CIRJE-F-9

Towards a Theory of Subjective Games

Hitoshi Matsushima
The University of Tokyo

June 1998

Towards a Theory of Subjective Games*

Hitoshi Matsushima+

Faculty of Economics, University of Tokyo

June 4, 1998

(The First Version: May 17, 1997)

* This paper is the revised version of a part of the article entitled "Procedural Rationality and Inductive Learning I: Toward a Theory of Subjective Games", which was written on the basis of the seminar at the Tokyo Center of Economic Research (TCER), Tokyo, January 1997. This paper has a lot of new insights which has never been discussed in the elder versions. Especially, Proposition 3 and Theorem 4, which are the main results of this paper, are newly introduced and proved. The first version was presented at the TCER International Summer Conference, Tokyo, May 1997, and the Eastern Meeting of the Japan Association of Economics and Econometrics, Shiga, May 1997. The brief summary of the first version was given by Section 5 of the article "Bounded Rationality in Economics: A Game Theorist's View," in *the Japanese Economic Review* Vol. 48, No. 3 .

+ Faculty of Economics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan. Fax: (03) 3818-7082. E-mail: hitoshi@e.u-tokyo.ac.jp

Abstract

The repeated situation of two-person symmetric games with random matching is considered, where an individual does not know the objective payoff function, and therefore, formulates the subjective payoff function in every period according to a learning rule defined by Matsushima (1998). We assume that the objective game satisfies a property of *strategic coordination*, which implies that the opponents' choosing the *same* action as the individual is beneficial to the latter. It is shown that, in the long run, the individual comes to misperceive that there exists *no* strategic conflict with the opponents with respect to *fairness* as well as *efficiency*, i.e., formulate the *subjective game* which has the *unique efficient* action vector, and succeeds to implement it as the *strictly dominant* action vector.

JEL Classification Numbers: C70, C90, D43, D80.

1. Introduction

This paper investigates the repeated situation of two-person *symmetric* games with *random matching*, where an individual in a society does *not* know the objective payoff function. In every period, this individual *subjectively* evaluates the payoff function, and the *subjective game* is defined by the combination of the set of actions and this subjective payoff function. In distinction with the subjective game, the true component game will be called the *objective game*.

The individual is modeled as an *adaptive learning rule* which translates past experiences into evaluations for the payoff function, evaluations for the opponent's decisions, and her decisions. In every period, the individual can not observe the opponent's choice of action directly, but can observe the realization of some *random signal* which has information as to the opponent's actual choice of action. The individual is mainly motivated by the maximization of the subjective expected payoff in a myopic way, and *never* experiments with the actions other than the subjective best responses and the action chosen in the last period. We require that the learning rule satisfies the assumptions introduced by the companion paper Matsushima (1998).

We will show that, *in the long run*, the individual may come to misperceive that there exists *no* strategic conflict with the randomly matched opponents with respect to, not only *efficiency*, but also *fairness*: We require that the objective game satisfies a property of *strategic coordination*, which expresses the degree to which the opponent's choosing the *same* action as the individual is beneficial to the latter, and which is satisfied by many popular games in the textbooks for game theory such as the coordination game, the stag-hunt game, and the hawk-dove game, but is not satisfied by the prisoner-dilemma game. With this property, the individual comes to formulate the subjective games in the long run which has the *unique efficient* action vector, and the individual and the opponent succeed to choose it as the *strictly dominant* action vector in the subjective sense.

Most real economic situations are uncertain, complex, and not even well-structured. A real economic agent spends most time to visualize and perceive the situation. As Thomas Schelling (1960) has stressed, formulating the subjective game would be regarded as the most important step for an actual agent in reaching a decision. Selten (1978) has presented an informal model of the human reasoning process which takes into account the cognitive steps such as perception, problem solving, investigation, implementation, and learning, and has emphasized also that the step of perceiving the situation and formulating the subjective model is the most important. Applied game theorists in the 1970's and 1980's have *never* dealt with the question of how players formulate the subjective game and have interpreted game theory in a *naïve* way that the objective game as a full description of a state of the physical world and a state of mind is assumed to be *common knowledge* among players.[1] This naïve interpretation of game theory is sometimes criticized because players are required to be *ideally rational* in an unrealistic way. Rubinstein (1991) presented the *perceptive* interpretation of game theory as the alternative to this naïve interpretation, in which a combination of a game and a strategy profile is viewed as a *common perception* among players.

Despite its unquestionable importance, the investigation of the subjective game is at this time very immature.[2] It must be admitted that we have little knowledge about several important questions such as how the class of possible subjective games is restricted, in what way the subjective game is connected with the objective game, and so on. This paper would be regarded as the first attempt to give clear answers to these questions.

