t \equiv \mu'(x_t)\, x_t$, (A7) can be reduced to equation (4) in the text if it holds that $v_j'(\mu'(x)\, x) \equiv \pi_j'(x)/\mu'(x)$. It is not difficult to see that $v_j'(\mu'(x)\, x) \equiv \pi_j'(x)/\mu'(x)$ for reasonable utility functions. For example, when $\mu(x) \equiv x^{1-\rho}/(1-\rho)$ and $\pi_t(x) \equiv x^{1-\lambda}/(1-\lambda)$, it holds that $v_j'(\mu'(x)\, x) \equiv \pi_j'(x)/\mu'(x)$ if $v_j(m) \equiv [(1-\rho)/(1-\lambda)]m^{(1-\lambda)/(1-\rho)}$ where $(1-\lambda)/(1-\rho) < 1$. This verifies that for some reasonable utility functions, all results in the text can carry through in the above cash-in-advance model.

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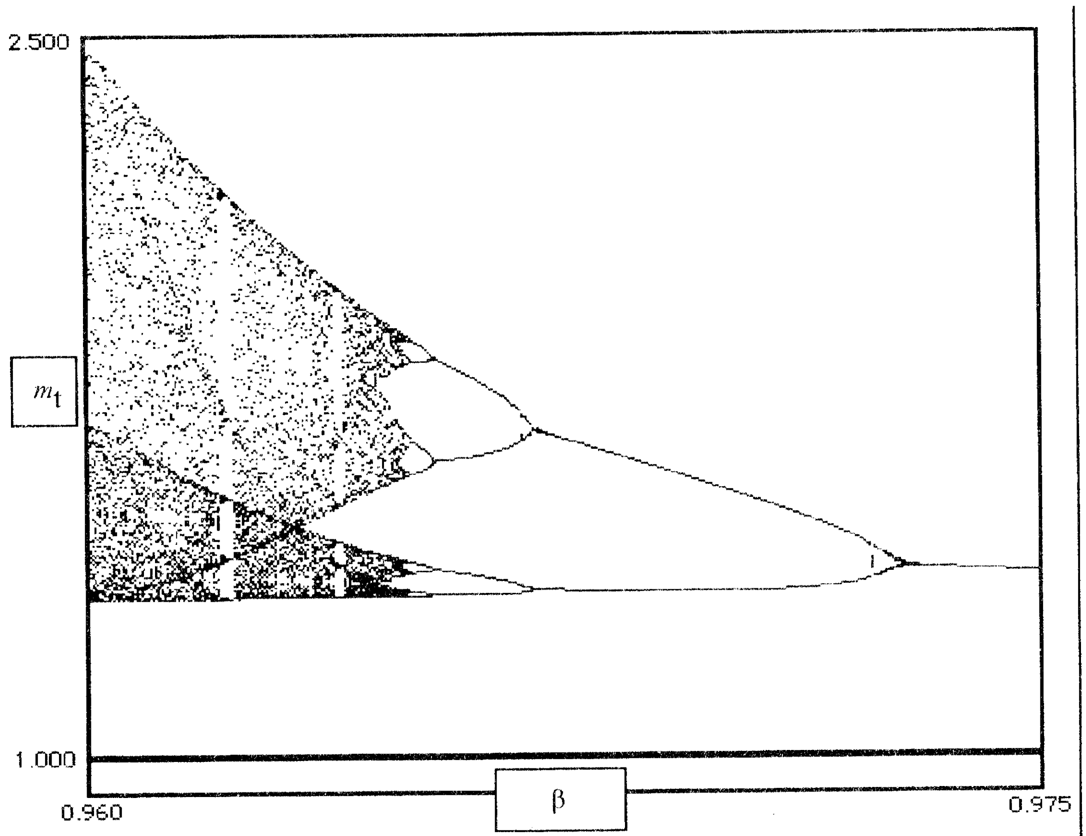
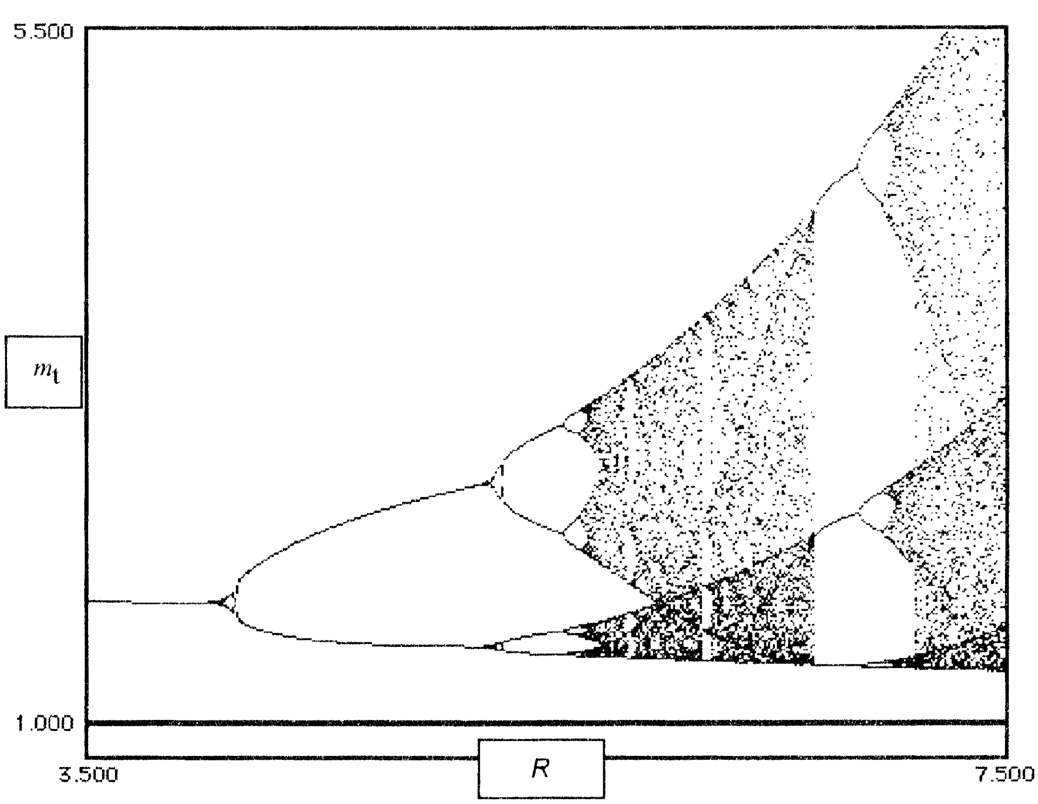
Figure 1. Bifurcation Maps for Two Alternative Parameters