The organization of this paper is as follows. Section 2 gives the model and the basic results. On the basis of the analysis of long-run single-person decision making problems in Matsushima (1998), we derive the following results: (i) The subjective games in the long run essentially *differ* from the objective game. (ii) These subjective games have the *strictly dominant* action vector. (iii) This action vector is equivalent to the objective *maximin* action vector among the set of pure action vectors, and is *not* necessarily a Nash equilibrium in the objective game. (iv) *All* of the popular games in the textbooks for game theory such as the zero-sum game, the prisoner-dilemma game, the coordination game, the stag-hunt game, and the hawk-dove game are *never* viewed as the subjective games.

Section 3 is the main part of this paper. We present approximately a necessary and sufficient condition under which the individual comes to misperceive that there exists no strategic conflict with the opponents with respect to *efficiency*. This is regarded as a *weak* property of strategic coordination, which implies that the objective payoff for the maximin action vector is larger than the average of the minimal payoff and the maximal payoff for the maximin action. Moreover, we present approximately a necessary and sufficient condition under which the individual comes to misperceive that there exists no strategic conflict with respect to *fairness* as well as efficiency. This is regarded as a

---

[1] See Harsanyi (1967,1968). See also Matsushima (1997).

[2] An exception is the work by Kalai and Lehrer (1995), which investigated the related problem according to the Bayesian framework.

stronger property of strategic coordination, which implies that the payoff for the maximin action vector is equal to the maximal payoff for the maximin action with respect to the opponent's choice of action.

Section 4 investigates four examples of the objective games, i.e., *the hawk-dove game, the stag-hunt game, the coordination game*, and *the prisoner-dilemma game*. We explain that the first three examples satisfy the stronger property of strategic coordination, but the last example does not satisfy even the weak property. Moreover, we explain that if the objective game is a coordination game, the individual comes to choose the action which is *"spiteful"* in the objective game. We also introduce *generalized coordination games* as an important subset of objective games which satisfy the stronger property of strategic coordination.

The results of this paper may be regarded as being *negative* by the orthodox game theorists, because all of the popular games such as the zero-sum game, the prisoner-dilemma game, the coordination game, the stag-hunt game, and the hawk-dove game can not be viewed as the subjective games. One of the main reasons why we fails to describe these games as the subjective games is that we may too much simplify the learning rules. Some discussions as to the degree to which the learning rules are simplified have already been given in Matsushima (1998), and will been given in Section 2 of this paper also. We may be requested to give more intensive discussions from different view-points, but we must admit that it is beyond the purpose of this paper. We would like to emphasize that this paper should be regarded as the *benchmark* for the possible progress toward a theory of subjective games in the future.

2. Subjective Games

We consider an individual in a society who is randomly matched with an opponent and plays a two-person *symmetric* noncooperative game (A, u) infinitely many times, where A is the finite set of actions and $u: A^2 \to R$ is the *objective* payoff function. Each of the individual and the randomly matched opponent chooses among A. When the individual chooses action a and the opponent chooses action a', the individual obtains payoff $u(a,a')$ and the opponent obtains payoff $u(a',a)$. The individual knows that the game is symmetric and A is the set of actions, but she does *not* know the objective payoff function u. We define

$$\underline{v}(a) \equiv \min V(a),$$

and

$$\bar{v}(a) \equiv \max V(a),$$

where $\underline{v}(a)$ is the *minimal* payoff for action a and $\bar{v}(a)$ is the *maximal* payoff for action a.

In every period t, the individual chooses action $a(t) \in A$, the matched opponent chooses action $b(t) \in A$, and the individual obtains payoff $v(t) \in R$. The individual can *not directly* observe the opponent's choice of action $b(t)$, but she instead can observe the realization of a *random signal* $\phi(t) \in \Phi$ at the end of period t, where Φ is the finite set of possible signals.

Let h^0 be *the null history* and $H^0 \equiv \{h^0\}$. For every $t \geq 1$, let $h^t \equiv (a(\tau), b(t), v(\tau), \phi(\tau))_{\tau=1}^{t}$ be *a history up to period* t, where $v(\tau) = u(a(\tau), b(\tau))$ for all $\tau = 1,\dots,t-1$. Let H^t be the set of all histories h^t up to period t and $H \equiv \bigcup_{t=0}^{\infty} H^t$.

For every $t \geq 1$, every $h^{t-1} \in H^{t-1}$ and every $b \in A$, the probability that the opponent chooses action b is given by $p^{(h^{t-1})}(b)$. For every $(a,b) \in A^2$ and every $\phi \in \Phi$, the probability that the individual observes signal ϕ when h^{t-1} was realized, the individual chose action a, and the opponent chose action b, is given by $q^{(h^{t-1})}(\phi | a,b)$. The individual does *not* know these probability functions $p^{(\cdot)}$ and $q^{(\cdot)}$.

The individual is modeled by a *learning rule* (d, Γ). Here $\Delta(A)$ is the set of mixed actions and $d: H \to \Delta(A)$ is *a decision rule*. For every $t \geq 1$, every $h^{t-1} \in H^{t-1}$ and every $a \in A$, the individual chooses a with probability $d(h^{t-1})(a)$. It might be appropriate to assume that $d(h^{t-1})$ is *independent* of $(b(\tau))_{\tau=1}^{t-1}$, because the individual can not observe the opponent's choice of action.