(i) The case where β is a bifurcation parameter



parameters: $u'(y) = 1$, $B = 1$, $D = 0.01$, $n = 12$, $R = 10$.

(ii) The case where R is a bifurcation parameter



parameters: $u'(y) = 1$, $B = 1$, $n = 12$, $\beta(1+D) = 0.95$, $\beta = 0.9$.

Figure 2 The Movements of Real Money Balances with Seasonal Fluctuations

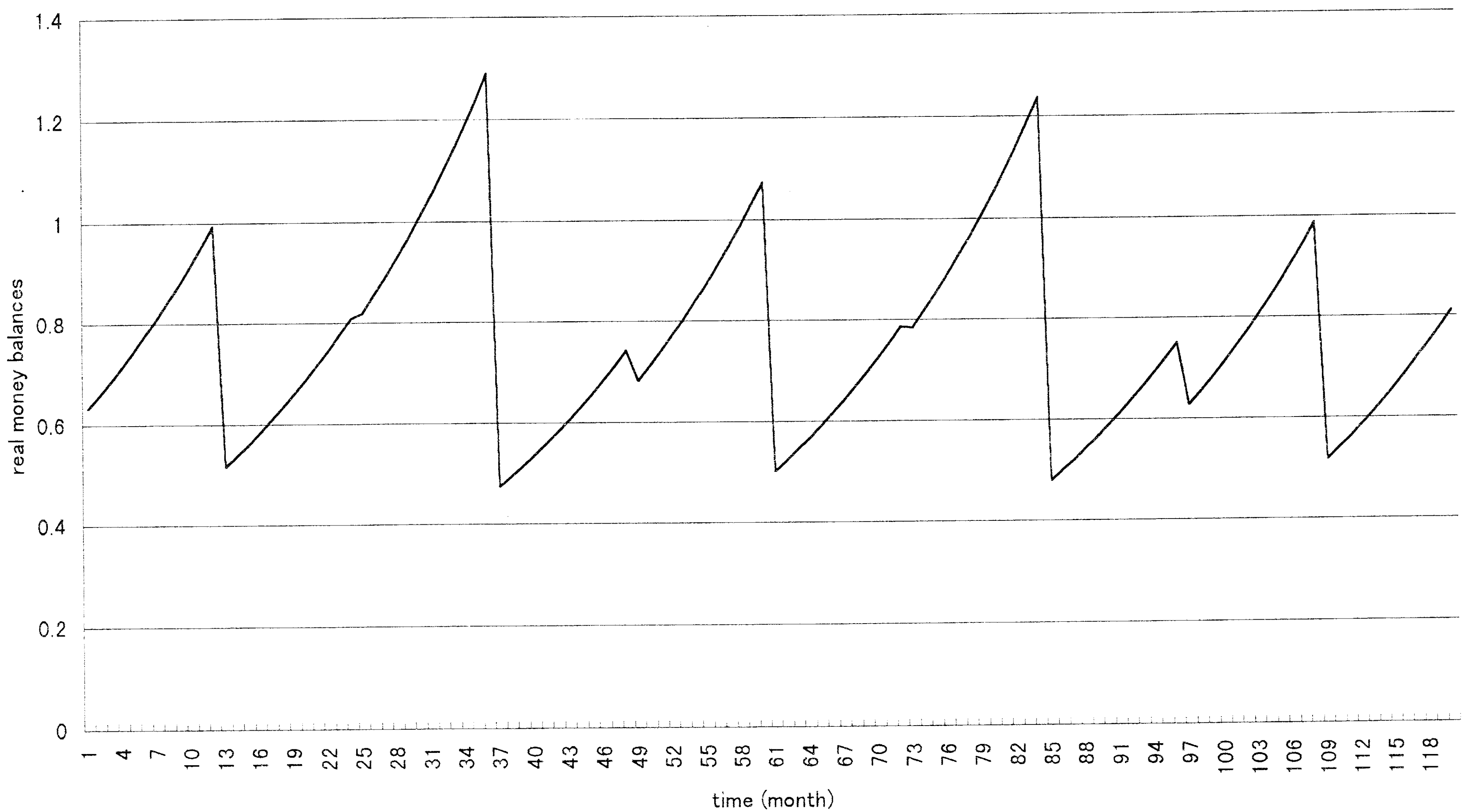


Figure 3 The Movements of Real Money Balances in Season 1

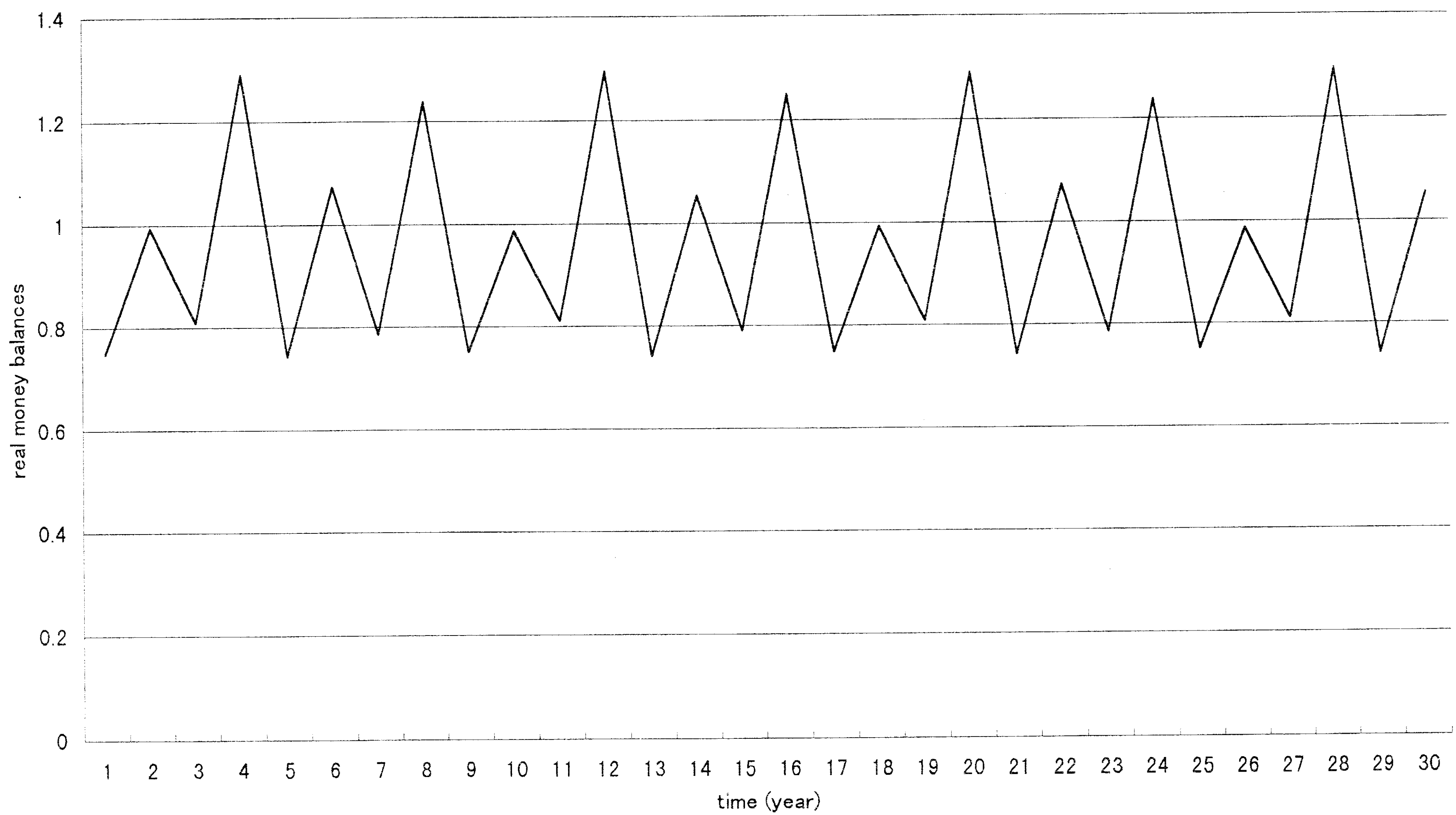


Figure 4 Real Money Balances for Alternative Parameters (1)

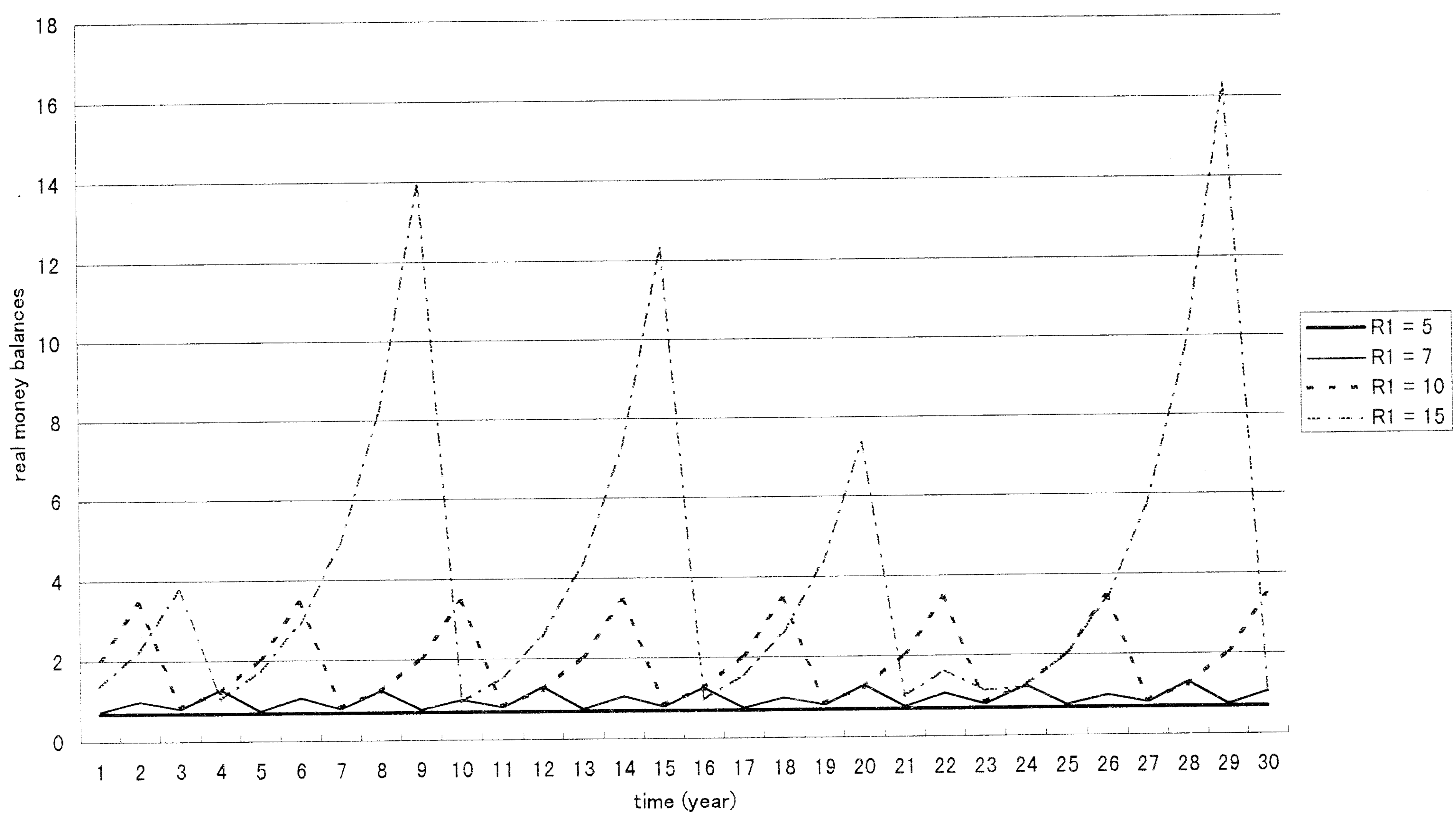


Figure 5 Real Money Balances for Alternative Parameters (2)