$\Gamma \equiv (\Gamma^{(a)})_{a \in A}$ is *an evaluation rule*, where $\Gamma^{(a)} \equiv ((v^{(a,\phi)})_{\phi \in \Phi}, \delta^{(a)})$ is *an evaluation* *rule for action* a, $v^{(a,\phi)}: H \to R$ is *a payoff evaluation rule for* (a,ϕ), $\delta^{(a)}: H \to \Delta(\Phi)$ is called *a probability evaluation rule for action* a, and $\Delta(\Phi)$ is the set of probability functions on Φ. The individual anticipates that she obtains payoff $v^{(a,\phi)}(h^{t-1})$ in period t, when h^{t-1} was realized, she chooses action $a(t) = a$, and she observes signal $\phi(t) = \phi$. The individual also anticipates that she observes signal $\phi(t) = \phi$ with probability $\delta^{(a)}(h^{t-1})(\phi)$ in period t, when h^{t-1} was realized and she chooses action $a(t) = a$. For every $a \in A$, every $t \geq 1$, and every $h^{t-1} \in H^{t-1}$, *the subjective expected payoff* is defined by

$$V^{(a)}(h^{t-1}) \equiv \sum_{\phi\in\Phi} \delta^{(a)}(h^{t-1})(\phi) v^{(a,\phi)}(h^{t-1}).$$

For every $t' \geq 1$ and every $t > t'$, a history $h^t \in H^t$ up to period t is said to be *reachable from* a history $h^{t'} \in H^{t'}$ up to period t' if for every $\tau \in \{t'+1,\ldots,t\}$,

$$d(h^{\tau-1})(a(\tau)) > 0.$$

The model presented above is a special case of long-run single-person decision making problems introduced by the companion paper Matsushima (1998). Similarly to Matsushima (1998), we will require the following seven assumptions.

Assumption 1: For every $t > 1$ and every $h^{t-1} \in H^{t-1}$,

$$d(h^{t-1})(a(t-1)) > 0,$$

$$[V^{(a)}(h^{t-1}) \geq V^{(a')}(h^{t-1}) \text{ for all } a' \in A] \Rightarrow [d(h^{t-1})(a) > 0],$$

and

$$[a \neq a(t-1) \textit{ and } V^{(a)}(h^{t-1}) < V^{(a')}(h^{t-1}) \text{ for some } a' \in A]$$
$$\Rightarrow [d(h^{t-1})(a) = 0].$$

Assumption 2: For every $t \geq 1$, every $h^t \in H^t$, and every $(a,\phi) \in A \times \Phi$,

$$v^{(a,\phi)}(h^t) \geq \min\left[v(t),\ v^{(a,\phi)}(h^{t-1})\right] \text{ whenever } a(t) = a.$$

Assumption 3: For every $t \geq 1$, every $h^t \in H^t$, and every $(a,\phi) \in A \times \Phi$,

$$[(a(t),\phi(t)) \neq (a,\phi)] \Rightarrow [v^{(a,\phi)}(h^t) = v^{(a,\phi)}(h^{t-1})].$$

Assumption 4: For every $\mu > 0$, there exists a positive integer s^* such that for every $(a,v,\phi) \in A \times V \times \Phi$, every $t > s^*$ and every $h^{t-1} \in H^{t-1}$, if

$$(a(\tau), v(\tau), \phi(\tau)) = (a,v,\phi) \text{ for all } \tau = t-s^*,\ldots,t-1,$$

then

$$|v^{(a,\phi)}(h^{t-1}) - v| < \mu,$$

and

$$\delta^{(a')}(h^{t-1})(\phi) > 1 - \mu \text{ for all } a' \in A.$$

Assumption 5: There exists a positive real number $\varepsilon > 0$ such that for every $t \geq 1$, every $h^{t-1} \in H^{t-1}$, and every $(a,b,\phi) \in A^2 \times \Phi$,

$$p^{(h^{t-1})}(b) > \varepsilon,$$

$$q^{(h^{t-1})}(\phi | a,b) > \varepsilon,$$

$$[d(h^{t-1})(a) > 0] \Leftrightarrow [d(h^{t-1})(a) > \varepsilon] \text{ for all } a \in A,$$

and

$$| v^{(a,\phi)}(h^t) - \underline{v}(a^*)| > \varepsilon \text{ for all } a \neq a^*.$$

Assumptions 1 through 4 are the same as Assumptions 1 through 4 presented in Matsushima (1998) respectively. Assumption 5 is the same as Assumption 6 presented in Matsushima (1998).

Let $\hat{a} \in A$ be the *unique maximin action* among pure actions in the sense that

$$\underline{v}(\hat{a}) > \underline{v}(a) \text{ for all } a \neq \hat{a}.$$

As Matsushima (1998) has explained, the uniqueness of maximin action is guaranteed by Assumptions 2 and the fourth inequalities of Assumption 5.

Assumption 6: There exists $\phi \in \Phi$ such that

$$v^{(\hat{a},\phi)}(h^0) \geq \underline{v}(a) \text{ for all } a \neq \hat{a}.$$

The inequalities in Assumption 6 are the same as inequalities (3) presented in Matsushima (1998).

Among Assumptions 1 through 6, Assumptions 1 and 3 will play the crucial roles in deriving the results of this paper. Assumption 1 implies that the individual is mainly motivated by the maximization of the subjective expected payoff in a myopic way, and *never* experiments with the actions other than the subjective best responses and the action chosen in the last period. Assumption 3 implies that the payoff evaluation for a combination of an action and a signal is influenced only by the experiences which the individual obtains when actually choosing this action and observing this signal.

The more detailed discussion about the implications of Assumptions 1 through 6 have been discussed in Matsushima (1998). We will omit them in this paper. I would recommend readers to read Section 3 of Matsushima (1998).

The following assumption is newly required in the paper.

Assumption 7: $\Phi = A$.

On the basis of Assumption 7, we assume that in every period t, *the individual is convinced that the opponent has actually chosen action* b *whenever she observes signal* $\phi(t)=b$. It is needless to say that the observed signal $\phi(t)=b$ may be different from the action $b(t)$ actually chosen by the opponent.

Throughout this paper, let the real number $\varepsilon > 0$ in Assumption 5 be chosen close to zero. It might be the case that the individual can, not perfectly, but *almost perfectly*, monitor the actions which the opponents have chosen, i.e., the probability of observing signal b when the opponent has actually chosen action b, $p^{(h^{t-1})}(b|b)$, is close to unity. In the next section, we will explicitly require this kind of the almost perfection of monitoring ability.

We will define the *subjective game* $(A, \hat{u}^{(h^{t-1})})$ in period t associated with history $h^{t-1} \in H^{t-1}$ by

$$\hat{u}^{(h^{t-1})}: A^2 \to R,$$

and

$$\hat{u}^{(h^{t-1})}(a,b) = v^{(a,b)}(h^{t-1}) \text{ for all } (a,b) \in A^2.$$

In distinction with the subjective games, we will call (A,u) the *objective game*.

The basic proposition of this paper is presented as follows.

Proposition 1: *For every* $\xi \in (0,1]$, *there exists a positive integer* s *such that for every* $t \geq s$, *the following property holds with at least probability* $1-\xi$:

$$d(h^t)(\hat{a}) = 1,$$

$$\hat{u}^{(h^t)}(\hat{a},b) \geq \underline{v}(\hat{a}) - \varepsilon > \hat{u}^{(h^t)}(a,b) \textit{ for all } b \in A \textit{ and all } a \neq \hat{a},$$

and for every $t' > t$ *and every* $h^{t'} \in H^{t'}$ *that is reachable from* h^t,

$$d(h^{t'})(\hat{a}) = 1,$$

$$\hat{u}^{(h^{t'})}(\hat{a},b) \geq \underline{v}(\hat{a}) - \varepsilon,$$

and

$$\hat{u}^{(h^{t'})}(a,b) = \hat{u}^{(h^t)}(a,b) \textit{ for all } b \in A \textit{ and all } a \neq \hat{a}.$$

Proof: Assumption 5 in Matsushima (1998) is automatically satisfied, because the objective payoff function u is independent of time and history. Hence, Theorems 1 and 5 in Matsushima (1998) and the definition of $(A, \hat{u}^{(h^{t-1})})$ prove this proposition.

Q.E.D.

Proposition 1 says that:

1) The individual comes to choose only the objective maximin action $\hat{a}$ in the long run, irrespective of what she anticipates the opponents' choices are.[3]

2) The individual *subjectively* regards the objective maximin action $\hat{a}$ as being *strictly dominant*, i.e.,

$$\hat{u}^{(h')}(\hat{a},b) > \hat{u}^{(h')}(a,b) \text{ for all } b \in A \text{ and all } a \neq \hat{a}.$$

3) All popular games in the text books for game theory such as the zero-sum game, the prisoner-dilemma game, the stag-hunt game, the hawk-dove game and the coordination game do not satisfy the properties in Proposition 1, and therefore, can *not* be perceived as subjective games in the long run.

The companion paper Matsushima (1998) has given several discussions about how restrictive Assumptions 1 through 6 are. Assumption 7 says that the individual can monitor only the opponent's choices of action. In a real environment, however, the individual may monitor the opponent's payoffs also. Suppose, instead of Assumption 7, that $\Phi = A \times V$, and *the individual is convinced that the opponent has actually chosen action a and obtained payoff v whenever she observes signal $\phi(t) = (a, v)$*. Then, the individual may regard v as being relevant to the payoff evaluation for the case that the individual chooses action a and the opponent chooses action a' conversely, and she may be willing to modify this payoff evaluation even though she has never chosen action a. Needless to say, this contradicts Assumption 3 and violates the properties of Proposition 1.

[3] Sarin and Vahid (1997) also explained the similar property.

3. Subjective Efficiency

In this section, we consider the society in which, not only the individual who is the object of the study in this paper, but also many other individuals behave according to their respective learning rules which satisfy Assumptions 1 through 7. We assume that:

(i) The individual can *almost perfectly* monitor which actions the opponents has chosen.

(ii) After continuing to choose the objective maximin action $\hat{a}$ sufficiently many times, its subjective payoff vector $(\hat{u}^{(h^{t-1})}(\hat{a},a))_{a\in A}$ can be approximated by the objective payoff vector $(u(\hat{a},a))_{a\in A}$ with high probability.

We shall confine our attention to histories h^{t-1} such that the properties of Proposition 1 hold and the subjective payoff vector when the decision maker chooses the objective maximin action $\hat{a}$, $(\hat{u}^{(h^{t-1})}(\hat{a},a))_{a\in A}$, is approximated by the objective payoff vector $(u(\hat{a},a))_{a\in A}$, that is,

$$|\hat{u}^{(h^{t-1})}(\hat{a},a)-u(\hat{a},a)|<\varphi \text{ for all } a\in A, \qquad (1)$$

where $\varphi>0$ is close to unity. Such histories are realized with high probability in the long run, because the randomly matched opponents as well as the individual, according to learning rules satisfying Assumptions 1 through 7, typically choose action $\hat{a}$ in the long run.[4]

We define

$$V\equiv \textit{The convex hull of } \{v\in R^2 | \, v=u(a,b) \textit{ for some } (a,b)\in A^2\}.$$

An action vector $(a,b)\in A^2$ is said to be *subjectively efficient* in $(A,\hat{u}^{(h^{t-1})})$, if there exists no $v\in V$ such that $v\neq(\hat{u}^{(h^{t-1})}(a,b),\hat{u}^{(h^{t-1})}(b,a))$ and

$$v\geq(\hat{u}^{(h^{t-1})}(a,b),\hat{u}^{(h^{t-1})}(b,a)).$$

The main purpose of this paper is to clarify *whether the individual perceives the strategic conflict with the opponents to be substantial or not*, that is, to clarify whether the individual regards $(\hat{a},\hat{a})$ as the *unique* subjectively efficient action or not.

Lemma 2: *Suppose that subjective game* $(A,\hat{u}^{(h^{t-1})})$ *satisfies the properties in Proposition 1. Then, for every* $a\neq\hat{a}$ *and every* $b\in A$, (a,b) *is not subjectively efficient in* $(A,\hat{u}^{(h^{t-1})})$.

---

[4] In order to be consistent with the requirement that the learning rules satisfy Assumption 1, we may assume that a non-negligible number of individuals in the society do not allow learning rules satisfying Assumptions 1 through 6, and therefore, choose actions other than action $\hat{a}$.

Proof: The second inequalities of Proposition 1 imply

$$\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) > \hat{u}^{(h^{t-1})}(a,b) \text{ for all } b \in A \text{ and all } a \neq \hat{a},$$

which imply that (a,b) is not subjectively efficient in $(A,\hat{u}^{(h^{t-1})})$.

Q.E.D.

Lemma 2, together with Proposition 1, implies that the objective maximin action vector $(\hat{a},\hat{a})$ is subjectively efficient *among* the set of pure action vectors.

Proposition 3: *The maximin action vector* $(\hat{a},\hat{a})$ *is subjectively efficient in* $(A,\hat{u}^{(h^{t-1})})$, *if*

$$2u(\hat{a},\hat{a}) - 3\varphi > \underline{v}(\hat{a}) + \overline{v}(\hat{a}) - \varepsilon. \tag{2}$$

Moreover, $(\hat{a},\hat{a})$ *is not subjectively efficient in* $(A,\hat{u}^{(h^{t-1})})$, *if the following inequalities and equalities hold:*

$$2u(\hat{a},\hat{a}) < \underline{v}(\hat{a}) + \overline{v}(\hat{a}) - 2\varepsilon - 3\varphi, \tag{3}$$

$$d(h^0)(\hat{a}) = 1,$$

$$v^{(a,b)}(h^0) = \underline{v}(\hat{a}) - 2\varepsilon \textit{ for all } a \neq \hat{a} \textit{ and all } b \in A,$$

and

$$v^{(\hat{a},a)}(h^0) > \underline{v}(\hat{a}) - 2\varepsilon \textit{ for all } a \in A.$$

Proof: First, we prove the former part of this proposition below. The second inequalities of Proposition 1 imply

$$\hat{u}^{(h^{t-1})}(a,\hat{a}) < \underline{v}(\hat{a}) - \varepsilon,$$

Inequalities (1) imply

$$\hat{u}^{(h^{t-1})}(\hat{a},a) < u(\hat{a},a) + \varphi \leq \overline{v}(\hat{a}) + \varphi,$$

and

$$2\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) > 2u(\hat{a},\hat{a}) - 2\varphi.$$

Hence, these inequalities and inequality (2) says

$$2\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) > 2u(\hat{a},\hat{a}) - 2\varphi > \underline{v}(\hat{a}) + \overline{v}(\hat{a}) - \varepsilon + \varphi$$
$$\geq \hat{u}^{(h^{t-1})}(\hat{a},a) + \hat{u}^{(h^{t-1})}(a,\hat{a}).$$

From Lemma 2, we have proven that $(\hat{a},\hat{a})$ is subjectively efficient in $(A,\hat{u}^{(h^{t-1})})$.

Second, we prove the latter part of this proposition below. The inequalities and equalities in this proposition imply that the individual chooses only action $\hat{a}$ from the initial period. Assumption 3 says that

$$u^{(h^{t-1})}(a,b) = \underline{v}(\hat{a}) - 2\varepsilon \text{ for all } a \neq \hat{a} \text{ and all } b \in A.$$

Let $\bar{b} \in A$ be the action such that $u(\hat{a}, \bar{b}) = \bar{v}(\hat{a})$. Inequalities (1) and (3) imply

$$
\begin{aligned}
&2\hat{u}^{(h'^{-1})}(\hat{a}, \hat{a}) < 2u(\hat{a}, \hat{a}) + 2\varphi \\
&< \underline{v}(\hat{a}) + \bar{v}(\hat{a}) - 2\varepsilon - 3\varphi + 2\varphi \\
&< \{\hat{u}^{(h'^{-1})}(\hat{a}, \bar{b}) + \varphi\} + \{\hat{u}^{(h'^{-1})}(\hat{a}, \bar{b}) + 2\varepsilon\} - 2\varepsilon - 3\varphi + 2\varphi \\
&= \hat{u}^{(h'^{-1})}(\hat{a}, \bar{b}) + \hat{u}^{(h'^{-1})}(\hat{a}, \bar{b}).
\end{aligned}
$$

Hence, $(\hat{a}, \hat{a})$ is not subjectively efficient in $(A, \hat{u}^{(h'^{-1})})$.

Q.E.D.

Since the real numbers ε and φ are sufficiently close to zero, inequality (2) is approximately the same as the following inequality:

$$u(\hat{a}, \hat{a}) > \frac{\underline{v}(\hat{a}) + \bar{v}(\hat{a})}{2}. \tag{4}$$

Inequality (4) says that the objective payoff for the maximin action vector is larger than the average of the minimal payoff and the maximal payoff for the maximin action. Inequality (4) is regarded as a *weak property of strategic coordination*, because the opponent's choosing the *same* action as the individual guarantees the latter at least the average of the minimal and maximal payoffs.

Inequality (3) is approximated by the requirement that inequality (4) does not hold. Hence, Proposition 3 says that inequality (4) is approximately a necessary and sufficient condition under which the individual subjectively believes that the individual and the opponent succeeded to implement an efficient allocation in the noncooperative way, and therefore, misperceives that there is *no* strategic conflict with respect to *efficiency*.

Many examples of games popular in the textbooks for game theory such as *the hawk-dove game, the stag-hunt game, and the coordination game* satisfy inequality (4), and therefore, the individual never perceives any strategic conflict with the randomly matched opponents with respect to efficiency, whenever the underlying objective game is some of these games.

An exception is *the prisoner-dilemma game*, which does *not* satisfy inequality (4), and therefore, the individual may perceive that she has a substantial strategic conflict with the opponent with respect to efficiency, provided that the objective game is the prisoner-dilemma game. In the next section, we will investigate these examples in detail.

We will present the main theorem of this paper as follows.

Theorem 4: *The maximin action vector* $(\hat{a}, \hat{a})$ *is the unique action which is subjectively efficient in* $(A, \hat{u}^{(h'^{-1})})$, *if inequality (2) holds and*

$$u(\hat{a},\hat{a}) = \bar{v}(\hat{a}). \quad (5)$$

Moreover, $(\hat{a},\hat{a})$ *is not the unique subjectively efficient action in* $(A,\hat{u}^{(h^{t-1})})$, *if*

$$u(\hat{a},\hat{a}) < \bar{v}(\hat{a}) - \varphi. \quad (6)$$

Proof: First, we prove the former part of this theorem as follows. Inequalities (1) and (5) imply that for every $a \neq \hat{a}$,

$$\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) > \bar{v}(\hat{a}) - \varphi > u(\hat{a},a) + \varphi > \hat{u}^{(h^{t-1})}(\hat{a},a).$$

The second inequalities in Proposition 1imply

$$\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) \geq \underline{v}(\hat{a}) - \varepsilon > \hat{u}^{(h^{t-1})}(a,\hat{a}).$$

These inequalities and Lemma 2 imply that $(\hat{a},\hat{a})$ is the unique subjectively efficient action in $(A,\hat{u}^{(h^{t-1})})$.

Next, we prove the latter part of this theorem as follows. Let $\bar{b} \in A$ be the action such that $u(\hat{a},\bar{b}) = \bar{v}(\hat{a})$. Inequality (6), together with inequalities (1), implies

$$\hat{u}^{(h^{t-1})}(\hat{a},\hat{a}) < u(\hat{a},\hat{a}) + \varphi < \bar{v}(\hat{a}) - \varphi < \hat{u}^{(h^{t-1})}(\hat{a},\bar{b}),$$

which means that $(\hat{a},\hat{a})$ is not the unique subjectively efficient action in $(A,\hat{u}^{(h^{t-1})})$.

Q.E.D.

Inequality (5) implies that the payoff for the maximin action vector is equal to the maximal payoff for the maximin action. Inequality (5) is regarded as *stronger property of strategic coordination* than inequality (4), because the opponent's choosing the same action as the individual *maximizes* the latter individual's objective payoff. Since the real number φ is sufficiently close to zero, inequality (5) implies inequality (2), and inequality (6) is approximated by the requirement that inequality (5) does not hold. Hence, Theorem 4 implies that inequality (5) is approximately a necessary and sufficient condition under which the individual subjectively believes that the opponent and she succeeded to implement the *unique* subjectively efficient allocation in the noncooperative way, and therefore, she misperceives that there is no strategic conflict with respect to, not only efficiency, but also *fairness*.

Many popular games such as *the hawk-dove game, the stag-hunt game, and the coordination game* satisfy, not only inequality (4), but also inequality (5), and therefore, the individual never perceives any strategic conflict with respect to fairness as well as efficiency, whenever the objective game is some of these games.

4. Examples

In this section, we investigate four examples of objective games. The first three examples, i.e., the hawk-dove game, the stag-hunt game, and the coordination game, satisfy inequality (5). In any of these three examples, the individual subjectively believes that she succeeds to implement the unique efficient allocation very frequently. On the other hand, the fourth example, i.e., the prisoner-dilemma game, does not satisfy (4). Hence, the individual may perceive that she fails to implement any efficient allocation, provided that the objective game is the prisoner-dilemma game.

4.1. Hawk-Dove Game

In the hawk-dove game as the objective game presented in Figure 1.1, the mixed action assigning "dove" probability $1/3$ and "hawk" probability $2/3$ is the unique *evolutionary stable strategy* (See Maynard Smith (1982)). The subjective game in the long run is approximated by Figure 1.2. Players come to choose only "dove", which is the maximin action in the objective game. The action profile (dove, dove) is strictly dominant and uniquely efficient in this subjective game, but is neither Nash equilibrium nor efficient in the objective game.

[Figure 1.1]

[Figure 1.2]

4.2. Stag-Hunt Game

In the stag-hunt game in Figure 2.1, (stag, stag) is the *payoff-dominant* Nash equilibrium.[5] If $x < 3/2$, (stag, stag) is *risk-dominant* in the sense of Harsanyi and Selten (1988).[6] If $x > 3/2$, (hare, hare) is the risk-dominant equilibrium. The subjective game is approximated by Figure 2.2, where (hare, hare) is strictly dominant and uniquely efficient.

[Figure 2.1]

[Figure 2.2]

4.3. Coordination Game

In the coordination game presented in Figure 3.1, there are multiple Nash equilibria, i.e., (c,c), (d,d), and the mixed action profile which assigns c probability $\frac{1+2\beta-\gamma}{3-\gamma}$ and d probability $\frac{2-2\beta}{3-\gamma}$. Clearly, (c,c) is the payoff-dominant equilibrium. If $\beta-\gamma>1$, (d,d) is the risk-dominant equilibrium. If $\beta-\gamma<1$, (c,c) is the risk-dominant equilibrium.

If $\beta>\gamma$, the associated subjective game is approximated by Figure 3.2 in which the payoff-dominated equilibrium (d,d) in the objective game is strictly dominant and uniquely efficient. On the other hand, if $\beta<\gamma$, the associated subjective game is approximated by Figure 3.3 in which the payoff-dominant equilibrium (c,c) in the

---

[5] A Nash equilibrium $(a_i) \in \times_{i\in N} A$ in $G=(N,(A_i,u_i))$ is *payoff-dominant* if for every Nash equilibrium $(a'_i) \in \times_{i\in N} A_i$ and every $i \in N$, $u_i((a_j)) \geq u_i((a_j))$.

[6] A Nash equilibrium (a,a) in a 2×2 symmetric game such that $A_1 = A_2 = \{a,b\}$ is *risk-dominant* if $u_1(b,b)-u_1(a,b) < u_1(a,a)-u_1(b,a)$.

objective game is strictly dominant and uniquely efficient. If $0 < \beta - \gamma < 1$, the strictly dominant equilibrium in the subjective game is not equal to the risk-dominant equilibrium in the objective game. In the coordination game, the *"spiteful"* action in the sense that a player's choosing this action makes the opponent's payoff worse than her own payoff will survive.

[Figure 3.1]

[Figure 3.2]

[Figure 3.3]

We will generalize the class of coordination games in the following way. An objective game (A, u) is said to be a *generalized coordination game*, if for every $a \in A$ and every $a' \neq a$,

$$u(a,a) > u(a,a').$$

This inequalities imply that, given that the individual chooses any action, the opponent's choosing the same action as the individual maximizes the individual's objective payoff. Clearly, the coordination game in Figure 3.1 is a generalized coordination game.[7] We must note that any generalized coordination game satisfies inequality (5), and therefore, the individual comes to misperceive that there exists no strategic conflict with the opponents with respect to fairness as well as efficiency, provided that the underlying objective game is a generalized coordination game.

---

[7] We must note that a generalized coordination game is not necessarily a coordination game in the ordinary sense, because players' choosing the same action may not necessarily be a Nash equilibrium.

4.4. Prisoner-Dilemma Game

In the prisoner-dilemma game in Figure 4.1, (defection, defection) is the *strictly dominant* action vector. The subjective game is approximated by Figure 4.2 where (defection, defection) is the strictly dominant equilibrium also. If $L_2 > -2$, then the average of $u(collusion,\ defection)$ and $u(defection,\ collusion)$, i.e., $\frac{2+L_2}{2}$, is more than zero, and therefore, (defection, defection) is not subjectively efficient.

[Figure 4.1]

[Figure 4.2]

References

Harsanyi, J. (1967, 1968): "Games with Incomplete Information Played by Bayesian Players," *Management Science* 14, 159-182, 320-334, 486-502.

Harsanyi, J. and R. Selten (1988): *A General Theory of Equilibrium Selection in Games*, Cambridge: MIT Press.

Kalai, E. and E. Lehrer (1995): "Subjective Games and Equilibria," *Games and Economic Behavior* 15, 1-26.

Matsushima, H. (1997a): "Bounded Rationality in Economics: A Game Theorist's View," *Japanese Economic Review* Vol. 48, No. 3.

Matsushima, H. (1997b): "Procedural Rationality and Inductive Learning I: Towards a Theory of Subjective Games," Discussion Paper 97-F-21, Faculty of Economics, University of Tokyo.

Matsushima (1998): "Learning about Stochastic Payoff Structure," mimeo.

Maynard Smith, J. (1982): *Evolution and the Theory of Games*, Cambridge University Press.

Rubinstein, A. (1991): "Comments on the Interpretation of Game Theory," *Econometrica* 59, 909-924.

Sarin, R. and F. Vahid (1997): "Payoff Assessments without Probabilities: A Simple Dynamic Model of Choice," mimeo.

Schelling, T. (1960): *The Strategy of Conflict*, Cambridge: Harvard University Press.

Selten, R. (1978): "The Chain Store Paradox," *Theory and Decision* 9, 127-159.

	dove	hawk
dove	1, 1	0, 3
hawk	3, 0	-1, -1

Figure 1.1: Hawk-Dove Game as the Objective Game

	dove	hawk
dove	1, 1	0, L_1
hawk	L_1, 0	L_2, L_2

Figure 1.2: The Subjective Game
($L_1 < 0$ *and* $L_2 < 0$)

	stag	hare
stag	3, 3	0, x
hare	x 0	x, x

Figure 2.1: Stag-Hunt Game as the Objective Game

$(0 < x < 3)$

	stag	hare
stag	L_1, L_1	L_2, x
hare	x, L_2	x, x

Figure 2.2: The Subjective Game

$(L_1 < x \text{ and } L_2 < x)$

	c	d
c	2, 2	γ, β
d	β, γ	1, 1

Figure 3.1: Coordination Game as the Objective Game
$(\beta < 1, \gamma < 1)$

	c	d
c	L_1, L_1	L_2, β
d	β, L_2	1, 1

Figure 3.2: The Subjective Game
$(\gamma < \beta < 1, L_1 < \beta \text{ and } L_2 < \beta)$

	c	d
c	2, 2	γ, L_1
d	L_1, γ	L_2, L_2

Figure 3.3: The Subjective Game
$(\beta < \gamma < 1, L_1 < \gamma \textit{ and } L_2 < \gamma)$

	collusion	defection
collusion	1, 1	-3, 2
defection	2, -3	0, 0

Figure 4.1: Prisoner-Dilemma Game as the Objective Game

	collusion	defection
collusion	L_1, L_1	L_2, 2
defection	2, L_2	0, 0

Figure 4.1: The Subjective Game

($L_1 < 0$ *and* $L_2 < 0$)